AN EXPLICIT REALIZATION OF LOGARITHMIC MODULES FOR THE VERTEX OPERATOR ALGEBRA $\mathcal{W}_{p,p'}$

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ABSTRACT. By extending the methods used in our earlier work, in this paper, we present an explicit realization of logarithmic $\mathcal{W}_{p,p'}$–modules that have $L(0)$ nilpotent rank three. This was achieved by combining the techniques developed in [AM3] with the theory of local systems of vertex operators [LL]. In addition, we also construct a new type of extension of $\mathcal{W}_{p,p'}$, denoted by $\mathcal{V}$. Our results confirm several claims in the physics literature regarding the structure of projective covers of certain irreducible representations in the principal block. This approach can be applied to other models defined via a pair screenings.

1. INTRODUCTION

Let p, p' be coprime ≥ 2 and $c_{p,p'} = 1 - \frac{6(p-p')^2}{pp'}$ the minimal central charge for the Virasoro algebra. As shown in [FGST1], [FGST2], there exists an important extension of the rational vertex operator algebra $L(c_{p,p'}, 0)$, usually denoted by $\mathcal{W}_{p,p'}$, whose representation theory shares many similarities with representations of certain finite dimensional quantum group $\mathfrak{g}_{p,p'}$ at roots of unity. In the same series of paper, it was predicted that these vertex algebras are in fact $\mathcal{W}$-algebras with finitely many irreps. From the vertex algebra point of view, many claims in [FGST1], [FGST2] are still conjectures, including the part on the classification of irreducible representations. Also, conjecturally, $\mathcal{W}_{p,p'}$ is expected to be a C_2-cofinite vertex algebra, combining into an exact sequence

$$0 \longrightarrow \overline{\mathcal{W}} \longrightarrow \mathcal{W}_{p,p'} \longrightarrow L(c_{p,p'}, 0) \longrightarrow 0,$$

where $\overline{\mathcal{W}}$ is the maximal simple ideal in $\mathcal{W}_{p,p'}$. The vertex algebra $\mathcal{W}_{p,p'}$ has some common features with a better understood (simple) triplet vertex algebra $\mathcal{W}_p$ studied intensively in many works (see [AM2], [NT] and references therein), but there are many differences. In addition to $\mathcal{W}_{p,p'}$ not being simple, it is also expected that (cf. [FGST1], [W])

(i) The category $\mathcal{W}_{p,p'} - \mathrm{Mod}$ contains $p + p' - 2$ "thin" blocks, $\frac{(p-1)(p'-1)}{2}$ "thick" blocks and two semi-simple blocks.
(ii) Characters (i.e. modified graded dimensions) of irreducible $\mathcal{W}_{p,p'}$-modules involve modular forms of weight two [FGST1].

In our recent works [AM4, AM5], motivated primarily by [FGST1] and [FGST2], we started to investigate the representation theory of vertex algebra $\mathcal{W}_{p,p'}$. There our focus was mainly on $(p, 2)$–minimal models, which enjoy some special properties absent for general p'. In [AM5], we

showed that $\mathcal{W}_{p,2}$ is C_2–cofinite and irrational (for every odd $p \geq 3$). We also classified all irreducible $\mathcal{W}_{p,2}$-modules and described the structure of the corresponding Zhu algebras. As a consequence we give evidence that $\mathcal{W}_{p,2}$ contains logarithmic modules which have $L(0)$–nilpotent rank two or three, but not higher. We expect that the vertex algebra $\mathcal{W}_{p,p'}$ shares similar properties, but this is still an open problem. Both [AM4] and [AM5] deal with explicit realizations of irreducible modules, but no logarithmic $\mathcal{W}_{p,p'}$-modules were constructed there. In fact, the only rigorous result for rank 3 logarithmic modules comes from [AM6], where we used Zhu's algebra to prove the existence of logarithmic module of rank 3 but only for $\mathcal{W}_{3,2}$-algebra.

Motivated by a similar circle of ideas, in [AM3] (and shortly afterwords independently in [NT]) we used vertex operator algebra theory to give an explicit realization of some projective covers of irreducible modules for the triplet vertex algebra $\mathcal{W}_p$ which are logarithmic of $L(0)$–nilpotent rank two (we stress that the paper [AM3] deals only with special projective covers - the rest was constructed in [NT]). The key idea was to realize logarithmic modules (in non-semisimple blocks) on the direct sum of two irreducible modules for the lattice vertex algebra containing the triplet vertex algebra. The methods in [AM3], [NT] can be also used to handle other models defined via a single screening. Thin blocks in $\mathcal{W}_{p,p'} - \mathrm{Mod}$ are expected to have properties similar to those for the triplet vertex algebra $\mathcal{W}_p$, while thick blocks - including the principal block - should involve more complicated indecomposable modules. From the point of view of construction of projective covers, thin blocks presumably can be handled by the methods developed in [AM3] and [NT].

In the present work, generalizing [AM3], we shall present an explicit construction of some logarithmic $\mathcal{W}_{p,p'}$-modules in certain thick blocks which have $L(0)$–nilpotent rank three.

Let us explain the main results and concepts of our construction. In Section 2 we recall the definition of the vertex algebra $\mathcal{W}_{p,p'}$ as the intersection of kernels of two commuting screening operators. This realization is useful for construction of irreducible representations, but in order to construct logarithmic representations we need to embed $\mathcal{W}_{p,p'}$ into more complicated vertex algebra. In Section 3 we present certain results on extended vertex algebras. We construct the extended vertex algebra

$$\mathcal{W}_{p,p'} \hookrightarrow V(p,p'),$$

and its natural module $MV(p,p')$. The vertex algebra $V(p,p')$ is realized as a direct sum of four irreducible modules for the lattice vertex algebra V_L (cf. Section 3). In Section 5 we construct a non-trivial homomorphism $\Phi : \mathcal{W}_{p,p'} \to \mathcal{V}$ where $\mathcal{V}$ is a vertex algebra realized as local fields on $V(p,p')$. In Proposition 5.3 we describe the structure of the vertex algebra $\mathcal{V}$ as a $\mathcal{W}(p,p')$–module. It is interesting to notice that the vertex algebra $\mathcal{V}$, as a $\mathcal{W}_{p,p'}$–module, is not finitely generated, and gives a new type of extension of $\mathcal{W}_{p,p'}$. In the case $p' = 2$ we are able to describe the structure of $\mathcal{V}$. By applying the construction from [AM3], in Theorem 6.4 we get construction of logarithmic $\mathcal{V}$–modules. Important consequence is:

Theorem 1.1. *The* $\mathcal{W}_{p,p'}$*–modules* $\overline{V(p,p')}$ *and* $\overline{MV(p,p')}$ *have* $L(0)$*–nilpotent rank* 3.

When specialized to the case of $\mathcal{W}_{3,2}$-algebra, we found precise agreements with the structure of projective covers $\mathcal{P}(1)$ and $\mathcal{P}(5)$ proposed in [GRW2]. Hence

Conjecture 1.2. *The logarithmic module $\overline{V(3,2)}$ (resp. $\overline{MV(3,2)}$) is a projective cover of the irreducible module $\mathcal{W}(1)$ (resp.$\mathcal{W}(5)$), where the notation $\mathcal{W}(i)$ was borrowed from* [AM4].

Two modules in Theorem 1.1 are distinguished in many aspects, and are expected to be intertwined through a simple current module. To explain this, recall that in [AM4] we found out that $\mathcal{W}_{p,2}$ has an important vertex superalgebra extension generated from two primary vectors inside

(1) $$M = \mathrm{Ker}_{V_{L+\alpha/2}} Q \cap \mathrm{Ker}_{V_{L+\alpha/2}} \widetilde{Q},$$

For general p and p', the space M is a module for $\mathcal{W}_{p,p'}$. Our next result is

Theorem 1.3. *There is a nonzero intertwining operator of type*

$$\begin{pmatrix} \overline{MV(p,p')} \\ M \ \overline{V(p,p')} \end{pmatrix}.$$

Again, in the case of $(3,2)$-algebra the above theorem can be made even more precise, because we have $\mathcal{W}(7) = M$.

Corollary 1.4. *Assuming Conjecture 1.2, the intertwining operator in Theorem 1.3 gives a non-trivial intertwining operator of type*

$$\begin{pmatrix} \mathcal{P}(5) \\ \mathcal{W}(7) \ \mathcal{P}(1) \end{pmatrix}.$$

Intertwining operators of this type were predicted in [GRW2]. We also refer to [W], [PR], [R1], [R2] for more about fusion rings of general $\mathcal{W}_{p,p'}$-modules. Result in those papers are in agreement with our Theorem 1.3.

Although we are still far from constructing all indecomposable projective modules as predicted in (say) [W], we believe that the methods here together with the technique of "powers" of screenings [NT], [AM5] will be sufficient to produce the remaining indecomposable modules and related intertwining operators.

We finish here by noting that almost all techniques in this paper can be used to study indecomposable modules for other vertex operator (super)algebras defined through a pair of commuting screenings.

Acknowledgment: We thank J. Rasmussen for a correspondence.

2. THE VERTEX ALGEBRA $\mathcal{W}_{p,p'}$

In this part we briefly recall the construction of $\mathcal{W}_{p,p'}$ via screening operators.

Define the rank one even lattice

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2pp',$$

where $p, p' \in \mathbb{Z}_{>0}$, $p, p' \geq 2$ and p and p' are relatively prime. Let V_L be the associated (rank one) lattice vertex algebra [LL]. No central extension is needed for the lattice part so we have

$$V_L = M(1) \otimes \mathbb{C}[L].$$

The vertex algebra V_L is a subalgebra of the generalized vertex algebra $V_{\widetilde{L}}$, where

$$\widetilde{L} = \mathbb{Z}\frac{\alpha}{2pp'}$$

is the dual lattice of L. Let us denote the (generalized) vertex operator map in $V_{\widetilde{L}}$ by $Y_{V_{\widetilde{L}}}$, so that

$$Y_{V_{\widetilde{L}}}(a,x) = \sum_{n\in\mathbb{Q}} a_n x^{-n-1}.$$

Define

$$\omega = \frac{1}{4pp'}\alpha(-1)^2\mathbf{1} + \frac{p-p'}{2pp'}\alpha(-2)\mathbf{1},$$

and $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. It can be easily showed that ω is a conformal vector with central charge

$$c_{p,p'} = 1 - \frac{6(p-p')^2}{pp'}.$$

We invoke the relevant screening operators [FGST1], [AM4]:

$$Q = e_0^{\alpha/p'} = \mathrm{Res}_z Y_{V_{\widetilde{L}}}(e^{\alpha/p'}, z), \quad \widetilde{Q} = e_0^{-\alpha/p} = \mathrm{Res}_z Y_{V_{\widetilde{L}}}(e^{-\alpha/p}, z).$$

The screening operators Q and $\widetilde{Q}$ enable us to define certain vertex subalgebras of V_L. Define as in [AM4], [FGST1]

$$\boxed{\mathcal{W}_{p,p'} = \mathrm{Ker}_{V_L}\widetilde{Q} \cap \mathrm{Ker}_{V_L}Q.}$$

Remark 1. *We still do not have a proof that a strongly generating set for $\mathcal{W}_{p,p'}$ consists of the conformal vector ω and three primary fields of conformal weight $(2p-1)(2p'-1)$ (there is conjecture in [FGST1] about this structure). For $p' = 2$ a strong generating set was found in [AM4].*

Several $\mathcal{W}_{p,p'}$-modules in this paper will be *logarithmic*, that is they are not diagonalizable with respect to the $L(0)$ Virasoro generator [AM1], [HLZ]. We also say that a $\mathcal{W}_{p,p'}$-module M is of $L(0)$-nilpotent rank k if $L(0)$ admits a Jordan block of size k, but not higher than k. Equivalently,

$$(L(0) - L_{ss}(0))^k = 0, \quad \text{and} \quad (L(0) - L_{ss}(0))^{k-1} \neq 0$$

on M, where L_{ss} is the semisimple part of $L(0)$.

Motivated by (conjectural) formulas of characters of irreducible $\mathcal{W}_{p,p'}$-modules [FGST1] we are naturally led to

Conjecture 2.1. *Every f.g. $\mathcal{W}_{p,p'}$-module is of $L(0)$-nilpotent rank at most 3.*

3. EXTENDED VERTEX ALGEBRA $V(p, p')$

We first define semi-direct product of a vertex algebra (V, Y) and its module (M, Y_M, d). Let

$$\overline{V} = V \oplus M.$$

We let

$$Y(v_1 + m_1, z)(v_2 + m_2) = Y_V(v_1, z)v_2 + Y_M(v_1, z)m_2 + e^{zd}Y_M(v_2, -z)m_1, \quad v_1, v_2 \in V, \ m_1, m_2 \in M.$$

The next results is known [LL], [AM3].

Lemma 3.1. *The space $\overline{V}$ has a natural vertex algebra structure.*

If in addition, V is a vertex operator algebra and M is integrally graded, then $\overline{V}$ is vertex operator algebra.

The vertex operator algebra just described (which is certainly not simple), has potentially many indecomposable modules. The next result gives a reasonable large source of examples.

Lemma 3.2. *Let $\overline{V}$ be as above, M_2 and M_3 be V-modules, and $\mathcal{Y}(\cdot, z) \in I\binom{M_3}{M M_2}$ an intertwining operator with integral powers of z. Then*

$$Y_{M_2 \oplus M_3}(v + m, z)(m_2 + m_3) = Y(v, z)(m_2 + m_3) + \mathcal{Y}(m, z)m_2,$$

$v \in V$, $m_i \in M_i$, $m \in M$, defines an $\overline{V}$-module structure on the space $M_2 \oplus M_3$.

Remark 2. *Observe that on the same vector space $M_2 \oplus M_3$ can have several nonisomorphic $\overline{V}$-module structures.*

Consider now a $\overline{V}$-module $M_2 \oplus M_3$ constructed as in Lemma 3.2 (possibly with the additional assumption that the weights of M_2 and M_3 are integral). Then we can apply again the first lemma to get a vertex (operator) algebra structure on

$$\overline{\overline{V}} = V \oplus M_1 \oplus M_2 \oplus M_3.$$

We are also interested in $\overline{\overline{V}}$-modules. One way to do produce modules is to consider the space

$$\overline{\overline{M}} = M_4 \oplus M_5 \oplus M_6 \oplus M_7,$$

where $M_4 \oplus M_5$ and $M_6 \oplus M_7$ are $\overline{V}$-modules and to employ Lemma 3.2 via an intertwining operator

$$(2) \qquad \mathcal{Y} \in I\begin{pmatrix} M_6 \oplus M_7 \\ M_2 \oplus M_3 \ \ M_4 \oplus M_5 \end{pmatrix}$$

where $\mathcal{Y}$ is viewed as intertwining operator among $\overline{V}$-modules. It would be desirable to get an intertwining operator of this type directly from V-modules. The following result comes handy.

Lemma 3.3. *Let $\mathcal{Y}_1 \in I\begin{pmatrix} M_6 \oplus M_7 \\ M_2 \ \ M_4 \oplus M_5 \end{pmatrix}$ and $\mathcal{Y}_2 \in I\begin{pmatrix} M_6 \oplus M_7 \\ M_3 \ \ M_4 \oplus M_5 \end{pmatrix}$ be V-intertwining operators with integral powers such that:*

$$\mathcal{Y}_2(\cdot, z)(M_5) = 0 \ \text{ and } \ \mathcal{Y}_2(\cdot, z)(M_4 \oplus M_5) \subset M_7((z)),$$

Then $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2$ defines an intertwining operator of type as in (2). Consequently $\overline{\overline{M}}$ has a $\overline{\overline{V}}$-module structure.

The next goal is to apply all previous results in the setup of V_L-modules. First observe that V_L is a vertex operator algebra with central charge $c_{p,p'}$ and that $V_{L-\alpha/p}$, $V_{L+\alpha/p'}$ and $V_{L+\alpha/p'-\alpha/p}$ are simple V_L-modules with integral weights.

Then we equip

$$V_L \oplus V_{L-\alpha/p}$$

with a vertex operator algebra structure as in Lemma 3.1. Because of a generalized vertex operator algebra structure on $(V_{\widetilde{L}}, Y_{V_{\widetilde{L}}}, \mathbf{1})$, the restriction of Y on $V_{L+\lambda_1}$ when acting on $V_{L+\lambda_2}$ gives an intertwining operator of type $\binom{V_{L+\lambda_1+\lambda_2}}{V_{L+\lambda_1} V_{L+\lambda_2}}$. Moreover, it is known that this space is one-dimensional. By using this fact and Lemma 3.2, we equip

$$V_{L+\alpha/p'} \oplus V_{L+\alpha/p'-\alpha/p}$$

with a $V_L \oplus V_{L-\alpha/p}$–module. Therefore on the space

$$\boxed{V(p,p') = V_L \oplus V_{L-\alpha/p} \oplus V_{L+\alpha/p'} \oplus V_{L+\alpha/p'-\alpha/p},}$$

there is a structure of a vertex operator algebra, such that the associated vertex operators can be reconstructed from the generalized vertex algebra $(V_{\widetilde{L}}, Y_{V_{\widetilde{L}}})$ as follows: Let $u, v \in V(p,p')$,

$$u = u_1 + u_2 + u_3 + u_4, \quad v = v_1 + v_2 + v_3 + v_4$$

$$u_1, v_1 \in V_L, \quad u_2, v_2 \in V_{L-\alpha/p}, \quad u_3, v_3 \in V_{L+\alpha/p'}, \quad u_4, v_4 \in V_{L+\alpha/p'-\alpha/p}.$$

Then

$$Y(u,z)v = Y_{V_{\widetilde{L}}}(u_1,z)v + Y_{V_{\widetilde{L}}}(u_2,z)(v_1+v_3) + Y_{V_{\widetilde{L}}}(u_3,z)(v_1+v_2) + Y_{V_{\widetilde{L}}}(u_4,z)v_1.$$

Now we discuss $V(p,p')$-module. Since we would like to invoke Lemma 3.3 we are heavily constrained with the choice of M_i, $4 \leq i \leq 7$. After a short analysis, besides $V(p,p')$ we narrow down another "interesting" [1] example.

Proposition 3.4. *The space*

$$MV(p,p') = V_{L+\alpha/2} \oplus V_{L+\alpha/2-\alpha/p} \oplus V_{L+\alpha/2+\alpha/p'} \oplus V_{L+\alpha/2+\alpha/p'-\alpha/p}.$$

has a natural $V(p,p')$-module structure.

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[1]We can certainly construct more examples by taking $\mathcal{Y}_1$ or $\mathcal{Y}_2$ to act trivially.

4. Weak vertex operators and "local subsets"

In this section we briefly outline basic facts about "local subsets" or "local systems" following [LL], but with some modifications. Let W be a complex vector space, equipped with an action of a derivation operator d. In [LL], a weak vertex operator (i.e. a "field") on (W, d) is a formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \mathrm{End}(W)[[z, z^{-1}]]$$

such that $a(z)b \in \mathbb{C}((z))$ for all $b \in W$ and

$$(3) \qquad [d, a(z)] = \frac{d}{dx}a(x).$$

In many constructions of vertex algebras and their modules, the operator d comes naturally. But in this paper we do not assume (3). Actually, as we shall see shortly, there is no (obvious) operator d satisfying (3), so we only work on W and not on (W, d).

Denote by $\mathcal{E}(W) = Hom(W, W((z)))$ and set

$$Y_{\mathcal{E}}(a(z), x_0)b(z) = \sum_{n \in \mathbb{Z}} a(z)_n b(z) z_0^{-n-1},$$

where n–th product of the fields $a(z)$ and $b(z)$ is defined by

$$(4) \qquad a(z)_n b(z) = \mathrm{Res}_{z_1} \left((z_1 - z)^n a(z_1)b(z) - (-z + z_1)^n b(z)a(z_1) \right).$$

We should refer to the triple $(\mathcal{W}, Y_{\mathcal{E}}, 1_W)$ as the canonical weak vertex algebra associated to W (or (W, d) if there is d such that (3) holds).

Weak vertex operators $a(z)$ and $b(z)$ are said to be mutually local if there exists a nonnegative integer k such that

$$(z_1 - z_2)^k a(z_1)b(z_2) = (z_1 - z_2)^k b(z_2)a(z_1).$$

If a weak vertex operator is local with itself it is called vertex operator.

Definition 4.1. *A subset or a subspace S of $\mathcal{E}(W)$ is said to be local if all weak vertex operators in S are mutually local. A local subalgebra $\mathcal{E}(W)$ is a weak vertex subalgebra which is local.*

In fact, any local subalgebra of $\mathcal{E}(W)$ is a vertex algebra. In particular, any maximal local subspace of $\mathcal{E}(W)$ is a vertex algebra with W as a faithful module. Finally, we have a result form [LL]:

Theorem 4.2. *Suppose S is a set of mutually local vertex operator on W, that is a local subset of $\mathcal{E}(W)$. Then the weak vertex subalgebra $\langle S \rangle$ generated by S is a vertex algebra, with W as a natural faithful $\langle S \rangle$-module, and the set*

$$\{u^{(1)}(z)_{n_1} \cdots u^{(r)}(z)_{n_r} I_W \mid r \in \mathbb{Z}_{>0}, \ u^{(1)}(z), \ldots, u^{(r)}(z) \in S, \ n_1, \ldots, n_r \in \mathbb{Z}\},$$

spans all of $\langle S \rangle$.

The module structure in the previous theorem is given by

$$(5) \qquad Y_W(a(z), x_0) = a(x_0).$$

5. AN EXTENSION OF $\mathcal{W}_{p,p'}$

In this section we shall construct an extension on the vertex algebra $\mathcal{W}_{p,p'}$ on a vertex algebra of local fields acting on a logarithmic representations constructed in [AM3].

Consider again the vertex algebra

$$V(p,p') = V_L \oplus V_{L-\alpha/p} \oplus V_{L+\alpha/p'} \oplus V_{L+\alpha/p'-\alpha/p}$$

and a family of weak vertex operators (i.e. fields)

$$S = \{e^{\alpha/p'}(z), \widetilde{Y}(v,z) \mid v \in \mathcal{W}_{p,p'}\},$$

acting on $V(p,p')$, where

(6) $$\widetilde{Y}(v,z) = Y(\Delta(e^{-\alpha/p}, z)v, z)$$

(7) $$\Delta(v,z) = z^{v_0} \exp\left(\sum_{n=1}^{\infty} \frac{v_n}{-n}(-z)^{-n}\right),$$

and

$$e^{\alpha/p'}(z) = Y(e^{\alpha/p'}, z),$$

as in Section 2. By using result from [Li] and [AM3], we can easily prove

Proposition 5.1. S is a local subset on $V(p,p')$.

Therefore, according to [LL], the set S generates a vertex algebra

$$\mathcal{V} := \langle S \rangle.$$

Let $Y_{\mathcal{V}}$ be the corresponding vertex operator map.

Remark 3. *The vertex algebra $\mathcal{V}$ is invariant under the differentiation operator $D = \frac{d}{dz}$ such that*

$$a(z) \mapsto a(z)_{-2} \cdot I(z) \quad (= Da(z)),$$

where $I(z)$ is the identity map.

Then

$$\Phi : \mathcal{W}_{p,p'} \longrightarrow \mathcal{V}$$
$$v \mapsto \widetilde{Y}(v,z), \quad v \in \mathcal{W}_{p,p'}$$

is an injective vertex algebra homomorphism. Moreover, $V(p,p')$ is a $\mathcal{V}$–module, and therefore $V(p,p')$ is a $\mathcal{W}_{p,p'}$–module.

Recall that (cf. [AM3])

$$\widetilde{L(z)} = \widetilde{Y}(\omega, z) = \sum_{n \in \mathbb{Z}} \widetilde{L(n)} z^{-n-2}$$

$$= Y(\omega, z) + z^{-1} e^{-\alpha/p}(z) = L(z) + z^{-1} e^{-\alpha/p}(z).$$

Remark 4. *Observe that (3) does not hold in general if we take $d = \widetilde{L(-1)}$ or perhaps $d = L(-1)$, and $a(z) \in S$.*

Lemma 5.2. *We have:*

(8) $$e^{\alpha/p'}(z)_1 \widetilde{L(z)} = e^{\alpha/p'}(z) + z^{-1}e^{\alpha/p'-\alpha/p}(z)$$

(9) $$(z^{-1}e^{\alpha/p'-\alpha/p}(z))_1 \widetilde{L(z)} = 0,$$

(10) $$e^{\alpha/p'}(z)_0 \widetilde{L(z)} = \frac{p}{p-p'}(z^{-1}De^{\alpha/p'-\alpha/p}(z)),$$

(11) $$(z^{-1}e^{\alpha/p'-\alpha/p}(z))_0 \widetilde{L(z)} = -z^{-1}(De^{\alpha/p'-\alpha/p}(z)).$$

Proof. The proof follows easily by using (4). We only prove the first formula for illustration; other formulas are proven the same way.

$$
\begin{aligned}
e^{\alpha/p'}(z)_1 \widetilde{L(z)} &= \mathrm{Res}_{z_1}(z-z_1)[\widetilde{L(z)}, e^{\alpha/p'}(z_1)] \\
&= \mathrm{Res}_{z_1}(z-z_1)[L(z) + z^{-1}e^{-\alpha/p}(z), e^{\alpha/p'}(z_1)] \\
&= \mathrm{Res}_{z_1}z[L(z) + z^{-1}e^{-\alpha/p}(z), e^{\alpha/p'}(z_1)] - \mathrm{Res}_{z_1}z_1[L(z) + e^{-\alpha/p}(z), e^{\alpha/p'}(z_1)] \\
&= \mathrm{Res}_{z_1}z[Y(e^{-\alpha/p}, z), Y(e^{\alpha/p'}, z_1)] - \mathrm{Res}_{z_1}z_1[L(z) + z^{-1}e^{-\alpha/p}(z), e^{\alpha/p'}(z_1)] \\
&= -Y(e_0^{\alpha/p'}e^{-\alpha/p}, z) + z^{-1}Y(e_1^{\alpha/p'}e^{-\alpha/p}, z) + Y(e_0^{\alpha/p'}e^{-\alpha/p}, z) - [L(z), e_1^{\alpha/p'}] \\
&= z^{-1}Y(e^{\alpha/p'-\alpha/p}, z) + Y(e^{\alpha/p'}, z) \\
\end{aligned}
$$

(12) $$= z^{-1}e^{\alpha/p'-\alpha/p}(z) + e^{\alpha/p'}(z).$$

$\square$

Set $\nu_{p,p'} = \frac{p}{p-p'}$. Define

$$\widetilde{H(z)} = e^{\alpha/p'}(z) + \nu_{p,p'}z^{-1}e^{\alpha/p'-\alpha/p}(z);$$
$$H(z) = e^{\alpha/p'}(z) + z^{-1}e^{\alpha/p'-\alpha/p}(z).$$

Then Lemma 5.2 implies that

(13) $$\widetilde{H(z)}_0 \widetilde{L(z)} = 0,$$

(14) $$\widetilde{H(z)}_1 \widetilde{L(z)} = H(z),$$

(15) $$\widetilde{H(z)}_n \widetilde{L(z)} = 0 \qquad (n \geq 2),$$

(16) $$\widetilde{L(z)}_1 H(z) = H(z).$$

(17)

Note also that $\widetilde{L(z)}_0 H(z) \neq DH(z)$.

By using commutator formulae in the vertex algebra $\mathcal{V}$ we get

(18) $$[\widetilde{L(z)}_{n+1}, \widetilde{H(z)}_m] = H(z)_{m+n}.$$

The vertex algebra $\mathcal{V}$ is an extension of $\mathcal{W}_{p,p'}$. Let us give a description of $\mathcal{V}$ as an $\mathcal{W}_{p,p'}$–module.

Let $\mathcal{I}$ be the ideal in $\mathcal{V}$ generated by $\mathcal{W}_{p,p'}.z^{-1}e^{\alpha/p'-\alpha/p}(z)$.

Proposition 5.3. *As a $\mathcal{W}_{p,p'}$–module, $\mathcal{I}$ is isomorphic to a direct sum of infinitely many cyclic modules generated by singular vectors $\{z^{-k}e^{\alpha/p'-\alpha/p}(z), \quad k \in \mathbb{Z}_{>0}\}$, i.e.,*

$$\mathcal{I} = \bigoplus_{k=1}^{\infty} \mathcal{I}_k, \quad \mathcal{I}_k = \mathcal{W}_{p,p'}.z^{-k}e^{\alpha/p'-\alpha/p}(z).$$

Proof. The operator $U = L(z)_0 - D$ commutes with the $\mathcal{W}_{p,p'}$-action, so it defines an intertwining operator between $\mathcal{I}$ and itself. Notice first

$$\widetilde{L(z)}_0(z^{-1}e^{\alpha/p'-\alpha/p}(z)) = z^{-1}De^{\alpha/p'-\alpha/p}(z) \in \mathcal{I}.$$

Therefore

$$z^{-2}e^{\alpha/p'-\alpha/p}(z) = -D(z^{-1}e^{\alpha/p'-\alpha/p}(z)) + \widetilde{L(z)}_0(z^{-1}e^{\alpha/p'-\alpha/p}(z)) \in \mathcal{I}$$
$$= U(z^{-1}e^{\alpha/p'-\alpha/p}(z)),$$

where we used Remark 3. So all singular vectors are generated by application of U^k on $z^{-1}e^{\alpha/p'-\alpha/p}(z)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the case $p' = 2$, by using results and methods developed in [AM5] one proves the following result:

Proposition 5.4. *Assume that $p' = 2$. Then for every $k \in \mathbb{Z}_{>0}$ we have $\mathcal{I}_k \cong Ker_{V_{L+\alpha/2-\alpha/p}}Q$. Moreover,*

$$\mathcal{V}/\mathcal{I} \cong \mathcal{W}_{p,2} \oplus Ker_{V_{L+\alpha/2}}Q.$$

6. LOGARITHMIC $\mathcal{W}_{p,p'}$-MODULES

In this section we shall construct certain logarithmic modules for $\mathcal{W}_{p,p'}$. Our main technique will be a deformation of the vertex algebra $\mathcal{V}$ constructed in Section 5.

We start with the following simple result.

Proposition 6.1. *Assume that V is a vertex algebra with conformal vector ω and Virasoro field $L(z) = Y(\omega, z)$. Assume that $b \in V$ such that*

$$L(n)b = \delta_{n,0}b \ (n \in \mathbb{Z}_{\geq 0}), \quad [b_i, b_j] = 0 \ (i, j \in \mathbb{Z}).$$

Then

$$\overline{L(z)} = Y(\omega, z) + z^{-1}Y(b, z)$$

is a Virasoro field acting on any V–module.

Proof. Let $\overline{L}(n) = L(n) + b_n$. We have:

$$\begin{aligned}
[\overline{L(n)}, \overline{L(m)}] &= [L(n) + b_n, L(m) + b_m] \\
&= [L(n), L(m)] + (n - m)(L(0)b)_{n+m} - (L(-1)b)_{n+m+1} + (L(-1)b)_{n+m+1} \\
&= (n - m)(L(n + m) + b_{n+m}) + \frac{n^3 - n}{12}c\delta_{n+m,0}.
\end{aligned}$$

The proof follows. $\square$

Remark 5. *Note that we do not require $L(-1)b = Db$, where D is the canonical derivation on V.*

Corollary 6.2. *Define*

$$\overline{L(z)} = \widetilde{L(z)} + z^{-1}H(z).$$

Then $\overline{L(z)}$ is a Virasoro field acting on $V(p, p')$ and every $V(p, p')$–module.

Proof. Inside the vertex algebra $\mathcal{V}$, we have conformal vector $\omega = \widetilde{L(z)}$ and vector $b = H(z)$, and the associated vertex operators $Y_{\mathcal{V}}(\widetilde{L(z)}, z_0) = \sum_{n \in \mathbb{Z}} \widetilde{L(z)}_n z_0^{-n-1}$, $Y_{\mathcal{V}}(H(z), z_0) = \sum_{n \in \mathbb{Z}} H(z)_n z_0^{-n-1}$ acting on $\mathcal{V}$. We apply Proposition 6.1 in this case and get the Virasoro field $Y_{\mathcal{V}}(\widetilde{L(z)}, z_0) + z_0^{-1} Y_{\mathcal{V}}(H(z), z_0)$. Applying this field on $V(p, p')$-modules we get the Virasoro field $\overline{L(z)}$, as required. $\square$

Lemma 6.3. $\widetilde{H(z)}_0 = 0$ *on* $\mathcal{V}$.

Proof. As in Lemma 5.1 of [AM3] we see that

$$[\widetilde{H(z)}_0, \widetilde{a}_n] = \widetilde{(Qa)}_n, \qquad (a \in \mathcal{W}_{p,p'}).$$

Since $Qa = 0$, we conclude that

$$[\widetilde{H(z)}_0, \widetilde{Y}(a, z)] = 0, \qquad (a \in \mathcal{W}_{p,p'}).$$

This easily implies the proof. $\square$

Therefore $\widetilde{H(z)}_n$, $n \in \mathbb{Z}$, define on $\mathcal{V}$ the structure of a module for the Heisenberg algebra such that $\widetilde{H(z)}_0$ acts trivially. Therefore the field

$$\Delta(\widetilde{H(z)}, z_1) = z_1^{\widetilde{H(z)}_0} \exp\left(\sum_{n=1}^{\infty} \frac{\widetilde{H(z)}_n}{-n} z_1^{-n}\right)$$

is well defined on $\mathcal{V}$. As in [AM3] (cf. [Li], [H]; see also [FFHST]) we have the following result:

Theorem 6.4.
(1) *For every $v(z) \in \mathcal{V}$ we define*

$$\overline{Y}(v(z), z_1) = Y_{\mathcal{V}}(\Delta(\widetilde{H(z)}, z_1)v(z), z_1).$$

Then $(\mathcal{V}, \overline{Y})$ is a $\mathcal{V}$–module.
(2) *Assume that $(M, Y_M(\cdot, z_1)$ is a weak $\mathcal{V}$–module. Define the pair $(\overline{M}, \overline{Y}_{\overline{M}}(\cdot, z_1))$ such that*

$$\overline{M} = M \qquad \text{as a vector space,}$$

$$\overline{Y}_{\overline{M}}(v(z), z_1) = Y_M(\Delta(\widetilde{H(z)}, z_1)v(z), z_1).$$

Then $(\overline{M}, \overline{Y}_{\overline{M}}(\cdot, z_1))$ is a weak $\mathcal{V}$–module. In particular, $(\overline{M}, \overline{Y}_{\overline{M}}(\cdot, z_1))$ is a $\mathcal{W}_{p,p'}$–module.

Recall that $V(p, p')$ is a module for the vertex algebra $\mathcal{V}$ with the vertex operator map

$$Y(v(z), z_0) = v(z_0), \qquad v(z) \in \mathcal{V}.$$

Applying the above construction we get a new explicit realization of logarithmic modules for $\mathcal{W}_{p,p'}$ of $L(0)$ nilpotent rank 3.

Theorem 6.5. $(\overline{V(p, p')}, \overline{Y})$ *is a* $\mathcal{W}_{p,p'}$–*module such that*

$$\overline{Y}(\omega, z) = \overline{L}(z).$$

Operator $\overline{L(0)}$ *acts on* $\overline{V(p, p')}$ *as*

$$\boxed{\overline{L(0)} = L(0) + Q + \widetilde{Q} + e_{-1}^{\alpha/p' - \alpha/p}}$$

and it has $L(0)$-*nilpotent rank* 3.

Proof. To prove the formula for $\overline{L(0)}$, we simply use

$$\overline{Y}_{\overline{M}}(v(z), z_1) = Y_M(\Delta(\widetilde{H(z)}, z_1)v(z), z_1),$$

and apply the formulas $\widetilde{H(z)}_0 \widetilde{L(z)} = 0$, $\widetilde{H(z)}_1 \widetilde{L(z)} = H(z)$, and $\widetilde{H(z)}_n \widetilde{L(z)} = 0$, $n \geq 2$ proven earlier.

From the definition of $\overline{V(p, p')}$ and of ω, we easily see that the (generalized) eigenspace $\overline{V(p, p')}_0$ is 2-dimensional spanned by $\mathbf{1}$ and $e^{\alpha/p - \alpha/p'}$ so Jordan block of length 3 does not appear on the lowest weight subspace. Since $L(0)$ acts as identity on the top component of $V_{L-\alpha/p}$, and the same way on top subspace of $V_{L+\alpha/p'}$, we first analyze $\overline{V(p, p')}_1$, which is four-dimensional spanned by $\alpha(-1)\mathbf{1}$, $e^{-\alpha/p}$, $e^{\alpha/p'}$ and $e_{-1}^{\alpha/p' - \alpha/p}\mathbf{1}$. We compute

$$\overline{L(0)} \cdot \alpha(-1)\mathbf{1} = \alpha(-1)\mathbf{1} + (Q + \tilde{Q})\alpha(-1)\mathbf{1} + e_{-1}^{\alpha/p - \alpha/p'}\alpha(-1)\mathbf{1},$$

$$= \alpha(-1)\mathbf{1} + 2p'e^{-\alpha/p} - 2pe^{\alpha/p'} + \alpha(2p - 2p')(\frac{1}{p} - \frac{1}{p'})\alpha(-1)e^{\alpha/p - \alpha/p'},$$

$$\overline{L(0)} \cdot e^{-\alpha/p} = e^{-\alpha/p} - \frac{\alpha(-1)}{p'}e^{\alpha/p - \alpha/p'},$$

$$\overline{L(0)} \cdot e^{\alpha/p'} = e^{\alpha/p'} + \frac{\alpha(-1)}{p}e^{\alpha/p - \alpha/p'},$$

$$\overline{L(0)} \cdot \alpha(-1)e^{\alpha/p - \alpha/p'} = \alpha(-1)e^{\alpha/p - \alpha/p'}.$$

Observe that $L(0) = \overline{L(0)}_{ss}$. Consequently,

$$(\overline{L(0)} - L(0))^2 \alpha(-1) = -4\alpha(-1)e^{\alpha/p - \alpha/p'} \neq 0.$$

It is easy to see that $(L(0) - \overline{L(0)})^3 = 0$. But $L(0)$ is the semisimple part of $\overline{L(0)}$, so the assertion follows. $\square$

In Proposition 3.4 we constructed a $V(p,p')$-module, which we denoted by $MV(p,p')$. By using the above method we obtain a $\mathcal{W}_{p,p'}$-module structure on $\overline{MV(p,p')}$.

Theorem 6.6. *The module $\overline{MV(p,p')}$ is of $L(0)$-nilpotent rank 3.*

Proof. By using the theory of Feigin-Fuchs modules (cf. [FF], [FGST2], [IK]) we see that there is a subsingular vector w in $M(1) \otimes e^{\alpha/2}$ such that

$$Qw = e^{\alpha/2 + \alpha/p'}, \quad L(0)w = \frac{(p+2)(p'+2)}{4}w.$$

Then

$$(\overline{L(0)} - L(0))^2 w = 2\,\widetilde{Q}Qw = 2e_0^{-\alpha/p}e^{\alpha/2+\alpha/p'} \neq 0.$$

The proof follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7. CONSTRUCTION OF CERTAIN INTERTWINING OPERATORS

So far we actually constructed intertwining operators of types

$$\begin{pmatrix} \overline{V(p,p')} \\ \mathcal{V}\ \ V(p,p') \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \overline{MV(p,p')} \\ \mathcal{V}\ \ MV(p,p') \end{pmatrix}.$$

Now we shall construct intertwining operators acting between $\overline{V(p,p')}$ and $\overline{MV(p,p')}$.

The vertex algebra $\mathcal{V}$ can be treated as a vertex algebra of fields acting on $V(p,p') \oplus MV(p,p')$. Let $\mathcal{W}$ be the maximal local subspace of fields acting on $V(p,p') \bigoplus MV(p,p')$ containing $\mathcal{V}$. Then $\mathcal{W}$ is a vertex algebra which can be treated as a $\mathcal{V}$–module.

We now take intertwining operator $\mathcal{Y}$ of type

$$\begin{pmatrix} MV(p,p') \\ MV(p,p')\ \ V(p,p') \end{pmatrix}$$

in the category of $V(p,p')$–modules (existence of such intertwining operator easily follows from the considerations in Section 3).

Consider the $\mathcal{W}_{p,p'}$–module

$$M = \mathrm{Ker}_{V_{L+\alpha/2}}Q \cap \mathrm{Ker}_{V_{L+\alpha/2}}\widetilde{Q} \subset V_{L+\alpha/2}.$$

Applying the operator $\Delta(e^{-\alpha/p}, z)$ we get the intertwining operator

$$\widetilde{\mathcal{Y}}(v, z) = \mathcal{Y}(\Delta(e^{-\alpha/p}, z)v, z)$$

of type

$$\begin{pmatrix} MV(p,p') \\ M\ \ V(p,p') \end{pmatrix}$$

in the category of $\mathcal{W}_{p,p'}$–module.

Next, we assume that $\widetilde{\mathcal{Y}}(v, z)$ acts trivially on $MV(p,p')$, so $\widetilde{\mathcal{Y}}(v, z)$ can be considered as field on $V(p,p') \bigoplus MV(p,p')$. One can see the following important lemma.

Lemma 7.1. *The fields*

$$\{\widetilde{\mathcal{Y}}(v,z),\ v \in M\} \cup S$$

are mutually local on $V(p,p') \bigoplus MV(p,p')$.

Therefore, the fields $\widetilde{\mathcal{Y}}(v,z)$, $v \in M$, belong to the vertex algebra $\mathcal{W}$. They generate the following $\mathcal{V}$–module:

$$\mathcal{MV} := \mathcal{V}.M = \mathrm{span}_{\mathbb{C}}\{a(z)_n \widetilde{\mathcal{Y}}(v,z) \mid a(z) \in \mathcal{V},\ v \in M, n \in \mathbb{Z}\}.$$

Define the intertwining operator $\mathcal{Y}_1(\cdot,z)$ by

$$\mathcal{Y}_1(u(z),z_0) = u(z_0), \quad u(z) \in \mathcal{MV}.$$

One can easily see that $\mathcal{Y}_1(\cdot,z)$ is an intertwining operator of type

$$\begin{pmatrix} MV(p,p') \\ \mathcal{MV}\ V(p,p') \end{pmatrix}$$

in the category of $\mathcal{V}$–modules.

Applying the operator $\Delta(\widetilde{H(z)},z_0)$, we get intertwining operator

$$\widetilde{\mathcal{Y}}_1(u(z),z_0) = \mathcal{Y}_1(\Delta(\widetilde{H(z)},z_0)u(z),z_0)$$

of type

(19) $$\begin{pmatrix} \overline{MV(p,p')} \\ \mathcal{MV}\ \overline{V(p,p')} \end{pmatrix}.$$

(Note that $\widetilde{H(z)}_0$ acts trivially on $\mathcal{MV}$.)

By considering (19) as intertwining operator in the category of $\mathcal{W}_{p,p'}$–modules, we get:

Theorem 7.2. *There is an non-zero intertwining operator of type*

$$\begin{pmatrix} \overline{MV(p,p')} \\ M\ \overline{V(p,p')} \end{pmatrix}$$

in the category of $\mathcal{W}_{p,p'}$–*modules.*

Remark 6. *In the case* $p' = 2$, *M is isomorphic to the irreducible* $\mathcal{W}_{p,2}$–*module with lowest weight* $h' = 3p - 2$ *(see* [AM5] *for details).*

8. $\mathcal{W}_{3,2}$-ALGEBRA

The vertex algebra $\mathcal{W}_{3,2}$ has attracted considerable interest in the physics literature, primarily because its central charge is zero. The main ingredients of $\mathcal{W}_{3,2}$ representation theory were worked out in [AM4]. In [GRW1] [GRW2], among many other things, M. Gaberdiel, I. Runkel

and S. Wood investigated indecomposable projective $\mathcal{W}_{3,2}$-modules by using various tools coming from tensor categories. According to [GRW2], based on known properties of projective modules on the quantum group side [FGST2], the internal structure of projective cover $\mathcal{P}(1)$ should exhibit embedding structure as on the figure
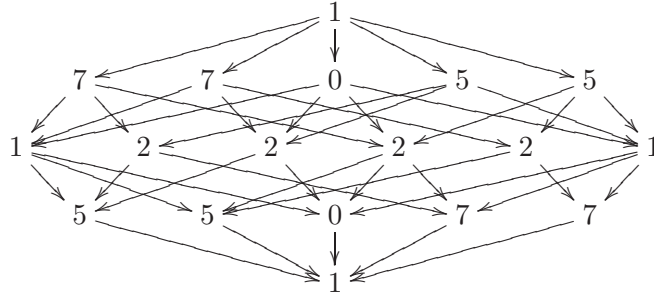

where we abbreviated i for the irreducible module $\mathcal{W}(i)$ (we are using the notation from [AM4] here). The arrows indicate the standard way of representing embeddings and quotients (for example, the module with only incoming arrows forms the socle of $\mathcal{P}(1)$).

Now, we compare this diagram with the logarithmic module $\overline{V(3,2)}$ constructed in the previous section on the direct sum of four irreducible V_L-modules $V_L \oplus V_{L-\alpha/3} \oplus V_{L+\alpha/2} \oplus V_{L+\alpha/6}$. By using classification of irreps in [AM4], and the embedding diagrams of Feigin-Fuchs modules we easily find embedding structure for V_L and $V_{L+\alpha/6}$
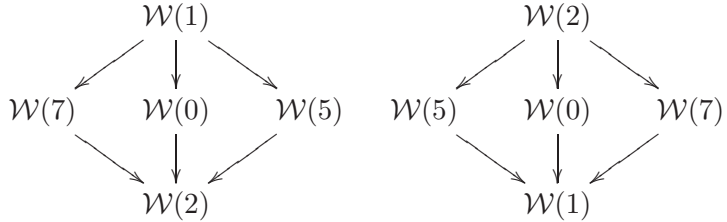
The other two V_L-modules have embedding structure:
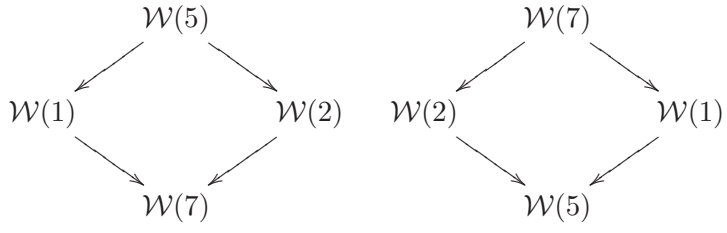
Now it is clear that these four diagrams, when put together, and by adding appropriate arrows coming from the deformed structure, should give the module $\mathcal{P}(1)$. So our work is in perfect agreement with [GRW2] and [FGST2].

Next, we analyze the logarithmic module $\overline{MV(3,2)}$ from Theorem 6.6. It is interesting to observe that $\overline{MV(3,2)}$ is *identical* to $\overline{V(3,2)}$ as vector spaces. Yet,

Proposition 8.1. $\overline{MV(3,2)} \not\cong \overline{V(3,2)}$

The proof of the proposition follows immediately by looking at the *socle* of the module in question. For instance $\mathcal{W}(5)$ belongs in the socle of $\overline{MV(3,2)}$, but not in the socle of $\overline{V(3,2)}$. Again, if we put four diagrams above diagrams to form $\overline{MV(3,2)}$, then - modulo some arrows - the internal structure of a (hypothetical) projective cover $\mathcal{W}(5)$ from [GRW2]:
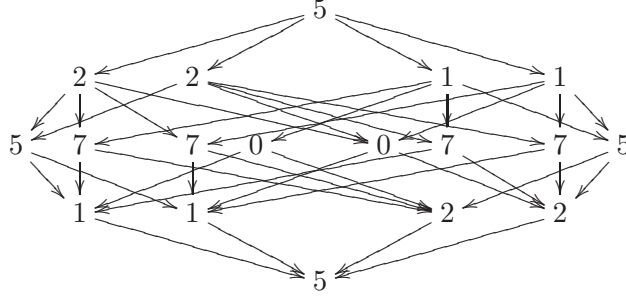
where it was denoted by $\mathcal{P}(5)$.

Remark 7. It is very likely that a combination of methods from [NT], [AM7] based on "powers" of screening operators can be used to construct other two projective modules $\mathcal{P}(2)$ and $\mathcal{P}(7)$ on the *same* direct product of four irreducible V_L-modules. The remaining projective cover in the principal block , namely $\mathcal{P}(0)$, seems to be the hardest to construct explicitly (although its structure was conjectured in [GRW2]). We do not pursue this direction in the present paper.

From everything being said we end with a hypothesis.

Conjecture 8.2. *Modules* $\overline{V(p,p')}$ *and* $\overline{MV(p,p')}$ *above are projective covers of particular irreducible* $\mathcal{W}_{p,p'}$*-modules of lowest conformal weight* 1 *and* $\frac{(p+2)(p'+2)}{4}$*, respectively.*

Even in the $c = 0$ case we cannot prove projectivity of either of the modules because we lack information about other indecomposable modules in the block and the related Ext-groups.

Remark 8. *In Appendix A1 of* [GRW2]*, the authors studied intertwiners between projective* $\mathcal{W}(2,3)$*– modules* $\mathcal{P}(1)$ *and* $\mathcal{P}(5)$*. In the language of logarithmic tensor product* [HLZ]*, relation between these modules should be interpreted as*

(20) $$\mathcal{P}(5) = \mathcal{P}(1) \widehat{\otimes} \mathcal{W}(7).$$

In Section 7 we have constructed an intertwining operator of type

$$\begin{pmatrix} \mathcal{P}(5) \\ \mathcal{W}(7) \ \ \mathcal{P}(1) \end{pmatrix}.$$

The existence of this intertwining operator is in agreement with (20).

We also get some intertwining operators of the form

$$\begin{pmatrix} \mathcal{P}(1) \\ \mathcal{W}(h) \ \ \mathcal{P}(1) \end{pmatrix} \qquad and \qquad \begin{pmatrix} \mathcal{P}(5) \\ \mathcal{W}(h) \ \ \mathcal{P}(5) \end{pmatrix}.$$

Existence of such intertwining operators was predicted by the fusion rules analysis from in [GRW1], [GRW2], [W].

9. GENERALIZED TWISTED MODULES AND LOGARITHMIC INTERTWINING OPERATORS

Let us finish with a brief comment on logarithmic intertwining operators among $\mathcal{W}_{p,p'}$-modules and generalized twisted modules associated to automorphisms of infinite order. In [AM3], Theorem 9.1, we already constructed examples of logarithmic intertwining operators [AM1], [HLZ] among triples of logarithmic $\mathcal{W}_{p,p'}$ modules, involving at most linear logarithmic factor $\log(z)$. Shortly after, Huang in [H], Theorem 5.8, connected our construction with his notion of generalized twisted modules, and based on [AM1] provided examples coming from automorphisms $e^{2\pi i Q}$ and $e^{2\pi i \tilde{Q}}$.

Results from the previous section raise the issue of construction of more general intertwining operators and related generalized twisted $e^{2\pi i Q}$ and $e^{2\pi i \tilde{Q}}$-modules, which in addition to linear logarithmic terms also involve the quadratic term $\log^2(z)$. This of course is supposed to capture the ubiquitous rank 3 nilpotency in $\mathcal{W}_{p,p'}$-Mod. We should say that for the triplet vertex algebra $\mathcal{W}_p$, due to rank 2 nilpotency, linear logarithmic intertwiners are sufficient.

Because our approach relies heavily on local systems, we propose to study a certain logarithmic extension of $\mathcal{V}$ denoted by $\mathcal{V}_{log}$ which can be viewed as a logarithmic $\mathcal{V}$-module in the weakest sense. This kind of logarithmic extension will be further deformed via $\Delta(\widetilde{H(z)}, z_1)$ to construct desired intertwining operators and generalized twisted modules. This and similar constructions will be pursued in the forthcoming publication.

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