# MODULE CATEGORIES OVER GRADED FUSION CATEGORIES

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ABSTRACT. Let $\mathcal{C}$ be a fusion category which is an extension of a fusion category $\mathcal{D}$ by a finite group $G$. We classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data $(c, M, \alpha)$ of $\mathcal{C}$. We also describe functor categories over $\mathcal{C}$ (and in particular the dual categories of $\mathcal{C}$). We use this in order to classify module categories over the Tambara Yamagami fusion categories, and their duals.

CONTENTS



## 1. INTRODUCTION

Let $\mathcal{C}$ be a fusion category. We say that $\mathcal{C}$ is an extension of the fusion category $\mathcal{D}$ by a finite group $G$ if $\mathcal{C}$ is graded by the group $G$ in such a way that $\mathcal{C}_1 = \mathcal{D}$. In [5] Etingof et. al. classified extension of a given fusion category $\mathcal{D}$ by a given finite group $G$. Their classification is given by a triple $(c, M, \alpha)$, where $c : G \to Pic(\mathcal{D})$ is a homomorphism, $M$ belongs to a torsor over $H^2(G, inv(Z(\mathcal{D})))$, and $\alpha$ belongs to a torsor over $H^3(G, k^*)$. The group $Pic(\mathcal{D})$ is the group of invertible $\mathcal{D}$-bimodules (up to equivalence), and the group $inv(Z(\mathcal{D}))$ is the group of (isomorphism classes of) invertible objects in the center $Z(\mathcal{D})$ of $\mathcal{D}$.

Let us recall briefly the construction from [5]. Suppose that we are given a classification data $(c, M, \alpha)$. The corresponding category $\mathcal{C}$ will be $\bigoplus_{g\in G} c(g)$ as a $\mathcal{D}$-bimodule category. If we choose arbitrary isomorphisms $c(g) \boxtimes_{\mathcal{D}} c(h) \to c(gh)$ for the tensor product in $\mathcal{C}$, the multiplication will not necessarily be associative. This non associativity is encoded in a cohomological obstruction $O_3(c) \in Z^3(G, inv(Z(\mathcal{D})))$. The element $M$ belongs to $C^2(G, inv(Z(\mathcal{D})))$, and should satisfy $\partial M = O_3(c)$ (that is- it should be a "solution" to the obstruction $O_3(c)$). If we change $M$ by a coboundary, we get an equivalent solution. Therefore, the choice of $M$ is equivalent to choosing an element from a torsor over $H^2(G, inv(Z(\mathcal{D})))$. Given $c$ and $M$, we still have one more obstruction in order to furnish from $\mathcal{C}$ a fusion category. This obstruction is the commutativity of the pentagon diagram, and is given by a four cocycle $O_4(c, M) \in Z^4(G, k^*)$. The element $\alpha$ belongs to $C^3(G, k^*)$, and should satisfy $\partial\alpha = O_4(c, M)$. We think of $\alpha$ as a solution to the obstruction $O_4(c, M)$. Again, if we change $\alpha$ by a coboundary, we will get an equivalent solution. Therefore, the choice of $\alpha$ can be seen as a choice from a torsor over $H^3(G, k^*)$.

We shall write $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$ to indicate the fact that $\mathcal{C}$ is an extension of $\mathcal{D}$ by $G$ given by the extension data $(c, M, \alpha)$, and we shall assume from now on that $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$.

In this paper we shall classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data $(\mathcal{C}, M, \alpha)$.

Our classification of module categories will follow the lines of the classification of [5]. We will begin by proving the following structure theorem for module categories over $\mathcal{C}$.

**Proposition 1.1.** *Let $\mathcal{L}$ be an indecomposable module category over $\mathcal{C}$. There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H = \bigoplus_{a\in H} \mathcal{C}_a$ module category $\mathcal{N}$ which remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \cong Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{N}) \triangleq \mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}$.*

This proposition enables us to reduce the classification of $\mathcal{C}$-module categories to the classification of $\mathcal{C}_H$-module categories which remains indecomposable over $\mathcal{D}$, where $H$ varies over subgroups of $G$.

In order to classify such categories we will go, in some sense, the other way around. We will begin with an indecomposable $\mathcal{D}$-module category $\mathcal{N}$, and we will ask how can we equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$ module category.

As in the classification in [5], the answer will also be based upon choosing solutions to certain obstruction (in case it is possible). We will begin with the observation, in Section 2, that we have a natural action of $G$ on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. This action is given by the following formula

$$g \cdot \mathcal{N} = \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}.$$

If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, then the action of $\mathcal{C}_H$ on $\mathcal{N}$ will give an equivalence of $\mathcal{D}$-module categories $h \cdot \mathcal{N} \cong \mathcal{N}$ for every $h \in H$. In other words- $\mathcal{N}$ will be $H$-invariant. We may think of the fact that $\mathcal{N}$ should be $H$-invariant as the "zeroth obstruction" we have in order to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category.

In case $\mathcal{N}$ is $H$-invariant, we choose equivalences $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ for every $a \in H$. We would like these equivalences to give us a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$. As one might expect, not every choice of equivalences will do that. If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, we will see in Section 4 that we have a natural action of $H$ on the group

$\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$. In case we only know that $\mathcal{N}$ is $H$-invariant, we only have an *outer* action of $H$ on $\Gamma$ (i.e. a homomorphism $\rho : H \to Out(\Gamma)$). The first obstruction will thus be the possibility to lift this outer action to a proper action.

Once we overcome this obstruction (and choose a lifting $\Phi$ for the outer action), our second obstruction will be the fact that the two functors

$$F_1, F_2 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \boxtimes \mathcal{N} \to \mathcal{N}$$

defined by

$$F_1(X \boxtimes Y \boxtimes N) = (X \otimes Y) \otimes N$$

and

$$F_2(X \boxtimes Y \boxtimes N) = X \otimes (Y \otimes N)$$

should be isomorphic. We will see that this obstruction is given by a certain two cocycle $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(Aut_{\mathcal{D}}(\mathcal{N})))$. A solution for this obstruction is an element $v \in C^1(H, Z(Aut_{\mathcal{D}}(\mathcal{N})))$ that should satisfy $\partial v = O_2(\mathcal{N}, c, H, M, \Phi)$.

Our last obstruction will be the fact that the above functors should be not only isomorphic, but they should be isomorphic in a way which will make the pentagon diagram commutative. This obstruction is encoded by a three cocycle $O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) \in Z^3(H, k^*)$. A solution $\beta$ for this obstruction will be an element of $C^2(H, k^*)$ such that $\partial\beta = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha)$.

We can summarize our main result in the following proposition:

**Proposition 1.2.** *An indecomposable module category over $\mathcal{C}$ is given by a tuple $(\mathcal{N}, H, \Phi, v, \beta)$, where $\mathcal{N}$ is an indecomposable module category over $\mathcal{D}$, $H$ is a subgroup of $G$ which acts trivially on $\mathcal{N}$, $\Phi : H \to Aut(Aut_{\mathcal{D}}(\mathcal{N}))$ is a homomorphism, $v$ belongs to a torsor over $H^1(H, Z(Aut_{\mathcal{D}}(\mathcal{N})))$, and $\beta$ belongs to a torsor over $H^2(H, k^*)$.*

We shall denote the indecomposable module category which corresponds to the tuple $(\mathcal{N}, H, \Phi, v, \beta)$ by $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$. In order to classify module categories, we need to give not only a list of all indecomposable module categories, but also to explain when does two elements in the list define equivalent module categories. We will see in Section 6 that if $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ is any indecomposable module category, $g \in G$ is an arbitrary element and $F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv \mathcal{N}'$ is an equivalence of $\mathcal{D}$-module categories (where $\mathcal{N}'$ is another indecomposable $\mathcal{D}$-module category), then $F$ gives rise to a tuple $(\mathcal{N}', gHg^{-1}, \Phi', v', \beta')$ which defines an equivalent $\mathcal{C}$-module category. Our second main result is the following:

**Proposition 1.3.** *Two tuples $(\mathcal{N}, H, \Phi, v, \beta)$ and $(\mathcal{N}', H', \Phi', v', \beta')$ determine equivalent $\mathcal{C}$-module categories if and only if the second tuple is defined by the first tuple and by some equivalence $F$ as above.*

We shall prove proposition 1.3 in Section 6. We will also decompose this condition into a few simpler ones: we will see, for example, by considering the case $g = 1$, that we can change $\Phi$ to be $t\Phi t^{-1}$, where $t$ is any conjugation automorphism of $Aut_{\mathcal{D}}(\mathcal{N})$.

In Section 7 we will describe the category of functors $Fun_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ where $\mathcal{N}$ and $\mathcal{M}$ are two module categories over $\mathcal{C}$. We will see that we can view this category as the equivariantization of the category $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{M})$ with respect to an action of $G$. We will also be able to prove the following criterion of $\mathcal{C}$ to be group theoretical. $\mathcal{C}$ is a group theoretical if and only if there is a pointed $\mathcal{D}$-module category $\mathcal{N}$, i.e., $\mathcal{D}^*_{\mathcal{N}}$ is pointed, stable under the $G$-action , i.e., for every $g \in G$, $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ as $\mathcal{D}$-module categories.

A theorem of Ostrik says that any indecomposable module category over a fusion category $\mathcal{D}$ is equivalent to a category of the form $Mod_{\mathcal{D}} - A$, of right $A$-modules in the category $\mathcal{D}$, where $A$ is some semisimple indecomposable algebra in the category $\mathcal{D}$. In other words- any module category has a description by objects which lie inside the fusion category $\mathcal{D}$. In Section 8 we will explain how we can understand the obstructions and their solutions, and also the functor categories, by intrinsic description; that is- by considering algebras and modules inside the categories $\mathcal{D}$ and $\mathcal{C}$.

This description will be much more convenient for calculations. It will also enables us to view the first and the second obstruction in a unified way. Indeed, in Section 8 we will show that we have a natural short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1$$

and that a solution for the first two obstructions is equivalent to a choice of a splitting of this sequence (and therefore, we can solve the first two obstructions if and only if this sequence splits). We will also show, following the results of Section 8, that two splittings which differ by conjugation by an element of $\Gamma$ will give us equivalent module categories. We mention that we could have derive this short exact sequence and this result without using algebras and modules inside our categories, but it would have been more complicated.

In Section 9 we shall give a detailed example. We will consider the Tambara Yamagami fusion categories, $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$. In this case $\mathcal{C}$ is an extension of the category $Vec_A$, where $A$ is an abelian group, by the group $\mathbb{Z}_2$.

## 2. Preliminaries

In this section, $\mathcal{C}$ will be a general fusion category and $\mathcal{D}$ a sub-fusion category of $\mathcal{C}$. We recall some basic facts about module categories over $\mathcal{C}$ and $\mathcal{D}$. For a more detailed discussion on these notions, we refer the reader to [1] and to [4]. Let $\mathcal{N}$ be a module category over $\mathcal{C}$. If $X, Y \in Ob\mathcal{N}$, then the *internal hom* of $X$ and $Y$ is the unique object of $\mathcal{C}$ which satisfies the formula

$$Hom_{\mathcal{C}}(W, \underline{Hom}_{\mathcal{C}}(X, Y)) = Hom_{\mathcal{N}}(W \otimes X, Y)$$

for every $W \in Ob\mathcal{C}$. For every $X \in Ob\mathcal{N}$ the object $\underline{Hom}_{\mathcal{C}}(X, X)$ has a canonical algebra structure. We say that $X$ *generates* $\mathcal{N}$ (over $\mathcal{C}$) if $\mathcal{N}$ is the smallest sub $\mathcal{C}$-module-category of $\mathcal{N}$ which contains $X$. For every algebra $A$ in $\mathcal{C}$, $mod_{\mathcal{C}} - A$, the category of right $A$-modules in $\mathcal{C}$, has a structure of a left $\mathcal{C}$-module category.

A theorem of Ostrik says that all module categories are of this form:

**Theorem 2.1.** *(see [1]) Let $\mathcal{N}$ be a module category, and let $X$ be a generator of $\mathcal{N}$ over $\mathcal{C}$. We have an equivalence of $\mathcal{C}$-module categories $\mathcal{N} \cong Mod_{\mathcal{C}} - \underline{Hom}(X, X)$ given by $F(Y) = \underline{Hom}(X, Y)$.*

Next, we recall the definition of the induced module category. If $\mathcal{N}$ is a $\mathcal{D}$-module category, $Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N})$ is a module category over $\mathcal{C}$ which satisfies Frobenius reciprocity. This means that for every $\mathcal{C}$-module category $\mathcal{R}$ we have that

$$Fun_{\mathcal{C}}(Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}), \mathcal{R}) \cong Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{R}).$$

The next lemma proves that the induced module category always exists. It will also gives us some idea about how the induced module category "looks like".

**Lemma 2.2.** *Suppose that $\mathcal{N} \cong mod_{\mathcal{D}} - A$ for some algebra $A \in Ob\mathcal{D}$. Then $A$ can also be considered as an algebra in $\mathcal{C}$, and $Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}) \cong mod_{\mathcal{C}} - A$.*

*Proof.* Let us prove that Frobenius reciprocity holds. For this, we first need to represent $\mathcal{R}$ in an appropriate way. We choose a generator $X$ of $\mathcal{R}$ **over** $\mathcal{D}$. It is easy to see that $X$ is also a generator over $\mathcal{C}$. Then, by Ostrik's Theorem we have that $\mathcal{R} \cong mod_{\mathcal{C}} - \underline{Hom}_{\mathcal{C}}(X, X)$ over $\mathcal{C}$, and $\mathcal{R} \cong mod_{\mathcal{D}} - \underline{Hom}_{\mathcal{D}}(X, X)$ over $\mathcal{D}$. If we denote $\underline{Hom}_{\mathcal{C}}(X, X)$ by $B$, then it is easy to see by the definition of $\underline{Hom}$ that $\underline{Hom}_{\mathcal{D}}(X, X) \cong B_{\mathcal{D}}$, where $B_{\mathcal{D}}$ is the largest subobject of $B$ which is also an object of $\mathcal{D}$ (since $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$, this is also a subalgebra of $B$). By another theorem of Ostrik (see [1]), we know that $Fun_{\mathcal{C}}(mod_{\mathcal{C}} - A, mod_{\mathcal{C}} - B) \cong bimod_{\mathcal{C}} - A - B$. Using the theorem of Ostrik again, we see that $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{R}) \cong bimod_{\mathcal{D}} A - B_{\mathcal{D}}$. One can verify that the functor which sends an $A - B_{\mathcal{D}}$ bimodule $Z$ in $\mathcal{D}$ to $Z \otimes_{B_{\mathcal{D}}} B$ gives an equivalence between the two categories. □

*Remark* 2.3. The fact that the induction functor is an equivalence of categories arise from the fact that for such a $B$, the equivalence between the categories $mod_{\mathcal{D}} - B_{\mathcal{D}}$ and $mod_{\mathcal{C}} - B$ is given by $X \mapsto X \otimes_{B_{\mathcal{D}}} B$.

One can show that the induced module category is also equivalent to $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$.

In particular, we have the following:

**Corollary 2.4.** *Let $\mathcal{N}$ be a module category over $\mathcal{C}$. Suppose that $X$ is a generator of $\mathcal{N}$ over $\mathcal{C}$, and that the algebra $A = \underline{Hom}(X, X)$ is supported on $\mathcal{D}$. Then $\mathcal{N} \cong Ind_{\mathcal{D}}^{\mathcal{C}}(mod_{\mathcal{D}} - A)$.*

## 3. Decomposition of the module category over the trivial component subcategory. The zeroth obstruction

We begin by considering the action of $G$ on $\mathcal{D}$-module categories. For every $g \in G$, $\mathcal{C}_g$ is an invertible $\mathcal{D}$-bimodule category. Therefore, if $\mathcal{N}$ is an indecomposable $\mathcal{D}$-module category, the category $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ is also indecomposable. It is easy to see that we get in this way an action of $G$ on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. Let now $\mathcal{L}$ be an indecomposable $\mathcal{C}$-module category. We can consider $\mathcal{L}$ also as a module category over $\mathcal{D}$. We claim the following:

**Lemma 3.1.** *As a $\mathcal{D}$-module category, $\mathcal{L}$ is $G$-invariant.*

*Remark* 3.2. For this lemma, we do not need to assume that $\mathcal{L}$ is indecomposable.

*Proof.* We have the following equivalence of $\mathcal{D}$-module categories

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L}) \cong$$
$$(\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong (\mathcal{C}_g \boxtimes_{\mathcal{D}} \oplus_{a \in G} \mathcal{C}_a) \boxtimes_{\mathcal{C}} \mathcal{L} \cong$$
$$(\oplus_{a \in G} \mathcal{C}_{ga}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{L}.$$

This proves the claim.

□

If $H$ is a subgroup of $G$, we have the subcategory $\mathcal{C}_H = \bigoplus_{h \in H} C_h$ of $\mathcal{C}$, which is an extension of $\mathcal{D}$ by $H$. We claim the following:

**Proposition 3.3.** *There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H$ module category $\mathcal{N}$ which remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \equiv Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{N})$.*

*Proof.* Suppose that $\mathcal{L}$ decomposes over $\mathcal{D}$ as

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{L}_i.$$

For every $g \in G$, we have seen that the action functor defines an equivalence of categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{L}$. Since

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \bigoplus_{i=1}^{n} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L}_i,$$

we see that $G$ permutes the index set $\{1, \ldots, n\}$. This action is transitive, as otherwise $\mathcal{L}$ would not have been indecomposable over $\mathcal{C}$. Let $H < G$ be the stabilizer of $L_1$. Then $\mathcal{N} = \mathcal{L}_1$ is a $\mathcal{C}_H$-module category which remains indecomposable over $\mathcal{D}$. Let $X \in Ob\mathcal{L}_1$ be a generator of $\mathcal{L}$ over $\mathcal{C}$ (any nonzero object would be a generator, as $\mathcal{L}$ is indecomposable over $\mathcal{C}$). By the fact that the stabilizer of $\mathcal{L}_1$ is $H$, it is easy to see that $\underline{Hom}_{\mathcal{C}}(X, X)$ is contained in $\mathcal{C}_H$. The rest of the lemma now follows from corollary 2.4.

□

So in order to classify indecomposable module categories over $\mathcal{C}$, we need to classify, for every $H < G$, the indecomposable module categories over $\mathcal{C}_H$ which remain indecomposable over $\mathcal{D}$. For every indecomposable module category $\mathcal{L}$ over $\mathcal{C}$, we have attached a subgroup $H$ of $G$ and an indecomposable $\mathcal{C}_H$ module category $\mathcal{L}_1$ which remains indecomposable over $\mathcal{D}$. The subgroup $H$ and the module category $\mathcal{L}_1$ will be the first two components of the tuple which corresponds to $\mathcal{L}$. Notice that we could have chosen any conjugate of $H$ as well.

## 4. The first two obstructions

Let $\mathcal{L}$, $\mathcal{N} = \mathcal{L}_1$ and $H$ be as in the previous section. For every $a \in H$ we have an equivalence of $\mathcal{D}$-module categories $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \cong N$ given by the action of $\mathcal{C}_H$ on $\mathcal{N}$. Suppose on the other hand that we are given an $H$-invariant indecomposable module category $\mathcal{N}$ over $\mathcal{D}$. Let us fix a family of equivalences $\{\psi_a\}_{a \in H}$, where $\psi_a : \mathcal{C}_a \boxtimes \mathcal{N} \to \mathcal{N}$. Let us see when does this family comes from an action of $\mathcal{C}_H$ on $\mathcal{N}$.

We know that the two functors

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \stackrel{m\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_H \boxtimes \mathcal{N} \dot{\to} \mathcal{N}$$

and

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \stackrel{1_{\mathcal{C}_H}\boxtimes(\cdot)}{\to} \mathcal{C}_H \boxtimes \mathcal{N} \dot{\to} \mathcal{N}$$

should be isomorphic. It is easy to see that the two functors

$$\mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{m\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{N} \dot{\to} \mathcal{N}$$

and

$$\mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_H}\boxtimes(\cdot)}{\to} \mathcal{C}_H \boxtimes_{\mathcal{D}} \mathcal{N} \dot{\to} \mathcal{N}$$

should also be isomorphic. Since the action of $\mathcal{C}_H$ on $\mathcal{N}$ is given by the action of $\mathcal{D}$ together with the $\psi_a$'s, this condition translates to the fact that for every $a, b \in H$ the two functors

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_{ab}}{\to} \mathcal{N}$$

and

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes\psi_b}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_a}{\to} \mathcal{N}$$

should be isomorphic. We can express this condition in the following equivalent way- for every $a, b \in H$, the autoequivalence of $\mathcal{N}$ as a $\mathcal{D}$-module category

$$\begin{aligned} Y_{a,b} = \mathcal{N} \stackrel{\psi_a^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} &\stackrel{1_{\mathcal{C}_a}\boxtimes\psi_b^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \\ &\stackrel{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_{ab}^{-1}}{\to} \mathcal{N} \end{aligned}$$

should be isomorphic to the identity autoequivalence. We shall decompose this condition into two simpler ones.

Consider the group $\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$, where by $Aut_{\mathcal{D}}$ we mean the group of $\mathcal{D}$-autoequivalences (up to isomorphism) of $\mathcal{N}$. For $a \in H$ and $F \in \Gamma$ define $a \cdot F \in \Gamma$ as the composition

$$\mathcal{N} \stackrel{\psi_a^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes F}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_a}{\to} \mathcal{N}.$$

We get a map $\Phi : H \to Aut(\Gamma)$ given by $\Phi(g)(F) = h \cdot F$. This map depends on the choice of the $\psi_a$'s and is not necessary a group homomorphism. However, the following equation does hold for every $a, b \in H$:

$$\Phi(a)\Phi(b) = \Phi(ab)C_{Y_{a,b}}, \tag{4.1}$$

where we write $C_x$ for conjugation by $x \in \Gamma$.

Notice that $\psi_a$ is determined up to composition with an element in $\Gamma$, and that by changing $\psi_a$ to be $\psi'_a = \gamma\psi_a$, for $\gamma \in \Gamma$, we change $\Phi(a)$ to be

$$\Phi(a)c_\gamma,$$

where by $c_\gamma$ we mean conjugation by $\gamma$. Equation 4.1 shows that the composition $\rho = \pi\Phi$, where $\pi$ is the quotient map $\pi : Aut(\Gamma) \to Out(\Gamma)$ does give a group homomorphism. Notice that by the observation above, $\rho$ does not depend on the choice of the $\psi_a$'s, but only on $c$, $\mathcal{N}$ and $H$. We have the following

**Lemma 4.1.** *If the $\psi_a$'s arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then the map $\Phi$ is a group homomorphism.*

*Proof.* This follows from the fact that by the discussion above, if the $\psi_a$'s arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then $Y_{a,b}$ is trivial for every $a, b \in H$, and by Equation 4.1 we see that $\Phi$ is a group homomorphism. $\square$

So $c, \mathcal{N}$ and $H$ determines a homomorphism $\rho : H \to Out(\Gamma)$. We thus see that in order to give $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category, we need to give a lifting of $\rho$ to a homomorphism to $Aut(\Gamma)$. The first obstruction is thus the possibility to lift $\rho$ in such a way.

Suppose then that we have a lifting, that is- a homomorphism $\Phi : H \to Aut(\Gamma)$ such that $\pi\Phi = \rho$. To say that $\Phi$ is a homomorphism is equivalent to say that we have chosen the $\psi_a$'s in such a way that $C_{Y_{a,b}} = Id$, or in other words- in such a way that for every $a, b \in H$, $Y_{a,b}$ is in $Z(\Gamma)$, the center of $\Gamma$.

Notice that after choosing $\Phi$, we still have some liberty in changing the $\psi_a$'s. Indeed, if we choose $\psi'_a = \gamma_a\psi_a$, where $\gamma_a \in Z(\Gamma)$ for every $a \in H$, we still get the same $\Phi$, and it is easy to see that every $\psi'_a$ that will give us the same $\Phi$ is of this form.

In order to furnish a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$, we need $Y_{a,b}$ to be not only central, but trivial. A straightforward calculation shows now that the function $H \times H \to Z(\Gamma)$ given by $(a, b) \mapsto Y_{a,b}$ is a two cocycle. If we choose a different set of isomorphisms $\psi'_a = \gamma_a\psi_a$ where $\gamma_a \in Z(\Gamma)$, we will get a cocycle $Y'$ which is cohomologous to $Y$. So the second obstruction is the cohomology class of the two cocycle $(a, b) \mapsto Y_{a,b}$. We shall denote this obstruction by $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. Notice that this obstruction depends linearly on $M$ in the following sense: we have a natural homomorphism of groups $\xi : inv(Z(\mathcal{D})) \to \Gamma$, given by the formula

$$\xi(T)(N) = T \otimes N$$

(that is- $\xi(T)$ is just the autoequivalence of acting by $T$, or if we identify $Z(\mathcal{D})$ with the category of $\mathcal{D}$-bimodule endofunctors of the regular $\mathcal{D}$-bimodule and thus $inv(Z(\mathcal{D})$ is identified with the group of isomorphisms classes of $\mathcal{D}$-automorphisms, then $\xi(T)(N)$ is given by $\mathcal{N} \xrightarrow{\cong} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{T \boxtimes_{\mathcal{D}} N} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\cong} \mathcal{N}$ ). It can be seen that if we would have chosen $M' = M\zeta$, where $\zeta \in Z^2(G, Z(\mathcal{D}))$, then we would have changed $O_2$ to be $O_2 res^G_H(\xi_*(\zeta))$.

In conclusion- we saw that if $\mathcal{N}$ is a $\mathcal{D}$-module category upon which $H$ acts trivially, then we have an induced homomorphism $\rho : H \to Out(\Gamma)$. The first obstruction to define on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the fact that $\rho$ should be of the form $\pi\Phi$ where $\Phi : H \to Aut(Aut_{\mathcal{D}}(\mathcal{N}))$ is a homomorphism. After choosing such a lifting $\Phi$ we get the second obstruction, which is a two cocycle $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. A solution to this obstruction will be an element $v \in C^1(H, Z(\Gamma))$ which satisfies

$$\partial v = O_2(\mathcal{N}, c, H, M, \Phi).$$

We will see later, in Section 8, that to find a solution for the first and for the second obstruction is the same thing as to find a splitting for a certain short exact sequence. We will also see why two solutions $v$ and $v'$ which differs by a coboundary give equivalent module categories (and therefore we can view the set of possible solutions, in case it is not empty) as a torsor over $H^1(H, Z(\Gamma))$.

## 5. The third obstruction

So far we have almost defined a $\mathcal{C}_H$-action on $\mathcal{N}$, by means of the equivalences $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$. The solutions for the first and for the second obstruction ensures us that for every $a, b \in H$ the two functors

$$F_1 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \overset{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \overset{\psi_{ab}}{\to} \mathcal{N}$$

and

$$F_2 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \overset{1_{\mathcal{C}_a}\boxtimes \psi_b}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \overset{\psi_a}{\to} \mathcal{N}$$

are isomorphic.

For every $a, b \in H$, let us fix an isomorphism $\eta(a, b) : F_1 \to F_2$ between the two functors. In other words, for every $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$ and $N \in \mathcal{N}$ we have a natural isomorphism

$$\eta(a, b)_{X,Y,N} : (X \otimes Y) \otimes N \to X \otimes (Y \otimes N).$$

Since $F_1$ and $F_2$ are simple as objects in the relevant functor category (they are equivalences), the choice of the isomorphism $\eta(a, b)$ is unique up to scalar, for every $a, b \in H$.

The final condition for $\mathcal{N}$ to be a $\mathcal{C}_H$-module category is the commutativity of the pentagonal diagram. In other words, for every $a, b, d \in H$, and every $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$, $Z \in \mathcal{C}_d$ and $N \in \mathcal{N}$, the following diagram should commute:

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes N \arrow[rr, "\eta(a,bd)_{X,Y\otimes Z,N}"] & & X \otimes ((Y \otimes Z) \otimes N) \arrow[d, "\eta(b,d)_{Y,Z,N}"] \\
((X \otimes Y) \otimes Z) \otimes N \arrow[u, "\alpha_{X,Y,Z}"] \arrow[dr, "\eta(ab,d)_{X\otimes Y,Z,N}"] & & X \otimes (Y \otimes (Z \otimes N)) \\
& (X \otimes Y) \otimes (Z \otimes N) \arrow[ur, "\eta(a,b)_{X,Y,Z\otimes N}"] &
\end{tikzcd}$$

This diagram will always be commutative up to a scalar $O_3(a, b, d)$ which depends only on $a, b$ and $d$, and not on the particular objects $X, Y, Z$ and $N$. One can also see that the function $(a, b, d) \mapsto O_3(a, b, d)$ is a three cocycle on $H$ with values in $k^*$, and that choosing different $\eta(a, b)$'s will change $O_3$ by a coboundary. We call $O_3 = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) \in H^3(H, k^*)$ the third obstruction. A solution to this obstruction is equivalent to giving a set of $\eta(a, b)$'s such that the pentagon diagram will be commutative. We will see in the next section that by altering $\eta$ by a coboundary we will get equivalent module categories. Thus, we see that the set of solutions for this obstruction will be a torsor over the group $H^2(H, k^*)$ (in case a solution exists). Notice that this obstruction depends "linearly" on $\alpha$, in the sense that if we would have change $\alpha$ to be $\alpha\zeta$ where $\zeta \in H^3(G, k^*)$, then we would have changed the obstruction by $\zeta$. In other words:

$$O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha\zeta) = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) res^G_H(\zeta).$$

This ends the proof of Proposition 1.2.

## 6. The isomorphism condition

In this section we answer the question of when does the $\mathcal{C}$-module categories $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$ are equivalent.

Assume then that we have an equivalence of $\mathcal{C}$-module categories

$$F : \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta').$$

Let us denote these categories by $\mathcal{M}$ and $\mathcal{M}'$ respectively. Then $F$ is also an equivalence of $\mathcal{D}$-module categories. Recall that as $\mathcal{D}$-module categories, $\mathcal{M}$ splits as

$$\bigoplus_{g \in G/H} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}.$$

A similar decomposition holds for $\mathcal{M}'$.

By considering these decompositions, it is easy to see that $F$ induces an equivalence of $\mathcal{D}$-module categories between $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and $\mathcal{N}'$ for some $g \in G$. Let us denote the restriction of $F$ to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ as a functor of $\mathcal{D}$-module categories by $t_F$. We can reconstruct the tuple $(\mathcal{N}', H', \Phi', v', \beta')$ from $t_F$ in the following way: We have already seen that $\mathcal{N}'$ is equivalent to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and that the stabilizer subgroup of the category $\mathcal{N}'$ will be $H' = gHg^{-1}$.

Let us denote by $\Gamma'$ the group $Aut_{\mathcal{D}}(N')$. We have a natural isomorphism $\nu : \Gamma \to \Gamma'$ given by the formula

$$\nu(t) : \mathcal{N}' \xrightarrow{F^{-1}} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{1 \boxtimes t} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F} \mathcal{N}'.$$

Using the functor $t_F$ and the map $\nu$ we can see that the map

$$\rho' : gHg^{-1} \to Out(\Gamma')$$

which appears in the construction of the second module category is the composition

$$gHg^{-1} \xrightarrow{c_g} H \xrightarrow{\rho} Out(\Gamma) \to Out(\Gamma'),$$

where the last morphism is induced by $\nu$. The map $\Phi'$ which lifts $\rho'$ will depend on $\Phi$ in a similar fashion. The same holds for the second obstruction and its solution.

For the third obstruction, the situation is a bit more delicate. Since $F$ is a functor of $\mathcal{C}$-module categories, we have, for each $a \in H$, a natural isomorphism between the functors

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \xrightarrow{1 \boxtimes F} \mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} \mathcal{N}' \dot{\to} \mathcal{N}'$$

and

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \dot{\to} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F} \mathcal{N}'$$

For any $a \in H$, the choice of the natural isomorphism is unique up to a scalar. A direct calculation shows that if we change the natural isomorphisms by a set of scalars $\zeta_a$, we will get an equivalence $\mathcal{M}(N, H, \Phi, v, \beta) \to \mathcal{M}(N', H', \Phi', v', \beta'')$ where $\beta'' = \beta' \partial \zeta$. This is the reason that cohomologous solutions for the third obstruction will give us equivalent module categories.

In conclusion, we have the following:

**Proposition 6.1.** *Assume that we have an isomorphism $F : \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$ Then there is a $g \in G$ such that $F$ will induce an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$, and the data $(\mathcal{N}', H', \Phi', v', \beta')$*

*can be reconstructed from $t_F$ in the way described above ($\beta'$ will be reconstructable only up to a coboundary) .*

*Remark* 6.2. We do not have any restriction on $t_F$. In other words, given $t_F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$ we can always reconstruct the tuple $(\mathcal{N}', H', \Phi', v', \beta')$ in the way described above.

We would like now to "decompose" the equivalence in the theorem into several steps. The first ingredient that we need in order to get an equivalence is an element $g \in G$ such that $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv \mathcal{N}'$.

Consider now the case where this ingredient is trivial, that is- $g = 1$, $\mathcal{N} = \mathcal{N}'$ and $H = H'$. In that case $t_F$ is an autoequivalence of the $\mathcal{D}$-module category $\mathcal{N}$. Let us denote by $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ and by $\psi'_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ the structural equivalences of the two categories (where $a \in H$). Since $F$ is an equivalence of $\mathcal{C}$-module categories, we see that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi_a"] \arrow[d, "1 \boxtimes t_F"] & \mathcal{N} \arrow[d, "t_F"] \\
\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi'_a"] & \mathcal{N}
\end{tikzcd}$$

and a direct calculation shows that $\Phi$ and $\Phi'$ satisfy the following formula:

$$\Phi'(a)(V) = t_F \Phi(a)(t_F^{-1} V t_F) t_F^{-1} \tag{6.1}$$

where $V$ is any element in $\Gamma$.

Another way to write Equation 6.1 is $\Phi' = c_{t_F} \Phi c_{t_F}^{-1}$, where by $c_{t_F}$ we mean the automorphism of $\Gamma$ of conjugation by $t_F$. In other words- this shows that we have some freedom in choosing $\Phi$, and if we change $\Phi$ in the above fashion, we will still get equivalent categories.

Consider now the case where also $\Phi = \Phi'$. This means that for every $a \in H$ the element $t_F \Phi(a)(t_F)^{-1}$ is central in $\Gamma$. A direct calculation shows that the function $r$ defined by $r(a) = t_F \Phi(a)(t_F)^{-1}$ is a one cocycle with values in $Z(\Gamma)$, and that $v/v' = r$. Notice in particular that by choosing arbitrary $t_F \in Z(\Gamma)$ we see that cohomologous solutions to the second obstruction will give us equivalent categories. However, we see that more is true, and it might happen that non cohomologous $v$ and $v'$ will define equivalent categories.

Last, if the situation is that $t_F = \Phi(a)(t_F)$ for every $a \in H$, the only way in which we can alter the tuple (and still get an equivalent category) $(\mathcal{N}, H, \Phi, v, \beta)$ will be, as we have seen earlier, to change $\beta$ by a coboundary.

## 7. Functor categories as equivariantization categories

In this section we shall describe the category of $\mathcal{C}$-functors between the module categories $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$. For simplicity we shall denote these categories as $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Consider the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$. This is a $k$-linear category which by Theorem 2.16 in [4] is semisimple.

**Lemma 7.1.** *There is a natural $G$-action $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ induced by the structure $\mathcal{C}$-module categories on $\mathcal{M}_1$ and $\mathcal{M}_2$.*

*Proof.* There are $\mathcal{D}$-module equivalences $\psi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \cong \mathcal{M}_1$ and $\phi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \cong \mathcal{M}_2$. Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module morphism, we define $T_g \in Aut^{\otimes}(Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ to be the following functor $\mathcal{M}_1 \xrightarrow{\psi_g^{-1}} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \xrightarrow{Id_{\mathcal{C}_g} \boxtimes_{\mathcal{D}} F} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \xrightarrow{\phi_g} \mathcal{M}_2$. The composition of the 2-arrows in the following diagram defines a natural isomorphism $\mu_{g_1,g_2} : T_{g_1} \circ T_{g_2} \cong T_{g_1 g_2}$

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "\psi_{g_1}^{-1}"] \arrow[d, "="] & \mathcal{C}_{g_1} \boxtimes_{\mathcal{D}} \mathcal{M}_1 \arrow[r, "\psi_{g_2}^{-1}"] & \mathcal{C}_{g_1} \boxtimes_{\mathcal{D}} \mathcal{C}_{g_2} \boxtimes_{\mathcal{D}} \mathcal{M}_1 \arrow[r, "F"] \arrow[d, "\cong"] \arrow[dll, Rightarrow] & \mathcal{C}_{g_1} \boxtimes_{\mathcal{D}} \mathcal{C}_{g_2} \boxtimes_{\mathcal{D}} \mathcal{M}_2 \arrow[r, "\phi_{g_2}"] \arrow[d, "\cong"] \arrow[dl, Rightarrow] & \mathcal{C}_{g_1} \boxtimes_{\mathcal{D}} \mathcal{M}_2 \arrow[r, "\phi_{g_2}"] & \mathcal{M}_2 \arrow[d, "="] \arrow[dll, Rightarrow] \\
\mathcal{M}_1 \arrow[rr, "\psi_{g_1 \cdot g_2}^{-1}"] & & \mathcal{C}_{g_1 \cdot g_2} \boxtimes_{\mathcal{D}} \mathcal{M}_1 \arrow[r, "F"] & \mathcal{C}_{g_1 \cdot g_2} \boxtimes_{\mathcal{D}} \mathcal{M}_2 \arrow[rr, "\phi_{g_1 \cdot g_2}"] & & \mathcal{M}_2
\end{tikzcd}$$

□

Since we have a $G$-action on $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$, we can talk about the equivariantization $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$. By definition, an object in $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$ is a pair $(F, \{T_g\}_{g \in G})$, where $T_g : g \cdot F \to F$ are natural equivalences which satisfy a certain coherence condition (for the exact definition, see for example [3]). Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module functor. A structure on $\mathcal{C}$-module functor on $F$ induces a structure a $G$-equivariant object, are vica versa.

Let us conclude this discussion by the following lemma:

**Lemma 7.2.** *The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the equivariantization $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$ of the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ with respect to the aforementioned $G$-action.*

*Remark* 7.3. Assume that the ground field $k$ is of characteristic 0. Let $\mathcal{M}$ be an indecomposable $\mathcal{C}$-module category. Although $\mathcal{C}^*_{\mathcal{M}} \triangleq Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a fusion category, $Fun_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is, in general, only a multifusion category, because, $\mathcal{M}$ may decompose as $\mathcal{D}$-module category. Equivariantization has only been defined in context of fusion categories, however, the definition in context of multifusion categories is mutatis mutandis. Notice that is case of multifusion equivariantization we don't always have the *Rep*$(G)$ subcategory supported on the trivial object.

In the next section we will give an intrinsic description of the functor categories, as categories of bimodules.

## 8. An intrinsic description by algebras and modules

The goal of this Section is to explain more concretely the action of the grading group on indecomposable module categories, the action of the grading group on $Aut_{\mathcal{D}}(\mathcal{N})$, the obstructions and their solutions.

In [1] Ostrik showed that any indecomposable module category over a fusion category $\mathcal{C}$ is equivalent as a module category to the category $Mod_{\mathcal{C}}-A$ for some semisimple indecomposable algebra $A$ in $\mathcal{C}$. In this section we will realize all the objects described in the previous sections by using algebras and modules inside $\mathcal{C}$. As before, we assume that $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, we denote $\mathcal{C}_1$ by $\mathcal{D}$ and $Aut_{\mathcal{D}}(\mathcal{N})$ by $\Gamma$.

8.1. **The action of $G$ on indecomposable module categories.** Assume that $A$ is a semisimple indecomposable algebra inside $\mathcal{D}$. Let $\mathcal{N} = Mod_{\mathcal{D}} - A$ be the category of right $A$-modules inside $\mathcal{D}$. We denote by $Mod_{\mathcal{C}_g} - A$ the category of $A$-modules with support in $\mathcal{C}_g$. We claim the following:

**Lemma 8.1.** *We have an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv Mod_{\mathcal{C}_g} - A$.*

*Proof.* We have already seen in Section 2 that we have an equivalence of $\mathcal{C}$-module categories

$$\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N} \equiv Mod_{\mathcal{C}}(A)$$

which is given by $X \boxtimes M \mapsto X \otimes M$. As a $\mathcal{D}$-modules category, the left hand category decomposes as $\bigoplus_{g\in G} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and the right hand category decomposes as $\bigoplus_{g\in G} Mod_{\mathcal{C}_g} - A$. It is easy to see that the above equivalence translates one decomposition into the other, and therefore the functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to Mod_{\mathcal{C}_g} - A$ given by $X \boxtimes M \mapsto X \otimes M$ is an equivalence of $\mathcal{D}$-module categories.

□

Next, we understand how we can describe functors by using bimodules.

**Lemma 8.2.** *Let $\mathcal{N} = Mod_{\mathcal{D}} - A$ and $\mathcal{N}' = Mod_{\mathcal{D}} - A'$, and let $g \in G$. Then every functor $F : \mathcal{N} \to \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ is of the form $F(T) = T \otimes_A Y$ for some $A - A'$ bimodule $Y$ with support in $\mathcal{C}_g$, here we identify $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ with $Mod_{\mathcal{C}_g} - A'$ as above.*

*Proof.* The proof follows the lines of the remark after Proposition 2.1 of [2]. We simply consider $F(A)$. The multiplication map $A \otimes A \to A$ gives us a map $A \otimes F(A) \to F(A)$, thus equipping $F(A)$ with a structure of a left $A$-module. We now see that $F(A)$ is indeed an $A - A'$ bimodule. Since the category $\mathcal{N}$ is semisimple the functor $F$ is exact. Since every object in $\mathcal{N}$ s a quotient of an object of the form $X \otimes A$ for some $X \in \mathcal{C}$, we see that $F$ is given by $F(T) = T \otimes_A F(A)$.

□

*Remark.* Notice that by applying the (2-)functor $\mathcal{C}_{g^{-1}} \boxtimes_{\mathcal{D}} -$ we see that every functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}' \to \mathcal{N}$ is given by tensoring with some $A' - A$ bimodule with support in $C_{g^{-1}}$.

8.2. **The outer action of $H$ on the group $Aut_{\mathcal{D}}(\mathcal{N})$. The first two obstructions.** Assume, as in the rest of the paper, that we have a subgroup $H < G$ and a module category $\mathcal{N} = Mod_{\mathcal{D}} - A$, and assume that $F_h : \mathcal{N} \cong \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$ for every $h \in H$. It follows from Lemma 8.2 that this equivalence is of the form $F_h(M) = M \otimes_A A_h$ for some $A - A$ bimodule $A_h$ with support in $\mathcal{C}_h$. The fact that this functor is an equivalence simply means that the bimodule $A_h$ is an invertible $A - A$ bimodule. In other words- there is another $A - A$ bimodule $B_h$ (whose support will necessary be in $\mathcal{C}_{h^{-1}}$) such that $A_h \otimes_A B_h \cong B_h \otimes_A A_h \cong A$. By Lemma 8.2 we can identify the group $\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$ with the group of isomorphisms classes of invertible $A - A$ bimodules with support in $\mathcal{D}$.

Denote by $\Lambda$ the group of isomorphisms classes all invertible $A - A$ bimodules with support in $\mathcal{C}_H$. Since every invertible $A - A$ bimodule is supported on a single grading component, we have a map $p : \Lambda \to H$ which assigns to an invertible $A - A$ bimodule the graded component it is supported on. We thus have a short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1. \tag{8.1}$$

Using this sequence, we can understand the outer action of $H$ on $Aut_{\mathcal{D}}(\mathcal{N})$, and the first and the second obstruction. The outer action is given in the following way: for $h \in H$, choose an invertible $A - A$ bimodule $A_h$ with support in $\mathcal{C}_h$. Choose an inverse to $A_h$ and denote it by $A_h^{-1}$. Then the action of $h \in H$ on some invertible bimodule $M$ with support in $\mathcal{D}$ is the following conjugation:

$$h \cdot M = A_h \otimes_A M \otimes_A A_h^{-1}.$$

This action depends on the choice we made of the invertible bimodule $A_h$.

The first obstruction is the possibility to lift this outer action to a proper action. In other words, it says that we can choose the $A_h$'s in such a way that conjugation by $A_h \otimes A_{h'}$ is the same as conjugation by $A_{hh'}$, or in other words, in such a way that for every $h, h' \in H$, the invertible bimodule

$$B_{h,h'} = A_h \otimes_A A_{h'} \otimes_A A_{hh'}^{-1}$$

will be in the center of $\Gamma$ (again- we identify $\Gamma$ with the group of invertible bimodules with support in $\mathcal{D}$). A solution for the first obstruction will be a choice of a set of such bimodules $A_h$.

The second obstruction says that the cocycle $(h, h') \mapsto B_{h,h'}$ is trivial in $H^2(H, Z(Aut_{\mathcal{D}}(\mathcal{N}))$. This simply says that we can change $A_h$ to be $A_h \otimes_A D_h$ for some $D_h \in Z(Aut_{\mathcal{D}}(\mathcal{N}))$, in such a way that

$$(A_h \otimes_A D_h) \otimes_A (A_{h'} \otimes_A D_{h'}) \otimes_A (A_{hh'} \otimes_A D_{hh'})^{-1} \cong A$$

as $A$-bimodules. A solution for the second obstruction will be a choice of such a set $D_h$ of bimodules.

It is easier to understand the first and the second obstruction together: we have one big obstruction- the sequence 8.1 should split, and we need to choose a splitting. First, if the sequence splits, then we can lift the outer action into a proper action, and we need to choose such a lifting. Then, the obstruction to the splitting *with the chosen action* is given by a two cocycle with values in the center of $\Gamma$. Thus, a solution for both the first and the second obstruction will be a choice of bimodules $A_h$ for every $h \in H$ such that the support of $A_h$ is in $\mathcal{C}_h$ and such that $A_h \otimes_A A_{h'} \cong A_{hh'}$ for every $h, h' \in H$. Following the line of Section 6, we see that we are interested in splittings only up to conjugation by an element of $\Gamma$.

8.3. **The third obstruction.** Assume then that we have a set of bimodules $A_h$ as in the end of the previous subsections. We would like to understand now the third obstruction.

Recall that we are trying to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category. By Ostrik's Theorem (see [1]), there is an object $\mathcal{N} \in \mathcal{N}$ such that $A \cong \underline{Hom}_{\mathcal{D}}(N, N)$ where by $\underline{Hom}_{\mathcal{D}}$ we mean the internal Hom of $\mathcal{N}$, where we consider $\mathcal{N}$ as a $\mathcal{D}$-module category. So far we gave equivalences

$F_h : \mathcal{N} \to \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$. If $\mathcal{N}$ were a $\mathcal{C}_H$-module category via the choices of these equivalences, then the internal $\mathcal{C}_H$-Hom, $\tilde{A} = \underline{Hom}_{\mathcal{C}_H}(N, N)$ would be

$$\tilde{A} = \bigoplus_{h \in H} A_h.$$

We thus see that to give on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the same as to give on $\tilde{A}$ a structure of an associative algebra. For every $h, h' \in H$, choose an isomorphism of $A - A$ bimodules $A_h \otimes_A A_{h'} \to A_{hh'}$. Notice that since these are invertible $A - A$ bimodules, there is only one such isomorphism up to a scalar.

Now for every $h, h', h'' \in H$, we have two isomorphisms $(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'h''}$, namely

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'} \otimes_A A_{h''} \to A_{hh'h''}$$

and

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_h \otimes_A (A_{h'} \otimes_A A_{h''}) \to A_h \otimes_A A_{h'h''} \to A_{hh'h''}.$$

This two isomorphisms differ by a scalar $b(h, h', h'')$. The function $(h, h', h'') \mapsto b(h, h', h'')$ is a three cocycle which is the third obstruction. A solution to the third obstruction will thus be a choice of isomorphisms $A_h \otimes_A A_{h'} \to A_{hh'}$ which will make $\tilde{A}$ an associative algebra. Once we have such a choice, we can change it by some two cocycle to get another solution.

8.4. **Functor categories.** We end this section by giving an intrinsic description of functor categories. Assume that we have two module categories $\mathcal{M}_1 = \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$, and $\mathcal{M}_2 = \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$. Let us denote $H'$ by $K$. As we have seen in the previous subsections, if $\mathcal{N} \cong Mod_{\mathcal{D}} - A_1$ and $\mathcal{N}' \cong Mod_{\mathcal{D}} - B_1$, then $\mathcal{M}_1 \cong Mod_{\mathcal{C}} - A$ and $\mathcal{M}_2 \cong Mod_{\mathcal{C}} - B$, where $A$ is an algebra of the form $\oplus_{h \in H} A_h$, and a similar description holds for $B$.

The functor category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A-B$-bimodules in $\mathcal{C}$. Since $A$ and $B$ have a graded structure, we will be able to say something more concrete on this category.

Let $X$ be an indecomposable $A - B$-bimodule in $\mathcal{C}$. It is easy to see that the support of $X$ will be contained inside a double coset of the form $HgK$ for some $g \in G$. Since the bimodules $A_h$ and $B_k$ are invertible, it is easy to see that the support will be exactly this double coset.

Consider now the $g$-component $X_g$ of $X$. As can easily be seen, this is an $A_1 - B_1$-bimodule. Actually, more is true. Consider the category $\mathcal{C} \boxtimes \mathcal{C}^{op}$. Inside this category we have the algebra

$$(AB)_g = \oplus_{x \in H \cap gKg^{-1}} A_x \boxtimes B_{g^{-1}x^{-1}g}$$

with the multiplication defined by the restricting the multiplication from $A \boxtimes B \in \mathcal{C} \boxtimes \mathcal{C}^{op}$ . The category $\mathcal{C}$ is a $\mathcal{C} \boxtimes \mathcal{C}^{op}$-module category in the obvious way, and we have a notion of an $(AB)_g$-module inside $\mathcal{C}$.

**Lemma 8.3.** *The category of $(AB)_g$-modules inside $\mathcal{C}$ is equivalent to the category of $A - B$-bimodules with support in the double coset $HgK$.*

*Proof.* If $X$ is an $A - B$-bimodule with support in $HgK$, then $X_g$ is an $(AB)_g$-module via restriction of the left $A$-action and the right $B$-action.

Conversely, if $V$ is an $(AB)_g$-module inside $\mathcal{C}$, we can consider the induced module

$$(A \boxtimes B) \otimes_{(AB)_g} V.$$

This is an $A-B$-bimodule, and one can see that the two constructions gives equivalences in both directions.

□

*Remark.* This is a generalization of Proposition 3.1 of [2], where the same situation is considered for the special case that $\mathcal{C} = Vec_G^{\omega}$ and $\mathcal{D} = 1$.

In conclusion, we have the following

**Proposition 8.4.** *The functor category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A-B$-bimodules. Each such simple bimodule is supported on a double coset of the form $HgK$, and the subcategory of bimodules with support in $HgK$ is equivalent to the category of $(AB)_g$-modules inside $\mathcal{C}$.*

## 9. An example: classification of modules categories over the Tambara Yamagami fusion categories and dual categories

Our second example will be $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$, the Tambara Yamagami fusion categories. Let $A$ be a finite group. Let $R_A$ be the fusion ring with basis $A \cup \{m\}$ whose multiplication is given by the following formulas:
$g \cdot h = gh$, for every $g, h \in A$
$g \cdot m = m \cdot g = m$ and
$m \cdot m = \sum_{g \in A} g$.
In [7] Tambara and Yamagami classified all fusion categories with the above fusion ring. They showed that if there is a fusion category $\mathcal{C}$ whose fusion ring is $R_A$ then $A$ must be abelian. They also showed that for a given $A$ such fusion categories can be parametrized (up to equivalence) by pairs $(\chi, \tau)$ where $\chi : A \times A \to k^*$ is a nondegenerate symmetric bicharacter, and $\tau$ is a square root (either positive or negative) of $\frac{1}{|A|}$. We denote the corresponding fusion category by $\mathcal{TY}(A, \chi, \tau)$.

The category $\mathcal{TY}(A, \chi, \tau)$ is naturally graded by $\mathbb{Z}_2$. The trivial component is $Vec_A$ and the nontrivial component, which we shall denote by $\mathcal{M}$, has one simple object $m$. $\mathcal{M}$ with its left(right) module structure over the trivial component is rank one, hence, the associativity constraints of $Vec_A$ are trivial. In [5] the authors described how the Tambara Yamagami fusion categories corresponds to an extension data of $Vec_A$ by the group $\mathbb{Z}_2$. We shall explain now the classification of module categories over $\mathcal{TY}(A, \chi, \tau)$ given by our construction.

### 9.1. Getting started.

We begin by recalling how module categories over $Vec_A$ look like. Since $A$ is abelian and the associativity constraints are trivial, module categories over $Vec_A$ are parametrized (up to equivalence) by pairs $(H, \psi)$ where $H < A$ and $\psi \in H^2(H, k^*)$. We can think of the corresponding module category $\mathcal{M}(H, \psi)$ as the category of right $k^{\psi}H$-modules inside $Vec_A$.

The first two parameters for a module category over $\mathcal{TY}(A, \chi, \tau)$ are an indecomposable module category $\mathcal{M}(H, \psi)$ over $Vec_A$ and a subgroup of $\mathbb{Z}_2$. The group $\mathbb{Z}_2$ has only two subgroups- itself and the trivial subgroup. If we take the trivial subgroup, then all other parameters and obstructions are trivial. We thus get in this way all module categories of the form $Ind^{\mathcal{C}}_{Vec_A}(\mathcal{M}(H,\psi))$. By Section 6 it is easy to see that two such distinct module categories $\mathcal{M}(H,\psi)$ and $\mathcal{M}(H',\psi')$ will be equivalent if and only if $\sigma\cdot\mathcal{M}(H,\psi)\equiv\mathcal{M}(H',\psi')$, where we denote by $\sigma$ the nontrivial element of the grading group $\mathbb{Z}_2$.

If we have a module category whose parametrization begins with $(\mathcal{M}(H,\psi),\mathbb{Z}_2,\ldots)$ then $\sigma\cdot\mathcal{M}(H,\psi)\equiv\mathcal{M}(H,\psi)$. In order to classify module categories over $\mathcal{C}$ we thus need to understand what is $\sigma\cdot\mathcal{M}(H,\psi)$ and when does $\sigma\cdot\mathcal{M}(H,\psi)\equiv\mathcal{M}(H,\psi)$. We shall do so by using the intrinsic description from Section 8. We shall also use the intrinsic description in order to understand the obstructions and their solutions.

9.2. **The action of $\sigma$ on indecomposable module categories and representations of twisted abelian group algebras.** Let $\mathcal{N}=\mathcal{M}(H,\psi)=Mod_{Vec_A}k^{\psi}H$. As explained in Section 8, the module category $\mathcal{M}\boxtimes_{Vec_A}\mathcal{N}$ can be described as the category of right $k^{\psi}H$-modules with support in the category $\mathcal{M}$, here $\mathcal{M}$ is the nontrivial grading component of $\mathcal{TY}(A,\chi,\tau)$. A $k^{\psi}H$-module with support in $\mathcal{M}$ is of the form $m\otimes V$ where $V$ is a vector space which is a $k^{\psi}H$-module in the usual sense. So the category $\mathcal{M}\boxtimes_{Vec_A}\mathcal{N}$ is equivalent, as an abelian category, to the category of $k^{\psi}H$-modules in the usual sense.

We shall describe now a parametrization of the simple $k^{\psi}H$-modules. Let $k^{\psi}H=\oplus_{h\in H}U_h$. The multiplication in $k^{\psi}H$ is given by the rule $U_hU_k=\psi(h,k)U_{hk}$. Denote by $R=Rad(\psi)$ the subgroup of all $h\in H$ such that $U_h$ is central in $k^{\psi}H$.

As the field $k$ is algebraically closed of characteristic zero and $H$ is abelian, the data that stored in the cocycle $\psi$ is simply the way in which the $U_h$'s commute. More precisely- define the following alternating form on $H$:

$$\xi_{\psi}(a,b)=\psi(a,b)/\psi(b,a).$$

It turns out (see [6]) that the assignment $\psi\mapsto\xi_{\psi}$ depends only on the cohomology class of $\psi$, and that it gives a bijection between $H^2(H,k^*)$ and the set of all alternating forms on $H$. The elements of $R$ can be described as those $h\in H$ such that $\xi_{\psi}(h,-)=1$. As can easily be seen, $\xi_{\psi}$ is the inflation of an alternating form on $H/R$. It follows easily that $\psi$ is the inflation of a two cocycle $\bar{\psi}$ on $H/R$.

It can also be seen that $\xi_{\bar{\psi}}$ is nondegenerate on $H/R$ and that $k^{\bar{\psi}}H/R\cong M_n(k)$ where $n=\sqrt{|H/R|}$. It follows that $k^{\bar{\psi}}H/R$ has only one simple module (up to isomorphism) which we shall denote by $V_1$ (i.e, $\bar{\psi}$ is non degenerate on $H/R$). By inflation, $V_1$ is also a $k^{\psi}H$-module. Let $\zeta$ be a character of $H$, and let $k^{\zeta}$ be the corresponding one dimensional representation of $H$. Then $k^{\zeta}\otimes V_1$ is also a simple module of $k^{\psi}H$, where $H$ acts diagonally. It turns out that these are all the simple modules of $k^{\psi}H$, and that $V_{\zeta_1}\cong V_{\zeta_2}$ if and only if the restrictions of $\zeta_1$ and $\zeta_2$ to $R$ coincide.

The simple modules of $k^{\psi}H$ are thus parametrized by the characters of $R$ (we use here the fact that the restriction from the character group of $H$ to that of $R$ is onto). For every character $\zeta$ of $R$, we denote by $V_{\zeta}$ the unique simple module of $k^{\psi}H$ upon which $R$ acts via the character $\zeta$. So the simple $k^{\psi}H$-modules with support in $\mathcal{M}$ are of the form $m \otimes V_{\zeta}$.

Let us describe $V_a \otimes (m \otimes V_{\zeta})$. It can easily be seen that this is also a simple module, so we just need to understand via which character $R$ acts on it. Using the associativity constraints in $\mathcal{TY}(A, \chi, \tau)$, we see that for $v \in V_{\zeta}$ and $r \in R$ we have

$$(V_a \otimes m \otimes v) \cdot U_r = \chi(a, r) V_a \otimes (m \otimes v \cdot U_r) = \chi(a, r)\zeta(r) V_a \otimes m \otimes v.$$

This means that $V_a \otimes (m \otimes V_{\zeta}) = m \otimes V_{\zeta\chi(a,-)}$. So the stabilizer of $V_{\zeta}$ is the subgroup of all $a \in A$ such that $\chi(a, r) = 1$ for all $r \in R$, i.e., it is $R^{\perp}$ (by $\perp$ we mean with respect to $\chi$. It follows that $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ is equivalent to a category of the form $\mathcal{M}(R^{\perp}, \tilde{\psi})$. Where $\tilde{\psi}$ is some two cocycle.

Let us figure out $\tilde{\psi}$. If $a \in R^{\perp}$, then the restriction of $\chi(a, -)$ to $H$ is a character which vanishes on $R$. Therefore, there is a unique (up to multiplication by an element of $R$) element $t_a \in H$ such that $\xi_{\psi}(t_a, -) = \chi(a, -)$. It follows that there is an isomorphism $r_a : V_a \otimes (m \otimes V_1) \to m \otimes V_1$ which is given by the formula $V_a \otimes (m \otimes v) \mapsto m \otimes (v \cdot U_{t_a})$. Now for every $a, b \in R^{\perp}$, $\tilde{\psi}(a, b)$ should make the following diagram commute:

$$\begin{tikzcd}
(V_a \otimes V_b) \otimes (m \otimes V_1) \arrow[r] \arrow[d, "r_{ab}"] & V_a \otimes (V_b \otimes (m \otimes V_1)) \arrow[r, "r_b"] & V_a \otimes (m \otimes V_1) \arrow[d, "r_a"] \\
m \otimes V_1 \arrow[rr, "\tilde{\psi}(a,b)"] & & m \otimes V_1
\end{tikzcd}$$

An easy calculation shows that this means that $\tilde{\psi}(a, b) = \psi(t_b, t_a)$. We thus have the following result:

**Lemma 9.1.** *We have $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(R^{\perp}, \tilde{\psi})$ where $R$ is the radical of $\psi$ and $\tilde{\psi}$ is described above.*

Suppose now that $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(H, \psi)$. This means that $Rad(\psi) = H^{\perp}$. The bicharacter $\chi$ defines by restriction a pairing on $H \times H$, and by dividing out by $H^{\perp}$, we get a nondegenerate symmetric bicharacter $\bar{\chi} : H/H^{\perp} \times H/H^{\perp} \to k^*$. It is easy to see that the assignment $h \mapsto t_h$ that was described above induces an automorphism $s$ of $H/H^{\perp}$ which satisfies $\bar{\chi}(a, b) = \xi_{\bar{\psi}}(s(a), b)$. The fact that $\tilde{\psi} = \psi$ means that $\xi_{\bar{\psi}}(s(b), s(a)) = \xi_{\bar{\psi}}(a, b)$. Equivalently, this means that $\bar{\chi}(a, b) = \xi_{\bar{\psi}}(s(a), b) = \xi_{\bar{\psi}}(s(b), s^2(a)) = \bar{\chi}(b, s^2(a))$ and since $\bar{\chi}$ is nondegenerate, this is equivalent to the fact that $s^2 = Id$.

In summary:

**Lemma 9.2.** *We have $\sigma \cdot \mathcal{M}(H, \psi) \equiv \mathcal{M}(H, \psi)$ if and only if the following two conditions hold:*

*1.$H^{\perp} < H$.*

*2.There is an automorphism $s$ of order 2 of $H/H^{\perp}$ such that $(a, b) \mapsto \bar{\chi}(s(a), b)$ is an alternating form. The inflation of this alternating form to $H$ will be $\xi_{\psi}$.*

9.3. **The vanishing of the first obstruction and invertible bimodules with support in $Vec_A$.** Assume now that we have a module category $\mathcal{M}(H,\psi)$ such that $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(H,\psi)$. Let $s$ be an automorphism as in Lemma 9.2. In order to explain the first and the second obstruction for furnishing a $\mathcal{TY}(A,\chi,\tau)$-module category structure on $\mathcal{M}(H,\psi)$, we need to consider the group of invertible $k^\psi H$-bimodules in $\mathcal{C}$. As we have seen in Section 8, such an invertible bimodule with support in $Vec_A$ ($\mathcal{M}$) corresponds to a functor equivalence $F:\mathcal{N}\to\mathcal{N}$ ($F:\mathcal{M}\boxtimes_{Vec_A}\mathcal{N}\to\mathcal{N}$). The functor is given by tensoring with the invertible bimodule.

Let us first classify invertible $k^\psi H$-bimodules with support in $Vec_A$. Their description was given in Ostrik's paper [2]. We recall it briefly.

If $a\in A$ and $\lambda$ is a character on $H$, we define the bimodule $M_{a,\lambda}$ to be

$$\oplus_{h\in H} V_{ah},$$

where the action of $k^\psi H$ is given by

$$U_h \cdot V_{ah'} \cdot U_{h''} = \psi(h,h')\lambda(h)\psi(hh',h'')V_{ahh'h''}.$$

Choose now coset representatives $a_1,\ldots,a_r$ of $H$ in $A$. Proposition 3.1 of [2] tells us that the modules $M_{a_i,\lambda}$ where $i=1,\ldots r$ and $\lambda\in\hat{H}$ are all the invertible $k^\psi H$ bimodules, and each invertible bimodule with support in $Vec_A$ appears in this list exactly once.

By a more careful analysis we can get to the following description of the group of invertible bimodules: we have a homomorphism $\xi: H\to\hat{H}$ given by $h\mapsto \xi_\psi(h,-)$. Then the group $E$ of all invertible bimodules with support in $Vec_A$ can be described as the pushout which appears in the following diagram:

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\xi"] & A \arrow[d] \\ \hat{H} \arrow[r] & E \end{tikzcd}$$

The group $E$ is thus also isomorphic to the group $Aut_{Vec_A}(\mathcal{M}(H,\psi))$. Notice that the group $E$ *abelian*. This means that $Aut(E)=Out(E)$ and the first obstruction vanishes; to give a homomorphism $\mathbb{Z}_2\to Out(E)$ is the same thing as to give an automorphism $\mathbb{Z}_2\to Aut(E)$. This also means that we do not have a choice of a "solution" in here, and that we have a proper, not outer, action of $\mathbb{Z}_2$ on $E$ from the beginning.

9.4. **The group of all invertible bimodules.** Since $\mathcal{M}(H,\psi)$ is $\sigma$-invariant, we see by Section 8 that the group $\tilde{E}$ of (isomorphism classes of) invertible $k^\psi H$ bimodules in $\mathcal{C}$ is given as an extension

$$1\to E\to\tilde{E}\to\mathbb{Z}_2\to 1.$$

Moreover, we have seen that the second obstruction is the cohomology class of this extension in $H^2(\mathbb{Z}_2,E)$, and that a solution to the second obstruction is a splitting of this sequence, up to conjugation by an element of $E$.

Now suppose that we have an invertible $k^\psi H$ bimodule $X$ with support in $\mathcal{M}$. Then any other invertible $k^\psi H$ bimodule with support in $\mathcal{M}$ will be of the form $X\otimes_{k^\psi H} e$, where $e\in E$. The action of $\sigma$ on $E$ will be conjugation by $X$ (as we have seen earlier, this is well defined and not depend on $X$ as the group $E$ is abelian), and the second obstruction will be the possibility to choose $X$ in such a way that $X \otimes_{k^\psi H} X \cong k^\psi H$ (that is- $X^2$ is the identity in the group of invertible bimodules).

We begin by choosing $X$ explicitly. It should be of the form $X = m \otimes V$, where $V$ is both a left and a right $k^\psi H$ bimodule. The interaction between the left structure and the right structure is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h, h')U_h \cdot (v \cdot U_{h'}). \tag{9.1}$$

The fact that $X$ is invertible implies that $V$ has to be simple as a left and as a right $k^\psi H$-module. Take $V$ to be $V_1$ from Subsection 9.2. we need to define on $V$ a structure of a left $k^\psi H$-module. We know that

$$(v \cdot U_{t_h}) \cdot U_{h'} = \chi(h, h')(v \cdot U_{h'})U_{t_h}.$$

By Equation 9.1 and by the simplicity of $V$, we see that this means that we must have $U_h \cdot v = \nu(h)v \cdot U_{t_h}$ for some set of scalars $\{\nu(h)\}_{h\in H}$. An easy calculation shows that these scalars should satisfy the equation

$$\nu(ab)\psi(a, b) = \nu(a)\nu(b)\psi(t_b, t_a)$$

for every $a, b \in H$. In other words-

$$\partial\nu(a, b) = \psi(a, b)/\psi(t_b, t_a). \tag{9.2}$$

Since $\mathcal{N}$ is $\sigma$-invariant, we do know that the cocycles $\psi(a, b)$ and $\psi(t_b, t_a)$ are cohomologous, and therefore such a function $\nu$ exists. Notice that we have some freedom in choosing $\nu$- we can change it to be $\nu\eta$ where $\eta$ is some character on $H$. It is easy to see by this construction that the invertible $k^\psi H$ bimodules are parametrized by pairs $(\phi, \nu)$ where $\phi$ is a character of $H^\perp$ by which it acts from the right on the module, and $\nu$ is a function which satisfy the equation

$$\partial\nu(a, b) = \psi(a, b)/\psi(t_a, t_b)\phi(t_{ab}t_a^{-1}t_b^{-1}).$$

When $\phi = 1$, this equation is exactly Equation 9.2. We denote the corresponding invertible bimodule by $X(\phi, \nu)$. It is easy to see that the restriction of $\nu$ to $H^\perp$ is a character. We fix an invertible bimodule $X$ for which $\phi = 1$, and for which the restriction of $\nu$ to $H^\perp$ is the trivial character (we use here the fact that we can alter $\nu$ by a character of $H$ and the fact that any character of $H^\perp$ can be extended to a character of $H$).

9.5. **The action of $\sigma$ on $E$, and the second obstruction.** We would like to understand now what is the bimodule $\sigma(U_{a_i,\lambda})$. We have the equation

$$X \otimes_{k^\psi H} U_{a_i,\lambda} = \sigma(U_{a_i,\lambda}) \otimes_{k^\psi H} X.$$

A similar calculation to the calculations we had so far reveals the fact that if $X$ is given by $(1, \nu)$ then $U_{a_i,\lambda} \otimes_{k^\psi H} X$ is given by $(\chi(a_i, -), \nu\lambda\chi^{-1}(a_i, t_-))$, while $X \otimes_{k^\psi H} U_{a_i,\lambda}$ is given by $(\lambda^{-1}, \nu\chi^{-1}(a_i, -)\lambda(t(-)))$. From these two formulas we can derive an explicit formula for the action of $\sigma$ on $E$. It follows that if $\sigma(U_{a_i,\lambda}) = U_{a_j,\mu}$ then $j$ is the unique index which satisfies $\lambda^{-1} = \chi(a_j, -)$ on $H^\perp$, and $\mu$ is given by the formula $\mu = \chi^{-1}(a_i, -)\lambda(t_-)\chi(a_i, t_-)$.

Let us find now the second obstruction. For this, we just need to calculate $X \otimes_{k^\psi H} X$. Consider first $X \otimes X$. It is isomorphic to $V \otimes V \otimes \bigoplus_{a\in A}(V_a)$ Let us divide this out first by the action of $H^\perp$. If $h \in H^\perp$ we see that we divide $V \otimes V \otimes V_a$ by $v \otimes w - \chi(a,h) v \otimes w$. If $a \notin H$ then there is an $h \in H^{\perp}$ such that $\chi(a,h) \neq 1$. Therefore the support of $X \otimes_{k^{\psi}H} X$ will be $Vec_H$. Since $V$ is simple as a left and as a right $k^{\psi}H$-module, it is easy to see that $V \otimes_{k^{\psi}H} V$ is one dimensional. We thus see that $X \otimes_{k^{\psi}H} X \cong U_{1,\lambda}$ for some character $\lambda$. A direct calculation shows that $\lambda(h) = \nu(h)\nu(t_h)$. This means that the second obstruction is the character $\lambda$, as an element of $H^2(\mathbb{Z}_2, E) = E^{\sigma}/im(1+\sigma)$ (recall that $\hat{H}$ is a subgroup of $E$).

Suppose that the second obstruction does vanish, and suppose that we have a solution $X(\omega,\eta)$. In other words $X(\omega,\eta) \otimes_{k^{\psi}H} X(\omega,\eta) \cong k^{\psi}H$. A direct calculation similar to the one we had above shows that the restriction of $\omega$ to $H^{\perp}$ coincide with $\eta$. Recall from Section 8 that if $U_{a_i,\lambda}$ is any invertible $k^{\psi}H$-bimodule with support in $Vec_A$, then this solution is equivalent to the solution $U_{a_i,\lambda} \otimes_{k^{\psi}H} X(\omega,\eta) \otimes_{k^{\psi}H} U^{-1}_{a_i,\lambda}$. A direct calculation shows that $U_{1,\eta} \otimes_{k^{\psi}H} X(\omega,\eta) \otimes_{k^{\psi}H} U^{-1}_{1,\eta} = X(\omega',1)$ where $\omega(H^{\perp}) = 1$. It follows that we can assume without loss of generality that the solution is of the form $X(\omega,1)$.

As we have seen above, $X(\omega,1)^{\otimes 2} \cong k^{\psi}H$ if and only if $\omega(h)\omega(t_h) = 1$ for every $h \in H$. So the second obstruction vanishes if and only if there is a function $\nu$ which satisfies equations 9.2 and also the equation

$$\nu(h)\nu(t_h) = 1 \tag{9.3}$$

for every $h \in H$. It might happen, however, that we will have two different solutions $\nu$ and $\nu'$, that will be equivalent- that is, there will be an invertible $k^{\psi}H$ bimodule $U_{a_i,\lambda}$ such that $U_{a_i,\lambda} \otimes_{k^{\psi}H} X(1,\nu) \otimes_{k^{\psi}H} U^{-1}_{a_i,\lambda} \cong X(1,\nu')$. A careful analysis shows that this happen if and only if the following condition holds: there is a character $\eta$ on $H$ which vanishes on $H^{\perp}$, such that

$$\nu(h)/\nu'(h) = \eta(h)/\eta(t_h). \tag{9.4}$$

In conclusion- the second obstruction is the existence of a function $\nu$ which satisfy Equations 9.2 and 9.3. and two such functions $\nu$ and $\nu'$ give equivalent solutions if and only if there is a character $\eta$ of $H$ which vanishes on $H^{\perp}$ and which satisfies Equation 9.4.

9.6. **The third obstruction.** As explained in Section 8, after solving the second obstruction, we can think about the third obstruction in the following way: we have an invertible $k^{\psi}H$ bimodule $X$ with support in $\mathcal{M}$, and $X \otimes_{k^{\psi}H} X \cong k^{\psi}H$. We would like to turn $k^{\psi}H \oplus X$ into an algebra. The only obstruction for that (and this is the third obstruction) is that the multiplication on $X \otimes X \otimes X$ might be associative only up to a scalar. This scalar is the third obstruction, considered as an element of $H^3(\mathbb{Z}_2, k^*) = \{1,-1\}$. Following the work of Tambara (see [6]), we see that this sign is the sign of the following expression

$$\Sigma_{h\in H}\nu(h)\tau.$$

If the third obstruction vanishes, we only have one possible solution, as $H^2(\mathbb{Z}_2, k^*) = 1$, since we have assumed that $k$ is algebraically closed.

9.7. **Dual categories.** In this subsection we shall give a general description of the dual categories of $\mathcal{TY}(A,\chi,\tau)$.

We begin with module categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N}, 1, \Phi, v, \beta)$. In this case, $\mathcal{L} \cong Mod_{\mathcal{C}} - k^{\psi}H$ for some $H < A$ and some two cocycle $\psi$. We have described above the category of $k^{\psi}H$-bimodules with support in $Vec_A$. We have seen that it will be a pointed category with an abelian group of invertible objects, which we have described in Subsection 9.3. Consider now the $k^{\psi}H$-bimodules with support in $\mathcal{M}$. Following previous calculations, we see that such a bimodule is given by a vector space $V$ which is both a left and a right $k^{\psi}H$-module, and the interaction between the left and the right structure is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h, h')U_h \cdot (v \cdot U_{h'}). \tag{9.5}$$

We can think of such modules as $k^{\theta}[H \times H]$-modules, where $\theta$ is a suitable two cocycle. By this point of view, the isomorphism classes of indecomposable modules is in bijection with the characters of $Rad(\theta) < H \times H$. Let us denote the indecomposable module which corresponds to a character $\zeta$ of $Rad(\theta)$ by $V_{\zeta}$. A routine and tedious calculation shows us that the group of invertible $k^{\psi}H$-bimodules with support in $Vec_A$ acts on the modules with support in $\mathcal{M}$ via the following formulas:

$$U_{a_i,\lambda} \otimes_B V_{\zeta} = V_{(\lambda, \chi(a_i,-))\zeta}$$

$$V_{\zeta} \otimes_B U_{a_i,\lambda} = V_{(\chi^{-1}(a_i,-),\lambda^{-1})\zeta}$$

We know that the dual category is graded by $\mathbb{Z}_2$ in the obvious sense. We use this fact in order to conclude the following multiplication formula:

$$V_{\zeta} \otimes_B V_{\eta} = \bigoplus_{(a_i,\lambda)} U_{a_i,\lambda}$$

where by $t^*(\eta)$ we mean the composition of $\eta$ with the map $H \times H \to H \times H$ given by $(h_1, h_2) \mapsto (h_2, h_1)$. Notice that by the analysis done in Section 8 and by the observation that the group of invertible bimodules with support in $Vec_A$ acts transitively on the set $\{V_{\zeta}\}$, we see that the dual is pointed if and only if the category $\mathcal{L}$ is $\sigma$-invariant.

We consider now module categories of the second type. By this we mean categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N}, \langle\sigma\rangle, \Phi, v, \beta)$. Assume that $\mathcal{L} = Mod - Vec_A\mathcal{M}(H, \psi)$ over $Vec_A$. Then $\sigma(H, \psi) = (H, \psi)$ and we have an action of $\sigma$ on the abelian group $E$ of invertible bimodules with support in $Vec_A$. We have an equivalence of fusion categories $(Vec_A)^*_{\mathcal{L}} \cong Vec^{\omega}_E$ for some three cocycle $\omega \in H^3(E, k^*)$

We have seen in Section 7 that the dual $(\mathcal{C})^*_{\mathcal{L}}$ will be the equivariantization of this category with respect to the action of $\mathbb{Z}_2$. If, for example, we would have known that $\omega = 1$, then this equivariantization would have been equivalent to the representation category of the group $\mathbb{Z}_2 \ltimes \hat{E}$ In general, the description of this category is not much harder.

We conclude by observing that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if there is a pair $(H, \psi)$ such that $\sigma(H, \psi) = (H, \psi)$. This gives an alternative proof of the fact that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if the metric group $(A, \chi)$ has a Lagrangian subgroup[6, Corollary 4.9].

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# MODULE CATEGORIES OVER GRADED FUSION CATEGORIES

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Abstract. Let $\mathcal{C}$ be a fusion category which is an extension of a fusion category $\mathcal{D}$ by a finite group $G$. We classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data $(c, M, \alpha)$ of $\mathcal{C}$. We also describe functor categories over $\mathcal{C}$ (and in particular the dual categories of $\mathcal{C}$). We use this in order to classify module categories over the Tambara Yamagami fusion categories, and their duals.

## 1. Introduction

Let $\mathcal{C}$ be a fusion category. We say that $\mathcal{C}$ is an extension of the fusion category $\mathcal{D}$ by a finite group $G$ if $\mathcal{C}$ is faithfully graded by the group $G$ in such a way that $\mathcal{C}_e = \mathcal{D}$. In [4] Etingof et. al. classified extension of a given fusion category $\mathcal{D}$ by a given finite group $G$. Their classification is given by a triple $(c, M, \alpha)$, where $c : G \to Pic(\mathcal{D})$ is a homomorphism, $M$ belongs to a torsor over $H^2(G, inv(Z(\mathcal{D})))$, and $\alpha$ belongs to a torsor over $H^3(G, k^*)$. The group $Pic(\mathcal{D})$ is the group of invertible $\mathcal{D}$-bimodules (up to equivalence), and the group $inv(Z(\mathcal{D}))$ is the group of (isomorphism classes of) invertible objects in the center $Z(\mathcal{D})$ of $\mathcal{D}$.

Let us recall briefly the construction from [4]. Suppose that we are given a classification data $(c, M, \alpha)$. The corresponding category $\mathcal{C}$ will be $\bigoplus_{g\in G} c(g)$ as a $\mathcal{D}$-bimodule category. If we choose arbitrary isomorphisms $c(g)\boxtimes_{\mathcal{D}} c(h) \to c(gh)$ for the tensor product in $\mathcal{C}$, the multiplication will not necessarily be associative. This non associativity is encoded in a cohomological obstruction $O_3(c) \in Z^3(G, inv(Z(\mathcal{D})))$. The element $M$ belongs to $C^2(G, inv(Z(\mathcal{D})))$, and should satisfy $\partial M = O_3(c)$ (that is- it should be a "solution" to the obstruction $O_3(c)$). If we change $M$ by a coboundary, we get an equivalent solution. Therefore, the choice of $M$ is equivalent to choosing an element from a torsor over $H^2(G, inv(Z(\mathcal{D})))$. Given $c$ and $M$, we still have one more obstruction in order to furnish from $\mathcal{C}$ a fusion category. This obstruction is the commutativity of the pentagon diagram, and is given by a four cocycle $O_4(c, M) \in Z^4(G, k^*)$. The element $\alpha$ belongs to $C^3(G, k^*)$, and should satisfy $\partial\alpha = O_4(c, M)$. We think of $\alpha$ as a solution to the obstruction $O_4(c, M)$. Again, if we change $\alpha$ by a coboundary, we will get an equivalent solution. Therefore, the choice of $\alpha$ can be seen as a choice from a torsor over $H^3(G, k^*)$.

We shall write $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$ to indicate the fact that $\mathcal{C}$ is an extension of $\mathcal{D}$ by $G$ given by the extension data $(c, M, \alpha)$, and we shall assume from now on that $\mathcal{C} = \mathcal{D}(G, c, M, \alpha)$.

In this paper we shall classify module categories over $\mathcal{C}$ in terms of module categories over $\mathcal{D}$ and the extension data $(\mathcal{C}, M, \alpha)$.

Our classification of module categories will follow the lines of the classification of [4]. We will begin by proving the following structure theorem for module categories over $\mathcal{C}$.

**Theorem 1.** *Let $\mathcal{L}$ be an indecomposable module category over $\mathcal{C}$. There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H = \bigoplus_{a\in H}\mathcal{C}_a$ module category $\mathcal{N}$ which remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \cong Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{N}) \triangleq \mathcal{C}\boxtimes_{\mathcal{C}_H}\mathcal{N}$.*

This proposition enables us to reduce the classification of $\mathcal{C}$-module categories to the classification of $\mathcal{C}_H$-module categories which remains indecomposable over $\mathcal{D}$, where $H$ varies over subgroups of $G$.

In order to classify such categories we will go, in some sense, the other way around. We will begin with an indecomposable $\mathcal{D}$-module category $\mathcal{N}$, and we will ask how can we equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$ module category.

As in the classification in [4], the answer will also be based upon choosing solutions to certain obstruction (in case it is possible). We will begin with the observation, in Section 3, that we have a natural action of $G$ on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. This action is given by the following formula

$$g\cdot\mathcal{N} = \mathcal{C}_g\boxtimes_{\mathcal{D}}\mathcal{N}.$$

If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, then the action of $\mathcal{C}_H$ on $\mathcal{N}$ will give an equivalence of $\mathcal{D}$-module categories $h\cdot\mathcal{N}\cong\mathcal{N}$ for every $h\in H$. In other words- $\mathcal{N}$ will be $H$-invariant. We may think of the fact that $\mathcal{N}$ should be $H$-invariant as the "zeroth obstruction" we have in order to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category.

In case $\mathcal{N}$ is $H$-invariant, we choose equivalences $\psi_a : \mathcal{C}_a\boxtimes_{\mathcal{D}}\mathcal{N}\to\mathcal{N}$ for every $a\in H$. We would like these equivalences to give us a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$. As one might expect, not every choice of equivalences will do that. If $\mathcal{N}$ has a structure of a $\mathcal{C}_H$-module category, we will see in Section 4 that we have a natural action of $H$ on the group $\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$. In case we only know that $\mathcal{N}$ is $H$-invariant, we only have an *outer* action of $H$ on $\Gamma$ (i.e. a homomorphism $\rho : H\to Out(\Gamma)$). The first obstruction will thus be the possibility to lift this outer action to a proper action.

Once we overcome this obstruction (and choose a lifting $\Phi$ for the outer action), our second obstruction will be the fact that the two functors

$$F_1, F_2 : \mathcal{C}_a\boxtimes_{\mathcal{D}}\mathcal{C}_b\boxtimes_{\mathcal{D}}\boxtimes\mathcal{N}\to\mathcal{N}$$

defined by

$$F_1(X\boxtimes Y\boxtimes N) = (X\otimes Y)\otimes N$$

and

$$F_2(X\boxtimes Y\boxtimes N) = X\otimes(Y\otimes N)$$

should be isomorphic. We will see that this obstruction is given by a certain two cocycle $O_2(\mathcal{N},c,H,M,\Phi)\in Z^2(H,Z(Aut_{\mathcal{D}}(\mathcal{N})))$. A solution for this obstruction is an element $v\in C^1(H,Z(Aut_{\mathcal{D}}(\mathcal{N})))$ that should satisfy $\partial v = O_2(\mathcal{N},c,H,M,\Phi)$.

Our last obstruction will be the fact that the above functors should be not only isomorphic, but they should be isomorphic in a way which will make the pentagon diagram commutative. This obstruction is encoded by a three cocycle $O_3(\mathcal{N},c,H,M,\Phi,v,\alpha)\in Z^3(H,k^*)$. A solution $\beta$ for this obstruction will be an element of $C^2(H,k^*)$ such that $\partial\beta = O_3(\mathcal{N},c,H,M,\Phi,v,\alpha)$.

We can summarize our main result in the following theorem:

**Theorem 2.** *An indecomposable module category over $\mathcal{C}$ is given by a tuple $(\mathcal{N},H,\Phi,v,\beta)$, where $\mathcal{N}$ is an indecomposable module category over $\mathcal{D}$, $H$ is a subgroup of $G$ which acts trivially on $\mathcal{N}$, $\Phi : H\to Aut(Aut_{\mathcal{D}}(\mathcal{N}))$ is a homomorphism, $v$ belongs to a torsor over $H^1(H,Z(Aut_{\mathcal{D}}(\mathcal{N})))$, and $\beta$ belongs to a torsor over $H^2(H,k^*)$.*

We shall denote the indecomposable module category which corresponds to the tuple $(\mathcal{N}, H, \Phi, v, \beta)$ by $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$. In order to classify module categories, we need to give not only a list of all indecomposable module categories, but also to explain when does two elements in the list define equivalent module categories. We will see in Section 6 that if $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ is any indecomposable module category, $g \in G$ is an arbitrary element and $F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv \mathcal{N}'$ is an equivalence of $\mathcal{D}$-module categories (where $\mathcal{N}'$ is another indecomposable $\mathcal{D}$-module category), then $F$ gives rise to a tuple $(\mathcal{N}', gHg^{-1}, \Phi', v', \beta')$ which defines an equivalent $\mathcal{C}$-module category. Our second main result is the following:

**Theorem 3.** *Two tuples $(\mathcal{N}, H, \Phi, v, \beta)$ and $(\mathcal{N}', H', \Phi', v', \beta')$ determine equivalent $\mathcal{C}$-module categories if and only if the second tuple is defined by the first tuple and by some equivalence $F$ as above.*

We shall prove Theorem 3 in Section 6. We will also decompose this condition into a few simpler ones: we will see, for example, by considering the case $g = 1$, that we can change $\Phi$ to be $t\Phi t^{-1}$, where $t$ is any conjugation automorphism of $Aut_{\mathcal{D}}(\mathcal{N})$.

In Section 7 we will describe the category of functors $Fun_{\mathcal{C}}(\mathcal{N}, \mathcal{M})$ where $\mathcal{N}$ and $\mathcal{M}$ are two module categories over $\mathcal{C}$. We will prove a Mackey type decomposition theorem, and we will also see that we can view this category as the equivariantization of the category $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{M})$ with respect to an action of $G$. We will also be able to prove the following criterion of $\mathcal{C}$ to be group theoretical: $\mathcal{C}$ is a group theoretical if and only if there is a pointed $\mathcal{D}$-module category $\mathcal{N}$ (i.e., $\mathcal{D}^*_{\mathcal{N}}$ is pointed), stable under the $G$-action, i.e., for every $g \in G$, $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ as $\mathcal{D}$-module categories. We shall also explain why this is a reformulation of the criterion which appears in [5].

A theorem of Ostrik says that any indecomposable module category over a fusion category $\mathcal{D}$ is equivalent to a category of the form $Mod_{\mathcal{D}} - A$, of right $A$-modules in the category $\mathcal{D}$, where $A$ is some semisimple indecomposable algebra in the category $\mathcal{D}$. In other words- any module category has a description by objects which lie inside the fusion category $\mathcal{D}$. In Section 8 we will explain how we can understand the obstructions and their solutions, and also the functor categories, by intrinsic description; that is- by considering algebras and modules inside the categories $\mathcal{D}$ and $\mathcal{C}$.

This description will be much more convenient for calculations. It will also enables us to view the first and the second obstruction in a unified way. Indeed, in Section 8 we will show that we have a natural short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1$$

and that a solution for the first two obstructions is equivalent to a choice of a splitting of this sequence (and therefore, we can solve the first two obstructions if and only if this sequence splits). We will also show, following the results of Section 8, that two splittings which differ by conjugation by an element of $\Gamma$ will give us equivalent module categories.

In Section 9 we shall give a detailed example. We will consider the Tambara Yamagami fusion categories, $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$. In this case $\mathcal{C}$ is an extension of the category $Vec_A$, where $A$ is an abelian group, by the group $\mathbb{Z}_2$.

*Remark.* During the final stages of the writing of this paper it came to our attention that Cesar Galindo is working on a paper with similar results. We would like to remark that our results and his were obtained independently.

## 2. Preliminaries

In this section, $\mathcal{C}$ will be a general fusion category and $\mathcal{D}$ a fusion subcategory of $\mathcal{C}$. We recall some basic facts about module categories over $\mathcal{C}$ and $\mathcal{D}$. For a more detailed discussion on these notions, we refer the reader to [7] and to [3]. Let $\mathcal{N}$ be a module category over $\mathcal{C}$. If $X, Y \in Ob\mathcal{N}$, then the *internal hom* of $X$ and $Y$ is the unique object of $\mathcal{C}$ which satisfies the formula

$$Hom_{\mathcal{C}}(W, \underline{Hom}_{\mathcal{C}}(X,Y)) = Hom_{\mathcal{N}}(W \otimes X, Y)$$

for every $W \in Ob\mathcal{C}$. For every $X \in Ob\mathcal{N}$ the object $\underline{Hom}_{\mathcal{C}}(X,X)$ has a canonical algebra structure. We say that $X$ *generates* $\mathcal{N}$ (over $\mathcal{C}$) if $\mathcal{N}$ is the smallest sub $\mathcal{C}$-module-category of $\mathcal{N}$ which contains $X$. For every algebra $A$ in $\mathcal{C}$, $mod_{\mathcal{C}} - A$, the category of right $A$-modules in $\mathcal{C}$, has a structure of a left $\mathcal{C}$-module category.

A theorem of Ostrik says that all module categories are of this form:

**Theorem 4.** *(see [7]) Let $\mathcal{N}$ be a module category, and let $X$ be a generator of $\mathcal{N}$ over $\mathcal{C}$. We have an equivalence of $\mathcal{C}$-module categories $\mathcal{N} \cong Mod_{\mathcal{C}} - \underline{Hom}(X,X)$ given by $F(Y) = \underline{Hom}(X,Y)$.*

Next, we recall the definition of the induced module category. If $\mathcal{N}$ is a $\mathcal{D}$-module category, $Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N})$ is a module category over $\mathcal{C}$ which satisfies Frobenius reciprocity. This means that for every $\mathcal{C}$-module category $\mathcal{R}$ we have that

$$Fun_{\mathcal{C}}(Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}), \mathcal{R}) \cong Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{R}).$$

The next lemma proves that the induced module category always exists. It will also gives us some idea about how the induced module category "looks like".

**Lemma 5.** *Suppose that $\mathcal{N} \cong mod_{\mathcal{D}} - A$ for some algebra $A \in Ob\mathcal{D}$. Then $A$ can also be considered as an algebra in $\mathcal{C}$, and $Ind_{\mathcal{D}}^{\mathcal{C}}(\mathcal{N}) \cong mod_{\mathcal{C}} - A$.*

*Proof.* Let us prove that Frobenius reciprocity holds. For this, we first need to represent $\mathcal{R}$ in an appropriate way. We choose a generator $X$ of $\mathcal{R}$ **over** $\mathcal{D}$. It is easy to see that $X$ is also a generator over $\mathcal{C}$. Then, by Ostrik's Theorem we have that $\mathcal{R} \cong mod_{\mathcal{C}} - \underline{Hom}_{\mathcal{C}}(X,X)$ over $\mathcal{C}$, and $\mathcal{R} \cong mod_{\mathcal{D}} - \underline{Hom}_{\mathcal{D}}(X,X)$ over $\mathcal{D}$. If we denote $\underline{Hom}_{\mathcal{C}}(X,X)$ by $B$, then it is easy to see by the definition of $\underline{Hom}$ that $\underline{Hom}_{\mathcal{D}}(X,X) \cong B_{\mathcal{D}}$, where $B_{\mathcal{D}}$ is the largest subobject of $B$ which is also an object of $\mathcal{D}$ (since $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$, this is also a subalgebra of $B$). By another theorem of Ostrik (see [7]), we know that $Fun_{\mathcal{C}}(mod_{\mathcal{C}} - A, mod_{\mathcal{C}} - B) \cong bimod_{\mathcal{C}} - A - B$. Using the theorem of Ostrik again, we see that $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{R}) \cong bimod_{\mathcal{D}}(A - B_{\mathcal{D}})$. One can verify that the functor which sends an $A - B_{\mathcal{D}}$ bimodule $Z$ in $\mathcal{D}$ to $Z \otimes_{B_{\mathcal{D}}} B$ gives an equivalence between the two categories. □

*Remark* 6. The fact that the induction functor is an equivalence of categories arise from the fact that for such a $B$, the equivalence between the categories $mod_{\mathcal{D}} - B_{\mathcal{D}}$ and $mod_{\mathcal{C}} - B$ is given by $X \mapsto X \otimes_{B_{\mathcal{D}}} B$.

One can show that the induced module category is also equivalent to $\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N}$.

In particular, we have the following:

**Corollary 7.** *Let $\mathcal{C}$ be a fusion category and let $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$. Let $\mathcal{N}$ be a module category over $\mathcal{C}$. Suppose that $X$ is a generator of $\mathcal{N}$ over $\mathcal{C}$, and that the algebra $A = \underline{Hom}(X,X)$ is supported on $\mathcal{D}$. Then $\mathcal{N} \cong Ind_{\mathcal{D}}^{\mathcal{C}}(mod_{\mathcal{D}} - A)$.*

## 3. Decomposition of the module category over the trivial component subcategory. The zeroth obstruction

We begin by considering the action of $G$ on $\mathcal{D}$-module categories. For every $g \in G$, $\mathcal{C}_g$ is an invertible $\mathcal{D}$-bimodule category. Therefore, if $\mathcal{N}$ is an indecomposable

$\mathcal{D}$-module category, the category $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ is also indecomposable. It is easy to see that we get in this way an action of $G$ on the set of (equivalence classes of) indecomposable $\mathcal{D}$-module categories. Let now $\mathcal{L}$ be an indecomposable $\mathcal{C}$-module category. We can consider $\mathcal{L}$ also as a module category over $\mathcal{D}$. We claim the following:

**Lemma 8.** *As a $\mathcal{D}$-module category, $\mathcal{L}$ is $G$-invariant.*

*Remark* 9. For this lemma, we do not need to assume that $\mathcal{L}$ is indecomposable.

*Proof.* We have the following equivalences of $\mathcal{D}$-module categories

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L}) \cong$$
$$(\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{C}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong (\mathcal{C}_g \boxtimes_{\mathcal{D}} \oplus_{a\in G}\mathcal{C}_a) \boxtimes_{\mathcal{C}} \mathcal{L} \cong$$
$$(\oplus_{a\in G}\mathcal{C}_{ga}) \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{L} \cong \mathcal{L}.$$

This proves the claim. □

If $H$ is a subgroup of $G$, we have the subcategory $\mathcal{C}_H = \bigoplus_{h\in H} C_h$ of $\mathcal{C}$, which is an extension of $\mathcal{D}$ by $H$. We claim the following:

**Proposition 10.** *There is a subgroup $H < G$, and an indecomposable $\mathcal{C}_H$ module category $\mathcal{N}$ which remains indecomposable over $\mathcal{D}$ such that $\mathcal{L} \equiv Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{N})$.*

*Proof.* Suppose that $\mathcal{L}$ decomposes over $\mathcal{D}$ as

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{L}_i.$$

For every $g \in G$, we have seen that the action functor defines an equivalence of categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \mathcal{L}$. Since

$$\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L} \cong \bigoplus_{i=1}^{n} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{L}_i,$$

we see that $G$ permutes the index set $\{1, \ldots, n\}$. This action is transitive, as otherwise $\mathcal{L}$ would not have been indecomposable over $\mathcal{C}$. Let $H < G$ be the stabilizer of $L_1$. Then $\mathcal{N} = \mathcal{L}_1$ is a $\mathcal{C}_H$-module category which remains indecomposable over $\mathcal{D}$. Let $X \in Ob\mathcal{L}_1$ be a generator of $\mathcal{L}$ over $\mathcal{C}$ (any nonzero object would be a generator, as $\mathcal{L}$ is indecomposable over $\mathcal{C}$). By the fact that the stabilizer of $\mathcal{L}_1$ is $H$, it is easy to see that $\underline{Hom}_{\mathcal{C}}(X, X)$ is contained in $\mathcal{C}_H$. The rest of the lemma now follows from corollary 7. □

So in order to classify indecomposable module categories over $\mathcal{C}$, we need to classify, for every $H < G$, the indecomposable module categories over $\mathcal{C}_H$ which remain indecomposable over $\mathcal{D}$. For every indecomposable module category $\mathcal{L}$ over $\mathcal{C}$, we have attached a subgroup $H$ of $G$ and an indecomposable $\mathcal{C}_H$ module category $\mathcal{L}_1$ which remains indecomposable over $\mathcal{D}$. The subgroup $H$ and the module category $\mathcal{L}_1$ will be the first two components of the tuple which corresponds to $\mathcal{L}$. Notice that we could have chosen any conjugate of $H$ as well.

## 4. The first two obstructions

Let $\mathcal{L}$, $\mathcal{N} = \mathcal{L}_1$ and $H$ be as in the previous section. For every $a \in H$ we have an equivalence of $\mathcal{D}$-module categories $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \cong N$ given by the action of $\mathcal{C}_H$ on $\mathcal{N}$. Suppose on the other hand that we are given an $H$-invariant indecomposable module category $\mathcal{N}$ over $\mathcal{D}$. Let us fix a family of equivalences $\{\psi_a\}_{a\in H}$, where $\psi_a : \mathcal{C}_a \boxtimes \mathcal{N} \to \mathcal{N}$. Let us see when does this family comes from an action of $\mathcal{C}_H$ on $\mathcal{N}$.

We know that the two functors

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \stackrel{m\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_H \boxtimes \mathcal{N} \dot{\to} \mathcal{N}$$

and

$$\mathcal{C}_H \boxtimes \mathcal{C}_H \boxtimes \mathcal{N} \stackrel{1_{\mathcal{C}_H}\boxtimes(\cdot)}{\to} \mathcal{C}_H \boxtimes \mathcal{N} \dot{\to} \mathcal{N}$$

should be isomorphic.

Since the action of $\mathcal{C}_H$ on $\mathcal{N}$ is given by the action of $\mathcal{D}$ together with the $\psi_a$'s, this condition translates to the fact that for every $a, b \in H$ the two functors

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_{ab}}{\to} \mathcal{N}$$

and

$$\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes\psi_b}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_a}{\to} \mathcal{N}$$

should be isomorphic. We can express this condition in the following equivalent way- for every $a, b \in H$, the autoequivalence of $\mathcal{N}$ as a $\mathcal{D}$-module category

$$Y_{a,b} = \mathcal{N} \stackrel{\psi_a^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes\psi_b^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N}$$
$$\stackrel{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_{ab}^{-1}}{\to} \mathcal{N}$$

should be isomorphic to the identity autoequivalence. We shall decompose this condition into two simpler ones.

Consider the group $\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$, where by $Aut_{\mathcal{D}}$ we mean the group of $\mathcal{D}$-autoequivalences (up to isomorphism) of $\mathcal{N}$. For $a \in H$ and $F \in \Gamma$ define $a \cdot F \in \Gamma$ as the composition

$$\mathcal{N} \stackrel{\psi_a^{-1}}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes F}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_a}{\to} \mathcal{N}.$$

We get a map $\Phi : H \to Aut(\Gamma)$ given by $\Phi(g)(F) = h \cdot F$. This map depends on the choice of the $\psi_a$'s and is not necessary a group homomorphism. However, the following equation does hold for every $a, b \in H$:

$$\Phi(a)\Phi(b) = \Phi(ab)C_{Y_{a,b}}, \tag{4.1}$$

where we write $C_x$ for conjugation by $x \in \Gamma$.

Notice that $\psi_a$ is determined up to composition with an element in $\Gamma$, and that by changing $\psi_a$ to be $\psi'_a = \gamma\psi_a$, for $\gamma \in \Gamma$, we change $\Phi(a)$ to be $\Phi(a)c_\gamma$, where by $c_\gamma$ we mean conjugation by $\gamma$. Equation 4.1 shows that the composition $\rho = \pi\Phi$, where $\pi$ is the quotient map $\pi : Aut(\Gamma) \to Out(\Gamma)$ does give a group homomorphism. Notice that by the observation above, $\rho$ does not depend on the choice of the $\psi_a$'s, but only on $c$, $\mathcal{N}$ and $H$. We have the following

**Lemma 11.** *Let $\mathcal{N}$ be $H$-invariant $\mathcal{D}$-module category. There is a well defined group homomorphism $\rho : H \to Out(\Gamma)$. If the $\psi_a$'s arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then the map $\Phi$ described above is a group homomorphism.*

*Proof.* This follows from the fact that by the discussion above, if the $\psi_a$'s arise from an action of $\mathcal{C}_H$ on $\mathcal{N}$, then $Y_{a,b}$ is trivial for every $a, b \in H$, and by Equation 4.1 we see that $\Phi$ is a group homomorphism. □

So $c, \mathcal{N}$ and $H$ determines a homomorphism $\rho : H \to Out(\Gamma)$. We thus see that in order to give $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category, we need to give a lifting of $\rho$ to a homomorphism to $Aut(\Gamma)$. The first obstruction is thus the possibility to lift $\rho$ in such a way.

Suppose then that we have a lifting, that is- a homomorphism $\Phi : H \to Aut(\Gamma)$ such that $\pi\Phi = \rho$. To say that $\Phi$ is a homomorphism is equivalent to say that we have chosen the $\psi_a$'s in such a way that $C_{Y_{a,b}} = Id$, or in other words- in such a way that for every $a, b \in H$, $Y_{a,b}$ is in $Z(\Gamma)$, the center of $\Gamma$.

Notice that after choosing $\Phi$, we still have some liberty in changing the $\psi_a$'s. Indeed, if we choose $\psi'_a = \gamma_a\psi_a$, where $\gamma_a \in Z(\Gamma)$ for every $a \in H$, we still get the same $\Phi$, and it is easy to see that every $\psi'_a$ that will give us the same $\Phi$ is of this form.

In order to furnish a structure of a $\mathcal{C}_H$-module category on $\mathcal{N}$, we need $Y_{a,b}$ to be not only central, but trivial. A straightforward calculation shows now that the function $H \times H \to Z(\Gamma)$ given by $(a, b) \mapsto Y_{a,b}$ is a two cocycle. If we choose a different set of isomorphisms $\psi'_a = \gamma_a\psi_a$ where $\gamma_a \in Z(\Gamma)$, we will get a cocycle $Y'$ which is cohomologous to $Y$. So the second obstruction is the cohomology class of the two cocycle $(a, b) \mapsto Y_{a,b}$. We shall denote this obstruction by $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. Notice that this obstruction depends linearly on $M$ in the following sense: we have a natural homomorphism of groups $\xi : inv(Z(\mathcal{D})) \to \Gamma$, given by the formula

$$\xi(T)(N) = T \otimes N$$

(that is- $\xi(T)$ is just the autoequivalence of acting by $T$) It can be seen that if we would have chosen $M' = M\zeta$, where $\zeta \in Z^2(G, Z(\mathcal{D}))$, then we would have changed $O_2$ to be $O_2 res^G_H(\xi_*(\zeta))$.

In conclusion- we saw that if $\mathcal{N}$ is a $\mathcal{D}$-module category upon which $H$ acts trivially, then we have an induced homomorphism $\rho : H \to Out(\Gamma)$. The first obstruction to define on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the fact that $\rho$ should be of the form $\pi\Phi$ where $\Phi : H \to Aut(Aut_{\mathcal{D}}(\mathcal{N}))$ is a homomorphism. After choosing such a lifting $\Phi$ we get the second obstruction, which is a two cocycle $O_2(\mathcal{N}, c, H, M, \Phi) \in Z^2(H, Z(\Gamma))$. A solution to this obstruction will be an element $v \in C^1(H, Z(\Gamma))$ which satisfies

$$\partial v = O_2(\mathcal{N}, c, H, M, \Phi).$$

We will see later, in Section 8, that to find a solution for the first and for the second obstruction is the same thing as to find a splitting for a certain short exact sequence. We will also see why two solutions $v$ and $v'$ which differs by a coboundary give equivalent module categories (and therefore we can view the set of possible solutions, in case it is not empty, as a torsor over $H^1(H, Z(\Gamma))$.

## 5. The third obstruction

So far we have almost defined a $\mathcal{C}_H$-action on $\mathcal{N}$, by means of the equivalences $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$. The solutions for the first and for the second obstruction ensures us that for every $a, b \in H$ the two functors

$$F_1 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{M_{a,b}\boxtimes 1_{\mathcal{N}}}{\to} \mathcal{C}_{ab} \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_{ab}}{\to} \mathcal{N}$$

and

$$F_2 : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{C}_b \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{1_{\mathcal{C}_a}\boxtimes\psi_b}{\to} \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{\psi_a}{\to} \mathcal{N}$$

are isomorphic.

For every $a, b \in H$, let us fix an isomorphism $\eta(a, b) : F_1 \to F_2$ between the two functors. In other words, for every $X \in \mathcal{C}_a$, $Y \in \mathcal{C}_b$ and $N \in \mathcal{N}$ we have a natural isomorphism

$$\eta(a, b)_{X,Y,N} : (X \otimes Y ) \otimes N \to X \otimes (Y \otimes N).$$

Since $F_1$ and $F_2$ are simple as objects in the relevant functor category (they are equivalences), the choice of the isomorphism $\eta(a, b)$ is unique up to a scalar, for every $a, b \in H$.

The final condition for $\mathcal{N}$ to be a $\mathcal{C}_H$-module category is the commutativity of the pentagonal diagram. In other words, for every $a, b, d \in H$, and every $X \in \mathcal{C}_a$,

$Y \in \mathcal{C}_b$, $Z \in \mathcal{C}_d$ and $N \in \mathcal{N}$, the following diagram should commute:

$$\begin{tikzcd}
(X \otimes (Y \otimes Z)) \otimes N \arrow[rr, "\eta(a,bd)_{X,Y\otimes Z,N}"] & & X \otimes ((Y \otimes Z) \otimes N) \arrow[d, "\eta(b,d)_{Y,Z,N}"] \\
((X \otimes Y) \otimes Z) \otimes N \arrow[u, "\alpha_{X,Y,Z}"] \arrow[dr, "\eta(ab,d)_{X\otimes Y,Z,N}"] & & X \otimes (Y \otimes (Z \otimes N)) \\
& (X \otimes Y) \otimes (Z \otimes N) \arrow[ur, "\eta(a,b)_{X,Y,Z\otimes N}"] &
\end{tikzcd}$$

This diagram will always be commutative up to a scalar $O_3(a, b, d)$ which depends only on $a, b$ and $d$, and not on the particular objects $X, Y, Z$ and $N$. One can also see that the function $(a, b, d) \mapsto O_3(a, b, d)$ is a three cocycle on $H$ with values in $k^*$, and that choosing different $\eta(a, b)$'s will change $O_3$ by a coboundary. We call $O_3 = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) \in H^3(H, k^*)$ the third obstruction. A solution to this obstruction is equivalent to giving a set of $\eta(a, b)$'s such that the pentagon diagram will be commutative. We will see in the next section that by altering $\eta$ by a coboundary we will get equivalent module categories. Thus, we see that the set of solutions for this obstruction will be a torsor over the group $H^2(H, k^*)$ (in case a solution exists). Notice that this obstruction depends "linearly" on $\alpha$, in the sense that if we would have change $\alpha$ to be $\alpha\zeta$ where $\zeta \in H^3(G, k^*)$, then we would have changed the obstruction by $\zeta$. In other words:

$$O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha\zeta) = O_3(\mathcal{N}, c, H, M, \Phi, v, \alpha) res^G_H(\zeta).$$

This ends the proof of Theorem 2.

## 6. The isomorphism condition

In this section we answer the question of when does the $\mathcal{C}$-module categories $\mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$ are equivalent.

Assume then that we have an equivalence of $\mathcal{C}$-module categories

$$F : \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta').$$

Let us denote these categories by $\mathcal{M}$ and $\mathcal{M}'$ respectively. Then $F$ is also an equivalence of $\mathcal{D}$-module categories. Recall that as $\mathcal{D}$-module categories, $\mathcal{M}$ splits as

$$\bigoplus_{g \in G/H} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}.$$

A similar decomposition holds for $\mathcal{M}'$.

By considering these decompositions, it is easy to see that $F$ induces an equivalence of $\mathcal{D}$-module categories between $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and $\mathcal{N}'$ for some $g \in G$. Let us denote the restriction of $F$ to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ as a functor of $\mathcal{D}$-module categories by $t_F$. We can reconstruct the tuple $(\mathcal{N}', H', \Phi', v', \beta')$ from $t_F$ in the following way: We have already seen that $\mathcal{N}'$ is equivalent to $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and that the stabilizer subgroup of the category $\mathcal{N}'$ will be $H' = gHg^{-1}$.

Let us denote by $\Gamma'$ the group $Aut_{\mathcal{D}}(N')$. We have a natural isomorphism $\nu : \Gamma \to \Gamma'$ given by the formula

$$\nu(t) : \mathcal{N}' \overset{F^{-1}}{\to} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \overset{1 \boxtimes t}{\to} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \overset{F}{\to} \mathcal{N}'.$$

Using the functor $t_F$ and the map $\nu$ we can see that the map

$$\rho' : gHg^{-1} \to Out(\Gamma')$$

which appears in the construction of the second module category is the composition

$$gHg^{-1} \stackrel{c_g}{\to} H \stackrel{\rho}{\to} Out(\Gamma) \to Out(\Gamma'),$$

where the last morphism is induced by $\nu$. The map $\Phi'$ which lifts $\rho'$ will depend on $\Phi$ in a similar fashion. The same holds for the second obstruction and its solution.

For the third obstruction, the situation is a bit more delicate. Since $F$ is a functor of $\mathcal{C}$-module categories, we have, for each $a \in H$, a natural isomorphism between the functors

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \stackrel{1\boxtimes F}{\to} \mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} \mathcal{N}' \dot{\to} \mathcal{N}'$$

and

$$\mathcal{C}_{gag^{-1}} \boxtimes_{\mathcal{D}} (\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}) \dot{\to} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \stackrel{F}{\to} \mathcal{N}'$$

For any $a \in H$, the choice of the natural isomorphism is unique up to a scalar. A direct calculation shows that if we change the natural isomorphisms by a set of scalars $\zeta_a$, we will get an equivalence $\mathcal{M}(N, H, \Phi, v, \beta) \to \mathcal{M}(N', H', \Phi', v', \beta'')$ where $\beta'' = \beta'\partial\zeta$. This is the reason that cohomologous solutions for the third obstruction will give us equivalent module categories.

In conclusion, we have the following:

**Proposition 12.** *Assume that we have an isomorphism $F : \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta) \to \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$ Then there is a $g \in G$ such that $F$ will induce an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$, and the data $(\mathcal{N}', H', \Phi', v', \beta')$ can be reconstructed from $t_F$ in the way described above ($\beta'$ will be reconstructible only up to a coboundary) .*

Notice that we do not have any restriction on $t_F$. In other words, given any $t_F : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}'$ we can always reconstruct the tuple $(\mathcal{N}', H', \Phi', v', \beta')$ in the way described above.

We would like now to "decompose" the equivalence in the theorem into several steps. The first ingredient that we need in order to get an equivalence is an element $g \in G$ such that $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv \mathcal{N}'$.

Consider now the case where this ingredient is trivial, that is- $g = 1$, $\mathcal{N} = \mathcal{N}'$ and $H = H'$. In that case $t_F$ is an autoequivalence of the $\mathcal{D}$-module category $\mathcal{N}$. Let us denote by $\psi_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ and by $\psi'_a : \mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \to \mathcal{N}$ the structural equivalences of the two categories (where $a \in H$). Since $F$ is an equivalence of $\mathcal{C}$-module categories, we see that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi_a"] \arrow[d, "1\boxtimes t_F"] & \mathcal{N} \arrow[d, "t_F"] \\
\mathcal{C}_a \boxtimes_{\mathcal{D}} \mathcal{N} \arrow[r, "\psi'_a"] & \mathcal{N}
\end{tikzcd}$$

and a direct calculation shows that $\Phi$ and $\Phi'$ satisfy the following formula:

$$\Phi'(a)(V) = t_F\Phi(a)(t_F^{-1}Vt_F)t_F^{-1} \tag{6.1}$$

where $V$ is any element in $\Gamma$.

Another way to write Equation 6.1 is $\Phi' = c_{t_F}\Phi c_{t_F}^{-1}$, where by $c_{t_F}$ we mean the automorphism of $\Gamma$ of conjugation by $t_F$. In other words- this shows that we have some freedom in choosing $\Phi$, and if we change $\Phi$ in the above fashion, we will still get equivalent categories.

Consider now the case where also $\Phi = \Phi'$. This means that for every $a \in H$ the element $t_F\Phi(a)(t_F)^{-1}$ is central in $\Gamma$. A direct calculation shows that the function $r$ defined by $r(a) = t_F\Phi(a)(t_F)^{-1}$ is a one cocycle with values in $Z(\Gamma)$, and that $v/v' = r$. Notice in particular that by choosing arbitrary $t_F \in Z(\Gamma)$ we see that cohomologous solutions to the second obstruction will give us equivalent categories. However, we see that more is true, and it might happen that non cohomologous $v$ and $v'$ will define equivalent categories.

Last, if the situation is that $t_F = \Phi(a)(t_F)$ for every $a \in H$, we will have the same $(\mathcal{N}, H, \Phi, v)$, but $\beta$ might be different. We have seen that if $\beta$ and $\beta'$ are cohomologous they will define equivalent categories, but it might happen that noncohomologous $\beta$ and $\beta'$ will define equivalent categories as well.

## 7. Functor categories

In this section we are going to describe the category of functors between module categories over an extension in terms of module categories over the trivial component of the extension. We prove a categorical analogue of Mackey's Theorem and we give a criterion for an extension to be group theoretical. In addition, given that $\mathcal{C}$ is a $G$-extension of $\mathcal{D}$, we describe the category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ of $\mathcal{C}$-module functors as an equivariantization of the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ of $\mathcal{D}$-module functors with respect to $G$.

7.1. **Mackey's Theorem for module categories.** Let $\mathcal{C}$ be a $G$-extension of $\mathcal{D}$. For any subset $S \subseteq G$ denote the subcategory $\bigoplus_{g\in S}\mathcal{C}_g$ by $\mathcal{C}_S$. If $S$ is a subgroup of $G$ then $\mathcal{C}_S$ is a fusion subcategory. Let $H$ and $K$ be subgroups of $G$ and let $\mathcal{N}$ be a $\mathcal{C}_K$-module category. We prove now a categorical version of Mackey's Theorem.

**Theorem 13.** *$(\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|\mathcal{C}_H} \cong \bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}^g$, where $H^g = H \cap gKg^{-1}$ and $\mathcal{N}^g = (\mathcal{C}_{gK} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|H^g}$ is $\mathcal{C}_{H^g}$-module category and the sum is over all the double cosets.*

*Proof.* First, consider the transitive $H \times K^{op}$-action on $HgK$. The stabilizer of $g$ is $\{(gkg^{-1}, k^{-1}) | k \in K, gkg^{-1} \in H\}$. Hence, $\mathcal{C}_{HgK}$ is isomorphic to $\mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{C}_{gK}$ as $(\mathcal{C}_H, \mathcal{C}_K)$-bimodule category. Next, $(\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N})_{|\mathcal{C}_H} \cong \bigoplus_{HgK} \mathcal{C}_{HgK} \boxtimes_{\mathcal{C}_K} \mathcal{N}$ where the sum is over all the double cosets. Finally $\mathcal{C}_{HgK} \boxtimes_{\mathcal{C}_K} \mathcal{N} \cong (\mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{C}_{gK}) \boxtimes_{\mathcal{C}_K} \mathcal{N} \cong \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} (\mathcal{C}_{gK} \boxtimes_{\mathcal{C}_K} \mathcal{N}) = \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}^g$. □

*Remark* 14. The above theorem could be stated in the original Mackey's Theorem language, namely $res^G_H ind^G_K(\mathcal{N}) \cong \bigoplus_{HgK} ind^H_{H^g} res^K_{H^g}(\mathcal{N}^g)$. One notices that the proof of the theorem uses only basic consideration about double cosets.

7.2. **Functor categories.** Assume that we have two module categories $\mathcal{M}_1 = \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$, and $\mathcal{M}_2 = \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$. Let us denote $H'$ by $K$. Our goal is to calculate $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ in terms of functor categories over $\mathcal{D}$. We have

$$Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) = Fun_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}, \mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}').$$

By Frobenious reciprocity

$$Fun_{\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{N}, \mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}') \cong Fun_{\mathcal{C}_H}(\mathcal{N}, (\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}).$$

Since a module category is, by definition, a semisimple category every functor has both a left adjoint and a right adjoint. Taking left adjoints (right adjoints) gives us an equivalence of the corresponding functor categories.

Thus we obtain the following equivalence by taking left adjoints

$$Fun_{\mathcal{C}_H}(\mathcal{N}, (\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}) \cong Fun_{\mathcal{C}_H}((\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}, \mathcal{N}).$$

By Mackey's Theorem for module categories we have

$$Fun_{\mathcal{C}_H}((\mathcal{C} \boxtimes_{\mathcal{C}_K} \mathcal{N}')_{|\mathcal{C}_H}, \mathcal{N}) \cong Fun_{\mathcal{C}_H}(\bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}'^g, \mathcal{N})$$

and

$$Fun_{\mathcal{C}_H}(\bigoplus_{HgK} \mathcal{C}_H \boxtimes_{\mathcal{C}_{H^g}} \mathcal{N}'^g, \mathcal{N}) \cong \bigoplus_{HgK} Fun_{\mathcal{C}_{H^g}}(\mathcal{N}'^g, \mathcal{N}_{|\mathcal{C}_{H^g}}).$$

Finally, by taking right adjoints, we end up with the following

**Proposition 15.** *In the above notations*

$$Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2) \cong \bigoplus_{HgK} Fun_{\mathcal{C}_{H^g}}(\mathcal{N}_{|\mathcal{C}_{H^g}}, \mathcal{N}'^g).$$

7.3. **A criterion for an extension to be group theoretical.** Let $\mathcal{C}$ be a fusion category. Recall that a $\mathcal{C}$-module category $\mathcal{M}$ is called *pointed* if $\mathcal{C}^*_{\mathcal{M}}$, the dual category with respect to $\mathcal{M}$ , is pointed. We say that $\mathcal{C}$ is group theoretical in case $\mathcal{C}$ has a pointed module category. As can easily be seen, $\mathcal{C}$ is pointed if and only if it has an indecomposable module category $\mathcal{N}$ such that any simple $\mathcal{C}$-linear functor $F : \mathcal{N} \to \mathcal{N}$ is invertible.

We now prove a criterion for an extension category to be group theoretical.

**Theorem 16.** *Let $\mathcal{C}$ be a $G$-extension of $\mathcal{D}$. $\mathcal{C}$ is group theoretical if and only if $\mathcal{D}$ has a pointed module category $\mathcal{N}$ which is $G$-stable, namely, for every $g \in G$, $\mathcal{N} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$.*

*Proof.* Suppose $\mathcal{N}$ is a pointed $G$-stable $\mathcal{D}$-module category. Consider $\mathcal{M} = \mathcal{C}\boxtimes_{\mathcal{D}}\mathcal{N}$. By Frobenius reciprocity we have

$$Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \cong \oplus_{g\in G} Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}).$$

Since for any $g \in G$ it holds that $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{N}$ and since all simple functors in $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ are invertible, we see that the same happens in $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, that is- $\mathcal{M}$ is pointed over $\mathcal{C}$ and $\mathcal{C}$ is group theoretical.

Conversely, suppose that $\mathcal{C}$ is group theoretical and suppose $\mathcal{M}$ is an indecomposable pointed $\mathcal{C}$-module category. We thus know that any simple functor $F : \mathcal{M} \to \mathcal{M}$ is invertible. We also know that there is a subgroup $H < G$ and an indecomposable $\mathcal{C}_H$-module category $\mathcal{N}$ such that $\mathcal{M} \cong \mathcal{C}\boxtimes_{\mathcal{C}_H}\mathcal{N} = \oplus_{gH\in G/H}\mathcal{C}_g\boxtimes_{\mathcal{D}}\mathcal{N}$ Since $\mathcal{M}$ is indecomposable, it is easy to see that for every $g \in G$ there is some simple $\mathcal{C}$-endofunctor $F : \mathcal{M} \to \mathcal{M}$ such that $F(\mathcal{N}) \subseteq \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$. But such a functor must be invertible, and it follows that $F$ induces an equivalence of $\mathcal{D}$ module categories $\mathcal{N} \cong \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$. Thus $\mathcal{N}$ is $G$-invariant.

Next, we would like to prove that $\mathcal{D}^*_{\mathcal{N}}$ is pointed. By Frobenius reciprocity we have $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \cong \oplus_{gH\in G/H} Fun_{\mathcal{C}_H}(\mathcal{N}, \mathcal{C}_g\boxtimes\mathcal{N})$ Thus the category $Fun_{\mathcal{C}_H}(\mathcal{N}, \mathcal{N})$ is a sub-fusion category of the pointed category $Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and is therefore pointed. We have a forgetful functor $Fun_{\mathcal{C}_H}(\mathcal{N}, \mathcal{N}) \to Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ which is known to be onto (see Proposition 5.3 of [3]). This implies that $Fun_{\mathcal{D}}(\mathcal{N}, \mathcal{N})$ is pointed, as required. □

*Remark* 17. The above criterion is actually equivalent to the one given in Corollary 3.10 of [5], namely, $\mathcal{C}$ is group theoretical if and only if $\mathcal{Z}(\mathcal{D})$ contains a $G$-stable Lagrangian subcategory. In order to explain why the two conditions are equivalent, recall first the definitions of a Lagrangian subcategory and of the action of $G$ on $\mathcal{Z}(\mathcal{D})$. A Lagrangian subcategory of $\mathcal{Z}(\mathcal{D})$ is a subcategory $\mathcal{E}$ such that $\mathcal{E}' = \mathcal{E}$ (see Section 3.2 of [1] for the definition of '). The action of $G$ on $\mathcal{Z}(\mathcal{D})$ is defined as follows: the center $\mathcal{Z}(\mathcal{D})$ can be considered as $Fun_{\mathcal{D}\boxtimes\mathcal{D}^{op}}(\mathcal{D}, \mathcal{D})$, the category of $\mathcal{D}$-bimodule endofunctors of $\mathcal{D}$. Given an element $g \in G$ and a $\mathcal{D}$-bimodule functor $F : \mathcal{D} \to \mathcal{D}$, the functor $g(F) : \mathcal{D} \to \mathcal{D}$ is defined via

$$\mathcal{D} \xrightarrow{\cong} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{C}_{g^{-1}} \xrightarrow{1\boxtimes_{\mathcal{D}} F \boxtimes_{\mathcal{D}} 1} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{D} \boxtimes_{\mathcal{D}} \mathcal{C}_{g^{-1}} \xrightarrow{\cong} \mathcal{D} .$$

In Theorem 4.66 of [1] it was proved that there is an equivalence between Lagrangian subcategory of $\mathcal{Z}(\mathcal{D})$ and pointed $\mathcal{D}$ module categories. Since $G$ acts on the center of $\mathcal{D}$, it also acts on the set of Lagrangian subcategories of $\mathcal{Z}(\mathcal{D})$. Let $\mathcal{N}$ be a $\mathcal{D}$-module category and let $\mathcal{L}$ be the corresponding Lagrangian subcategory of $\mathcal{Z}(\mathcal{D})$. From the above definition of the $G$ action on $\mathcal{Z}(\mathcal{D})$ it is possible to see that for any $g \in G$, the lagrangian subcategory which corresponds to $g \cdot \mathcal{N}$ is $g \cdot \mathcal{L}$. Therefore, $\mathcal{Z}(\mathcal{D})$ admit a $G$-stable Lagrangian subcategory if and only if $\mathcal{D}$ has a pointed $G$-stable module category.

7.4. **Functor categories as equivariantizations.** In this subsection we shall describe the category of $\mathcal{C}$-module functors between the module categories $\mathcal{M}(\mathcal{N}, H, \Phi, \upsilon, \beta)$ and $\mathcal{M}(\mathcal{N}', H', \Phi', \upsilon', \beta')$. For simplicity we shall denote these categories as $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Consider the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$. This is a $k$-linear category which by Theorem 2.16 of [3] is semisimple.

**Lemma 18.** *There is a natural $G$-action on $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ induced by the structure of $\mathcal{C}$-module categories on $\mathcal{M}_1$ and $\mathcal{M}_2$.*

*Proof.* There are $\mathcal{D}$-module equivalences $\psi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \cong \mathcal{M}_1$ and $\phi_g : \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \cong \mathcal{M}_2$, for every $g \in G$, defined by the $\mathcal{C}$-module structure on $\mathcal{M}_1$ and $\mathcal{M}_2$. Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module morphism, we define $g \cdot F$ to be the following functor

$$\mathcal{M}_1 \xrightarrow{\psi_g^{-1}} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \xrightarrow{Id_{\mathcal{C}_g} \boxtimes_{\mathcal{D}} F} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \xrightarrow{\phi_g} \mathcal{M}_2 \ .$$

One can easily check that this defines an action of the group $G$ on the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}2)$ in the sense of [2]

□

Since we have a $G$-action on $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$, we can talk about the equivariantization $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$. By definition, an object in $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$ is a pair $(F, \{T_g\}_{g \in G})$, where $T_g : g \cdot F \to F$ are natural equivalences which satisfy a certain coherence condition (for the exact definition, see [2]). Let $F : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{D}$-module functor.

To give $F : \mathcal{M}_1 \to \mathcal{M}_2$ a structure of a $\mathcal{C}$-module functor is the same thing as to give, for every $g \in G$, a natural isomorphism between the functors $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \to \mathcal{M}_1 \xrightarrow{F} \mathcal{M}_2$ and $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_1 \xrightarrow{1 \boxtimes F} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{M}_2 \to \mathcal{M}_2$. It can easily be seen that this is equivalent to give $F$ a structure of an object in the equivariantization category.

Let us conclude this discussion by the following

**Proposition 19.** *The category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the equivariantization $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)^G$ of the category $Fun_{\mathcal{D}}(\mathcal{M}_1, \mathcal{M}_2)$ with respect to the aforementioned $G$-action.*

*Remark* 20. Let $\mathcal{M}$ be an indecomposable $\mathcal{C}$-module category. Although $\mathcal{C}^*_{\mathcal{M}} \triangleq Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a fusion category, $Fun_{\mathcal{D}}(\mathcal{M}, \mathcal{M})$ is, in general, only a multifusion category because $\mathcal{M}$ might be decomposable as $\mathcal{D}$-module category. Equivariantization has only been defined in the context of fusion categories. However, the definition in context of multifusion categories is mutatis mutandis. Notice that is case of multifusion equivariantization we don't always have the $Rep(G)$ subcategory supported on the trivial object.

In the next section we will give an intrinsic description of the functor categories, as categories of bimodules.

## 8. An intrinsic description by algebras and modules

The goal of this section is to explain more concretely the action of the grading group on indecomposable module categories, the action of the grading group on $Aut_{\mathcal{D}}(\mathcal{N})$, the obstructions and their solutions.

In [7] Ostrik showed that any indecomposable module category over a fusion category $\mathcal{C}$ is equivalent as a module category to the category $Mod_{\mathcal{C}} - A$ for some semisimple indecomposable algebra $A$ in $\mathcal{C}$. In this section we will realize all the objects described in the previous sections by using algebras and modules inside $\mathcal{C}$. As before, we assume that $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$, we denote $\mathcal{C}_1$ by $\mathcal{D}$ and $Aut_{\mathcal{D}}(\mathcal{N})$ by $\Gamma$.

8.1. **The action of $G$ on indecomposable module categories.** Assume that $A$ is a semisimple indecomposable algebra inside $\mathcal{D}$. Let $\mathcal{N} = Mod_{\mathcal{D}} - A$ be the category of right $A$-modules inside $\mathcal{D}$. We denote by $Mod_{\mathcal{C}_g} - A$ the category of $A$-modules with support in $\mathcal{C}_g$. We claim the following:

**Lemma 21.** *We have an equivalence of $\mathcal{D}$-module categories $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \equiv Mod_{\mathcal{C}_g} - A$.*

*Proof.* We have already seen in Section 2 that we have an equivalence of $\mathcal{C}$-module categories

$$\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{N} \equiv Mod_{\mathcal{C}}(A)$$

which is given by $X \boxtimes M \mapsto X \otimes M$. As a $\mathcal{D}$-modules category, the left hand category decomposes as $\bigoplus_{g\in G} \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}$ and the right hand category decomposes as $\bigoplus_{g\in G} Mod_{\mathcal{C}_g} - A$. It is easy to see that the above equivalence translates one decomposition into the other, and therefore the functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N} \to Mod_{\mathcal{C}_g} - A$ given by $X \boxtimes M \mapsto X \otimes M$ is an equivalence of $\mathcal{D}$-module categories. □

Next, we understand how we can describe functors by using bimodules.

**Lemma 22.** *Let $\mathcal{N} = Mod_{\mathcal{D}} - A$ and $\mathcal{N}' = Mod_{\mathcal{D}} - A'$, and let $g \in G$. Then every functor $F : \mathcal{N} \to \mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ is of the form $F(T) = T \otimes_A Y$ for some $A - A'$ bimodule $Y$ with support in $\mathcal{C}_g$, here we identify $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}'$ with $Mod_{\mathcal{C}_g} - A'$ as above.*

*Proof.* The proof follows the lines of the remark after Proposition 2.1 of [8]. We simply consider $F(A)$. The multiplication map $A \otimes A \to A$ gives us a map $A \otimes F(A) \to F(A)$, thus equipping $F(A)$ with a structure of a left $A$-module. We now see that $F(A)$ is indeed an $A - A'$ bimodule. Since the category $\mathcal{N}$ is semisimple the functor $F$ is exact. Since every object in $\mathcal{N}$ s a quotient of an object of the form $X \otimes A$ for some $X \in \mathcal{C}$, we see that $F$ is given by $F(T) = T \otimes_A F(A)$. □

*Remark.* Notice that by applying the (2-)functor $\mathcal{C}_{g^{-1}} \boxtimes_{\mathcal{D}} -$ we see that every functor $\mathcal{C}_g \boxtimes_{\mathcal{D}} \mathcal{N}' \to \mathcal{N}$ is given by tensoring with some $A' - A$ bimodule with support in $C_{g^{-1}}$.

8.2. **The outer action of $H$ on the group $Aut_{\mathcal{D}}(\mathcal{N})$. The first two obstructions.** Assume, as in the rest of the paper, that we have a subgroup $H < G$ and a module category $\mathcal{N} = Mod_{\mathcal{D}} - A$, and assume that $F_h : \mathcal{N} \cong \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$ for every $h \in H$. It follows from Lemma 22 that this equivalence is of the form $F_h(M) = M \otimes_A A_h$ for some $A - A$ bimodule $A_h$ with support in $\mathcal{C}_h$. The fact that this functor is an equivalence simply means that the bimodule $A_h$ is an invertible $A - A$ bimodule. In other words- there is another $A - A$ bimodule $B_h$ (whose support will necessary be in $\mathcal{C}_{h^{-1}}$) such that $A_h \otimes_A B_h \cong B_h \otimes_A A_h \cong A$. By Lemma 22 we can identify the group $\Gamma = Aut_{\mathcal{D}}(\mathcal{N})$ with the group of isomorphisms classes of invertible $A - A$ bimodules with support in $\mathcal{D}$.

Denote by $\Lambda$ the group of isomorphisms classes all invertible $A - A$ bimodules with support in $\mathcal{C}_H$. Since every invertible $A - A$ bimodule is supported on a single grading component, we have a map $p : \Lambda \to H$ which assigns to an invertible $A - A$

bimodule the graded component it is supported on. We thus have a short exact sequence

$$1 \to \Gamma \to \Lambda \to H \to 1. \tag{8.1}$$

Using this sequence, we can understand the outer action of $H$ on $Aut_{\mathcal{D}}(\mathcal{N})$, and the first and the second obstruction. The outer action is given in the following way: for $h \in H$, choose an invertible $A-A$ bimodule $A_h$ with support in $\mathcal{C}_h$. Choose an inverse to $A_h$ and denote it by $A_h^{-1}$. Then the action of $h \in H$ on some invertible bimodule $M$ with support in $\mathcal{D}$ is the following conjugation:

$$h \cdot M = A_h \otimes_A M \otimes_A A_h^{-1}.$$

This action depends on the choice we made of the invertible bimodule $A_h$.

The first obstruction is the possibility to lift this outer action to a proper action. In other words, it says that we can choose the $A_h$'s in such a way that conjugation by $A_h \otimes A_{h'}$ is the same as conjugation by $A_{hh'}$, or in other words, in such a way that for every $h, h' \in H$, the invertible bimodule

$$B_{h,h'} = A_h \otimes_A A_{h'} \otimes_A A_{hh'}^{-1}$$

will be in the center of $\Gamma$ (again- we identify $\Gamma$ with the group of invertible bimodules with support in $\mathcal{D}$). A solution for the first obstruction will be a choice of a set of such bimodules $A_h$.

The second obstruction says that the cocycle $(h, h') \mapsto B_{h,h'}$ is trivial in $H^2(H, Z(Aut_{\mathcal{D}}(\mathcal{N}))$. This simply says that we can change $A_h$ to be $A_h \otimes_A D_h$ for some $D_h \in Z(Aut_{\mathcal{D}}(\mathcal{N}))$, in such a way that

$$(A_h \otimes_A D_h) \otimes_A (A_{h'} \otimes_A D_{h'}) \otimes_A (A_{hh'} \otimes_A D_{hh'})^{-1} \cong A$$

as $A$-bimodules. A solution for the second obstruction will be a choice of such a set $D_h$ of bimodules.

It is easier to understand the first and the second obstruction together: we have one big obstruction- the sequence 8.1 should split, and we need to choose a splitting. First, if the sequence splits, then we can lift the outer action into a proper action, and we need to choose such a lifting. Then, the obstruction to the splitting *with the chosen action* is given by a two cocycle with values in the center of $\Gamma$. Thus, a solution for both the first and the second obstruction will be a choice of bimodules $A_h$ for every $h \in H$ such that the support of $A_h$ is in $\mathcal{C}_h$ and such that $A_h \otimes_A A_{h'} \cong A_{hh'}$ for every $h, h' \in H$. Following the line of Section 6, we see that we are interested in splittings only up to conjugation by an element of $\Gamma$.

8.3. **The third obstruction.** Assume then that we have a set of bimodules $A_h$ as in the end of the previous subsections. We would like to understand now the third obstruction.

Recall that we are trying to equip $\mathcal{N}$ with a structure of a $\mathcal{C}_H$-module category. By Ostrik's Theorem (see [7]), there is an object $\mathcal{N} \in \mathcal{N}$ such that $A \cong \underline{Hom}_{\mathcal{D}}(N, N)$ where by $\underline{Hom}_{\mathcal{D}}$ we mean the internal Hom of $\mathcal{N}$, where we consider $\mathcal{N}$ as a $\mathcal{D}$-module category. So far we gave equivalences $F_h : \mathcal{N} \to \mathcal{C}_h \boxtimes_{\mathcal{D}} \mathcal{N}$. If $\mathcal{N}$ were a $\mathcal{C}_H$-module category via the choices of these equivalences, then the internal $\mathcal{C}_H$-Hom, $\tilde{A} = \underline{Hom}_{\mathcal{C}_H}(N, N)$ would be

$$\tilde{A} = \bigoplus_{h \in H} A_h.$$

We thus see that to give on $\mathcal{N}$ a structure of a $\mathcal{C}_H$-module category is the same as to give on $\tilde{A}$ a structure of an associative algebra. For every $h, h' \in H$, choose an isomorphism of $A-A$ bimodules $A_h \otimes_A A_{h'} \to A_{hh'}$. Notice that since these are invertible $A-A$ bimodules, there is only one such isomorphism up to a scalar.

Now for every $h, h', h'' \in H$, we have two isomorphisms $(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'h''}$, namely

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_{hh'} \otimes_A A_{h''} \to A_{hh'h''}$$

and

$$(A_h \otimes_A A_{h'}) \otimes_A A_{h''} \to A_h \otimes_A (A_{h'} \otimes_A A_{h''}) \to A_h \otimes_A A_{h'h''} \to A_{hh'h''}.$$

This two isomorphisms differ by a scalar $b(h, h', h'')$. The function $(h, h', h'') \mapsto b(h, h', h'')$ is a three cocycle which is the third obstruction. A solution to the third obstruction will thus be a choice of isomorphisms $A_h \otimes_A A_{h'} \to A_{hh'}$ which will make $\tilde{A}$ an associative algebra. Once we have such a choice, we can change it by some two cocycle to get another solution.

8.4. **Functor categories.** We end this section by giving an intrinsic description of functor categories. Assume that we have two module categories $\mathcal{M}_1 = \mathcal{M}(\mathcal{N}, H, \Phi, v, \beta)$, and $\mathcal{M}_2 = \mathcal{M}(\mathcal{N}', H', \Phi', v', \beta')$. Let us denote $H'$ by $K$. As we have seen in the previous subsections, if $\mathcal{N} \cong Mod_{\mathcal{D}} - A_1$ and $\mathcal{N}' \cong Mod_{\mathcal{D}} - B_1$, then $\mathcal{M}_1 \cong Mod_{\mathcal{C}} - A$ and $\mathcal{M}_2 \cong Mod_{\mathcal{C}} - B$, where $A$ is an algebra of the form $\oplus_{h \in H} A_h$, and a similar description holds for $B$.

The functor category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A - B$-bimodules in $\mathcal{C}$. Since $A$ and $B$ have a graded structure, we will be able to say something more concrete on this category.

Let $X$ be an indecomposable $A - B$-bimodule in $\mathcal{C}$. It is easy to see that the support of $X$ will be contained inside a double coset of the form $HgK$ for some $g \in G$. Since the bimodules $A_h$ and $B_k$ are invertible, it is easy to see that the support will be exactly this double coset.

Consider now the $g$-component $X_g$ of $X$. As can easily be seen, this is an $A_1 - B_1$-bimodule. Actually, more is true. Consider the category $\mathcal{C} \boxtimes \mathcal{C}^{op}$. Inside this category we have the algebra

$$(AB)_g = \oplus_{x \in H \cap gKg^{-1}} A_x \boxtimes B_{g^{-1}x^{-1}g}$$

with the multiplication defined by the restricting the multiplication from $A \boxtimes B \in \mathcal{C} \boxtimes \mathcal{C}^{op}$ . The category $\mathcal{C}$ is a $\mathcal{C} \boxtimes \mathcal{C}^{op}$-module category in the obvious way, and we have a notion of an $(AB)_g$-module inside $\mathcal{C}$.

**Lemma 23.** *The category of $(AB)_g$-modules inside $\mathcal{C}$ is equivalent to the category of $A - B$-bimodules with support in the double coset $HgK$.*

*Proof.* If $X$ is an $A - B$-bimodule with support in $HgK$, then $X_g$ is an $(AB)_g$-module via restriction of the left $A$-action and the right $B$-action. Conversely, if $V$ is an $(AB)_g$-module inside $\mathcal{C}$, we can consider the induced module

$$(A \boxtimes B) \otimes_{(AB)_g} V.$$

This is an $A - B$-bimodule, and one can see that the two constructions gives equivalences in both directions. □

*Remark.* This is a generalization of Proposition 3.1 of [8], where the same situation is considered for the special case that $\mathcal{C} = Vec_G^{\omega}$ and $\mathcal{D} = 1$. Also, notice that the decomposition to double cosets is the one which appears in Theorem [**?**]

In conclusion, we have the following

**Proposition 24.** *The functor category $Fun_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ is equivalent to the category of $A - B$-bimodules. Each such simple bimodule is supported on a double coset of the form $HgK$, and the subcategory of bimodules with support in $HgK$ is equivalent to the category of $(AB)_g$-modules inside $\mathcal{C}$.*

## 9. A detailed example: classification of modules categories over the Tambara Yamagami fusion categories and their dual categories

As an example of our results, we shall now describe the module categories over the Tambara Yamagami fusion categories $\mathcal{C} = \mathcal{TY}(A, \chi, \tau)$ and the corresponding dual categories. Let $A$ be a finite group. Let $R_A$ be the fusion ring with basis $A \cup \{m\}$ whose multiplication is given by the following formulas:

$$g \cdot h = gh, \forall g, h \in A$$

$$g \cdot m = m \cdot g = m$$

$$m \cdot m = \sum_{g \in A} g$$

In [10] Tambara and Yamagami classified all fusion categories with the above fusion ring. They showed that if there is a fusion category $\mathcal{C}$ whose fusion ring is $R_A$ then $A$ must be abelian. They also showed that for a given $A$ such fusion categories can be parameterized (up to equivalence) by pairs $(\chi, \tau)$ where $\chi : A \times A \to k^*$ is a nondegenerate symmetric bicharacter, and $\tau$ is a square root (either positive or negative) of $\frac{1}{|A|}$. We denote the corresponding fusion category by $\mathcal{C} := \mathcal{TY}(A, \chi, \tau)$.

The category $\mathcal{C}$ is naturally graded by $\mathbb{Z}_2 = \langle \sigma \rangle$. The trivial component is $Vec_A$ (with trivial associativity constraints) and the nontrivial component, which we shall denote by $\mathcal{M}$, has one simple object $m$. In [4] the authors described how the Tambara Yamagami fusion categories corresponds to an extension data of $Vec_A$ by the group $\mathbb{Z}_2$. We shall explain now the classification of module categories over $\mathcal{TY}(A, \chi, \tau)$ given by our parameterization.

Since $A$ is an abelian group and the associativity constraints in $Vec_A$ are trivial, module categories over $Vec_A$ are parameterized by pairs $(H, \psi)$ where $H < A$ is a subgroup and $\psi \in H^2(H, k^*)$. We shall denote the corresponding module category by $\mathcal{M}(H, \psi)$. As explained in Section 3, we have a natural action of $\mathbb{Z}_2 = \langle \sigma \rangle$ on the set of equivalence classes of module categories over $Vec_A$. We shall describe this action in Subsection 9.1.

Recall that the second component in the parameterization of a module category is a subgroup of the grading group. If this subgroup is the trivial subgroup, then we will just have a category which is induced from $Vec_A$. It is easy to see that such categories decompose over $Vec_A$ to the direct sum of two indecomposable module categories. In that case, all the obstructions and solutions will be trivial. If this subgroup is $\mathbb{Z}_2$ itself, we will have a $\mathcal{C}$-module category structure on $\mathcal{M}(H, \psi)$ for some $H$ and some $\psi$. In that case, it must hold that $\sigma(H, \psi) = (H, \psi)$, and we may have some nontrivial obstructions and solutions.

The rest of this section will be devoted to analyze the action of $\sigma$ and the obstructions and their solutions (for the case in which we have obstructions). We will also describe the relations of our result with the result of Tamabra on fiber functors on Tamabara Yamagami categories, and also describe the dual categories.

We would like now to describe the main result of this section. We will split our main classification result into two proposition, according to the subgroup of $\mathbb{Z}_2$ which appears in the parameterization. Our first proposition follows in a straight forward way from the discussion in the previous sections

**Proposition 25.** *Module categories over $\mathcal{C}$ whose parameterization begins with $(\mathcal{M}(H, \psi), 1, \ldots)$ are the induced categories $Ind^{\mathcal{C}}_{Vec_A}(\mathcal{M}(H, \psi))$. We will have an equivalence of $\mathcal{C}$ module categories $Ind^{\mathcal{C}}_{Vec_A}(\mathcal{M}(H, \psi)) \cong Ind^{\mathcal{C}}_{Vec_A}(\mathcal{M}(H', \psi'))$ if and only if $(H, \psi) = (H', \psi')$ or if $(H, \psi) = \sigma(H'\psi')$.*

In order to describe the other case, we need some notations. Suppose that $H < A$ is a subgroup which contains $H^{\perp}$ (the subgroup perpendicular to $H$ with respect to $\chi$). If we denote by $\bar{H} := H/H^{\perp}$, then $\chi$ induces a non-degenerate symmetric bicharacter $\bar{\chi} : \bar{H} \times \bar{H} \to k^*$. If $\psi \in H^2(H, k^*)$ satisfies $Rad(\psi) = H^{\perp}$ (the definition of $rad(\psi)$ is given in Subsection 9.1), then $\psi$ is the inflation of a nondegenerate two cocycle $\bar{\psi}$ on $\bar{H}$. We will usually not distinguish between $\psi$ and $\bar{\psi}$.

**Proposition 26.** *For $\mathcal{M}(H,\psi)$ to have a structure of a $\mathcal{C}$ module category, it is necessary that $\sigma(H,\psi) = (H,\psi)$. This implies that $Rad(\psi) = H^{\perp} < H$. If this holds, then $\mathcal{C}$-module categories structures on $\mathcal{M}(H,\psi)$ are parameterized by pairs $(s,\nu)$ where $s : H/H^{\perp} \to H/H^{\perp}$ is an involutive automorphism, and $\nu : H/H^{\perp} \to k^*$ is a function which satisfy for every $a, b \in H/H^{\perp}$*

$$\bar{\chi}(a,b) = \psi(s(a),b)/\psi(b,s(a))$$
$$\partial\nu(a,b) = \psi(a,b)/\psi(s(b),s(a))$$
$$\nu(a)\nu(s(a)) = 1$$
$$sign(\sum_{s(a)=a} \nu(a)) = sign(\tau)$$

*Two such pairs $(s,\nu)$ and $(s',\nu')$ will give equivalent module category structures on $\mathcal{M}(H,\psi)$ if and only if $s = s'$ and there exist a character $\phi : H/H^{\perp} \to k^*$ such that $\nu(h)/\nu'(h) = \eta(h)/\eta(s(h))$.*

9.1. **The action of $\sigma$ on indecomposable module categories and representations of twisted abelian group algebras.** Recall that the $Vec_A$ module category $\mathcal{N} = \mathcal{M}(H,\psi)$ is the category of right modules over the algebra $k^{P\psi}H$ inside $Vec_A$. We would like to understand the $Vec_A$ module category $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$.

As explained in Section 8, this module category can be described as the category of right $k^{\psi}H$-modules with support in the category $\mathcal{M}$, the nontrivial grading component of $\mathcal{C}$. A $k^{\psi}H$-module with support in $\mathcal{M}$ is of the form $m \otimes V$ where $V$ is a vector space which is a $k^{\psi}H$-module in the usual sense. So the category $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ is equivalent, at least as an abelian category, to the category of $k^{\psi}H$-modules in $Vec$.

We would like to describe $\mathcal{M}\boxtimes_{Vec_A}\mathcal{N}$ as a module category of the form $\mathcal{M}(H',\psi')$ for some $H' < A$ and some two cocycle $\psi' \in H^2(H', k^*)$. In order to do so, we begin by describing the simple $k^{\psi}H$ modules in $Vec$ (they will correspond to the simple objects in $M \boxtimes_{Vec_A} \mathcal{N}$).

Let $k^{\psi}H = \oplus_{h\in H} U_h$. The multiplication in $k^{\psi}H$ is given by the rule $U_hU_k = \psi(h,k)U_{hk}$. Denote by $R = Rad(\psi)$ the subgroup of all $h \in H$ such that $U_h$ is central in $k^{\psi}H$.

As the field $k$ is algebraically closed of characteristic zero and $H$ is abelian, the data that stored in the cocycle $\psi$ is simply the way in which the $U_h$'s commute. More precisely- let us define the following alternating form on $H$:

$$\xi_{\psi}(a,b) = \psi(a,b)/\psi(b,a).$$

It turns out (see [9]) that the assignment $\psi \mapsto \xi_{\psi}$ depends only on the cohomology class of $\psi$, and that it gives a bijection between $H^2(H, k^*)$ and the set of all alternating forms on $H$. The elements of $R$ can be described as those $h \in H$ such that $\xi_{\psi}(h, -) = 1$. As can easily be seen, $\xi_{\psi}$ is the inflation of an alternating form on $H/R$. It follows easily that $\psi$ is the inflation of a two cocycle $\bar{\psi}$ on $H/R$.

It can also be seen that $\xi_{\bar{\psi}}$ is nondegenerate on $H/R$ and that $k^{\bar{\psi}}H/R \cong M_n(k)$ where $n = \sqrt{|H/R|}$. It follows that $k^{\bar{\psi}}H/R$ has only one simple module (up to isomorphism) which we shall denote by $V_1$ (i.e, $\bar{\psi}$ is non degenerate on $H/R$). By inflation, $V_1$ is also a $k^\psi H$-module. Let $\zeta$ be a character of $H$, and let $k^\zeta$ be the corresponding one dimensional representation of $H$. Then $k^\zeta \otimes V_1$ is also a simple module of $k^\psi H$, where $H$ acts diagonally. It turns out that these are all the simple modules of $k^\psi H$, and that $V_{\zeta_1} \cong V_{\zeta_2}$ if and only if the restrictions of $\zeta_1$ and $\zeta_2$ to $R$ coincide.

The simple modules of $k^\psi H$ are thus parameterized by the characters of $R$ (we use here the fact that the restriction from the character group of $H$ to that of $R$ is onto). For every character $\zeta$ of $R$, we denote by $V_\zeta$ the unique simple module of $k^\psi H$ upon which $R$ acts via the character $\zeta$. So the simple $k^\psi H$-modules with support in $\mathcal{M}$ are of the form $m \otimes V_\zeta$.

In order to understand the structure of $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ as a $Vec_A$ module category, let us describe $V_a \otimes (m \otimes V_\zeta)$ for $a \in A$. It can easily be seen that this is also a simple module, so we just need to understand via which character $R$ acts on it. Using the associativity constraints in $\mathcal{TY}(A, \chi, \tau)$, we see that for $v \in V_\zeta$ and $r \in R$ we have

$$(V_a \otimes m \otimes v) \cdot U_r = \chi(a,r) V_a \otimes (m \otimes v \cdot U_r) = \chi(a,r)\zeta(r) V_a \otimes m \otimes v.$$

This means that $V_a \otimes (m \otimes V_\zeta) = m \otimes V_{\zeta\chi(a,-)}$. So the stabilizer of $V_\zeta$ is the subgroup of all $a \in A$ such that $\chi(a,r) = 1$ for all $r \in R$, i.e., it is $R^\perp$. It follows that $\mathcal{M} \boxtimes_{Vec_A} \mathcal{N}$ is equivalent to a category of the form $\mathcal{M}(R^\perp, \tilde{\psi})$. Where $\tilde{\psi}$ is some two cocycle.

Let us figure out what is $\tilde{\psi}$. If $a \in R^\perp$, then the restriction of $\chi(a,-)$ to $H$ is a character which vanishes on $R$. Therefore, there is a unique (up to multiplication by an element of $R$) element $t_a \in H$ such that $\xi_\psi(t_a, -) = \chi(a,-)$. It follows that there is an isomorphism $r_a : V_a \otimes (m \otimes V_1) \to m \otimes V_1$ which is given by the formula $V_a \otimes (m \otimes v) \mapsto m \otimes (v \cdot U_{t_a})$. Now for every $a, b \in R^\perp$, $\tilde{\psi}(a,b)$ should make the following diagram commute:

$$\begin{tikzcd}
(V_a \otimes V_b) \otimes (m \otimes V_1) \arrow[r] \arrow[d, "r_{ab}"] & V_a \otimes (V_b \otimes (m \otimes V_1)) \arrow[r, "r_b"] & V_a \otimes (m \otimes V_1) \arrow[d, "r_a"] \\
m \otimes V_1 \arrow[rr, "\tilde{\psi}(a,b)"] & & m \otimes V_1
\end{tikzcd}$$

An easy calculation shows that this means that $\tilde{\psi}(a,b) = \psi(t_b, t_a)$. We thus have the following result:

**Lemma 27.** *We have $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(R^\perp, \tilde{\psi})$ where $R$ is the radical of $\psi$ and $\tilde{\psi}$ is described above.*

Suppose now that $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(H,\psi)$. This implies that $Rad(\psi) = H^\perp$. The bicharacter $\chi$ defines by restriction a pairing on $H \times H$, and by dividing out by $H^\perp$, we get a nondegenerate symmetric bicharacter $\bar{\chi} : H/H^\perp \times H/H^\perp \to k^*$. It is easy to see that the assignment $h \mapsto t_h$ that was described above induces an automorphism $s$ of $H/H^\perp$ which satisfies

$$\bar{\chi}(a,b) = \xi_{\bar{\psi}}(s(a), b). \tag{9.1}$$

The fact that $\tilde{\psi} = \psi$ means that $\xi_{\bar{\psi}}(s(b), s(a)) = \xi_{\bar{\psi}}(a,b)$. Equivalently, this means that $\bar{\chi}(a,b) = \xi_{\bar{\psi}}(s(a),b) = \xi_{\bar{\psi}}(s(b), s^2(a)) = \bar{\chi}(b, s^2(a))$ and since $\bar{\chi}$ is nondegenerate, this is equivalent to the fact that $s^2 = Id$.

In summary:

**Lemma 28.** *We have $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(H,\psi)$ if and only if the following two conditions hold:*

*1.$H^\perp < H$.*

*2. There is an automorphism $s$ of order 2 of $H/H^{\perp}$ such that $(a,b) \mapsto \bar{\chi}(s(a),b)$ is an alternating form, and the inflation of this alternating form to $H$ is $\xi_{\psi}$.*

9.2. **The vanishing of the first obstruction and invertible bimodules with support in $Vec_A$.** Assume now that we have a module category $\mathcal{M}(H,\psi)$ such that $\sigma \cdot \mathcal{M}(H,\psi) \equiv \mathcal{M}(H,\psi)$. We would like to describe all module categories whose classification data begins with $(\mathcal{M}(H,\psi), \mathbb{Z}_2, \ldots)$. In other words- we would like to describe all possible ways (if any) to furnish a structure of a $\mathcal{C}$ module category on $\mathcal{M}(H,\psi)$.

So let $s$ be an automorphism as in Lemma 28. In order to explain the first obstruction for furnishing a $\mathcal{TY}(A,\chi,\tau)$-module category structure on $\mathcal{M}(H,\psi)$, we need to consider the group of invertible $k^{\psi}H$-bimodules in $\mathcal{TY}(A,\chi,\tau)$. As we have seen in Section 8, such an invertible bimodule with support in $Vec_A$ ($\mathcal{M}$) corresponds to a functor equivalence $F : \mathcal{N} \to \mathcal{N}$ ($F : \mathcal{M} \boxtimes_{Vec_A} \mathcal{N} \to \mathcal{N}$). The functor is given by tensoring with the invertible bimodule.

Let us first classify invertible $k^{\psi}H$-bimodules with support in $Vec_A$. Their description was given in Ostrik's paper [8]. We recall it briefly.

If $a \in A$ and $\lambda$ is a character on $H$, we define the bimodule $M_{a,\lambda}$ to be

$$\oplus_{h \in H} V_{ah},$$

where the action of $k^{\psi}H$ is given by

$$U_h \cdot V_{ah'} \cdot U_{h''} = \psi(h,h')\lambda(h)\psi(hh',h'')V_{ahh'h''}.$$

Choose now coset representatives $a_1, \ldots, a_r$ of $H$ in $A$. Proposition 3.1 of [8] tells us that the modules $M_{a_i,\lambda}$ where $i = 1, \ldots r$ and $\lambda \in \hat{H}$ are all the invertible $k^{\psi}H$ bimodules, and each invertible bimodule with support in $Vec_A$ appears in this list exactly once.

By a more careful analysis we can get to the following description of the group of invertible bimodules: we have a homomorphism $\xi : H \to \hat{H}$ given by $h \mapsto \xi_{\psi}(h,-)$. Then the group $E$ of all invertible bimodules with support in $Vec_A$ can be described as the pushout which appears in the following diagram: (see Theorem 5.2 of [6] for a more general result)

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\xi"] & A \arrow[d] \\ \hat{H} \arrow[r] & E \end{tikzcd}$$

The group $E$ is thus also isomorphic to the group $Aut_{Vec_A}(\mathcal{M}(H,\psi))$. Notice that the group $E$ is *abelian*. A solution to the first obstruction is a lifting of the natural map (see Section 4) $\mathbb{Z}_2 \to Out(E)$ to a map $\mathbb{Z}_2 \to Aut(E)$ But since $E$ is abelian, $Out(E) = Aut(E)$, so this problem is trivial, and it has only one solution. So we have a proper (and not just outer) action of $\mathbb{Z}_2$ on $E$.

9.3. **The group of all invertible bimodules and the second obstruction.** Since $\mathcal{M}(H,\psi)$ is $\sigma$-invariant, we see by Section 8 that the group $\tilde{E}$ of (isomorphism classes of) invertible $k^{\psi}H$ bimodules in $\mathcal{C}$ is given as an extension

$$\Sigma : 1 \to E \to \tilde{E} \to \mathbb{Z}_2 \to 1.$$

Moreover, we have seen that the second obstruction is the cohomology class of this extension in $H^2(\mathbb{Z}_2, E)$, and that a solution to the second obstruction is a splitting of this sequence, up to conjugation by an element of $E$.

So our next goal is to understand if the sequence $\Sigma$ splits. For this, we would like to understand the structure of the group $\tilde{E}$ better, and for this reason, we will describe now the invertible $k^{\psi}H$ bimodules with support in $\mathcal{M}$ (these are the elements of $\tilde{E}$ which goes to the nontrivial element in $\mathbb{Z}_2$). We begin by choosing such an invertible bimodule $X$ explicitly. It should be of the form $X = m \otimes V$, where $V$ is both a left and a right $k^{\psi}H$ module. The interaction between the left structure and the right structure follows from the associativity constraints and is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h, h')U_h \cdot (v \cdot U_{h'}). \tag{9.2}$$

The fact that $X$ is invertible implies that $V$ has to be simple as a left and as a right $k^{\psi}H$-module. Assume that $V$ is $V_{\phi}$ from Subsection 9.1 as a right $k^{\psi}H$ module, where $\phi$ is some character of $H^{\perp}$. We need to define on $V$ a structure of a left $k^{\psi}H$-module. By Equation 9.1 we know that

$$(v \cdot U_{t_h}) \cdot U_{h'} = \chi(h, h')(v \cdot U_{h'})U_{t_h}.$$

By Equation 9.2 and by the simplicity of $V$, we see that this means that we must have

$$U_h \cdot v = \nu(h)v \cdot U_{t_h} \tag{9.3}$$

for some set of scalars $\{\nu(h)\}_{h\in H}$. An easy calculation shows that these scalars should satisfy the equation

$$\nu(ab)\psi(a, b) = \nu(a)\nu(b)\psi(t_b, t_a)\phi(t_a t_b t_{ab}^{-1})$$

for every $a, b \in H$. In other words-

$$\partial(\nu\phi(t_-)) = \psi(a, b)/\psi(t_b, t_a). \tag{9.4}$$

Since $\mathcal{N}$ is $\sigma$-invariant, we do know that the cocycles $\psi(a, b)$ and $\psi(t_b, t_a)$ are cohomologous, and therefore such a function $\nu$ exists. Notice that we have some freedom in choosing $\nu$- we can change it to be $\nu\eta$ where $\eta$ is some character on $H$. It is easy to see by this construction that the invertible $k^{\psi}H$ bimodules with support in $\mathcal{M}$ are parameterized by pairs $(\phi, \nu)$ where $\phi$ is a character of $H^{\perp}$ by which it acts from the right on the module, and $\nu$ is a function which satisfy the equation

$$\partial\nu(a, b) = \psi(a, b)/\psi(t_a, t_b)\phi(t_{ab}t_a^{-1}t_b^{-1}).$$

We denote the corresponding invertible bimodule by $X(\phi, \nu)$. It is possible to choose $\psi$ and $t_h$ in such a way that will assure us that $\nu|_{H^{\perp}}$ is a character (for example- take $\psi$ an inflation of a cocycle on $H/H^{\perp}$ and take $t_h = 1$ for $h \in H^{\perp}$. We will thus assume henceforth that this is the case.

We fix an invertible bimodule $X$ for which $\phi = 1$, and for which the restriction of $\nu$ to $H^{\perp}$ is the trivial character (we use here the fact that we can alter $\nu$ by a character of $H$ and the fact that any character of $H^{\perp}$ can be extended to a character of $H$). It is also easy to see that we can choose $\phi$ as we wish because for every choice of $\phi$, Equation 9.4 will have a solution. One last remark- notice that in that case, where $H^{\perp}$ acts trivially from the left and from the right, Equation 9.3 implies that $\nu(h)$ depends only on the coset of h in $H^{\perp}$. We can thus consider $\nu$ also as a function from $H/H^{\perp}$ to $k^*$.

In conclusion- we have fixed an invertible bimodule $X$ with support in $\mathcal{M}$ upon which $H^{\perp}$ acts trivially from the left and from the right. Any other invertible bimodule with support in $\mathcal{M}$ will be of the form $X \otimes_{k^{\psi}H} e$ for some $e \in E$. The action of the nontrivial element $\sigma$ of $\mathbb{Z}_2$ on $E$ will be conjugation by $X$, and the second obstruction is the possibility to choose an $e \in E$ such that

$$(X \otimes e) \otimes_{k^{\psi}H} (X \otimes e) \cong k^{\psi}H.$$

9.4. **The action of $\sigma$ on $E$, and an explicit calculation of the second obstruction.** We would like to understand now the action of $\sigma$ on $E$. This in turn will help us to understand the second obstruction.

As we have seen, a general element in $E$ will be a bimodule of the form $U_{a_i,\lambda}$. So we would like to understand what is the bimodule $\sigma(U_{a_i,\lambda})$.

We have the equation

$$X \otimes_{k^\psi H} U_{a_i,\lambda} = \sigma(U_{a_i,\lambda}) \otimes_{k^\psi H} X.$$

A similar calculation to the calculations we had so far reveals the fact that if $X$ is given by $(1,\nu)$ then $U_{a_i,\lambda} \otimes_{k^\psi H} X$ is given by $(\chi(a_i,-),\nu\lambda\chi^{-1}(a_i,t_-))$, while $X \otimes_{k^\psi H} U_{a_i,\lambda}$ is given by $(\lambda^{-1},\nu\chi^{-1}(a_i,-)\lambda(t_-)$. From these two formulas we can derive an explicit formula for the action of $\sigma$ on $E$. It follows that if $\sigma(U_{a_i,\lambda}) = U_{a_j,\mu}$ then $j$ is the unique index which satisfies $\lambda^{-1} = \chi(a_j,-)$ on $H^\perp$, and $\mu$ is given by the formula $\mu = \chi^{-1}(a_i,-)\lambda(t_-)\chi(a_j,t_-)$.

Let us find now the second obstruction. For this, we just need to calculate $Q := X \otimes_{k^\psi H} X$. Consider first $X \otimes X$. It is isomorphic to $V \otimes V \otimes \bigoplus_{a\in A}(V_a)$. The bimodule $Q$ is the quotient of $X \otimes X$ when we divide out the action of $k^\psi H$.

Let us divide out first by the action of $H^\perp$. If $h \in H^\perp$ we see that we divide $V \otimes V \otimes V_a$ by $v \otimes w - \chi(a,h)v \otimes w$. If $a \notin H$ then there is an $h \in H^\perp$ such that $\chi(a,h) \neq 1$. Therefore the support of $X \otimes_{k^\psi H} X$ will be $Vec_H$. Since $V$ is simple as a left and as a right $k^\psi H$-module, it is easy to see that $V \otimes_{k^\psi H} V$ is one dimensional. We thus see that $X \otimes_{k^\psi H} X \cong U_{1,\lambda}$ for some character $\lambda$. A direct calculation shows that $\lambda(h) = \nu(h)\nu(t_h)$. This means that the second obstruction is the character $\lambda$, as an element of $H^2(\mathbb{Z}_2, E) = E^\sigma/im(1+\sigma)$ (recall that $\hat{H}$ is a subgroup of $E$).

Suppose that the second obstruction does vanish, and suppose that we have a solution $X(\phi,\nu)$. In other words $X(\phi,\nu) \otimes_{k^\psi H} X(\phi,\nu) \cong k^\psi H$. A direct calculation similar to the one we had above shows that the restrictions of $\phi$ and $\nu$ to $H^\perp$ coincide. Recall from Section 8 that if $U_{a_i,\lambda}$ is any invertible $k^\psi H$-bimodule with support in $Vec_A$, then this solution is equivalent to the solution $U_{a_i,\lambda} \otimes_{k^\psi H} X(\phi,\nu) \otimes_{k^\psi H} U^{-1}_{a_i,\lambda}$. Extend the character $\nu_{H^\perp}$ to a character $\eta$ of $H$. A direct calculation shows that $U_{1,\eta} \otimes_{k^\psi H} X(\phi,\nu) \otimes_{k^\psi H} U^{-1}_{1,\eta} = X(1,\nu')$. It follows that we can assume without loss of generality that $\phi = 1$.

As we have seen above, $X(\phi,\nu)^{\otimes 2} \cong k^\psi H$ if and only if $\nu(h)\nu(t_h) = 1$ for every $h \in H$. So the second obstruction vanishes if and only if there is a function $\nu$ which satisfies equations 9.4 and also the equation

$$\nu(h)\nu(t_h) = 1 \tag{9.5}$$

for every $h \in H$. It might happen, however, that we will have two different solutions $\nu$ and $\nu'$, that will be equivalent- that is, there will be an invertible $k^\psi H$ bimodule $U_{a_i,\lambda}$ such that $U_{a_i,\lambda} \otimes_{k^\psi H} X(1,\nu) \otimes_{k^\psi H} U^{-1}_{a_i,\lambda} \cong X(1,\nu')$. A careful analysis shows that this happen if and only if the following condition holds: there is a character $\eta$ on $H$ which vanishes on $H^\perp$, such that

$$\nu(h)/\nu'(h) = \eta(h)/\eta(t_h). \tag{9.6}$$

In conclusion- the second obstruction is the existence of a function $\nu : H \to H/H^\perp \to k^*$ which satisfy Equations 9.4 and 9.5. and two such functions $\nu$ and $\nu'$ give equivalent solutions if and only if there is a character $\eta$ of $H$ which vanishes on $H^\perp$ and which satisfies Equation 9.6.

9.5. **The third obstruction.** As explained in Section 8, after solving the second obstruction, we can think about the third obstruction in the following way: we have an invertible $k^\psi H$ bimodule $X$ with support in $\mathcal{M}$, and $X \otimes_{k^\psi H} X \cong k^\psi H$. We would like to turn $k^\psi H \oplus X$ into an algebra in $\mathcal{C}$. The only obstruction for that (and this is the third obstruction) is that the multiplication on $X \otimes X \otimes X$ might be associative only up to a scalar. This scalar is the third obstruction, considered as an element of $H^3(\mathbb{Z}_2, k^*) = \{1, -1\}$. Following the work of Tambara (see [9]), we see that this sign is the sign of the following expression

$$\sum_{a \in H/H^\perp} \nu(h)\tau.$$

If the third obstruction vanishes, we only have one possible solution, as $H^2(\mathbb{Z}_2, k^*) = 1$, since we have assumed that $k$ is algebraically closed. This finishes the proof of Proposition 26

9.6. **Relation to the Tambara's Work.** In [9], Tambara classified all fiber functors on $\mathcal{TY}(A, \chi, \tau)$. In the language of module categories, he classified all module categories over $\mathcal{TY}(A, \chi, \tau)$ of rank 1. In the language of our classification, he described all module categories whose parameterization begins with $(\mathcal{M}(A, \psi), \mathbb{Z}_2, \ldots)$ for some $\psi$.

There is a deeper connection between our result and the result of Tambara, as we will show now. Assume that we have a module category over $\mathcal{TY}(A, \chi, \tau)$ whose classification begins with $(\mathcal{M}(H, \psi), \mathbb{Z}_2, \ldots)$. Then, as we have seen, $H^\perp < H$, and $\chi$ induces a nondegenerate symmetric bicharacter $\bar{\chi}$ in $\bar{H} := H/H^\perp$. We thus have another Tambara Yamagami fusion category $\mathcal{D} := \mathcal{TY}(\bar{H}, \bar{\chi}, \bar{\tau})$, where $\bar{\tau}$ has the same sign as $\tau$. In order to explain the connection, we first recall the following theorem of Tambara (Proposition 3.2 in [9])

**Theorem 29.** *Fiber functors on $\mathcal{D}$ correspond to triples $(s, \psi, \nu)$ which satisfies the following coherence conditions:*

$$\bar{\chi}(a, b) = \xi_\psi(s(a), b)$$
$$\partial\nu(a, b) = \psi(a, b)/\psi(s(a), s(b))$$
$$\nu(a)\nu(s(a)) = 1$$
$$sign(\sum_{s(a)=a} \nu(a)) = sign(\tau)$$

*Two such triples $(s, \psi, \nu)$ and $(s', \psi', \nu')$ will give equivalent fiber functors if and only if $s = s'$ and there exist a function $\phi : H/H^\perp \to K$ such that $\psi = \partial\phi\psi'$ and $\nu(h)/\nu'(h) = \phi(h)/\phi(s(h))$*

*Remark* 30. This is not exactly the original formulation in Tambara's paper, but it is equivalent.

The following lemma is now an easy corollary from Proposition 26 and the above theorem.

**Lemma 31.** *There is a one to one correspondence between equivalence classes of fiber functors on $\mathcal{D}$ which corresponds to triples which contains the two cocycle $\psi$ and module categories over $\mathcal{C}$ whose parameterization begins with $(\mathcal{M}(H, \psi), \mathbb{Z}_2, \ldots)$.*

The Lemma says that we have a correspondence between fiber functors on one Tambara Yamagami category and some module categories oevr another Tambara Yamagami category. However, we do not know about a plausible explanation of *why* it happens.

We can now use the results of Tambara to obtain another description of our module categories. Indeed, in his paper Tamabara gave several description of fiber functors of $\mathcal{D}$. Applying Theorem 3.5 from [9], we get the following

**Corollary 32.** *Let $\mathcal{TY}(A,\chi,\tau)$, $\mathcal{M}(H,\psi)$ be as above. Assume that $H^{\perp} < H$ and that $Rad(\psi) = H^{\perp}$. Then the different ways to put on $\mathcal{M}(H,\psi)$ a $\mathcal{TY}(A,\chi,\tau)$-module structure are parameterized by pairs $(s,\mu)$ where $s$ is an involutive automorphism of $H/H^{\perp}$, and $\mu : \bar{H}^s/\bar{H}_s \to k^*$ satisfy*

$$\bar{\chi}(a,b) = \xi_{\psi}(s(a),b)$$

$$\mu(a)\mu(b)/\mu(ab) = \tilde{\chi}(a,b)$$

$$sign(\mu) = sign(\tau)$$

*Here $\bar{H}^s$ is the subgroup of $s$-invariant elements, $\bar{H}_s$ is the subgroup of elements of the form $as(a)$, The map $\tilde{\chi}$ is the induced bilinear form on $\bar{H}^s/\bar{H}_s$ (one of Tambara's result is the fact that this is indeed well defined), and $sign(\mu)$ is the sign of $\mu$ as a quadratic map (It is quite easy to show that $\bar{H}^s/\bar{H}_s$ is a vector space over $\mathbb{Z}_2$ and therefore we can talk about this sign). See Tambara's paper* [9] *for more details.*

9.7. **Dual categories.** In this subsection we shall give a general description of the dual categories of $\mathcal{TY}(A,\chi,\tau)$. First recall (see [3]) that if $\mathcal{L} \cong Mod_{\mathcal{C}} - L$ is a module category over a fusion category $\mathcal{C}$, where $L$ is an algebra in $\mathcal{C}$, then the dual category $(\mathcal{C})^*_{\mathcal{L}}$ is equivalent as a fusion category to the category of $L$-bimodules in $\mathcal{C}$.

We begin with duals with respect to module categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N},1,\Phi,v,\beta)$. In this case, $\mathcal{L} \cong Mod_{\mathcal{C}} - k^{\psi}H$ for some $H < A$ and some two cocycle $\psi$. We have described above the category of $k^{\psi}H$-bimodules with support in $Vec_A$. We have seen that it is a pointed category with an abelian group of invertible objects, which we have described in Subsection 9.2. Consider now the $k^{\psi}H$-bimodules with support in $\mathcal{M}$. Following previous calculations, we see that such a bimodule is given by a vector space $V$ which is both a left and a right $k^{\psi}H$-module, and the interaction between the left and the right structure is given by the formula

$$(U_h \cdot v) \cdot U_{h'} = \chi(h,h')U_h \cdot (v \cdot U_{h'}). \tag{9.7}$$

We can think of such modules as $k^{\theta}[H \times H]$-modules, where $\theta$ is a suitable two cocycle. By this point of view, the isomorphism classes of indecomposable modules is in bijection with the characters of $Rad(\theta) < H \times H$. Let us denote the indecomposable module which corresponds to a character $\zeta$ of $Rad(\theta)$ by $V_{\zeta}$. A routine and tedious calculation shows us that the group of invertible $k^{\psi}H$-bimodules with support in $Vec_A$ acts on the modules with support in $\mathcal{M}$ via the following formulas:

$$U_{a_i,\lambda} \otimes_B V_{\zeta} = V_{(\lambda,\chi(a_i,-))\zeta}$$

$$V_{\zeta} \otimes_B U_{a_i,\lambda} = V_{(\chi^{-1}(a_i,-),\lambda^{-1})\zeta}$$

We know that the dual category is graded by $\mathbb{Z}_2$ in the obvious sense. We use this fact in order to conclude the following multiplication formula:

$$V_{\zeta} \otimes_B V_{\eta} = \bigoplus_{(\lambda,\chi(a_i,-))t^*(\eta)=\zeta} U_{a_i,\lambda}$$

where by $t^*(\eta)$ we mean the composition of $\eta$ with the map $H \times H \to H \times H$ given by $(h_1,h_2) \mapsto (h_2,h_1)$. Notice that by the analysis done in Section 8 and by the observation that the group of invertible bimodules with support in $Vec_A$ acts transitively on the set $\{V_{\zeta}\}$, we see that the dual is pointed if and only if the category $\mathcal{L}$ is $\sigma$-invariant.

We consider now module categories of the second type. By this we mean categories of the form $\mathcal{L} = \mathcal{M}(\mathcal{N},\langle\sigma\rangle,\Phi,v,\beta)$. Assume that $\mathcal{L} = Mod - Vec_A\mathcal{M}(H,\psi)$ over $Vec_A$. Then $\sigma(H,\psi) = (H,\psi)$ and we have an action of $\sigma$ on the abelian group $E$ of invertible bimodules with support in $Vec_A$. We have an equivalence of fusion categories $(Vec_A)^*_{\mathcal{L}} \cong Vec^{\omega}_E$ for some three cocycle $\omega \in H^3(E,k^*)$.

We have seen in Section 7 that the dual $(\mathcal{C})^*_{\mathcal{L}}$ will be the equivariantization of this category with respect to the action of $\mathbb{Z}_2$. If, for example, we would have known that $\omega = 1$, then this equivariantization would have been equivalent to the representation category of the group $\mathbb{Z}_2 \ltimes \hat{E}$ In general, the description of this category is not much harder.

We conclude by observing that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if there is a pair $(H, \psi)$ such that $\sigma(H, \psi) = (H, \psi)$. This gives an alternative proof of the fact that $\mathcal{TY}(A, \chi, \tau)$ is group theoretical if and only if the metric group $(A, \chi)$ has a Lagrangian subgroup (see Corollary 4.9 of [5]).

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