TORUS MANIFOLDS WITH NON-ABELIAN SYMMETRIES

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ABSTRACT. Let G be a connected compact non-abelian Lie-group and T a maximal torus of G. A torus manifold with G-action is defined to be a smooth connected closed oriented manifold of dimension $2\dim T$ with an almost effective action of G such that $M^T \neq \emptyset$. We show that if there is a torus manifold M with G-action, then the action of a finite covering group of G factors through $\tilde{G} = \prod SU(l_i+1) \times \prod SO(2l_i+1) \times \prod SO(2l_i) \times T^{l_0}$. The action of $\tilde{G}$ on M restricts to an action of $\tilde{G}' = \prod SU(l_i+1) \times \prod SO(2l_i+1) \times \prod U(l_i) \times T^{l_0}$ which has the same orbits as the $\tilde{G}$-action.

We define invariants of torus manifolds with G-action which determine their $\tilde{G}'$-equivariant diffeomorphism type. We call these invariants admissible 5-tuples. A simply connected torus manifold with G-action is determined by its admissible 5-tuple up to $\tilde{G}$-equivariant diffeomorphism. Furthermore, we prove that all admissible 5-tuples may be realised by torus manifolds with $\tilde{G}''$-action, where $\tilde{G}''$ is a finite covering group of $\tilde{G}'$.

1. Introduction

A $2n$-dimensional smooth connected closed oriented manifold M with an almost effective action of an n-dimensional torus T is called *torus manifold* if $M^T \neq \emptyset$. If each point of M has an invariant open neighborhood, which is weakly equivariantly diffeomorphic to an open subset of the standard action of T on $\mathbb{C}^n$, then the orbit space M/T is an n-dimensional manifold with corners [15, p.720-721]. In this case M is said to be *quasitoric* if M/T is face preserving homeomorphic to a simple polytope P. In that case there are strong relations between the topology of M and the combinatorics of P [6, 5].

In this article we study torus manifolds, for which the T-action may be extended by an action of a connected compact non-abelian Lie-group G. To state our results, we introduce a bit more notations, which are used to describe the structure of torus manifolds.

A closed, connected submanifold M_i of codimension two of a torus manifold M, which is pointwise fixed by a one dimensional subtorus $\lambda(M_i)$ of T and which contains a T-fixed point, is called *characteristic* submanifold of M.

All characteristic submanifolds M_i are orientable and an orientation of M_i determines a complex structure on the normal bundle $N(M_i, M)$ of M_i.

We denote the set of unoriented characteristic submanifolds of M by $\mathfrak{F}$. If M is quasitoric the characteristic submanifolds of M are given by the preimages of the facets of P. In this case we identify $\mathfrak{F}$ with the set of facets of P.

Let G be a connected compact non-abelian Lie-group. We call a smooth connected closed oriented G-manifold M a *torus manifold with G-action* if G acts almost effectively on M, $\dim M = 2\operatorname{rank} G$ and $M^T \neq \emptyset$ for a maximal torus T of G. That means that M with the action of T is a torus manifold. Because all maximal tori of G are conjugated, M together with the action of any other maximal torus T' is also a torus manifold. Moreover, for all choices of a maximal torus of

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G, we get up to weakly equivariant diffeomorphism the same torus manifold. The G-action on M induces an action of the Weyl-group $W(G)$ of G on $\mathfrak{F}$ and the T-equivariant cohomology of M. Results of Masuda [14] and Davis-Januszkiewicz [6] make a comparison of these actions possible. From this comparison we get a description of the action on $\mathfrak{F}$ and the isomorphism type of $W(G)$. Namely there is a partition of $\mathfrak{F} = \mathfrak{F}_0 \amalg \dots \amalg \mathfrak{F}_k$ and a finite covering group $\tilde{G} = \prod_{j=1}^k G_j \times T^{l_0}$ of G such that each G_{j_0} is non-abelian and $W(G_{j_0})$ acts transitively on $\mathfrak{F}_{j_0}$ and trivially on $\mathfrak{F}_j$, $j \neq j_0$, and the orientation of each $M_i \in \mathfrak{F}_j$, $j \neq j_0$, is preserved by $W(G_{j_0})$ (see section 2).

We call such G_i the *elementary factors* of $\tilde{G}$.

By looking at the orbits of the T-fixed points, we find that we may assume without loss of generality that all elementary factors are isomorphic to $SU(l_i+1)$, $SO(2l_i)$ or $SO(2l_i+1)$ (see section 3). If M is quasitoric then all elementary factors are isomorphic to $SU(l_i+1)$.

Now assume $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then the restriction of the action of G_1 to $U(l_1)$ has the same orbits as the G_1-action (see section 6). The following theorem shows that the classification of simply connected torus manifolds with $\tilde{G}$-action reduces to the classification of torus manifolds with $U(l_1)\times G_2$-action.

Theorem 1.1 (Theorem 6.3). *Let M, M' be two simply connected torus manifolds with $\tilde{G}$-action, $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then M and M' are $\tilde{G}$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.*

By applying a blow up construction along the fixed points of an elementary factor of $\tilde{G}$ isomorphic to $SU(l_i+1)$ or $SO(2l_i+1)$, we get a fiber bundle over a complex or real projective space with some torus manifold as fiber.

This construction may be reversed and we call the inverse construction a blow down. With this notation we get:

Theorem 1.2 (Corollaries 5.6, 5.14, 7.2, Theorem 7.8). *Let $\tilde{G} = G_1 \times G_2$, M a torus manifold with G-action such that G_1 is elementary and $l_2 = \operatorname{rank} G_2$.*

- *If $G_1 = SU(l_1+1)$ and $\#\mathfrak{F}_1 = 2$ in the case $l_1 = 1$, then M is the blow down of a fiber bundle $\tilde{M}$ over $\mathbb{C}P^{l_1}$ with fiber some $2l_2$-dimensional torus manifold with G_2-action along an invariant submanifold of codimension two. Here the G_1-action on $\tilde{M}$ covers the standard action of $SU(l_i+1)$ on $\mathbb{C}P^{l_1}$.*
- *If $G_1 = SO(2l_1+1)$ and $\#\mathfrak{F}_1 = 1$ in the case $l_1 = 1$, then M is a blow down of a fiber bundle $\tilde{M}$ over $\mathbb{R}P^{2l_1}$ with fiber some $2l_2$-dimensional torus manifold with G_2-action along an invariant submanifold of codimension one or a Cartesian product of a $2l_1$-dimensional sphere and a $2l_2$-dimensional torus manifold with G_2-action. In the first case the G_1-action on $\tilde{M}$ covers the standard action of $SO(2l_1+1)$ on $\mathbb{R}P^{2l_1}$. In the second case G_1 acts in the usual way on S^{2l_1}.*

If all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i+1)$ or $SU(l_i+1)$, then we may iterate this construction. By this iteration we get a complete classification of torus manifolds with $\tilde{G}$-action up to $\tilde{G}$-equivariant diffeomorphism in terms of admissible 5-tuples (Theorem 8.5). For general G we have $\tilde{G} = \prod SU(l_i+1) \times \prod SO(2l_i+1) \times SO(2l_i) \times T^{l_0}$. We may restrict the action of $\tilde{G}$ to $\prod SU(l_i+1) \times \prod SO(2l_i+1) \times \prod U(l_i) \times T^{l_0}$. Therefore we get invariants for torus manifolds with G-action from the above classification. With Theorem 1.1, we see that these invariants determine the G-equivariant diffeomorphism type of simply connected torus manifolds with G-action.

At the end we apply our classification to get more explicit results in special cases. These are:

For the special case $G_2 = \{1\}$ we get:

Corollary 1.3 (Corollary 3.6)**.** *Assume that G is elementary and M a torus manifold with G-action. Then M is equivariantly diffeomorphic to S^{2l} or $\mathbb{C}P^l$ if $G = SO(2l+1), SO(2l)$ or $G = SU(l+1)$, respectively.*

We recover certain results of Kuroki [13, 11, 12] who gave a classification of torus manifolds with G-action and $\dim M/G \leq 1$ (see Corollaries 8.10 and 8.11).

For quasitoric manifolds we have the following result.

Theorem 1.4 (Corollary 8.9)**.** *If G is semi-simple and M a quasitoric manifold with G-action, then*

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1)$$

and M is equivariantly diffeomorphic to a product of complex projective spaces.

Furthermore, we give an explicit classification of simply connected torus manifolds with G-action such that $\tilde{G}$ is semi-simple and has two simple factors.

Theorem 1.5 (Corollaries 3.6, 8.12, 8.14)**.** *Let $\tilde{G} = G_1 \times G_2$ with G_i simple and M a simply connected torus manifold with G-action. Then M is one of the following:*

$$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \;\; \mathbb{C}P^{l_1} \times S^{2l_2}, \;\; \#_i(S^{2l_1} \times S^{2l_2})_i, \;\; S^{2l_1+2l_2}$$

The $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

The paper is organized as follows. In section 2 we investigate the action of the Weyl-group of G on $\mathfrak{F}$ and $H_T^*(M)$. In section 3 we determine the orbit-types of the T-fixed points in M and the isomorphism types of the elementary factors of G. In section 4 the basic properties of the blow up construction are established. In section 5 actions with elementary factor $G_1 = SU(l_1 + 1)$ are studied. In section 6 we give an argument which reduces the classification problem for actions with an elementary factor $G_1 = SO(2l_1)$ to that with an elementary factor $SU(l_1)$. In section 7 we classify torus manifolds with G-action with elementary factor $G_1 = SO(2l_1 + 1)$. In section 8 we iterate the classification results of the previous sections and illustrate them with some applications. There are two appendices with preliminary facts on Lie-groups and torus manifolds.

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2. The action of the Weyl-group on $\mathfrak{F}$

Let G be a compact connected Lie-group of rank n and T a maximal torus of G. Moreover, let M be a torus manifold with G-action. That means that G acts almost effectively on the $2n$-dimensional smooth closed connected oriented manifold M such that $M^T \neq \emptyset$. We call a closed connected submanifold M_i of codimension two of M, which is pointwise fixed by a one-dimensional subtorus $\lambda(M_i)$ of T and which contains a T-fixed point, a characteristic submanifold of M. If g is an element of the normalizer N_GT of T in G, then, for every characteristic submanifold M_i, gM_i is also a characteristic submanifold. Therefore there are actions of N_GT and the Weyl-group of G on $\mathfrak{F}$.

In this section we describe this action of the Weyl-group of G on $\mathfrak{F}$. At first we recall the definition of the equivariant cohomology of a G-space X. Let $EG \to BG$ be a universal principal G-bundle. Then EG is a contractible free right G-space. If T is a maximal torus of G, then we may identify $ET = EG$ and $BT = EG/T$.

The Borel-construction X_G of X is the orbit space of the right action $((e,x),g) \mapsto (eg, g^{-1}x)$ on $EG \times X$. The equivariant cohomology $H_G^*(X)$ of X is defined as the cohomology of X_G.

In this section we take all cohomology groups with coefficients in $\mathbb{Q}$.

The G-action on $EG \times X$ induces a right action of the normalizer of T on X_T. Therefore it induces a left action of the Weyl-group of G on the T-equivariant cohomology of X.

Now let $X = M$ be a torus manifold with G-action. Denote the characteristic submanifolds of M by M_i, $i = 1, \dots, m$. Then, for any $g \in N_GT$, $M_{g(i)} = gM_i$ is also a characteristic submanifold which depends only on the class $w = [g] \in W(G) = N_GT/T$. Therefore we get an action of the Weyl-group of G on $\mathfrak{F}$. Notice that $M_i \in \mathfrak{F}$ is a fixed point of the $W(G)$-action on $\mathfrak{F}$ if and only if it is invariant under the action of N_GT on M.

A choice of an orientation for each characteristic submanifold of M together with an orientation for M is called an *omniorientation* of M. If we fix an omniorientation for M, then the T-equivariant Poincaré-dual τ_i of M_i is well defined.

It is the image of the Thom-class of $N(M_i, M)_T$ under the natural map

$$\psi : H^2(N(M_i, M)_T, N(M_i, M)_T - (M_i)_T) \to H^2(M_T, M_T - (M_i)_T) \to H_T^2(M).$$

Because of the uniqueness of the Thom-class [17, p.110] and because ψ commutes with the action of $W(G)$, we have

$$\tau_{g(i)} = \pm g^* \tau_i. \tag{2.1}$$

Here the minus-sign occurs if and only if $g|_{M_i} : M_i \to M_{g(i)}$ is orientation reversing. We say that the class $[g] \in W(G)$ acts orientation preserving at M_i if this map is orientation preserving. If $[g]$ acts orientation preserving at all characteristic submanifolds, then we say that $[g]$ preserves the omniorientation of M.

Let $S = H^{>0}(BT)$ and $\hat{H}_T^*(M) = H_T^*(M)/S$-torsion. Because $M^T \neq \emptyset$, there is an injection $H^2(BT) \hookrightarrow H_T^2(M)$ and

$$H^2(BT) \cap S\text{-torsion} = \{0\}. \tag{2.2}$$

By [14, p. 240-241], the τ_i are linearly independent in $\hat{H}_T^*(M)$. By Lemma 3.2 of [14, p. 246], they form a basis of $\hat{H}_T^2(M)$.

The Lie-algebra LG of G may be endowed with an Euclidean inner product which is invariant for the adjoint representation. This allows us to identify the Weyl-group $W(G)$ of G with a group of orthogonal transformations on the Lie-algebra LT of T. It is generated by reflections in the walls of the Weyl-chambers of G [4, p. 192-193]. In the following we say that an element of $W(G)$ is a reflection if and only if it is a reflection in a wall of a Weyl-chamber of G. An element $w \in W(G)$ is a reflection if and only if it acts as a reflection on $H^2(BT)$.

Here we say that $A \in \mathrm{Gl}(L)$ acts as a reflection on the $\mathbb{Q}$-vector space L if there is a decomposition $L = L_+ \oplus L_-$ with $\dim_{\mathbb{Q}} L_- = 1$ and $A|_{L_\pm} = \pm \mathrm{Id}$. Notice that $A \in \mathrm{Gl}(L)$ acts as a reflection on L if and only if $\operatorname{ord} A = 2$ and $\operatorname{trace}(A, L) = \dim_{\mathbb{Q}} L - 2$.

Lemma 2.1. *Let $w \in W(G)$ be a reflection. Then there are the following possibilities for the action of w on $\mathfrak{F}$:*

1. *w fixes all except exactly two elements of $\mathfrak{F}$. It acts orientation preserving at all characteristic submanifolds.*
2. *w fixes all except exactly two elements of $\mathfrak{F}$. Denote the elements of $\mathfrak{F}$ which are not fixed by w by M_1, M_2. The action of w is orientation preserving at all characteristic submanifolds of M except M_1, M_2. It is orientation reversing at M_1, M_2.*

(3) *w fixes all elements of $\mathfrak{F}$. It acts orientation reversing at exactly one characteristic submanifold of M.*

Proof. Using the arguments given before Lemma 2.1, we have the following commutative diagram of $W(G)$-representations with exact rows and columns

$$\begin{array}{ccccccc} & & & & S\text{-torsion in } H_T^2(M) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & H^2(BT) & \longrightarrow & H_T^2(M) & \xrightarrow{\phi} & H^2(M) \\ & & & & \downarrow & & \\ & & & & \hat{H}_T^2(M) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Here ϕ denotes the natural map $H_T^2(M) \to H^2(M)$.

Because G is connected, the $W(G)$-action on $H^2(M)$ is trivial. By (2.2) the S-torsion in $H_T^2(M)$ injects into $H^2(M)$. Therefore $W(G)$ acts trivially on the S-torsion in $H_T^2(M)$.

Because w is a reflection, we have $\operatorname{trace}(w, H^2(BT)) = \dim_{\mathbb{Q}} H^2(BT) - 2$. From the exact row in the diagram we get

$$\begin{aligned} \operatorname{trace}(w, H_T^2(M)) &= \operatorname{trace}(w, H^2(BT)) + \operatorname{trace}(w, \operatorname{im}\phi) \\ &= \dim_{\mathbb{Q}} H^2(BT) - 2 + \dim_{\mathbb{Q}} \operatorname{im}\phi \\ &= \dim_{\mathbb{Q}} H_T^2(M) - 2. \end{aligned}$$

Similarly we get

$$\begin{aligned} \operatorname{trace}(w, \hat{H}_T^2(M)) &= \operatorname{trace}(w, H_T^2(M)) - \operatorname{trace}(w, S\text{-torsion in } H_T^2(M)) \\ &= \dim_{\mathbb{Q}} \hat{H}_T^2(M) - 2. \end{aligned}$$

Now the statement follows from (2.1) because the τ_i form a basis of $\hat{H}_T^2(M)$. □

Lemma 2.2. *An element $w \in W(G)$ acts as a reflection on $\hat{H}_T^2(M)$ if and only if it is a reflection.*

Proof. Because, by (2.2), $H^2(BT)$ injects into $\hat{H}_T^2(M)$, $W(G)$ acts effectively on $\hat{H}_T^2(M)$. Therefore we may identify $W(G)$ with a subgroup of $\operatorname{Gl}(\hat{H}_T^2(M))$.

If $w \in W(G)$, then, as in the proof of Lemma 2.1, we see that

$$\dim_{\mathbb{Q}} H^2(BT) - \operatorname{trace}(w, H^2(BT)) = \dim_{\mathbb{Q}} \hat{H}_T^2(M) - \operatorname{trace}(w, \hat{H}_T^2(M)).$$

Therefore, by the remark before Lemma 2.1, an element of $W(G)$ of order two is a reflection if and only if it acts as a reflection on $\hat{H}_T^2(M)$. □

Let $\mathfrak{F}_0$ be the set of characteristic submanifolds, which are fixed by the $W(G)$-action on $\mathfrak{F}$ and at which $W(G)$ acts orientation preserving. Furthermore let $\mathfrak{F}_i$, $i = 1, \dots, k$, be the other orbits of the $W(G)$-action on $\mathfrak{F}$ and V_i the subspace of $\hat{H}_T^2(M)$ spanned by the τ_j with $M_j \in \mathfrak{F}_i$. Then $W(G)$ acts trivially on V_0. For $i > 0$, let W_i be the subgroup of $W(G)$ which is generated by the reflections which act non-trivially on V_i. Then, by Lemma 2.1, W_i acts trivially on V_j, $j \neq i$.

By (2.2), $H^2(BT)$ injects into $\hat{H}_T^2(M)$. Therefore $W(G)$ acts effectively on $\hat{H}_T^2(M)$. This fact implies that the subgroups W_i, $i = 1, \dots, k$, of $W(G)$ pairwise commute and $\langle W_1, \dots, W_i \rangle \cap W_{i+1} = \{1\}$ for all $i = 1, \dots, k-1$. Here $\langle W_1, \dots, W_i \rangle$

denotes the subgroup of $W(G)$ which is generated by $W_1, \dots, W_i$. Hence, we have an injective group homomorphism $\prod W_i \rightarrow W(G)$, $(w_1, \dots, w_k) \mapsto w_1 \dots w_k$.

Lemma 2.3. *The group homomorphism* $\prod W_i \rightarrow W(G)$, $(w_1, \dots, w_k) \mapsto w_1 \dots w_k$ *is an isomorphism.*

Proof. Because $W(G)$ is generated by reflections and each reflection is contained in a W_i, the above homomorphism is surjective. As noted before, it is injective. Therefore it is an isomorphism. □

Lemma 2.4. *For each pair* $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, *with* $M_{j_1} \neq M_{j_2}$ *there is a reflection* $w \in W_i$ *with* $w(M_{j_1}) = M_{j_2}$.

Proof. Because $\mathfrak{F}_i$ is an orbit of the $W(G)$-action on $\mathfrak{F}$ and $W(G)$ is generated by reflections, there is a $M'_{j_1} \in \mathfrak{F}_i$ with $M'_{j_1} \neq M_{j_2}$ and a reflection $w \in W_i$ with $w(M'_{j_1}) = M_{j_2}$.

Because W_i is generated by reflections and acts transitively on $\mathfrak{F}_i$ the natural map $W_i \rightarrow S(\mathfrak{F}_i)$ to the permutation group $S(\mathfrak{F}_i)$ of $\mathfrak{F}_i$ is a surjection by Lemma 2.1 and Lemma 3.10 of [1, p. 51]. Therefore there is a $w' \in W_i$ with

$$w'(M_{j_1}) = M'_{j_1}, \qquad w'(M'_{j_1}) = M_{j_1}, \qquad w'(M_{j_2}) = M_{j_2}.$$

Now $w'^{-1}ww' \in W_i$ is a reflection with the required properties. □

It follows from Lemma 2.1 that for each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there are at most two reflections, which map M_{j_1} to M_{j_2}.

If $M_{j'_1}, M_{j'_2} \in \mathfrak{F}_i$ is another pair with $M_{j'_1} \neq M_{j'_2}$, then one sees as in the proof of Lemma 2.4 that there is a $w' \in W_i$ with

$$w'(M_{j'_1}) = M_{j_1}, \qquad w'(M_{j'_2}) = M_{j_2}.$$

Therefore there is a bijection

$$\{w \in W_i;\ w \text{ reflection}, w(M_{j_1}) = M_{j_2}\} \rightarrow \{w \in W_i;\ w \text{ reflection}, w(M_{j'_1}) = M_{j'_2}\}$$
$$w \mapsto w'^{-1}ww'.$$

In particular, the number of reflections which map M_{j_1} to M_{j_2} does not depend on the choice of $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$.

Lemma 2.5. *Assume* $\#\mathfrak{F}_i > 1$ *and* $i > 0$. *If for each pair* $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$ *with* $M_{j_1} \neq M_{j_2}$ *there is exactly one reflection in* W_i*, which maps* M_{j_1} *to* M_{j_2}*, then* W_i *is isomorphic to* $S(\mathfrak{F}_i) \cong W(SU(l_i + 1))$ *with* $l_i + 1 = \#\mathfrak{F}_i$.

Proof. First we show that there is no reflection of the third type as described in Lemma 2.1 in W_i. Assume that $w' \in W_i$ is a reflection of the third type. Then let $M_1 \in \mathfrak{F}_i$ be the characteristic submanifold at which w' acts orientation reversing. Furthermore, let $M_1 \neq M_2 \in \mathfrak{F}_i$.

Then by Lemma 2.4 there is a reflection $w \in W_i$ such that $wM_1 = M_2$. Hence, $w'ww'$ is a reflection with $w'ww'M_1 = M_2$. Because w and $w'ww'$ have a different orientation behaviour at M_1, we have $w \neq w'ww'$, contradicting our assumption.

To prove the lemma, it is sufficient to show that the kernel of the natural map $W_i \rightarrow S(\mathfrak{F}_i)$ is trivial. Let w be an element of this kernel. Then for each $\tau_j \in V_i$ we have

$$w\tau_j = \pm\tau_j.$$

If we have $w\tau_j = \tau_j$ for all $\tau_j \in V_i$, then $w = \mathrm{Id}$.

Now assume that $w\tau_{j_0} = -\tau_{j_0}$ for a $\tau_{j_0} \in V_i$. Then there are reflections $w_1, \dots, w_n \in W_i$, $n \geq 2$, with $-\tau_{j_0} = w\tau_{j_0} = w_1 \dots w_n \tau_{j_0}$. After removing some of the w_i, we may assume that

$$\begin{aligned} w_i \dots w_n \tau_{j_0} &\neq \pm \tau_{j_0} && \text{for all } i = 2, \dots, n, \\ w_{i+1} \dots w_n \tau_{j_0} &\neq \pm w_i \dots w_n \tau_{j_0} && \text{for all } i = 2, \dots, n. \end{aligned}$$

Therefore, by Lemma 2.1, we have $w_i \tau_{j_0} = \tau_{j_0}$ for $2 \leq i < n$. This equation together with $w\tau_{j_0} = -\tau_{j_0}$ implies

$$w_n \dots w_2 w_1 w_2 \dots w_n \tau_{j_0} = -w_n \tau_{j_0}.$$

Therefore $w_n \dots w_2 w_1 w_2 \dots w_n M_{j_0} = w_n M_{j_0}$.

But $w_n \dots w_2 w_1 w_2 \dots w_n$ is a reflection. Therefore, by assumption, we have

$$w_n \dots w_2 w_1 w_2 \dots w_n = w_n$$

and

$$w_n \tau_{j_0} = w_n w_{n-1} \dots w_2 w_1 w_2 \dots w_n \tau_{j_0} = -w_n \tau_{j_0}.$$

Because $w_n \tau_{j_0} \neq 0$, this is impossible. Hence, our assumption that $w\tau_{j_0} = -\tau_{j_0}$ is false.

Therefore the kernel is trivial. □

To get the isomorphism type of W_i in the case, where there is a pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ and exactly two reflections in W_i, which map M_{j_1} to M_{j_2}, we first give a description of the Weyl-groups of some Lie-groups.

Let L be an l-dimensional $\mathbb{Q}$-vector space with basis $e_1, \dots, e_l$. For $1 \leq i < j \leq l$ let $f_{ij\pm}, g_i \in \mathrm{Gl}(L)$ such that

$$f_{ij+} e_k = \begin{cases} e_i & \text{if } k = j \\ e_j & \text{if } k = i \\ e_k & \text{else} \end{cases}$$

$$f_{ij-} e_k = \begin{cases} -e_i & \text{if } k = j \\ -e_j & \text{if } k = i \\ e_k & \text{else} \end{cases}$$

$$g_i e_k = \begin{cases} -e_i & \text{if } k = i \\ e_k & \text{else.} \end{cases}$$

Then we have the following isomorphisms of groups [4, p. 171-172]:

$$\begin{aligned} W(SU(l-1)) \cong S(l) &\cong \langle f_{ij+}; 1 \leq i < j \leq l \rangle, \\ W(SO(2l)) &\cong \langle f_{ij\pm}; 1 \leq i < j \leq l \rangle, \\ W(SO(2l+1)) \cong W(Sp(l)) &\cong \langle f_{ij\pm}, g_1; 1 \leq i < j \leq l \rangle. \end{aligned}$$

From this description and Lemma 2.1, we get:

Lemma 2.6. *If for each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there are exactly two reflections in W_i which map M_{j_1} to M_{j_2}, then with $l_i = \#\mathfrak{F}_i$ we have*

(1) $W_i \cong W(SO(2l_i))$ *if there is no reflection of the third type as described in Lemma 2.1 in W_i.*
(2) $W_i \cong W(SO(2l_i + 1)) \cong W(Sp(l_i))$ *if there is a reflection of the third type in W_i.*

By [4, p. 233], G has a finite covering group $\tilde{G}$ such that $\tilde{G} = \prod_i G_i \times T^{l_0}$, where the G_i are simple simply connected compact Lie-groups. The Weyl-group of G is given by $W(G) = \prod_i W(G_i)$.

We call two reflections $w, w' \in W(G)$ equivalent if there are reflections $w_1, \dots, w_k \in W(G)$ such that

$$w = w_1, \qquad w' = w_k, \qquad [w_i, w_{i+1}] \neq 1.$$

Here $[w_i, w_{i+1}]$ denotes the commutator of w_i and w_{i+1}. Because the Dynkin-diagram of a simple Lie-group is connected, each $W(G_i)$ is generated by equivalent reflections. Therefore each $W(G_i)$ is contained in a W_j. Therefore we get $W_i = \prod_{j\in J_i} W(G_j)$. Using Lemmas 2.5 and 2.6, we deduce:

$$W_i = \begin{cases} W(G_j) & \text{for some } j \text{ if } W_i \not\cong W(SO(4)) \\ W(G_{j_1}) \times W(G_{j_2}) & \text{with } G_{j_1} \cong G_{j_2} \cong SU(2) \text{ if } W_i \cong W(SO(4)). \end{cases}$$

Therefore we may write $\tilde{G} = \prod_i G_i \times T^{l_0}$ with $W_i = W(G_i)$ and G_i simple and simply connected or $G_i = \operatorname{Spin}(4)$. In the following we will call these G_i the *elementary factors* of $\tilde{G}$.

We summarize the above discussion in the following lemma.

Lemma 2.7. *Let M be a torus manifold with G-action and $\tilde{G}$ as above. Then all G_i are non-exceptional, i.e. $G_i = SU(l_i+1), Spin(2l_i), Spin(2l_i+1), Sp(l_i)$.*

The Weyl-group of an elementary factor G_i of $\tilde{G}$ acts transitively on $\mathfrak{F}_i$ and trivially on $\mathfrak{F}_j$, $j \neq i$.

For a given isomorphism type of G_i, there are at most two possible values of $\#\mathfrak{F}_i$. The possible values of $\#\mathfrak{F}_i$ are listed in the following table.

G_i	$\#\mathfrak{F}_i$
$SU(2) = Spin(3) = Sp(1)$	$1, 2$
$Spin(4)$	2
$Spin(5) = Sp(2)$	2
$SU(4) = Spin(6)$	$3, 4$
$SU(l_i+1)$, $l_i \neq 1, 3$	$l_i + 1$
$Spin(2l_i+1)$, $l_i > 2$	l_i
$Spin(2l_i)$, $l_i > 3$	l_i
$Sp(l_i)$, $l_i > 2$	l_i

If we restrict our attention to quasitoric manifolds with G-action, then we get a much shorter list of possible isomorphism types of the elementary factors. In fact, if M is a quasitoric manifold with G-action, then, as shown in the next lemma, all elementary factors of G are isomorphic to $SU(l_i+1)$ for some $l_i \geq 1$.

Lemma 2.8. *Let M be a quasitoric manifold with G-action. Then there is a covering group $\tilde{G}$ of G with $\tilde{G} = \prod_{i=1}^{k_1} SU(l_i+1) \times T^{l_0}$.*

Proof. First we show for $i > 0$:

$$W_i \cong S(\mathfrak{F}_i). \tag{2.3}$$

To do so, it is sufficient to prove that there is an omniorientation on M which is preserved by the action of $W(G)$. This is true if for every characteristic submanifold M_i and $g \in N_GT$ such that $gM_i = M_i$, g preserves the orientation of M_i. Since G is connected, g preserves the orientation of M and acts trivially on $H^2(M)$.

Because each vertex of the orbit polytope P of M is the intersection of exactly n facets of P, every fixed point of the T-action on M is the transverse intersection of exactly n characteristic submanifolds. Thus, the Poincaré-dual $PD(M_i) \in H^2(M)$ of M_i is non-zero because $M_i \cap M^T \neq \emptyset$. Therefore g preserves the orientation of

M_i since otherwise

$$\begin{aligned} PD(M_i) &= \frac{1}{2}(PD(M_i) + PD(M_i)) \\ &= \frac{1}{2}(PD(M_i) + g^*PD(M_i)) \qquad (g \text{ acts trivially on } H^2(M)) \\ &= \frac{1}{2}(PD(M_i) - PD(M_i)) \qquad (g \text{ reverses the orientation of } M_i) \\ &= 0. \end{aligned}$$

This establishes (2.3). Recall that all simple compact simply connected Lie-groups having a Weyl-group isomorphic to some symmetric group are isomorphic to some $SU(l+1)$. Therefore all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i+1)$. From this the statement follows. □

Remark 2.9. In [15] Masuda and Panov show that the cohomology with coefficients in $\mathbb{Z}$ of a torus manifold M is generated by its degree-two part if and only if the torus action on M is locally standard and the orbit space M/T is a homology polytope. That means that all faces of M/T are acyclic and all intersections of facets of M/T are connected. In particular, each T-fixed point is the transverse intersection of n characteristic submanifolds. Therefore the above lemma also holds in this case.

For a characteristic submanifold M_i of M, let $\lambda(M_i)$ denote the one-dimensional subtorus of T which fixes M_i pointwise. The normalizer N_GT of T in G acts by conjugation on the set of one-dimensional subtori of T. The following lemma shows that

$$\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$$

is N_GT-equivariant.

Lemma 2.10. *Let M be a torus manifold with G-action, $g \in N_GT$ and $M_i \subset M$ be a characteristic submanifold. Then we have:*

1. $\lambda(gM_i) = g\lambda(M_i)g^{-1}$.
2. *If $gM_i = M_i$, then g acts orientation preserving on M_i if and only if*

$$\lambda(M_i) \to \lambda(M_i) \quad t \mapsto gtg^{-1}$$

is orientation preserving.

Proof. First we prove (1). Let $x \in M_i$ be a generic point. Then the identity component T_x^0 of the stabilizer of x in T is given by $T_x^0 = \lambda(M_i)$. Therefore we have

$$\lambda(gM_i) = T_{gx}^0 = gT_x^0g^{-1} = g\lambda(M_i)g^{-1}.$$

Now we prove (2). An orientation of M_i induces a complex structure on $N(M_i, M)$. We fix an isomorphism $\rho : \lambda(M_i) \to S^1$ such that the action of $t \in \lambda(M_i)$ on $N(M_i, M)$ is given by multiplication with $\rho(t)^m$, $m > 0$. The differential $Dg : N(M_i, M) \to N(M_i, M)$ is orientation preserving if and only if it is complex linear. Otherwise it is complex anti-linear. Therefore for $v \in N(M_i, M)$ we have

$$\begin{aligned} \rho(gtg^{-1})^m v &= (Dg)(Dt)(Dg)^{-1}v = (Dg)\rho(t)^m(Dg)^{-1}v \\ &= \rho(t)^{\pm m}(Dg)(Dg)^{-1}v = \rho(t^{\pm 1})^m v. \end{aligned}$$

This equation implies that $\rho(gtg^{-1}t^{\mp 1}) \in \mathbb{Z}/m\mathbb{Z}$. Because $\lambda(M_i)$ is connected and $\mathbb{Z}/m\mathbb{Z}$ is discrete, $gtg^{-1} = t^{\pm 1}$ follows, where the plus-sign arises if and only if g acts orientation preserving on M_i. □

3. G-action on M

In this section we consider torus manifolds with G-action such that $\tilde{G}$ has only one elementary factor G_1, i.e. $\tilde{G} = G_1 \times T^{l_0}$. There are two cases:

(1) There is a T-fixed point, which is not fixed by G_1.
(2) There is a G-fixed point.

We first discuss the case, where there is a T-fixed point which is not fixed by G_1.

Lemma 3.1. *Let $\tilde{G} = G_1 \times T^{l_0}$ with G_1 elementary,* $\operatorname{rank} G_1 = l_1$ *and M a torus manifold with G-action of dimension $2n = 2(l_0 + l_1)$. If there is an $x \in M^T$, which is not fixed by the action of G_1, then*

(1) *$G_1 = SU(l_1 + 1)$ or $G_1 = Spin(2l_1 + 1)$ and the stabilizer of x in G_1 is conjugated to $S(U(l_1) \times U(1))$ or $Spin(2l_1)$, respectively.*
(2) *The G_1-orbit of x equals the component of $M^{T^{l_0}}$ which contains x.*

Moreover, if $G_1 = SU(4)$, one has $\#\mathfrak{F}_1 = 4$.

Proof. The G_1-orbit of x is contained in the component N of $M^{T^{l_0}}$ containing x. Therefore we have

$$\operatorname{codim} G_{1x} = \dim G_1/G_{1x} = \dim G_1 x \leq \dim N \leq 2l_1.$$

Furthermore the stabilizer G_{1x} of x has maximal rank l_1. In particular, its identity component G_{1x}^0 is a closed connected maximal rank subgroup.

Next we use the theory of Lie-groups to determine the isomorphism types of G_1 and G_{1x}. At first we consider the case $G_1 \neq \operatorname{Spin}(4)$. From the classification of closed connected maximal rank subgroups of a compact Lie-group given in [2, p. 219] we get the following connected maximal rank subgroups H of maximal dimension:

G_1	H	$\operatorname{codim} H$
$SU(2) = \operatorname{Spin}(3) = Sp(1)$	$S(U(1) \times U(1))$	2
$\operatorname{Spin}(5) = Sp(2)$	$\operatorname{Spin}(4)$	4
$SU(4) = \operatorname{Spin}(6)$	$S(U(3) \times U(1))$	6
$SU(l_1 + 1)$, $l_1 \neq 1, 3$	$S(U(l_1) \times U(1))$	$2l_1$
$\operatorname{Spin}(2l_1 + 1)$, $l_1 > 2$	$\operatorname{Spin}(2l_1)$	$2l_1$
$\operatorname{Spin}(2l_1)$, $l_1 > 3$	$\operatorname{Spin}(2l_1 - 2) \times \operatorname{Spin}(2)$	$4l_1 - 4$
$Sp(l_1)$, $l_1 > 2$	$Sp(l_1 - 1) \times Sp(1)$	$4l_1 - 4$

Because H is unique up to conjugation and

$$\operatorname{codim} H \leq \operatorname{codim} G_{1x}^0 = \operatorname{codim} G_{1x} \leq 2l_1,$$

we see $G_1 = SU(l_1 + 1)$ or $G_1 = \operatorname{Spin}(2l_1 + 1)$. Moreover, G_{1x} is conjugated to a subgroup of G_1 which contains $S(U(l_1) \times U(1))$ or $\operatorname{Spin}(2l_1)$, respectively.

If $l_1 > 1$, then $S(U(l_1) \times U(1))$ is a maximal subgroup of $SU(l_1 + 1)$ by Lemma A.1. Therefore, if $G_1 = SU(l_1 + 1)$ and $l_1 > 1$, then G_{1x} is conjugated to $S(U(l_1) \times U(1))$. Because $\operatorname{codim} S(U(l_1) \times U(1)) = 2l_1 \geq \dim N \geq \operatorname{codim} G_{1x}$, we have $G_1 x = N$ in this case.

If $G_1 = \operatorname{Spin}(2l_1 + 1)$, $l_1 \geq 1$, then by Lemma A.4 there are two proper subgroups of G_1, which contain $\operatorname{Spin}(2l_1)$; $\operatorname{Spin}(2l_1)$ and its normalizer H_0. Because of dimension reasons we have $N = G_1 x$. Because $\operatorname{Spin}(2l_1 + 1)/H_0$ is not orientable and $M^{T^{l_0}}$ is orientable, $G_{1x} = \operatorname{Spin}(2l_1)$ follows. The case $G_1 = SU(2)$ is included in the discussion in this paragraph because $SU(2) = \operatorname{Spin}(3)$.

Now we prove the last statement of the lemma. If $G_1 = SU(4)$, then $G_1 x$ is G_1-equivariantly diffeomorphic to $\mathbb{C}P^3$ by the above discussion. Because $\mathbb{C}P^3$ has four characteristic submanifolds with pairwise non-trivial intersections, by Lemmas B.2

and B.3, there are four characteristic submanifolds $M_1, \dots, M_4$, which intersect transversely with $G_1x = N$. Because G_1x is a component of $M^{T^{l_0}}$ we have by Lemma B.1 that $\lambda(M_i) \not\subset T^{l_0}$. Therefore $\lambda(M_i)$ is not fixed pointwise by the action of $W(G_1)$ on T. Here $W(G_1)$ acts on T by conjugation. Now it follows with Lemma 2.10 that $M_1, \dots, M_4$ belong to $\mathfrak{F}_1$.

Now we turn to the case $G_1 = \mathrm{Spin}(4) = SU(2) \times SU(2)$.

Then there are the following proper closed connected maximal rank subgroups H of G_1 of codimension at most 4:

$$SU(2)\times S(U(1)\times U(1)),\ S(U(1)\times U(1))\times SU(2),\ S(U(1)\times U(1))\times S(U(1)\times U(1)).$$

The last has codimension four in G_1. The others have codimension two in G_1.

At first assume that G_1x has dimension four. Then we have $G_{1x}^0 = S(U(1) \times U(1)) \times S(U(1) \times U(1))$. There are five proper subgroups of Spin(4) which contain $S(U(1) \times U(1)) \times S(U(1) \times U(1))$ as a maximal connected subgroup, namely:

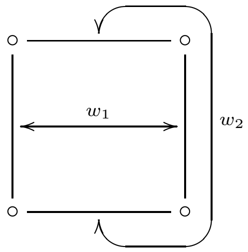
$$\begin{aligned}
H_1' &= S(U(1) \times U(1)) \times S(U(1) \times U(1))\\
H_2' &= N_{SU(2)}S(U(1) \times U(1)) \times S(U(1) \times U(1))\\
H_3' &= S(U(1) \times U(1)) \times N_{SU(2)}S(U(1) \times U(1))\\
H_4' &= N_{SU(2)}S(U(1) \times U(1)) \times N_{SU(2)}S(U(1) \times U(1))\\
H_5' &= \{(g_1, g_2) \in N_{SU(2)}S(U(1) \times U(1)) \times N_{SU(2)}S(U(1) \times U(1));\\
&\qquad g_1 \in S(U(1) \times U(1)) \Leftrightarrow g_2 \in S(U(1) \times U(1))\}
\end{aligned}$$

Therefore G_1x is G_1-equivariantly diffeomorphic to one of the following spaces:

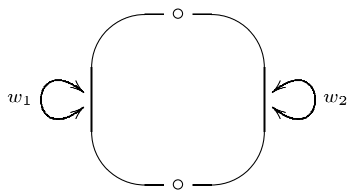
$$\begin{aligned}
&\mathrm{Spin}(4)/H_1' = S^2 \times S^2,\\
&\mathrm{Spin}(4)/H_5' = S^2 \times_{\mathbb{Z}_2} S^2 = \text{orientable double cover of } \mathbb{R}P^2 \times \mathbb{R}P^2,\\
&\mathrm{Spin}(4)/H_2' = \mathbb{R}P^2 \times S^2,\\
&\mathrm{Spin}(4)/H_3' = S^2 \times \mathbb{R}P^2,\\
&\mathrm{Spin}(4)/H_4' = \mathbb{R}P^2 \times \mathbb{R}P^2.
\end{aligned}$$

Since $G_1x = M^{T^{l_0}}$ is orientable, the latter three do not occur.

For $N = G_1x = S^2 \times S^2, S^2 \times_{\mathbb{Z}_2} S^2$, let $N^{(1)}$ be the union of the T-orbits in N of dimension less than or equal to one. Then $W(G_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the orbit space $N^{(1)}/T$. This space is given by one of the following graphs:



$(S^2 \times S^2)^{(1)}/T$



$(S^2 \times_{\mathbb{Z}_2} S^2)^{(1)}/T$

Where the edges correspond to orbits of dimension one and the vertices to the fixed points. The arrows indicate the action of the generators $w_1, w_2 \in W(G_1)$ on this space. Let M_1, M_2 be the two characteristic submanifolds of M which intersect transversely with N in x. Because N is a component of $M^{T^{l_0}}$, $\lambda(M_i)$, $i = 1, 2$, is not a subgroup of T^{l_0} by Lemma B.1. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 2.10, this fact implies $M_1, M_2 \in \mathfrak{F}_1$. Therefore there is a $w \in W(G_1)$ with $w(M_1) = M_2$. But from the pictures above we see that M_1 and

M_2 are not in the same $W(G_1)$-orbits. Therefore the case $\dim G_1x = 4$ does not occur.

Now assume that G_1x has dimension two. Then we may assume without loss of generality that $G_{1x}^0 = SU(2) \times S(U(1) \times U(1))$. Therefore $G_1x \subset M^{SU(2)\times 1}$. Because $G_1x \subset M^{T^{l_0}}$, G_1x is a component of $M^{S(U(1)\times U(1))\times 1 \times T^{l_0}}$ in this case. Therefore, by Lemmas B.1 and B.3, there are characteristic submanifolds $M_2, \dots, M_{l_0+2}$ of M such that G_1x is a component of $\bigcap_{i=2}^{l_0+2} M_i$. Furthermore, we may assume that $\lambda(M_2) \not\subset T^{l_0}$. Therefore, by Lemma 2.10, we have $M_2 \in \mathfrak{F}_1$.

But there is also a characteristic submanifold M_1 of M which intersects G_1x transversely in x. With the Lemmas B.1 and 2.10, we see $M_1 \in \mathfrak{F}_1$.

Therefore there is a $w \in W(G_1)$ with $w(M_2) = M_1$. But this is impossible because $M_2 \supset G_1x \not\subset M_1$.

Therefore $G_1 \neq \mathrm{Spin}(4)$ and the lemma is proved. □

Remark 3.2. If, in the situation of Lemma 3.1, $T \cap G_1$ is the standard maximal torus of G_1, then it follows by Proposition 2 of [8, p. 325] that G_{1x} is conjugated to the groups given in Lemma 3.1 (1) by an element of the normalizer of the maximal torus.

Lemma 3.3. *In the situation of the previous lemma x is contained in the intersection of exactly l_1 characteristic submanifolds belonging to $\mathfrak{F}_1$.*

Proof. Because $N = G_1x$ has dimension $2l_1$, x is contained in exactly l_1 characteristic submanifolds of N. By Lemmas B.2 and B.3, we know that they are components of intersections of characteristic submanifolds $M_1, \dots, M_{l_1}$ of M with N.

Because G_1x is a component of $M^{T^{l_0}}$, $\lambda(M_i)$ is not a subgroup of T^{l_0} for $i = 1, \dots, l_1$ by Lemmas B.1 and B.3. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 2.10, this implies that M_i belongs to $\mathfrak{F}_1$.

By Lemmas B.3 and B.1, G_1x is the intersection of l_0 characteristic submanifolds $M_{l_1+1}, \dots, M_n$ of M. We show that these manifolds do not belong to $\mathfrak{F}_1$. Assume that there is an $i \geq l_1 + 1$ such that M_i belongs to $\mathfrak{F}_1$. Because $W(G_1)$ acts transitively on $\mathfrak{F}_1$, there is a $w \in W(G_1)$ with $w(M_i) = M_j$, $j \leq l_1$. But this is impossible because $M_i \supset G_1x \not\subset M_j$. □

Now we turn to the case, where there is a T-fixed point which is fixed by G_1.

Lemma 3.4. *Let $\tilde{G} = G_1 \times T^{l_0}$ with G_1 elementary, $\operatorname{rank} G_1 = l_1$ and M a torus manifold with G-action of dimension $2n = 2(l_0 + l_1)$. If there is a T-fixed point $x \in M^T$, which is fixed by G_1, then $G_1 = SU(l_1+1)$ or $G_1 = Spin(2l_1)$.*

Moreover, if $G_1 \neq Spin(8)$ one has

$$T_xM = V_1 \oplus V_2 \otimes_{\mathbb{C}} W_1 \text{ if } G_1 = SU(l_1+1) \text{ and } \#\mathfrak{F}_1 = 4 \text{ in the case } l_1 = 3, \tag{3.1}$$

$$T_xM = V_3 \oplus W_2 \text{ if } G_1 = Spin(2l_1) \text{ and } \#\mathfrak{F}_1 = 3 \text{ in the case } l_1 = 3, \tag{3.2}$$

where W_1 is the standard complex representation of $SU(l_1+1)$ or its dual, W_2 is the standard real representation of $SO(2l_1)$ and the V_i are complex T^{l_0}-representations.

In the case $G_1 = Spin(8)$, one may change the action of G_1 on M by an automorphism of G_1, which is independent of x, to reach the situation described in (3.2).

Furthermore we have $x \in \bigcap_{M_i \in \mathfrak{F}_1} M_i$. If $l_1 = 1$, then we have $\#\mathfrak{F}_1 = 2$.

Proof. Let $M_1, \dots, M_n$ be the characteristic submanifolds of M, which intersect in x. Then the weight spaces of the $\tilde{G}$-representation T_xM are given by

$$N_x(M_1, M), \dots, N_x(M_n, M).$$

For $g \in N_G T$ we have $M_i = gM_j$ if and only if $N_x(M_i, M) = gN_x(M_j, M)$. Because G_1 acts non-trivially on T_xM, there is at least one M_i, $i \in \{1, \dots, n\}$, such that $M_i \in \mathfrak{F}_1$.

In the following a weight space of T_xM together with a choice of an orientation for this weight space is called an oriented weight space of T_xM. The action of G_1 on T_xM induces an action of $W(G_1)$ on the set of oriented weight spaces of T_xM.

Because $W(G_1)$ acts transitively on $\mathfrak{F}_1$ and x is a G-fixed point, we have

$$\frac{1}{2}\#\{\text{oriented weight spaces of } T_xM \text{ which are not fixed by } W(G_1)\} = \#\mathfrak{F}_1 \tag{3.3}$$

and $x \in \bigcap_{M_i \in \mathfrak{F}_1} M_i$.

For the $\tilde{G}$-representation T_xM we have

$$T_xM = N_x(M^{T^{l_0}}, M) \oplus T_xM^{T^{l_0}}. \tag{3.4}$$

If $l_0 = 0$, then we have $N_x(M^{T^{l_0}}, M) = \{0\}$. Otherwise the action of T^{l_0} induces a complex structure on $N_x(M^{T^{l_0}}, M)$. By [4, p. 68] and [4, p. 82], we have

$$N_x(M^{T^{l_0}}, M) = \bigoplus_i V_i \otimes_{\mathbb{C}} W_i, \tag{3.5}$$

where the V_i are one-dimensional complex T^{l_0}-representations and the W_i are irreducible complex G_1-representations. Since T^{l_0} acts almost effectively on M, there are at least $n - l_1$ summands in this decomposition. Therefore we get

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \sum_{j \neq i} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq n - (n - l_1 - 1) = l_1 + 1. \tag{3.6}$$

Furthermore

$$\dim_{\mathbb{R}} T_xM^{T^{l_0}} \leq 2(n - l_0) = 2l_1. \tag{3.7}$$

If there is a W_{i_0} with $\dim_{\mathbb{C}} W_{i_0} = l_1 + 1$, then from equation (3.5) we get, for all other W_i,

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \dim_{\mathbb{C}} V_{i_0} \otimes_{\mathbb{C}} W_{i_0} - \sum_{j \neq i, i_0} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq 1. \tag{3.8}$$

So they are one-dimensional. Therefore they are trivial. Furthermore we have

$$\dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) = \sum_i \dim_{\mathbb{C}} V_i \otimes_{\mathbb{C}} W_i \geq n$$

because there are at least $n - l_1$ summands in the decomposition (3.5). Therefore $T_xM^{T^{l_0}}$ is zero-dimensional in this case.

If $\dim_{\mathbb{R}} T_xM^{T^{l_0}} = 2l_1$, then we have

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \sum_{j \neq i} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq 1.$$

Therefore all W_i are one dimensional. So they are trivial in this case.

There are the following lower bounds $d_{\mathbb{R}}, d_{\mathbb{C}}$ for the dimension of real and complex non-trivial irreducible representations of G_1 [19, p. 53-54]:

G_1	$d_{\mathbb{R}}$	$d_{\mathbb{C}}$
$SU(2) = \mathrm{Spin}(3) = Sp(1)$	3	2
$\mathrm{Spin}(4)$	3	2
$\mathrm{Spin}(5) = Sp(2)$	5	4
$SU(4) = \mathrm{Spin}(6)$	6	4
$SU(l_1 + 1)$, $l_1 \neq 1, 3$	$2l_1 + 2$	$l_1 + 1$
$\mathrm{Spin}(2l_1 + 1)$, $l_1 > 2$	$2l_1 + 1$	$2l_1 + 1$
$\mathrm{Spin}(2l_1)$, $l_1 > 3$	$2l_1$	$2l_1$
$Sp(l_1)$, $l_1 > 2$	$2l_1 + 1$	$2l_1$

In [19, p. 53-54] the dominant weights of the G_1-representations realising these bounds are also given. They are important in the discussion below.

Because G_1 acts non-trivially on T_xM, one of the W_i's or $T_xM^{T^{l_0}}$ is a non-trivial G_1-representation. Therefore we have $d_{\mathbb{R}} \leq 2l_1$ or $d_{\mathbb{C}} \leq l_1 + 1$ by (3.6) and (3.7). Therefore $G_1 \neq Sp(l_1)$, $l_1 > 1$, and $G_1 \neq \mathrm{Spin}(2l_1 + 1)$, $l_1 > 1$.

If $G_1 = \mathrm{Spin}(2l_1), l_1 > 3$, then all W_i are trivial because

$$\dim_{\mathbb{C}} W_i \leq l_1 + 1 < 2l_1 = d_{\mathbb{C}}.$$

Moreover, $T_xM^{T^{l_0}}$ has dimension $2l_1$. Therefore it is the standard real $SO(2l_1)$-representation if $l_1 > 4$. If $l_1 = 4$, then there are three eight-dimensional real representations of Spin(8), namely the standard real $SO(8)$-representation and the two half spinor representations. They have three different kernels. Notice that the kernel of the G_1-representation $T_xM^{T^{l_0}}$ is equal to the kernel of the G_1-action on M. Therefore, if one of them is isomorphic to $T_xM^{T^{l_0}}$, then it is isomorphic to $T_yM^{T^{l_0}}$ for all $y \in M^T$. So we may – after changing the action of Spin(8) on M by an automorphism – assume that $T_xM^{T^{l_0}}$ is the standard real $SO(8)$-representation.

If $G_1 = SU(l_1 + 1), l_1 \neq 1, 3$, then only one W_i is non-trivial and $T_xM^{T^{l_0}}$ has dimension zero. The non-trivial W_i is the standard representation of $SU(l_1 + 1)$ or its dual depending on the complex structure of $N_x(M^{T^{l_0}}, M)$.

If $G_1 = SU(4)$, then there are one real representation of dimension 6 and two complex representations of dimension 4. If the first representation occurs in the decomposition of T_xM, then, by (3.3), we have $\#\mathfrak{F}_1 = 3$. If one of the others occurs, then $\#\mathfrak{F}_1 = 4$.

If $G_1 = SU(2)$, then there is one non-trivial W_i of dimension 2. Therefore, by (3.3), one has $\#\mathfrak{F}_1 = 2$.

If $G_1 = \mathrm{Spin}(4)$, then T_xM is an almost faithful representation. Because all almost faithful complex representations of Spin(4) have at least dimension four there is no W_i of dimension three.

If there is one W_{i_0} of dimension two, then we see as in (3.8) that all other W_i and $T_xM^{T^{l_0}}$ have dimension less than or equal to two. Because there is no non-trivial two-dimensional real Spin(4)-representation there is another W_i of dimension two. Therefore there are eight oriented weight spaces of T_xM which are not fixed by the action on $W(G_1)$. But this contradicts (3.3) because $\#\mathfrak{F}_1 = 2$.

Therefore all W_i are one-dimensional. Hence, they are trivial. $T_xM^{T^{l_0}}$ has to be the standard four-dimensional real representation of Spin(4). □

With the Lemmas 3.1 and 3.4, we see that there is no elementary factor of $\tilde{G}$, which is isomorphic to $Sp(l_1)$ for $l_1 > 2$.

Now let $G_1 = \mathrm{Spin}(2l)$. If $l = 3$, we assume $\#\mathfrak{F}_1 = 3$. Then, by looking at the G_1-representation T_xM, one sees with Lemma 3.4 that the G_1-action factors through $SO(2l)$.

Now let $G_1 = \mathrm{Spin}(2l+1)$, $l > 1$. Then, by Lemma 3.1, we have $G_{1x} \cong \mathrm{Spin}(2l)$. Because the G_{1x}-action on $N_x(G_1x, M)$ is trivial by Lemma 3.4, the G_1-action factors through $SO(2l+1)$.

In the case $G_1 = \mathrm{Spin}(3)$ and $\#\mathfrak{F}_1 = 1$ we have $G_1x = S^2$. The characteristic submanifold $M_1 \in \mathfrak{F}_1$ intersects G_1x transversely in x. Because $\#\mathfrak{F}_1 = 1$, $\lambda(M_1)$ is invariant under the action of $W(G_1)$ on the maximal torus of G. Because, by Lemma 2.10, the non-trivial element of $W(G_1)$ reverses the orientation of $\lambda(M_1)$, it is a maximal torus of G_1. Therefore the center of G_1 acts trivially on M. Hence, the G_1-action on M factors through $SO(3)$.

If, in the case $G_1 = \mathrm{Spin}(3)$ and $\#\mathfrak{F}_1 = 2$, the principal orbit type of the G_1-action is given by $\mathrm{Spin}(3)/\mathrm{Spin}(2)$, then the G_1-action factors through $SO(3)$.

Therefore in the following we may replace an elementary factor G_i of $\tilde{G}$ isomorphic to $\mathrm{Spin}(l)$, which satisfies the above conditions, by $SO(l)$.

Convention 3.5. If we say that an elementary factor G_i is isomorphic to $SU(2)$ or $SU(4)$, then we mean that $\#\mathfrak{F}_i = 2$ or $\#\mathfrak{F}_i = 4$, respectively. Conversely, if we say that G_i is isomorphic to $SO(3)$ we mean that $\#\mathfrak{F}_i = 1$ or $\#\mathfrak{F}_i = 2$ and the $SO(3)$-action has principal orbit type $SO(3)/SO(2)$. If we say $G_i = SO(6)$, then we mean $\#\mathfrak{F}_i = 3$.

Corollary 3.6. *Assume that G is elementary. Then M is equivariantly diffeomorphic to $\mathbb{C}P^{l_1}$ or $M = S^{2l_1}$ if $\tilde{G} = SU(l_1+1)$ or $\tilde{G} = SO(2l_1+1), SO(2l_1)$, respectively.*

Proof. If G is elementary, then we may assume that $G = \tilde{G} = SO(2l_1), SO(2l_1 + 1), SU(l_1+1)$ and $\dim M = 2l_1$.

If $G = SO(2l_1)$, then, by Lemmas 3.1 and 3.4, the principal orbit type of the $SO(2l_1)$-action is given by $SO(2l_1)/SO(2l_1-1)$, which has codimension one in M.

The group $S(O(2l_1-1)\times O(1))$ is the only proper subgroup of $SO(2l_1)$, which contains $SO(2l_1-1)$ properly. Because $SO(2l_1)/S(O(2l_1-1)\times O(1)) = \mathbb{R}P^{2l_1-1}$ is orientable all orbits of the $SO(2l_1)$-action are of types $SO(2l_1)/SO(2l_1-1)$ or $SO(2l_1)/SO(2l_1)$ by [3, p. 185].

By [3, p. 206-207], we have

$$M = D_1^{2l_1} \cup_\phi D_2^{2l_1},$$

where $SO(2l_1)$ acts on the disks $D_i^{2l_1}$ in the usual way and

$$\phi : S^{2l_1-1} = SO(2l_1)/SO(2l_1-1) \to S^{2l_1-1} = SO(2l_1)/SO(2l_1-1)$$

is given by $gSO(2l_1-1) \mapsto gnSO(2l_1-1)$, where $n \in N_{SO(2l_1)}SO(2l_1-1) = S(O(2l_1-1)\times O(1))$.

Therefore $\phi = \pm \mathrm{Id}_{S^{2l_1-1}}$ and $M = S^{2l_1}$.

If $G = SO(2l_1+1)$, then

$$M = SO(2l_1+1)/SO(2l_1) = S^{2l_1}$$

follows directly from Lemmas 3.1 and 3.4.

Now assume $G = SU(l_1+1)$. Because $\dim M = 2l_1$, the intersection of l_1+1 pairwise distinct characteristic submanifolds of M is empty. By Lemma 3.4, no T-fixed point is fixed by G. Therefore from Lemma 3.1 we get

$$M = SU(l_1+1)/S(U(l_1)\times U(1)) = \mathbb{C}P^{l_1}.$$

□

Remark 3.7. Another proof of this statement follows from the classification given in section 8.

4. Blowing up

In this section we describe blow ups of torus manifolds with G-action. They are used in the following sections to construct from a torus manifold M with G-action another torus manifold $\tilde{M}$ with G-action, such that an elementary factor of the covering group $\tilde{G}$ of G has no fixed point in $\tilde{M}$.

References for this construction are [7, p. 602-611] and [16, p. 269-270].

As before we write $\tilde{G} = \prod_{i=1}^{k} G_i \times T^{l_0}$ with G_i elementary and T^{l_0} a torus.

We will see in sections 5 and 7 that there are the following two cases:

(1) A component N of M^{G_1} has odd codimension in M.
(2) A component N of M^{G_1} has even codimension in M and there is a $g \in Z(\tilde{G})$ such that g acts trivially on N and g^2 acts as $-\operatorname{Id}$ on $N(N,M)$.

In the second case the action of g on $N(N,M)$ induces a G-invariant complex structure. We equip $N(N,M)$ with this structure. Let $E = N(N,M) \oplus \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ in the first case and $\mathbb{K} = \mathbb{C}$ in the second case.

In the following we call case (1) the real case and case (2) the complex case.

Lemma 4.1. *The projectivication $P_{\mathbb{K}}(E)$ is orientable.*

Proof. Because M is orientable the total space of the normal bundle of N in M is orientable. Therefore

$$E = N(N,M) \oplus \mathbb{K} = N(N,M) \times \mathbb{K}$$

and the associated sphere bundle $S(E)$ are orientable.

Let $Z_{\mathbb{K}} = \mathbb{Z}/2\mathbb{Z}$ if $\mathbb{K} = \mathbb{R}$ and $Z_{\mathbb{K}} = S^1$ if $\mathbb{K} = \mathbb{C}$. Then $Z_{\mathbb{K}}$ acts on E and $S(E)$ by multiplication on the fibers. Now $P_{\mathbb{K}}(E)$ is given by $S(E)/Z_{\mathbb{K}}$. If $\mathbb{K} = \mathbb{C}$, then $Z_{\mathbb{K}}$ is connected. Therefore it acts orientation preserving on $S(E)$.

If $\mathbb{K} = \mathbb{R}$, then $\dim E$ is even. Therefore the restriction of the $Z_{\mathbb{K}}$-action to a fiber of E is orientation preserving. Hence, it preserves the orientation of $S(E)$.

Because the action of $Z_{\mathbb{K}}$ is orientation preserving on $S(E)$, $P_{\mathbb{K}}(E)$ is orientable. □

Choose a G-invariant Riemannian metric on $N(N,M)$ and a G-equivariant closed tubular neighborhood B around N. Then one may identify

$$B = \{z_0 \in N(N,M); |z_0| \leq 1\} = \{(z_0 : 1) \in P_{\mathbb{K}}(E); |z_0| \leq 1\}.$$

By gluing the complements of the interior of B in M and $P_{\mathbb{K}}(E)$ along the boundary of B, we get a new torus manifold with G-action $\tilde{M}$, the *blow up* of M along N. It is easy to see, using isotopies of tubular neighborhoods, that the G-equivariant diffeomorphism-type of $\tilde{M}$ does not depend on the choices of the Riemannian metric and the tubular neighborhood.

$\tilde{M}$ is oriented in such a way that the induced orientation on $M - \mathring{B}$ coincides with the orientation induced from M. This forces the inclusion of $P_{\mathbb{K}}(E) - \mathring{B}$ to be orientation reversing. Because G_1 is elementary there is no one-dimensional G_1-invariant subbundle of $N(N,M)$. Therefore we have $\#\pi_0(\tilde{M}^{G_1}) = \#\pi_0(M^{G_1}) - 1$.

So by iterating this process over all components of M^{G_1} one ends up at a torus manifold $\tilde{M}'$ with G-action without G_1-fixed points. In the following we will call $\tilde{M}'$ the blow up of M along M^{G_1}.

Lemma 4.2. *There is a G-equivariant map $F : \tilde{M} \to M$ which maps the exceptional submanifold $M_0 = P_{\mathbb{K}}(N(N,M) \oplus \{0\})$ to N and is the identity on $M - B$. Moreover, F restricts to a diffeomorphism $\tilde{M} - M_0 \to M - N$. Its restriction to M_0 is the bundle projection $P_{\mathbb{K}}(N(N,M) \oplus \{0\}) \to N$.*

Proof. The G-equivariant map

$$f : P_{\mathbb{K}}(E) - \mathring{B} \to B \;\; (z_0 : z_1) \mapsto (z_0\bar{z}_1 : |z_0|^2) \;\; (z_0 \in N(N,M), z_1 \in \mathbb{K})$$

is the identity on ∂B. Therefore it may be extended to a continuous map $h : \tilde{M} \to M$, which is the identity outside of $P_{\mathbb{K}}(E) - \mathring{B}$.

Because $f|_{P_{\mathbb{K}}(E)-\mathring{B}-M_0} : P_{\mathbb{K}}(E) - \mathring{B} - M_0 \to B - N$ is a diffeomorphism there is a G-equivariant diffeomorphism $F' : \tilde{M} - M_0 \to M - N$, which is the identity outside $P_{\mathbb{K}}(E) - \mathring{B} - M_0$ and coincides with f near M_0 by [10, p. 24-25]. Therefore F' extends to a differentiable map $F : \tilde{M} \to M$ such that $F|_{M_0} = f|_{M_0}$ is the bundle projection. □

Lemma 4.3. *Let H be a closed subgroup of G. Then there is a bijection*

$$\{\text{components of } M^H \not\subset N\} \to \{\text{components of } \tilde{M}^H \not\subset M_0\}$$

such that

$$N' \mapsto \tilde{N}' = \left(P_{\mathbb{K}}(N(N \cap N', N') \oplus \mathbb{K}) - \mathring{B}\right) \cup_{\partial B \cap N'} \left(N' - \mathring{B}\right)$$

and its inverse is given by

$$F(N'') \leftarrow N'',$$

where N' is a component of M^H and N'' is one of $\tilde{M}^H$. Here $F(N'')$ is the image of N'' under the map F defined in Lemma 4.2. For a component N' of M^H, we call $\tilde{N}'$ the proper transform of N'.

Proof. At first we calculate the fixed point set of the H-action on $\tilde{M}$.

$$\begin{aligned}\tilde{M}^H &= \left(\left(P_{\mathbb{K}}(E) - \mathring{B}\right) \cup_{\partial B} \left(M - \mathring{B}\right)\right)^H \\ &= \left(P_{\mathbb{K}}(E) - \mathring{B}\right)^H \cup_{\partial B^H} \left(M - \mathring{B}\right)^H.\end{aligned}$$

Because H is compact, there are pairwise distinct i-dimensional non-trivial irreducible H-representations V_{ij} and H-vector bundles E_{ij} over N^H such that

$$N(N,M)|_{N^H} = N(N,M)|_{N^H}^H \oplus \bigoplus_i \bigoplus_j E_{ij},$$

and the H-representation on each fiber of E_{ij} is isomorphic to $\mathbb{K}^{d_{ij}} \otimes_{\mathbb{K}} V_{ij}$, where $\mathbb{K}^{d_{ij}}$ denotes the trivial H-representation of dimension d_{ij}.

Now the H-fixed points in $P_{\mathbb{K}}(E)$ are given by

$$\begin{aligned}P_{\mathbb{K}}(E)^H &= P_{\mathbb{K}}(N(N,M) \oplus \mathbb{K})|_{N^H}^H \\ &= P_{\mathbb{K}}(N(N,M)|_{N^H}^H \oplus \mathbb{K}) \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}).\end{aligned}$$

Because $N(N,M)|_{N^H}^H = N(N^H, M^H)$ we get

$$\begin{aligned}\tilde{M}^H &= \left(\left(P_{\mathbb{K}}(N(N^H, M^H) \oplus \mathbb{K}) - \mathring{B}^H\right) \cup_{\partial B^H} \left(M - \mathring{B}\right)^H\right) \\ &\qquad \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}) \\ &= \coprod_{N' \subset M^H} \tilde{N}' \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}),\end{aligned}$$

where N' runs through the connected components of M^H which are not contained in N. Thus the statement follows. □

By replacing H in Lemma 4.3 by an one-dimensional subtorus of T, we get:

Corollary 4.4. *There is a bijection between the characteristic submanifolds of M and the characteristic submanifolds of $\tilde{M}$, which are not contained in M_0.*

Proof. The only thing, that is to prove here, is that for a characteristic submanifold M_i of M, $\tilde{M}_i^T$ is non-empty. If $(M_i - N)^T \neq \emptyset$, then this is clear.

If $p \in (M_i \cap N)^T$, then $P_{\mathbb{K}}(N(M_i \cap N, M_i) \oplus \{0\})|_p$ is a T-invariant submanifold of $\tilde{M}_i$, which is diffeomorphic to $\mathbb{C}P^k$ or $\mathbb{R}P^{2k}$. Therefore it contains a T-fixed point. □

This bijection is compatible with the action of the Weyl-group of G on the sets of characteristic submanifolds of $\tilde{M}$ and M.

In the real case the exceptional submanifold M_0 has codimension one in $\tilde{M}$ and is G-invariant. Because there is no S^1-representation of real dimension one, M_0 does not contain a characteristic submanifold of $\tilde{M}$ in this case.

In the complex case M_0 is G-invariant and may be a characteristic submanifold of $\tilde{M}$.

Therefore there is a bijection between the non-trivial orbits of the $W(G)$-actions on the sets of characteristic submanifolds of M and $\tilde{M}$. Hence we get the same elementary factors for the G-actions on $\tilde{M}$ and M.

Corollary 4.5. *Let H be a closed subgroup of G and N' a component of M^H such that $N \cap N'$ has codimension one –in the real case– or two –in the complex case– in N'. Then F induces a $(N_G H)^0$-equivariant diffeomorphism of $\tilde{N}'$ and N'.*

Proof. Because of the dimension assumption the $(N_G H)^0$-equivariant map

$$f|_{P_{\mathbb{K}}(N(N\cap N',N')\oplus\mathbb{K})-\mathring{B}\cap N'} : P_{\mathbb{K}}(N(N \cap N', N') \oplus \mathbb{K}) - \mathring{B} \cap N' \to B \cap N'$$

from the proof of Lemma 4.2 is a diffeomorphism. Because the restriction of F to $\tilde{M} - M_0$ is an G-equivariant diffeomorphism the restriction $F|_{\tilde{N}'-M_0} : \tilde{N}' - M_0 \to N' - N$ is a $(N_G H)^0$-equivariant diffeomorphism. Therefore $F|_{\tilde{N}'} : \tilde{N}' \to N'$ is a diffeomorphism. □

Lemma 4.6. *In the complex case let $\bar{E} = N(N, M)^* \oplus \mathbb{C}$, where $N(N, M)^*$ is the normal bundle of N in M equipped with the dual complex structure. Then there is a G-equivariant diffeomorphism*

$$\tilde{M} \to P_{\mathbb{C}}(\bar{E}) - \mathring{B} \cup_{\partial B} M - \mathring{B}.$$

That means that the diffeomorphism type of $\tilde{M}$ does not change if we replace the complex structure on $N(N, M)$ by its dual.

Proof. We have $P_{\mathbb{C}}(E) = E/\sim$ and $P_{\mathbb{C}}(\bar{E}) = E/\sim'$, where

$$(z_0, z_1) \sim (z_0', z_1') \Leftrightarrow \exists t \in \mathbb{C}^* \quad (tz_0, tz_1) = (z_0', z_1'),$$
$$(z_0, z_1) \sim' (z_0', z_1') \Leftrightarrow \exists t \in \mathbb{C}^* \quad (tz_0, \bar{t}z_1) = (z_0', z_1').$$

Therefore

$$E \to E \qquad\qquad (z_0, z_1) \mapsto (z_0, \bar{z}_1)$$

induces a G-equivariant diffeomorphism $P_{\mathbb{C}}(E) - \mathring{B} \to P_{\mathbb{C}}(\bar{E}) - \mathring{B}$ which is the identity on ∂B. By [10, p. 24-25] the result follows. □

Lemma 4.7. *If in the complex case $G_1 = SU(l_1 + 1)$ and $\operatorname{codim} N = 2l_1 + 2$ or in the real case $G_1 = SO(2l_1 + 1)$ and $\operatorname{codim} N = 2l_1 + 1$, then $F : \tilde{M} \to M$ induces a homeomorphism $\bar{F} : \tilde{M}/G_1 \to M/G_1$.*

Proof. Because $F|_{\tilde{M}-M_0}: \tilde{M} - M_0 \to M - N$ is a equivariant diffeomorphism and $\tilde{M}/G_1$,M/G_1 are compact Hausdorff-spaces, the only thing, that has to be checked, is that

$$F|_{P_{\mathbb{K}}(N(N,M))}: P_{\mathbb{K}}(N(N,M)) \to N$$

induces a homeomorphism of the orbit spaces. But this map is just the bundle map $P_{\mathbb{K}}(N(N,M)) \to N$.

If $G_1 = SU(l_1+1)$, then, because of dimension reasons [19, p. 53-54], the G_1-representation on the fibers of $N(N,M)$ is the standard representation of G_1 or its dual. If $G_1 = SO(2l_1+1)$, then, by [19, p. 53-54], the G_1-representation on the fibers of $N(N,M)$ is the standard representation of G_1.

Thus, in both cases the G_1-action on the fibers of $P_{\mathbb{K}}(N(N,M)) \to N$ is transitive. Therefore the statement follows. □

Remark 4.8. All statements proved above also hold for non-connected groups of the form $G \times K$ where K is a finite group and G is connected if we replace N by a K-invariant union of components of M^{G_1}.

Now we want to reverse the construction of a blow up. Let A be a closed G-manifold and $E \to A$ be a G-vector bundle such that G_1 acts trivially on A. If E is even dimensional, we assume that there is a $g \in Z(G)$ such that g acts trivially on A and g^2 acts on E as $-\operatorname{Id}$. In this case we equip E with the complex structure induced by the action of g.

Assume that $\tilde{M}$ is a G-manifold and there is a G-equivariant embedding of $P_{\mathbb{K}}(E) \hookrightarrow \tilde{M}$ such that the normal bundle of $P_{\mathbb{K}}(E)$ is isomorphic to the tautological bundle over $P_{\mathbb{K}}(E)$.

Then one may identify a closed G-equivariant tubular neighborhood B^c of $P_{\mathbb{K}}(E)$ in $\tilde{M}$ with

$$B^c = \{(z_0 : 1) \in P_{\mathbb{K}}(E \oplus \mathbb{K}); |z_0| \geq 1\} \cup \{(z_0 : 0) \in P_{\mathbb{K}}(E \oplus \mathbb{K})\}.$$

By gluing the complements of the interior of B^c in $\tilde{M}$ and $P_{\mathbb{K}}(E \oplus \mathbb{K})$, we get a G-manifold M such that A is G-equivariantly diffeomorphic to a union of components of M^{G_1}.

We call M the *blow down* of $\tilde{M}$ along $P_{\mathbb{K}}(E)$.

It is easy to see that the G-equivariant diffeomorphism type of M does not depend on the choices of a metric on E and the tubular neighborhood of $P_{\mathbb{K}}(E)$ in $\tilde{M}$ if G_1 acts transitively on the fibers of $P_{\mathbb{K}}(E) \to A$.

It is also easy to see that the blow up and blow down constructions are inverse to each other.

5. The case $G_1 = SU(l_1 + 1)$

In this section we discuss actions of groups, which have a covering group of the form $G_1 \times G_2$, where $G_1 = SU(l_1+1)$ is elementary and G_2 acts effectively on M. It turns out that the blow up of M along M^{G_1} is a fiber bundle over $\mathbb{C}P^{l_1}$. This fact leads to our first classification result.

The assumption on G_2 is no restriction on G, because one may replace any covering group $\tilde{G}$ by the quotient $\tilde{G}/H$ where H is a finite subgroup of G_2 acting trivially on M. Following Convention 3.5, we also assume $\#\mathfrak{F}_1 = 2$ or $\#\mathfrak{F}_1 = 4$ in the cases $G_1 = SU(2)$ or $G_1 = SU(4)$, respectively. Furthermore, we assume after conjugating T with some element of G_1 that $T_1 = T \cap G_1$ is the standard maximal torus of G_1.

5.1. **The G_1-action on M.** We have the following lemma:

Lemma 5.1. *Let M be a torus manifold with G-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$ elementary. Then the $W(S(U(l_1) \times U(1)))$-action on $\mathfrak{F}_1$ has an orbit $\mathfrak{F}_1'$ with l_1 elements and there is a component N_1 of $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$, which contains a T-fixed point.*

Proof. We know that $W(SU(l_1+1)) = S_{l_1+1} = S(\mathfrak{F}_1)$ and $W(S(U(l_1) \times U(1))) = S_{l_1} \subset S_{l_1+1}$. Therefore the first statement follows. Let $x \in M^T$. Then, by Lemmas 3.3 and 3.4, x is contained in the intersection of l_1 characteristic submanifolds of M belonging to $\mathfrak{F}_1$. Because $W(G_1) = S(\mathfrak{F}_1)$ there is a $g \in N_{G_1}T_1$ such that $gx \in \bigcap_{M_i \in \mathfrak{F}_1'} M_i$. Therefore the second statement follows. □

Remark 5.2. We will see in Lemma 5.10 that $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$ is connected.

Lemma 5.3. *Let M be a torus manifold with G-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$ elementary. Furthermore, let N_1 as in Lemma 5.1. Then there is a group homomorphism $\psi_1 : S(U(l_1) \times U(1)) \rightarrow Z(G_2)$ such that, with*

$$H_0 = SU(l_1+1) \times \operatorname{im} \psi_1,$$
$$H_1 = S(U(l_1) \times U(1)) \times \operatorname{im} \psi_1,$$
$$H_2 = \{(g, \psi_1(g)) \in H_1; g \in S(U(l_1) \times U(1))\},$$

(1) $\operatorname{im} \psi_1$ *is the projection of $\lambda(M_i)$ to G_2, for all $M_i \in \mathfrak{F}_1$,*
(2) N_1 *is a component of M^{H_2},*
(3) N_1 *is invariant under the action of G_2,*
(4) $M = G_1 N_1 = H_0 N_1$.

Proof. Denote by T_2 the maximal torus $T \cap G_2$ of G_2. Let $x \in N_1^T$. If $x \in M^{SU(l_1+1)}$, then we have, by Lemma 3.4, the $SU(l_1+1) \times T_2$-representation

$$T_x M = W \otimes_{\mathbb{C}} V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i,$$

where W is the standard complex representation of $SU(l_1+1)$ or its dual and the V_i are one-dimensional complex representations of T_2. Because G_2 acts effectively on M the weights of the V_i form a basis of the integral lattice in LT_2^*. From the description of the weight spaces of $T_x M$ given in the proof of Lemma 3.4, we get that $T_x N_1$ is $S(U(l_1) \times U(1))$-invariant and that there is a one-dimensional complex representation W_1 of $S(U(l_1) \times U(1))$ such that

$$T_x N_1 = W_1 \otimes_{\mathbb{C}} V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i.$$

Now assume that x is not fixed by $SU(l_1+1)$. Because, by Lemma 3.1, $G_1 x \subset M^{T_2}$ is G_1-equivariantly diffeomorphic to $\mathbb{C}P^{l_1}$, we see by the definition of N_1 that $G_{1x} = S(U(l_1) \times U(1))$.

At the point x, we get a representation of $S(U(l_1) \times U(1)) \times T_2$ of the form

$$T_x M = T_x N_1 \oplus T_x G_1 x.$$

Since T_2 acts effectively on M and trivially on $G_1 x$, there is a decomposition

$$T_x N_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_{\mathbb{C}} W_i,$$

where the W_i are one-dimensional complex $S(U(l_1) \times U(1))$-representations and the V_i are one-dimensional complex T_2-representations whose weights form a basis of the integral lattice in LT_2^*.

Therefore, in both cases, there is a homomorphism $\psi_1 : S(U(l_1) \times U(1)) \to S^1 \to T_2$ such that, for all $g \in S(U(l_1) \times U(1))$, $(g, \psi_1(g))$ acts trivially on $T_xN_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_{\mathbb{C}} W_i$.

Hence the component of the identity of the isotropy subgroup of the torus T for generic points in N_1 is given by

$$H_3 = \{(t, \psi_1(t)) \in T_1 \times T_2\}. \tag{5.1}$$

With Lemma B.1, we see that

$$H_3 = \langle \lambda(M_i); M_i \in \mathfrak{F}_1, M_i \supset N_1 \rangle. \tag{5.2}$$

Because the Weyl-group of G_2 acts trivially and orientation preserving on $\mathfrak{F}_1$, $\lambda(M_i)$, $M_i \in \mathfrak{F}_1$, is pointwise fixed by the action of $W(G_2)$ on T by Lemma 2.10. It follows with (5.2) that H_3 is pointwise fixed by the action of $W(G_2)$ on T. Here $W(G_2)$ acts on T by conjugation. Therefore the image of ψ_1 is contained in the center of G_2. Furthermore $\operatorname{im} \psi_1$ is the projection of $\lambda(M_i)$, $M_i \in \mathfrak{F}_1$, to T_2.

Because H_3 commutes with G_2 it follows that N_1 is G_2-invariant. So we have proved the first and the third statement.

Now we turn to the second and fourth part.

Because $T_xN_1 = (T_xM)^{H_3} = (T_xM)^{H_2}$, N_1 is a component of M^{H_2}. Because, by Lemma A.2, H_1 is the only proper closed connected subgroup of H_0, which contains H_2 properly, for $y \in N_1$ there are the following possibilities

- $H_{0y}^0 = H_0$,
- $H_{0y}^0 = H_1$ and $\dim H_0 y = 2l_1$,
- $H_{0y}^0 = H_2$ and $\dim H_0 y = 2l_1 + 1$,

where H_{0y}^0 is the identity component of the stabilizer of y in H_0. If $g \in H_0$ such that $gy \in N_1$, then we have $H_{0gy}^0 = gH_{0y}^0 g^{-1} \in \{H_0, H_1, H_2\}$. Therefore

$$g \in N_{H_0} H_{0y}^0 = \begin{cases} H_0 & \text{if } y \in M^{H_0} \\ H_1 & \text{if } y \notin M^{H_0} \text{ and } l_1 > 1 \\ N_{G_1}T_1 \times \operatorname{im}\psi_1 & \text{if } H_{0y}^0 = H_1 \text{ and } l_1 = 1 \\ T_1 \times \operatorname{im}\psi_1 & \text{if } H_{0y}^0 = H_2, l_1 = 1 \text{ and } \operatorname{im}\psi_1 \neq \{1\}. \end{cases}$$

Now let $y \in N_1$ such that $H_{0y}^0 \neq H_0$. Because N_1 is a component of M^{H_2} and $H_0 y$ is H_2 invariant, $N_1 \cap H_0 y$ is a union of some components of $(H_0 y)^{H_2}$. Therefore $N_1 \cap H_0 y$ is a submanifold of M. Moreover,

$$T_yN_1 \cap T_yH_0y = (T_yM)^{H_2} \cap T_yH_0y = (T_yH_0y)^{H_2} = T_y(N_1 \cap H_0y).$$

Hence,

$$\begin{aligned} \dim T_yN_1 \cap T_yH_0y &= \dim N_1 \cap H_0y \leq \dim H_1 y \\ &= \dim H_1/H_{0y}^0 = \begin{cases} 0 & \text{if } H_{0y}^0 = H_1 \\ 1 & \text{if } H_{0y}^0 = H_2 \text{ and } \operatorname{im}\psi_1 \neq \{1\} \end{cases} \end{aligned}$$

follows. Therefore N_1 intersects $H_0 y$ transversely in y. It follows, by Lemma A.5, that $GN_1 - N_1^{H_0} = H_0N_1 - N_1^{H_0}$ is an open subset of M.

Because M is connected and $\operatorname{codim} M^{H_0} \geq 4$, $M - M^{H_0}$ is connected. Since $(M - M^{H_0}) \cap H_0N_1 = H_0N_1 - N_1^{H_0}$ is closed in $M - M^{H_0}$, we have $M - M^{H_0} =$

$H_0N_1 - N_1^{H_0}$. Hence

$$\begin{aligned} M &= \left(M - M^{H_0}\right) \amalg M^{H_0} = \left(H_0N_1 - N_1^{H_0}\right) \amalg M^{H_0} \\ &= \left(H_0N_1 - N_1^{H_0}\right) \amalg \left(M^{H_0} \cap N_1\right) \amalg \left(M^{H_0} - N_1^{H_0}\right) \\ &= H_0N_1 \amalg \left(M^{H_0} - N_1^{H_0}\right). \end{aligned}$$

Because N_1 is a component of M^{H_2}, $N_1^{H_0}$ is a union of components of M^{H_0}. Therefore $M^{H_0} - N_1^{H_0}$ is closed in M. Because H_0N_1 is closed in M it follows that $M = GN_1 = H_0N_1 = G_1N_1$. □

The following lemma guarantees together with Lemma A.3 that, if $l_1 > 1$, then the homomorphism ψ_1 is independent of all choices made in its construction namely the choice of N_1 and of $x \in N_1^T$.

Lemma 5.4. *In the situation of Lemma 5.3 let $T' = T_2$ or $T' = \operatorname{im}\psi_1$. Then the principal orbit type of the $G_1 \times T'$-action on M is given by $(G_1 \times T')/H_2$.*

Proof. Let $H \subset G_1 \times T'$ be a principal isotropy subgroup. Then, by Lemma 5.3, we may assume $H \supset H_2$. Consider the projection

$$\pi_1 : G_1 \times T' \to G_1$$

on the first factor.

At first we show that the restriction of π_1 to H is injective. Because $(G_1 \times T')_x \cap T' = T'_x$ for all $x \in M$ and the T'-action on M is effective there is an $x \in M$ such that

$$(G_1 \times T')_x \cap T' = \{1\}.$$

Furthermore, there is an $g \in G_1 \times T'$ such that $(G_1 \times T')_x \supset gHg^{-1}$.

Because T' is contained in the center of $G_1 \times T'$, we get

$$\begin{aligned} gHg^{-1} \cap T' &= \{1\}, \\ H \cap g^{-1}T'g &= \{1\}, \\ H \cap T' &= \{1\}. \end{aligned}$$

Therefore the restriction of π_1 to H is injective.

Furthermore, $\pi_1(H) \supset \pi_1(H_2) = S(U(l_1) \times U(1))$. Therefore, by Lemma A.1, we have

$$\pi_1(H) = \begin{cases} SU(l_1+1), S(U(l_1) \times U(1)) & \text{if } l_1 > 1 \\ SU(l_1+1), S(U(l_1) \times U(1)), N_{G_1}T_1 & \text{if } l_1 = 1. \end{cases}$$

There is a left inverse $\phi : \pi_1(H) \to H \hookrightarrow G_1 \times T'$ to $\pi_1|_H$. Therefore there is a group homomorphism $\psi' : \pi_1(H) \to T'$ such that

$$H = \phi(\pi_1(H)) = \{(g, \psi'(g)) \in G_1 \times T';\ g \in \pi_1(H)\}.$$

Because H_2 is a subgroup of H, we see that $\psi'|_{S(U(l_1)\times U(1))} = \psi_1$.

At first we discuss the cases $\pi_1(H) = SU(l_1+1)$ and $\pi_1(H) = S(U(l_1) \times U(1))$. Because T' is abelian we have in these cases

$$H = \phi(\pi_1(H)) = \begin{cases} G_1 & \text{if } \pi_1(H) = SU(l_1+1) \\ H_2 & \text{if } \pi_1(H) = S(U(l_1) \times U(1)). \end{cases}$$

The first case does not occur because G_1 acts non-trivially on M.

Now we discuss the case $l_1 = 1$ and $\pi_1(H) = N_{G_1}T_1$. Because for $t \in T_1$ and $g \in N_{G_1}T_1 - T_1$ we have

$$\psi'(t)^{-1} = \psi'(gtg^{-1}) = \psi'(g)\psi'(t)\psi'(g)^{-1} = \psi'(t),$$

it follows that ψ_1 is trivial in this case.

Let $x \in M^T$. Then it follows by the definition of ψ_1 in the proof of Lemma 5.3 that x is not a fixed point of G_1. By Lemma 3.1, we know that

$$G_{1x} = S(U(l_1) \times U(1)) = T_1.$$

Therefore $(G_1 \times T')_x = T_1 \times T'$ is abelian. But H is non-abelian if $\pi_1(H) = N_{G_1}T_1$. This is a contradiction because H is conjugated to a subgroup of $(G_1 \times T')_x$. □

If $l_1 = 1$, we have $\#\mathfrak{F}_1 = 2$ and $W(S(U(l_1) \times U(1))) = \{1\}$. Therefore there are two choices for N_1. Denote them by M_1 and M_2.

Lemma 5.5. *In the situation described above let ψ_i be the homomorphism constructed for M_i, $i = 1, 2$. Then we have $\psi_1 = \psi_2^{-1}$.*

Proof. By (5.1) and (5.2), we have

$$\lambda(M_i) = \{(t, \psi_i(t)) \in H_1;\ t \in S(U(1) \times U(1))\}.$$

Now, with Lemma 2.10, we see

$$\begin{aligned}\lambda(M_1) = g\lambda(M_2)g^{-1} &= \{(t^{-1}, \psi_2(t)) \in H_1;\ t \in S(U(1) \times U(1))\} \\ &= \{(t, \psi_2(t)^{-1}) \in H_1;\ t \in S(U(1) \times U(1))\},\end{aligned}$$

where $g \in N_{G_1}T_1 - T_1$. Therefore the result follows. □

Corollary 5.6. *If in the situation of Lemma 5.3 the G_1-action on M has no fixed point, then M is the total space of a G-equivariant fiber bundle over $\mathbb{C}P^{l_1}$ with fiber some torus manifold; more precisely $M = H_0 \times_{H_1} N_1$.*

Proof. $H_0 \times_{H_1} N_1$ is defined to be the space $H_0 \times N_1/\sim_1$, where

$$\begin{aligned} & (g_1, y_1) \sim_1 (g_2, y_2) \\ \Leftrightarrow \quad & \exists h \in H_1 \quad g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2. \end{aligned}$$

By Lemma 5.3 we have that $M = H_0 N_1 = (H_0 \times N_1)/\sim_2$, where

$$\begin{aligned} & (g_1, y_1) \sim_2 (g_2, y_2) \\ \Leftrightarrow \quad & g_1 y_1 = g_2 y_2. \end{aligned}$$

We show that the two equivalence relations $\sim_1, \sim_2$ are equal.

For $(g_1, y_1), (g_2, y_2) \in H_0 \times N_1$ we have

$$\begin{aligned} & g_1 y_1 = g_2 y_2 \\ \Leftrightarrow \quad & \exists h \in N_{H_0} H^0_{0y_1} \quad g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2 \\ \Leftrightarrow \quad & \exists h \in H_1 \quad g_1 h^{-1} = g_2 \text{ and } hy_1 = y_2. \end{aligned}$$

For the last equivalence we have to show the implication from the second to the third line. If $l_1 > 1$, $N_{H_0}H^0_{0y_1}$ is equal to H_1 because y_1 is not a H_0-fixed point. So we have $h \in H_1$.

If $l_1 = 1$, then N_1 is a characteristic submanifold of M belonging to $\mathfrak{F}_1$. If $H^0_{0y_1} = H_2$ we are done because $N_{H_0}H^0_{0y_1} = H_1$.

Now assume that $H^0_{0y_1} = H_1$ and there is an $h \in N_{G_1}T_1 \times \operatorname{im}\psi_1 - T_1 \times \operatorname{im}\psi_1$ such that $y_2 = hy_1 \in N_1$. Then $y_2 \in N_1 \cap N_2 \subset M^{T_1 \times \operatorname{im}\psi_1}$, where N_2 is the other characteristic submanifold of M belonging to $\mathfrak{F}_1$.

As shown in the proof of Lemma 5.3, N_1 intersects $H_0 y_2$ transversely in y_2. Therefore one has

$$T_{y_2}N_1 \oplus T_{y_2}H_0 y_2 = T_{y_2}M = T_{y_2}N_2 \oplus T_{y_2}H_0 y_2$$

as $T_1 \times \operatorname{im}\psi_1$-representations. This implies

$$T_{y_2}N_1 = T_{y_2}N_2$$

as $T_1 \times \operatorname{im}\psi_1$-representations. Therefore $T_1 \times \operatorname{im}\psi_1$ acts trivially on both N_1 and N_2. Therefore we have $\operatorname{im}\psi_1 = \{1\}$ and $\lambda(N_1) = \lambda(N_2) = T_1$. Hence, we get a contradiction because the intersection of N_1 and N_2 is non-empty. □

Corollary 5.7. *In the situation of Lemma 5.3 we have* $M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{F}_1} M_i$.

Proof. At first let $l_1 > 1$. By Lemma 5.3, we know $M^{H_0} \subset M^{G_1} \subset N_1$. Therefore $M^{G_1} \subset \bigcap_{g \in N_{G_1}T_1} gN_1 = \bigcap_{M_i \in \mathfrak{F}_1} M_i$. There is a $g \in N_{G_1}T_1 - T_1$ with $gH_2g^{-1} \not\subset H_1$. Thus, the subgroup $\langle H_2, gH_2g^{-1}\rangle$ of H_0, which is generated by H_2 and gHg^{-1}, contains H_2 as a proper subgroup. Therefore $\langle H_2, gH_2g^{-1}\rangle = H_0$ follows by Lemma A.2. Because H_2 acts trivially on N_1, this equation implies

$$M^{H_0} \supset \bigcap_{g \in N_{G_1}T_1} gN_1 = \bigcap_{M_i \in \mathfrak{F}_1} M_i.$$

Now let $l_1 = 1$. Then $\mathfrak{F}_1$ contains two characteristic submanifolds M_1 and M_2. As in the first case one can show that $M^{H_0} \subset M^{G_1} \subset M_1 \cap M_2$.

So $M^{H_0} \supset M_1 \cap M_2$ remains to be shown. Assume that there is an $y \in M_1 \cap M_2 - M^{H_0}$. Then we also have $y \in M^{H_1}$. Now the above assumption leads to a contradiction as in the proof of Corollary 5.6. □

Corollary 5.8. *If in the situation of Lemma 5.3* ψ_1 *is trivial, then* M^{G_1} *is empty. Otherwise the normal bundle of* $M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{F}_1} M_i$ *possesses a* G*-invariant complex structure. It is induced by the action of some element* $g \in \operatorname{im}\psi_1$. *Furthermore, it is unique up to conjugation.*

Proof. If ψ_1 is trivial, then $\langle \lambda(M_i); M_i \in \mathfrak{F}_1\rangle$ is contained in the l_1-dimensional maximal torus of G_1 by Lemma 5.3. By Corollary 5.7 and Lemma B.1, it follows that M^{H_0} is empty.

If ψ_1 is non-trivial, then for $y \in M^{H_0}$ we have

$$N_y(M^{H_0}, M) = V_{\mathbb{C}} \oplus V_{\mathbb{R}},$$

where $\operatorname{im}\psi_1$ acts non-trivially on the H_0-representation $V_{\mathbb{C}}$ and trivially on the H_0-representation $V_{\mathbb{R}}$. Clearly $V_{\mathbb{C}}$ has at least real dimension two and the action of $\operatorname{im}\psi_1$ induces a H_0-invariant complex structure on $V_{\mathbb{C}}$. Because M^{H_0} has codimension $2l_1 + 2$ by Corollary 5.7 and Lemma B.1, the dimension of $V_{\mathbb{R}}$ is at most $2l_1$. So it follows from [19, p. 53-54] that $V_{\mathbb{R}}$ is trivial if $l_1 \neq 3$.

If $l_1 = 3$, we have $SU(4) = \mathrm{Spin}(6)$, and there are two possibilities:

(1) $V_{\mathbb{R}}$ is trivial.
(2) $V_{\mathbb{R}}$ is the standard representation of $SO(6)$ and $V_{\mathbb{C}}$ a one-dimensional complex representation of $\operatorname{im}\psi_1$.

Because the principal orbits are dense in M, it follows with the slice theorem that the principal orbit types of the H_0-actions on $N_y(M^{H_0}, M)$ and M are equal. Therefore in the second case the principal orbit type of the H_0-action on M is given by $\mathrm{Spin}(6) \times S^1/\mathrm{Spin}(5) \times \{1\}$. Therefore we see with Lemma 5.4 that the second case does not occur.

Because of dimension reasons we get

$$N_y(M^{H_0}, M) = V_{\mathbb{C}} = W \otimes_{\mathbb{C}} V,$$

where W is the standard complex representation of $SU(l_1+1)$ or its dual and V is a complex one-dimensional $\operatorname{im}\psi_1$-representation. Because $\operatorname{im}\psi_1 \subset Z(G)$, we see that $N(M^{H_0}, M)$ has a G-invariant complex structure, which is induced by the action of some $g \in \operatorname{im}\psi_1$.

Next we prove the uniqueness of this complex structure. Assume that there is another $g' \in Z(G) \cap G_y$ whose action induces a complex structure on $N_y(M^{H_0}, M)$.

Then g' induces a – with respect to the complex structure induced by g – complex linear H_0-equivariant map

$$J : N_y(M^{H_0}, M) \to N_y(M^{H_0}, M)$$

with $J^2 + \mathrm{Id} = 0$. Because $N_y(M^{H_0}, M)$ is an irreducible H_0-representation it follows by Schur's Lemma that J is multiplication with $\pm i$. Therefore g' induces up to conjugation the same complex structure as g. □

Corollary 5.9. *If in the situation of Lemma 5.3 $M^{G_1} = M^{H_0} \neq \emptyset$, then* $\ker \psi_1 = SU(l_1)$.

Proof. Let $y \in M^{H_0}$. Then by the proof of Corollary 5.8 we have

$$N_y(M^{H_0}, M) = W \otimes_{\mathbb{C}} V,$$

where W is the standard complex $SU(l_1 + 1)$-representation or its dual and V is a one-dimensional complex $\operatorname{im} \psi_1$-representation. Furthermore, $\operatorname{im} \psi_1$ acts effectively on M.

Because the principal orbits are dense in M, it follows with the slice theorem that the principal orbit types of the H_0-actions on $N_y(M^{H_0}, M)$ and M are equal. Therefore a principal isotropy subgroup of the H_0-action on M is given by

$$H = \left\{ (g, g_{l+1}^{\pm 1}) \in H_1; g = \begin{pmatrix} A & 0 \\ 0 & g_{l_1+1} \end{pmatrix} \in S(U(l_1) \times U(1)) \text{ with } A \in U(l_1) \right\}.$$

Now the statement follows by the uniqueness of the principal orbit type and Lemmas 5.4 and A.3. □

Lemma 5.10. *In the situation of Lemma 5.1, the intersection $\bigcap_{M_i \in \mathfrak{F}_1'} M_i = N_1$ is connected.*

Proof. Let $\tilde{M}$ be the blow up of M along M^{G_1} and $\tilde{N}_1$ the proper transform of N_1 in $\tilde{M}$. By Corollary 5.6, we have $\tilde{M} = H_0 \times_{H_1} \tilde{N}_1$, which is a fiber bundle over $\mathbb{C}P^{l_1}$. The characteristic submanifolds of $\tilde{M}$, which are permuted by $W(G_1)$, are given by the preimages of the characteristic submanifolds of $\mathbb{C}P^{l_1}$ under the bundle map. By Corollary 4.4 and the discussion following this corollary, they are also given by the proper transforms $\tilde{M}_i$ of the characteristic submanifolds $M_i \in \mathfrak{F}_1$ of M. Because l_1 characteristic submanifolds of $\mathbb{C}P^{l_1}$ intersect in a single point we see $\bigcap_{M_i \in \mathfrak{F}_1'} \tilde{M}_i = \tilde{N}_1$. Therefore this intersection is connected. Because $\bigcap_{M_i \in \mathfrak{F}_1'} \tilde{M}_i$ is mapped by F to $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$, we see that $\bigcap_{M_i \in \mathfrak{F}_1'} M_i = N_1$ is connected. □

5.2. **Blowing up along M^{G_1}.** By blowing up a torus manifold M with G-action along M^{G_1} one gets a torus manifold $\tilde{M}$ without G_1-fixed points.

Denote by $\tilde{N}_1$ the proper transform of N_1 as defined in Lemma 5.1. Then by Corollary 4.5 there is a $\langle H_1, G_2 \rangle$-equivariant diffeomorphism $F : \tilde{N}_1 \to N_1$.

As in section 4, we denote by $M_0 = P_{\mathbb{C}}(N(M^{G_1}, M) \oplus \{0\})$ the exceptional submanifold of $\tilde{M}$. Because $M_0 \cap \tilde{N}_1$ is mapped by this diffeomorphism to $M^{G_1} = M^{H_0} = N_1^{H_0}$, H_1 acts trivially on $M_0 \cap \tilde{N}_1$. By Corollary 5.6 we know that $\tilde{M}$ is diffeomorphic to $H_0 \times_{H_1} \tilde{N}_1 = H_0 \times_{H_1} N_1$.

A natural question arising here is: When is a torus manifold of this form a blow up of another torus manifold with G-action?

We claim that this is the case if and only if N_1 has a codimension two submanifold, which is fixed by the H_1-action and $\ker \psi_1 = SU(l_1)$.

Lemma 5.11. *Let N_1 be a torus manifold with G_2-action, A a closed codimension two submanifold of N_1, $\psi_1 \in \operatorname{Hom}(S(U(l_1) \times U(1)), Z(G_2))$ such that $\operatorname{im} \psi_1$ acts* *trivially on A and* $\ker\psi_1 = SU(l_1)$. *Let also*

$$H_0 = SU(l_1+1)\times \operatorname{im}\psi_1,$$
$$H_1 = S(U(l_1)\times U(1))\times \operatorname{im}\psi_1,$$
$$H_2 = \{(g,\psi_1(g)); g\in S(U(l_1)\times U(1))\}.$$

(1) *Then H_1 acts on N_1 by $(g,t)x = \psi_1(g)^{-1}tx$, where $x\in N_1$ and $(g,t)\in H_1$.*
(2) *Assume that $Z(G_2)$ acts effectively on N_1 and let $y\in A$ and V the one-dimensional complex H_1-representation $N_y(A,N_1)$. Then V extends to an l_1+1-dimensional complex representation of H_0. Therefore there is an l_1+1-dimensional complex G-vector bundle E' over A which contains $N(A,N_1)$ as a subbundle.*
(3) *Then the normal bundle of $H_0/H_1\times A$ in $H_0\times_{H_1} N_1$ is isomorphic to the tautological bundle over $P_{\mathbb{C}}(E'\oplus\{0\})$.*

The lemma guarantees together with the discussion at the end of section 4 that one can remove $H_0/H_1\times A$ from $H_0\times_{H_1} N_1$ and replace it by A to get a torus manifold with G-action M such that $M^{H_0} = A$. The blow up of M along A is $H_0\times_{H_1} N_1$.

Proof. (1) is trivial.

(2) For $i = 1,\dots,l_1+1$ let

$$\lambda_i : T_1 \to S^1 \quad \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_{l_1+1}\end{pmatrix} \mapsto g_i$$

and $\mu : \operatorname{im}\psi_1 \to S^1$ the character of the $\operatorname{im}\psi_1$ representation $N_y(A,N_1)$. Then μ is an isomorphism.

And by [4, p. 176] the character ring of the maximal torus $T_1\times\operatorname{im}\psi_1$ of $H_1 = S(U(l_1)\times U(1))\times\operatorname{im}\psi_1$ is given by

$$R(T_1\times\operatorname{im}\psi_1) = \mathbb{Z}[\lambda_1,\dots,\lambda_{l_1+1},\mu,\mu^{-1}]/(\lambda_1\cdots\lambda_{l_1+1}-1).$$

With this notation, the character of V is given by $\mu\lambda_{l_1+1}^{\pm 1}$. Therefore the H_0-representation W with the character $\mu\sum_{i=1}^{l_1+1}\lambda_i^{\pm1}$ is l_1+1-dimensional and $V\subset W$.

Let $G_2 = G_2'\times\operatorname{im}\psi_1$ and $E'' = N(A,N_1)$ equipped with the action of G_2', but without the action of H_1. Then $E' = E''\otimes_{\mathbb{C}} W$ is a G-vector bundle with the required features.

Now we turn to (3). The normal bundle of $H_0/H_1\times A$ in $H_0\times_{H_1} N_1$ is given by $H_0\times_{H_1} N(A,N_1)$.

Consider the following commutative diagram

$$\begin{array}{ccc} H_0\times_{H_1} N(A,N_1) & \xrightarrow{f} & P_{\mathbb{C}}(E'\oplus\{0\})\times E' \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ H_0/H_1\times A & \xrightarrow{g} & P_{\mathbb{C}}(E'\oplus\{0\}) \end{array}$$

where the vertical maps are the natural projections and f,g are given by

$$f([(h_1,h_2):m]) = ([m\otimes h_2h_1e_1], m\otimes h_2h_1e_1)$$

and

$$g([h_1,h_2],q) = [m_q\otimes h_2h_1e_1],$$

where $e_1\in W-\{0\}$ is fixed such that for all $g'\in S(U(l_1)\times U(1))$ $\psi_1(g')g'e_1 = e_1$ and $m_q\neq 0$ some element of the fiber of $N(A,N_1)$ over $q\in A$.

The map f induces an isomorphism of the normal bundle of $H_0/H_1 \times A$ in $H_0 \times_{H_1} N_1$ and the tautological bundle over $P_{\mathbb{C}}(E' \oplus \{0\})$. □

5.3. **Admissible triples.** Now we are in the position to state our first classification theorem. To do so, we need the following definition.

Definition 5.12. Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. Then a triple (ψ, N, A) with

- $\psi \in \operatorname{Hom}(S(U(l_1) \times U(1)), Z(G_2))$,
- N a torus manifold with G_2-action,
- A the empty set or a closed codimension two submanifold of N, such that $\operatorname{im}\psi$ acts trivially on A and $\ker\psi = SU(l_1)$ if $A \neq \emptyset$,

is called *admissible for* $(\tilde{G}, G_1)$. We say that two admissible triples (ψ, N, A), (ψ', N', A') for $(\tilde{G}, G_1)$ are equivalent if there is a G_2-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$ and

$$\psi = \begin{cases} \psi' & \text{if } l_1 > 1 \\ \psi'^{\pm 1} & \text{if } l_1 = 1. \end{cases}$$

Theorem 5.13. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. There is a one-to-one-correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-action such that G_1 is elementary and the equivalence classes of admissible triples for $(\tilde{G}, G_1)$.*

Proof. Let M be a torus manifold with $\tilde{G}$-action such that G_1 is elementary. Then, by Corollaries 5.7 and 5.9, (ψ_1, N_1, M^{H_0}) is an admissible triple, where ψ_1 is defined as in Lemma 5.3 and N_1 is defined as in Lemma 5.1.

Let (ψ, N, A) be an admissible triple for $(\tilde{G}, G_1)$. If $A \neq \emptyset$, then, by Lemma 5.11, the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$ is a torus manifold with $\tilde{G}$-action. If $A = \emptyset$, then we have the torus manifold $H_0 \times_{H_1} N$.

We show that these two operations are inverse to each other. Let M be a torus manifold with $\tilde{G}$-action. If $M^{H_0} = \emptyset$, then, by Corollary 5.6, we have $M = H_0 \times_{H_1} N_1$. If $M^{H_0} \neq \emptyset$, then by the discussion before Lemma 5.11, M is the blow down of $H_0 \times_{H_1} N_1$ along $H_0/H_1 \times M^{H_0}$.

Now assume $l_1 > 1$. Let (ψ, N, A) be an admissible triple with $A \neq \emptyset$ and M the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11, we have $A = M^{H_0}$. By Lemma 5.10 and Corollary 4.5, we have $N = N_1$. With Lemmas 5.4 and A.3, one sees that $\psi = \psi_1$, where ψ_1 is the homomorphism defined in Lemma 5.3 for M.

Now let $(\psi, N, \emptyset)$ be an admissible triple and $M = H_0 \times_{H_1} N$. Then we have $M^{H_0} = \emptyset$. By Lemma 5.10 we have $N = N_1$. As in the first case one sees $\psi = \psi_1$.

Now assume $l_1 = 1$. Let (ψ, N, A) be an admissible triple with $A \neq \emptyset$ and M the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$. Then, by the remark after Lemma 5.11, $A = M^{H_0}$. By Lemma 5.5, we have two choices for N_1 and $\psi = \psi_1^{\pm 1}$. Because the two choices for N_1 lead to equivalent admissible triples we recover the equivalence class of (ψ, N, A). In the case $A = \emptyset$ a similar argument completes the proof of the theorem. □

Corollary 5.14. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$. Then the torus manifolds with $\tilde{G}$-action such that G_1 is elementary and $M^{G_1} \neq \emptyset$ are given by blow downs of fiber bundles over $\mathbb{C}P^{l_1}$ with fiber some torus manifold with G_2-action along a submanifold of codimension two.*

Now we specialise our classification result to special classes of torus manifolds.

Theorem 5.15. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then $H^*(M;\mathbb{Z})$ is generated by its degree two part if and only if $H^*(N;\mathbb{Z})$ is generated by its degree two part and A is connected.*

Proof. To make the notation simpler we omit the coefficients of the cohomology in the proof. If $H^*(M)$ is generated by its degree two part, then $H^*(N)$ is generated by its degree two part by [15, p. 716]. Moreover, A is connected by [15, p. 738] and Corollary 5.7.

Now assume that $H^*(N)$ is generated by its degree two part and $A = \emptyset$. Then by Poincaré duality $H_{\text{odd}}(N) = 0$. Therefore by an universal coefficient theorem $H^*(N) = \operatorname{Hom}(H_*(N), \mathbb{Z})$ is torsion free. By Corollary 5.6, M is a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber N. Because the Serre-spectral sequence of this fibration degenerates we have

$$H^*(M) \cong H^*(\mathbb{C}P^{l_1}) \otimes H^*(N)$$

as a $H^*(\mathbb{C}P^{l_1})$-modul. Because $H^*(N)$ is generated by its degree two part, it follows that the cohomology of M is generated by its degree two part.

Now we turn to the general case $A \neq \emptyset$. Then, by [15, p. 716], $H^*(A)$ is generated by its degree two part. Moreover, $H^*(N) \to H^*(A)$ is surjective. Let $\tilde{M}$ be the blow up of M along A and $F : \tilde{M} \to M$ the map defined in section 4.

Because, by Lemma 4.2, F is the identity outside some open tubular neighborhood of $A \times \mathbb{C}P^{l_1}$, the induced homomorphism $F^* : H^*(M, A) \to H^*(\tilde{M}, A \times \mathbb{C}P^{l_1})$ is an isomorphism by excision. Furthermore, the push forward $F_! : H^*(\tilde{M}) \to H^*(M)$ is a section of $F^* : H^*(M) \to H^*(\tilde{M})$. Therefore $F^* : H^*(M) \to H^*(\tilde{M})$ is injective and $H^{\text{odd}}(M)$ vanishes.

Because A is connected, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & H^2(\tilde{M}, A \times \mathbb{C}P^{l_1}) & \longrightarrow & H^2(N, A) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(\mathbb{C}P^{l_1}) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & H^2(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(\mathbb{C}P^{l_1}) & \longrightarrow & H^2(A \times \mathbb{C}P^{l_1}) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & H^3(\tilde{M}, A \times \mathbb{C}P^{l_1}) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Now from the snake lemma it follows that

$$H^2(M, A) \cong_{F^*} H^2(\tilde{M}, A \times \mathbb{C}P^{l_1}) \cong H^2(N, A)$$

and

$$H^3(M, A) \cong_{F^*} H^3(\tilde{M}, A \times \mathbb{C}P^{l_1}) \cong 0.$$

Because $\iota_{NM} = F \circ \iota_{N\tilde{M}}$, where $\iota_{NM}, \iota_{N\tilde{M}}$ are the inclusions of N in M and $\tilde{M}$, the left arrow in the following diagram is an isomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(M,A) & \longrightarrow & H^2(M) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_{NM}^*} & & \downarrow{\scriptstyle \iota_{NM}^*} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & H^2(N,A) & \longrightarrow & H^2(N) & \longrightarrow & H^2(A) & \longrightarrow & 0 \end{array}$$

Therefore it follows from the five lemma that

$$H^2(M) \cong H^2(N)$$

and

$$H^2(\tilde{M}) \cong H^2(\mathbb{C}P^{l_1}) \oplus H^2(N) \cong H^2(\mathbb{C}P^{l_1}) \oplus H^2(M).$$

Let $t \in H^2(\mathbb{C}P^{l_1})$ be a generator of $H^*(\mathbb{C}P^{l_1})$ and $x \in H^*(M)$. Then, because $H^*(\tilde{M})$ is generated by its degree two part, there are sums of products $x_i \in H^*(M)$ of elements of $H^2(M)$ such that

$$x = F_! F^*(x) = F_! \left(\sum F^*(x_i) t^i \right) = \sum x_i F_!(t^i).$$

Therefore it remains to show that $F_!(t^i)$ is a product of elements of $H^2(M)$.

The $l_1 + 1$ characteristic submanifolds $\tilde{M}_1, \dots, \tilde{M}_{l_1+1}$ of $\tilde{M}$ which are permuted by $W(G_1)$ are the preimages of the characteristic submanifolds of $\mathbb{C}P^{l_1}$ under the projection $\tilde{M} \to \mathbb{C}P^{l_1}$. Therefore they can be oriented in such a way that t is the Poincaré-dual of each of them.

Because F restricts to a diffeomorphism $\tilde{M} - A \times \mathbb{C}P^{l_1} \to M - A$ and $F(\tilde{M}_i) = M_i$, $F_!(t^i)$, $i \leq l_1$, is the Poincaré-dual $PD\left(\bigcap_{1\leq k\leq i} M_k\right)$ of the intersection $\bigcap_{1\leq k \leq i} M_k$ of characteristic submanifolds of M, which belong to $\mathfrak{F}_1$. Therefore for $i \leq l_1$ we have

$$F_!(t)^i = PD\left(\bigcap_{1\leq k\leq i} M_k \right) = F_!(t^i).$$

Because $t^i = 0$ for $i > l_1$, the statement follows. □

Theorem 5.16. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then M is quasitoric if and only if N is quasitoric and A is connected.*

Proof. At first assume that M is quasitoric. Then N is quasitoric and A connected because all intersections of characteristic submanifolds of M are quasitoric and connected.

Now assume that N is quasitoric and $A \subset N$ connected. Then, by Theorem 5.15 and [15, p. 738], the T-action on M is locally standard and M/T is a homology polytope. We have to show that M/T is face preserving homeomorphic to a simple polytope.

Let $T_2 = T \cap G_2$. Then the orbit space N/T_2 is face preserving homeomorphic to a simple polytope P. Because A is connected, A/T_2 is a facet F_1 of P.

With the notation from Lemma 5.11 let

$$B = \{(z_0 : 1) \in P_{\mathbb{C}}(E' \oplus \mathbb{C}); z_0 \in E', |z_0| \leq 1\}.$$

Then the orbit space of the T-action on B is given by $F_1 \times \Delta^{l_1+1}$.

Let B' be a closed $\tilde{G}$-invariant tubular neighborhood of $H_0/H_1 \times A$ in $H_0 \times_{H_1} N$. Then the bundle projection $\partial B' \to H_0/H_1 \times A$ extends to an equivariant map

$$H_0 \times_{H_1} N - \mathring{B}' \to H_0 \times_{H_1} N,$$

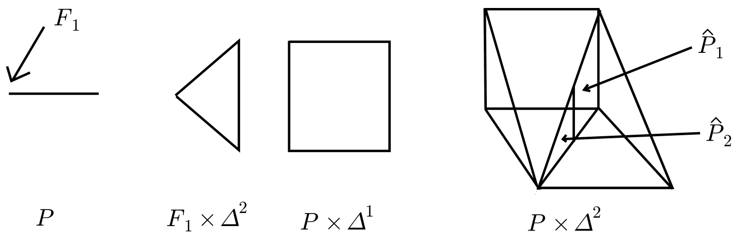


FIGURE 1. The orbit space of a blow down

which induces a face preserving homeomorphism

$$\left(H_0 \times_{H_1} N - \mathring{B}'\right)/T \cong P \times \Delta^{l_1}.$$

Now M is given by gluing B and $H_0 \times_{H_1} N - \mathring{B}'$ along the boundaries $\partial B, \partial B'$. The corresponding gluing of the orbit spaces is illustrated in Figure 1 for the case $\dim N = 2$ and $l_1 = 1$. Because the gluing map $f : \partial B \to \partial B'$ is $\tilde{G}$-equivariant and G_1 acts transitively on the fibers of $\partial B \to A$ and $\partial B' \to A$, it induces a map

$$\hat{f} : F_1 \times \Delta^{l_1} = \partial B/T \to \partial B'/T = F_1 \times \Delta^{l_1}, \qquad (x, y) \mapsto (\hat{f}_1(x), \hat{f}_2(x, y)),$$

where $\hat{f}_1 : F_1 \to F_1$ is a face preserving homeomorphism and $\hat{f}_2 : F_1 \times \Delta^{l_1} \to \Delta^{l_1}$ such that, for all $x \in F_1$, $\hat{f}_2(x, \cdot)$ is a face preserving homeomorphism of Δ^{l_1}.

Now fix embeddings

$$\Delta^{l_1+1} \hookrightarrow \mathbb{R}^{l_1+1} \text{ and } P \hookrightarrow \mathbb{R}^{n-l_1-1} \times [0, 1[$$

such that $\Delta^{l_1} \subset \mathbb{R}^{l_1} \times \{1\}$ and $\Delta^{l_1+1} = \operatorname{conv}(0, \Delta^{l_1})$ and $P \cap \mathbb{R}^{n-l_1-1} \times \{0\} = F_1$.

Denote by $p_1 : \mathbb{R}^{l_1+1} \to \mathbb{R}$ and $p_2 : \mathbb{R}^{n-l_1} \to \mathbb{R}$ the projections on the last coordinate. For $\epsilon > 0$ small enough, P and $P \cap \{p_2 \geq \epsilon\}$ are combinatorially equivalent. Therefore there is a face preserving homeomorphism

$$g_1 : P \to P \cap \{p_2 \geq \epsilon\}$$

such that $g_1(F_1) = P \cap \{p_2 = \epsilon\}$ and $g_1(F_i) = F_i \cap \{p_2 \geq \epsilon\}$ for the other facets of P. The map

$$\begin{aligned} g_2 : F_1 \times [0, 1] &\to P \cap \{p_2 \leq \epsilon\} \\ (x, y) &\mapsto x(1 - y) + y g_1(x) \end{aligned}$$

is a face preserving homeomorphism with $p_2 \circ g_2(x, y) = \epsilon y$ for all $(x, y) \in F_1 \times [0, 1]$. Now let

$$\begin{aligned} \hat{P} &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}, \\ \hat{P}_1 &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \geq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}, \\ \hat{P}_2 &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \leq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}. \end{aligned}$$

Then there are face preserving homeomorphisms

$$h_1 : P \times \Delta^{l_1} \to \hat{P}_1 \qquad (x, y) \mapsto (g_1(x), p_2(g_1(x))y)$$

and

$$h_2 : F_1 \times \Delta^{l_1+1} \to \hat{P}_2 \qquad (x, y) \mapsto (g_2(x, p_1(y)), \epsilon y).$$

We claim that $\hat{P}$ and M/T are face preserving homeomorphic. This is the case if

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2 : F_1 \times \Delta^{l_1} \to F_1 \times \Delta^{l_1}$$

extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. Now for $(x,y) \in F_1 \times \Delta^{l_1}$ we have

$$\begin{aligned}\hat{f}^{-1} \circ h_1^{-1} \circ h_2(x,y) &= \hat{f}^{-1} \circ h_1^{-1}(g_2(x,p_1(y)),\epsilon y)\\ &= \hat{f}^{-1} \circ h_1^{-1}(g_2(x,1),\epsilon y)\\ &= \hat{f}^{-1}(g_1^{-1} \circ g_2(x,1),y)\\ &= (\hat{f}_1^{-1}(x),(\hat{f}_2(x,\cdot))^{-1}(y)).\end{aligned}$$

Because Δ^{l_1+1} is the cone over Δ^{l_1} this map extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. □

Lemma 5.17. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then there is an isomorphism $\pi_1(N) \to \pi_1(M)$.*

Proof. Let $\tilde{M}$ be the blow up of M along A. Then, by [16, p. 270], there is a isomorphism $\pi_1(\tilde{M}) \to \pi_1(M)$.

Now, by Corollary 5.6, $\tilde{M}$ is the total space of a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber N. Therefore there is an exact sequence

$$\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \to \pi_1(N) \to \pi_1(\tilde{M}) \to 0.$$

Because the torus action on N has fixed points, there is a section in this bundle. Hence, $\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1})$ is surjective. □

6. The case $G_1 = SO(2l_1)$

In this section we study torus manifolds with G-action, where $\tilde{G} = G_1 \times G_2$ and $G_1 = SO(2l_1)$ is elementary. It turns out that the restriction of the action of G_1 to $U(l_1)$ on such a manifold has the same orbits as the action of $SO(2l_1)$. Therefore the results of the previous section may be applied to construct invariants for such manifolds. For simply connected torus manifolds with G-action these invariants determine their $\tilde{G}$-equivariant diffeomorphism type.

Let $\tilde{G} = G_1 \times G_2$, where $G_1 = SO(2l_1)$ is elementary, and M a torus manifold with G-action. Then, by Lemmas 3.1 and 3.4, one sees that the principal orbit type of the G_1-action is given by $SO(2l_1)/SO(2l_1-1)$. Therefore the G_1-action has only three orbit types $SO(2l_1)/SO(2l_1-1)$, $SO(2l_1)/S(O(2l_1-1) \times O(1))$ and $SO(2l_1)/SO(2l_1)$. The induced action of $U(l_1)$ has the same orbits, which are of type $U(l_1)/U(l_1-1)$, $U(l_1)/\langle U(l_1-1), \mathbb{Z}_2\rangle$ and $U(l_1)/U(l_1)$, respectively. Here $\langle U(l_1-1), \mathbb{Z}_2\rangle$ denotes the subgroup of $U(l_1)$, which is generated by $U(l_1-1)$ and the diagonal matrix with all entries equal to -1.

Let $S = S^1$. Then there is a finite covering

$$SU(l_1) \times S \to U(l_1) \qquad (A,s) \mapsto sA.$$

So we may replace the factor G_1 of $\tilde{G}$ by $SU(l_1)$ and G_2 by $S \times G_2$ to reach the situation of the previous section.

Let $x \in M^T$ and $T_2 = T \cap G_2$. Then we may assume by Lemma 3.4 that the $G_1 \times T_2$-representation T_xM is given by

$$T_xM = V \oplus W,$$

where V is a complex representation of T_2 and W is the standard real representation of G_1. Therefore

$$T_xM = V \oplus V_0 \otimes_{\mathbb{C}} W_0$$

as a $SU(l_1) \times S \times T_2$-representation, where V_0 is the standard complex one-dimensional representation of S and W_0 is the standard complex representation of $SU(l_1)$.

Therefore the group homomorphism ψ_1 and the groups H_0, H_1, H_2 introduced in Lemma 5.3 have the following form:

$$\operatorname{im}\psi_1 = S,$$

and

$$\begin{aligned} H_0 &= SU(l_1) \times S,\\ H_1 &= S(U(l_1-1)\times U(1)) \times S,\\ H_2 &= \left\{ (g, g_{l_1+1}^{-1}) \in H_1; g = \begin{pmatrix} A & 0\\ 0 & g_{l_1+1}\end{pmatrix} \text{ with } A \in U(l_1-1)\right\}. \end{aligned}$$

Let N_1 be the intersection of $l_1 - 1$ characteristic submanifolds of M belonging to $\mathfrak{F}_1$ as defined in Lemmas 5.1 and 5.10. Then, by Lemma 5.3, we know that N_1 is a component of M^{H_2} and $M = H_0 N_1$. Therefore we have $N_1 = M^{H_2}$ if, for all H_0-orbits O, O^{H_2} is connected. Because all orbits are of type H_0/H_0, H_0/H_2, $H_0/\langle H_2, \mathbb{Z}_2\rangle$ and

$$\begin{aligned} (H_0/H_2)^{H_2} &= N_{H_0}H_2/H_2 = H_1/H_2,\\ (H_0/\langle H_2,\mathbb{Z}_2\rangle)^{H_2} &= N_{H_0}H_2/\langle H_2,\mathbb{Z}_2\rangle = H_1/\langle H_2,\mathbb{Z}_2\rangle, \end{aligned}$$

it follows that $N_1 = M^{H_2}$.

The projection $H_1 \to H_1/H_2$ induces an isomorphism $S \to H_1/H_2$. Therefore S acts freely on $(H_0/H_2)^{H_2}$. Hence, S acts effectively on N_1.

By Corollary 5.7, $N_1^S = M^{H_0}$ has codimension two in N_1.

After these general remarks we first discuss the case, where there are no exceptional $SO(2l_1)$-orbits. That means the case, where there are no orbits of type $SO(2l_1)/S(O(2l_1-1)\times O(1))$. Then the induced $U(l_1)$-action has also no exceptional orbits. Moreover, by Corollary 5.7, M is a special $SO(2l_1)$-, $U(l_1)$-manifold in the sense of Jänich [9].

At first we discuss the question under which conditions the action of $U(l_1)\times G_2$ on a torus manifold satisfying the above conditions on the $U(l_1)$-orbits and having no exceptional $U(l_1)$-orbits extends to an action of $SO(2l_1)\times G_2$.

Let X be the orbit space of the $U(l_1)$-action on M. Then, by [9, p. 303], X is a manifold with boundary such that the interior $\mathring{X}$ of X corresponds to orbits of type $U(l_1)/U(l_1-1)$ and the boundary ∂X to the fixed points. The action of G_2 on M induces a natural action of G_2 on X.

Following Jänich [9] we may construct from M a manifold $M \odot M^{U(l_1)}$ with boundary, on which $U(l_1)\times G_2$ acts such that all orbits of the $U(l_1)$-action on $M\odot M^{U(l_1)}$ are of type $U(l_1)/U(l_1-1)$ and $\left(M\odot M^{U(l_1)}\right)/U(l_1) = X$. Denote by P_M the G_2-equivariant principal S^1-bundle

$$\left(M\odot M^{U(l_1)}\right)^{U(l_1-1)} \to X.$$

Lemma 6.1. *Let M be a torus manifold with $U(l_1)\times G_2$-action such that all $U(l_1)$-orbits are of type $U(l_1)/U(l_1-1)$ or $U(l_1)/U(l_1)$. Then the action of $U(l_1)\times G_2$ on M extends to an action of $SO(2l_1)\times G_2$ if and only if there is a G_2-equivariant $\mathbb{Z}_2$-principal bundle P'_M such that*

$$P_M = S^1 \times_{\mathbb{Z}_2} P'_M,$$

where the action of G_2 on S^1 is trivial.

Proof. If the action extends to a $SO(2l_1)\times G_2$-action, then $SO(2l_1)\times G_2$ acts on $M\odot M^{U(l_1)}$. Therefore $P'_M = \left(M\odot M^{U(l_1)}\right)^{SO(2l_1-1)} \to X$ is such a G_2-equivariant $\mathbb{Z}_2$-principal bundle.

If there is such a G_2-equivariant $\mathbb{Z}_2$-bundle P'_M, then by a G_2-equivariant version of Jänich's Klassifikationssatz [9] there is a torus manifold M' with $SO(2l_1) \times G_2$-action with $M'/U(l_1) = X$ and $P_M = S^1 \times_{\mathbb{Z}_2} P'_M = P_{M'}$. Therefore M' and M are $U(l_1) \times G_2$-equivariantly diffeomorphic. □

Lemma 6.2. *Let M, M' be torus manifolds with $SO(2l_1) \times G_2$-action such that there are no exceptional $SO(2l_1)$-orbits and $H_1(M;\mathbb{Z})$ and $H_1(M';\mathbb{Z})$ are torsion. If there is a $U(l_1) \times G_2$-equivariant diffeomorphism $f : M \to M'$, then there is a $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$. Moreover, g and f induce the same map on $M/U(l_1) - B$, where B is a collar of $\partial(M/U(l_1))$.*

Proof. The map f induces a G_2-equivariant diffeomorphism $\hat{f} : X = M/SO(2l_1) \to M'/SO(2l_1)$. We use this map to identify these spaces. It follows from [3, p. 91] and the equality $H_1(X;\mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ that $H_1(X;\mathbb{Z})$ is torsion. Hence, $H^1(X;\mathbb{Z}) = 0$.

Recall that for the universal principal $\mathbb{Z}_2$-bundle $P \to \mathbb{R}P^\infty$, the first Chern-class of the principal S^1-bundle $S^1 \times_{\mathbb{Z}_2} P \to \mathbb{R}P^\infty$ is given by $\delta w_1(P)$, where $\delta : H^1(\mathbb{R}P^\infty;\mathbb{Z}_2) \to H^2(\mathbb{R}P^\infty;\mathbb{Z})$ is the Bockstein-homomorphism and $w_1(P)$ is the first Stiefel-Whitney-class of P. By naturallity, this relation also holds for any principal $\mathbb{Z}_2$-bundle over X. Because $H^1(X;\mathbb{Z}) = 0$, the Bockstein-homomorphism $\delta : H^1(X;\mathbb{Z}_2) \to H^2(X;\mathbb{Z})$ is injective.

Hence, the principal S^1-bundle $P_M \to X$ has up to isomorphism at most one restriction of structure group to $\mathbb{Z}_2$. Therefore the two restrictions of the structure group induced by the $SO(2l_1)$-actions on M, M' are the same up to G_2-equivariant isomorphism.

Therefore, by the proof of Jänich's Klassifikationssatz, there is a $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M \to M'$, which induces the same map as f outside a neighbourhood of ∂X. □

Now we turn to the case where there are exceptional $SO(2l_1)$-orbits. Then we have:

Theorem 6.3. *Let M, M' be two simply connected torus manifolds with $SO(2l_1) \times G_2$-action. Then M and M' are $SO(2l_1) \times G_2$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.*

Proof. In this proof we take all cohomology groups with coefficients in $\mathbb{Z}$. Let $f : M \to M'$ be a $U(l_1) \times G_2$-equivariant diffeomorphism. Moreover, let A, A' be the union of the exceptional $U(l_1)$-orbits in M, M', respectively. Because the $U(l_1)$-representation $N_x(M^{U(l_1)}, M)$ is the standard representation for all $x \in M^{U(l_1)}$, there are invariant neighbourhoods of $M^{U(l_1)}$ and $M'^{U(l_1)}$ which do not contain any exceptional orbit. Hence, A, A' are closed submanifolds of M, M'.

Denote by D, D' the unit disc bundle in $N(A, M)$ and $N(A', M')$, respectively. Let $h : D \to B \subset M$ and $h' : D' \to B' \subset M'$ be $SO(2l_1) \times G_2$-equivariant tubular neighbourhoods of A and A'.

Then, by Theorems 4.6 and 8.3 of [10, p. 10,19] , we may assume that $f(B) = B'$ and that $h'^{-1} \circ f \circ h$ is a linear map.

It is sufficient to show the following two things:

1. There is a $SO(2l_1) \times G_2$-equivariant diffeomorphism $g : M - \mathring{B} \to M' - \mathring{B}'$ such that g and f induce the same maps on $(\partial B)/U(l_1)$.
2. The map g extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $M \to M'$.

If $H_1(M - \mathring{B})$ is torsion, we may apply the arguments from the proof of Lemma 6.2 to show (1). Therefore we show that $H_1(M - \mathring{B})$ is torsion.

Let $A_1, \dots, A_k$ be the orientable components of A of codimension two in M. We fix orientations for each of these components and for M. Let $\tau_1, \dots, \tau_k \in H^2(M)$ be the Poincaré duals for $A_1, \dots, A_k$. Because $H_1(M) = 0$, it follows from an universal coefficient theorem and Poincaré-duality that

$$H^2(M) \cong \operatorname{Hom}(H_2(M), \mathbb{Z}) \cong \operatorname{Hom}(H^{2n-2}(M), \mathbb{Z}),$$

where an isomorphism is given by

$$\alpha \mapsto (\beta \mapsto \langle \beta\alpha, [M] \rangle).$$

Here we have $\dim M = 2n$. In particular, $H^2(M)$ is torsion free.

We claim that the $\tau_1, \dots, \tau_k$ are linear independent. Let $a_1, \dots, a_k \in \mathbb{Z}$ such that

$$0 = \sum_{i=1}^{k} a_i \tau_i. \tag{6.1}$$

Then we have $0 = a_i \iota_{A_i}^* \tau_i$ where $\iota_{A_i} : A_i \to M$ is the inclusion. By restricting to an orbit O contained in A_i, we get

$$0 = a_i \iota_O^* \iota_{A_i}^* \tau_i \in H^2(SO(2l_1)/S(O(2l_1 - 1) \times O(1))) = \mathbb{Z}_2.$$

Because $N(A_i, M)|_O = SO(2l_1)/SO(2l_1 - 1) \times_{\mathbb{Z}_2} \mathbb{R}^2$ with $\mathbb{Z}_2$ acting on $\mathbb{R}^2$ by multiplication with -1, it follows that $\iota_O^* \iota_{A_i}^* \tau_i \neq 0$. Therefore a_i is divisible by two.

Hence, we may replace $a_i \mapsto \frac{1}{2} a_i$ in (6.1). Since the above arguments then hold for the new a_i, we see that the original a_i are divisible by arbitrary high powers of two. Therefore they must vanish.

There is an exact sequence

$$H^{2n-2}(M) \to H^{2n-2}(A) \to H^{2n-1}(M, A) \to 0.$$

Because, by [3, p. 185], there are no components of A, which have codimension one in M, there is an isomorphism

$$H^{2n-2}(A) \cong \mathbb{Z}^k \oplus (\mathbb{Z}_2)^{k_1},$$

where k_1 is the number of non-orientable components of codimension two of A. Let

$$\begin{aligned} \phi : H^{2n-2}(A) &\to \mathbb{Z}^k \\ \alpha &\mapsto (\langle \alpha, [A_1] \rangle, \dots, \langle \alpha, [A_k] \rangle). \end{aligned}$$

Because the $\tau_1, \dots, \tau_k$ are linear independent, it follows that $\phi \circ \iota^* : H^{2n-2}(M) \to \mathbb{Z}^k$ has rank k.

Therefore, from the exactness of the above sequence, it follows that $H^{2n-1}(M, A)$ is torsion. By Poincaré-duality and excision, it follows that $H_1(M - \mathring{B})$ is torsion. Hence we have proven (1).

Now we prove (2). By Theorem 9.4 of [10, p. 24], it is sufficient to show that

$$k = h'^{-1} \circ g \circ h : \partial D \to \partial D'$$

extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $D \to D'$.

Let O be an $SO(2l_1)$-orbit in A and $S \to O$ be the restriction of the sphere bundle $\partial D \to A$ to O. Because f and g induce the same maps on the orbit space $(\partial B)/U(l_1)$ and S is $SO(2l_1)$-invariant, we have $k(S) = h'^{-1} \circ f \circ h(S) = S'$. Because $h'^{-1} \circ f \circ h : D \to D'$ is a linear map, we see that S' is the restriction of the sphere bundle $\partial D' \to A'$ to an $SO(2l_1)$-orbit O'.

We may choose $SO(2l_1)$-equivariant bundle isomorphisms $k_1 : SO(2l_1)/SO(2l_1 - 1) \times_{\mathbb{Z}_2} S^m \to S$ and $k_1' : SO(2l_1)/SO(2l_1 - 1) \times_{\mathbb{Z}_2} S^m \to S'$. Because f and g induce the same maps on the orbit space $S/SO(2l_1) = S^m/\mathbb{Z}_2 = \mathbb{R}P^m$ and $h'^{-1} \circ f \circ h$ is a linear map, it follows that $k_1'^{-1} \circ g \circ k_1$ is of the form

$$[gSO(2l_1 - 1), x] \mapsto [gzSO(2l_1 - 1), \pm Ax] = [gSO(2l_1 - 1), \pm Ax],$$

where $z \in S(O(2l_1 - 1) \times O(1))/SO(2l_1 - 1) = \mathbb{Z}_2$ and $A \in O(m + 1)$. Therefore k is linear on each fiber. Hence, it extends to an $SO(2l_1) \times G_2$-equivariant diffeomorphism $D \to D'$. □

Let M be a simply connected torus manifold with $SO(2l_1) \times G_2$-action. By Theorem 5.13, there is an admissible triple (ψ, N, A) corresponding to M equipped with the action of $SU(l_1) \times S \times G_2$ as above. The admissible triple (ψ, N, A) determines the $SU(l_1) \times S \times G_2$-equivariant diffeomorphism type of M. With Theorem 6.3 we see that the $SO(2l_1) \times G_2$-equivariant diffeomorphism type of M is determined by (ψ, N, A).

Lemma 6.4. *Let M be a torus manifold with $G_1 \times G_2$-action, where $G_1 = SO(2l_1)$ is elementary and G_2 is a not necessary connected Lie-group. If $M^{SO(2l_1)}$ is connected then G_2 acts orientation preserving on $N(M^{SO(2l_1)}, M)$. Therefore G_2 acts orientation preserving on M if and only if it acts orientation preserving on $M^{SO(2l_1)}$.*

Proof. Let $g \in G_2$, $x \in M^{SO(2l_1)}$ and $y = gx \in M^{SO(2l_1)}$. Because $M^{SO(2l_1)}$ is connected there is a orientation preserving $SO(2l_1)$-invariant isomorphism

$$N_x(M^{SO(2l_1)}, M) \cong N_y(M^{SO(2l_1)}, M).$$

Therefore $g : N_x(M^{SO(2l_1)}, M) \to N_y(M^{SO(2l_1)}, M)$ induces an automorphism ϕ of the $SO(2l_1)$-representation $N_x(M^{SO(2l_1)}, M)$ which is orientation preserving if and only if g is orientation preserving.

Because, by Lemma 3.4, $N_x(M^{SO(2l_1)}, M)$ is just the standard real representation of $SO(2l_1)$, its complexification $N_x(M^{SO(2l_1)}, M) \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible complex representation. Therefore, by Schur's Lemma, there is a $\lambda \in \mathbb{C} - \{0\}$ such that for all $a \in N_x(M^{SO(2l_1)}, M)$

$$\phi(a) \otimes 1 = \phi_{\mathbb{C}}(a \otimes 1) = a \otimes \lambda.$$

This equation implies that $\lambda \in \mathbb{R} - \{0\}$ and $\phi(a) = \lambda a$. Therefore ϕ is orientation preserving. □

7. The case $G_1 = SO(2l_1 + 1)$

In this section we discuss actions of groups, which have a covering group, whose action on M factors through $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$ elementary. In the case $G_1 = SO(3)$ we also assume $\#\mathfrak{F}_1 = 1$ or that the principal orbit type of the $SO(3)$-action on M is given by $SO(3)/SO(2)$.

It is shown that a torus manifold with $\tilde{G}$-action is a product of a sphere and a torus manifold with G_2-action or the blow up along the fixed points of G_1 is a fiber bundle over a real projective space.

We assume that $T_1 = T \cap G_1$ is the standard maximal torus of G_1.

7.1. The G_1-action on M.

Lemma 7.1. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$, M a torus manifold with G-action such that G_1 is elementary. If $l_1 > 1$ there is, by Lemma 3.3, a component N_1 of $\bigcap_{M_i \in \mathfrak{F}_1} M_i$ with $N_1^T \neq \emptyset$. If $l_1 = 1$ let N_1 be a characteristic submanifold belonging to $\mathfrak{F}_1$. Then*

1. *N_1 is a component of $M^{SO(2l_1)}$.*
2. *$M = G_1 N_1$.*

Proof. Let $x \in N_1^T$. Then, by Lemmas 3.1, 3.4 and Remark 3.2, $G_{1x} = SO(2l_1)$. Let T_2 be the maximal torus $T \cap G_2$ of G_2. On the tangent space of M in x we have the $SO(2l_1) \times T_2$-representation

$$T_xM = N_x(G_1x, M) \oplus T_xG_1x.$$

By Lemma 3.1, T_2 acts trivially on G_1x. Moreover, T_2 acts almost effectively on $N_x(G_1x, M)$. Therefore it follows by dimension reasons that $N_x(G_1x, M)$ splits as a sum of complex one dimensional $SO(2l_1) \times T_2$-representations. If $l_1 > 1$, $SO(2l_1)$ has no non-trivial one-dimensional complex representation. Therefore we have

$$T_xM = \bigoplus_i V_i \oplus W, \tag{7.1}$$

where the V_i are one-dimensional complex representations of T_2 and W is the standard real representation of $SO(2l_1)$.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$, then $SO(2l_1)$ acts trivially on $N_x(G_1x, M)$ because $SO(3)/SO(2)$ is the principal orbit type of the $SO(3)$-action on M [3, p. 181].

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then, by the discussion leading to Convention 3.5, $SO(2)$ acts trivially on $N_x(G_1x, M)$. Therefore in these cases T_xM splits as in (7.1).

Because $N_x(G_1x, M)$ is the tangent space of N_1 in x the maximal torus T_1 of G_1 acts trivially on N_1. Therefore N_1 is the component of M^{T_1}, which contains x. Because $T_xN_1 = (T_xM)^{T_1} = (T_xM)^{SO(2l_1)}$, N_1 is a component of $M^{SO(2l_1)}$.

Now we prove (2). Let $y \in N_1$. Then there are the following possibilities:

- $G_{1y} = G_1$.
- $G_{1y} = S(O(2l_1) \times O(1))$ and $\dim G_1y = 2l_1$.
- $G_{1y} = SO(2l_1)$ and $\dim G_1y = 2l_1$.

If $g \in G_1$ such that $gy \in N_1$, then

$$gG_{1y}g^{-1} = G_{1gy} \in \{S(O(2l_1) \times O(1)), SO(2l_1), G_1\}$$

and

$$g \in N_{G_1}G_{1y} = \begin{cases} G_1 & \text{if } y \in M^{G_1} \\ S(O(2l_1) \times O(1)) & \text{if } y \notin M^{G_1}. \end{cases}$$

Therefore $G_1y \cap N_1 \subset S(O(2l_1) \times O(1))y$ contains at most two elements. If y is not fixed by G_1, then one sees as in the proof of Lemma 5.3 that G_1y and N_1 intersect transversely in y.

Therefore $G_1(N_1 - N_1^{G_1})$ is open in $M - M^{G_1}$ by Lemma A.5. Because M^{G_1} has codimension at least three, $M - M^{G_1}$ is connected. But

$$G_1\left(N_1 - N_1^{G_1}\right) = G_1N_1 \cap \left(M - M^{G_1}\right)$$

is also closed in $M - M^{G_1}$. Hence,

$$M - M^{G_1} = G_1\left(N_1 - N_1^{G_1}\right) = G_1N_1 - N_1^{G_1}.$$

Therefore one sees as in the proof of Lemma 5.3 that

$$M = G_1N_1 \amalg \left(M^{G_1} - N_1^{G_1}\right).$$

Because G_1N_1 and $M^{G_1} - N_1^{G_1}$ are closed in M the statement follows. □

Corollary 7.2. *If in the situation of Lemma 7.1 the G_1-action on M has no fixed point in M, then $M = SO(2l_1+1)/SO(2l_1) \times N_1$ or $M = SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$, where $\mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1)$.*

In the second case the $\mathbb{Z}_2$-action on N_1 is orientation reversing.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then we have $M = SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$. If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$, then we have $M = SO(2l_1+1)/SO(2l_1) \times N_1$.

Proof. Let $g \in S(O(2l_1) \times O(1)) = N_{G_1} SO(2l_1)$. Then gN_1 is a component of $M^{SO(2l_1)}$. Because $N_1 \subset M^{SO(2l_1)}$, gN_1 only depends on the class

$$gSO(2l_1) \in S(O(2l_1) \times O(1))/SO(2l_1) = \mathbb{Z}_2.$$

Therefore there are two cases

(1) There is a $g \in S(O(2l_1) \times O(1))$ such that $gN_1 \neq N_1$.
(2) The submanifold N_1 is $S(O(2l_1) \times O(1))$-invariant, i.e. $gN_1 = N_1$ for all $g \in S(O(2l_1) \times O(1))$.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then N_1 is the only characteristic submanifold of M belonging to $\mathfrak{F}_1$. Therefore only the second case occurs.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$, then there is a $g_1 \in N_{G_1} T_1$ such that $N_1 \neq g_1 N_1$. Therefore we are in the first case.

In general we have $M = G_1 \times N_1 / \sim$ with

$$\begin{aligned} & (g_1, y_1) \sim (g_2, y_2) \\ \Leftrightarrow \quad & g_1 y_1 = g_2 y_2 \\ \Leftrightarrow \quad & g_2^{-1} g_1 y_1 = y_2 \\ \Leftrightarrow \quad & g_2^{-1} g_1 \in S(O(2l_1) \times O(1)) \text{ and } g_2^{-1} g_1 y_1 = y_2 \end{aligned}$$

In case (1) the last statement is equivalent to

$$g_2^{-1} g_1 \in SO(2l_1) \text{ and } g_2^{-1} g_1 y_1 = y_2.$$

Therefore we get $M = SO(2l_1 + 1)/SO(2l_1) \times N_1$.

In case (2) we have as in the proof of Corollary 5.6

$$M = SO(2l_1 + 1) \times_{S(O(2l_1) \times O(1))} N_1 = SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1.$$

This equation implies that M is the orbit space of a diagonal $\mathbb{Z}_2$-action on $SO(2l_1 + 1)/SO(2l_1) \times N_1$. Because M is orientable this action has to be orientation preserving. But the $\mathbb{Z}_2$-action on $SO(2l_1 + 1)/SO(2l_1)$ is orientation reversing. Therefore the $\mathbb{Z}_2$-action on N_1 is also orientation reversing. □

Corollary 7.3. *In the situation of Lemma 7.1, $M^{G_1} \subset N_1$ is empty or has codimension one in N_1.*

Proof. By Lemma 7.1, it is clear that $M^{G_1} \subset N_1$. For $y \in M^{G_1}$ consider the G_1 representation $T_y M$. Because N_1 is a component of $M^{SO(2l_1)}$, the restriction of $T_y M$ to $SO(2l_1)$ equals the $SO(2l_1)$-representation $T_x M$, where $x \in N_1^T$.

Because, by Lemma 3.4, $T_x M$ is a direct sum of a trivial representation and the standard real representation of $SO(2l_1)$ and $T_1 \subset SO(2l_1)$, $T_y M$ is a sum of a trivial and the standard real representation of $SO(2l_1 + 1)$ by [4, p. 167]. Therefore $M^{G_1} \subset N_1$ has codimension one. □

7.2. **Blowing up along M^{G_1}.** As in section 5 we discuss the question when a manifold of the form given in Corollary 7.2 is a blow up.

If $\tilde{M}$ is the blow up of M along M^{G_1}, then there is an equivariant embedding of $P_{\mathbb{R}}(N(M^{G_1}, M))$ into $\tilde{M}$. Therefore the G_1-action on $\tilde{M}$ has an orbit of type $SO(2l_1 + 1)/S(O(2l_1) \times O(1))$. This fact shows that $\tilde{M}$ is of the form $SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} \tilde{N}_1$ where $\tilde{N}_1$ is the proper transform of N_1. By Lemma 4.5, $\tilde{N}_1$ and N_1 are G_2-equivariantly diffeomorphic. Because M^{G_1} has codimension one in N_1, the $\mathbb{Z}_2$-action on N_1 has a fixed point component of codimension one.

The following Lemma shows that these two conditions are sufficient.

Lemma 7.4. *Let N_1 be a torus manifold with G_2-action. Assume that there are a non-trivial orientation reversing action of $\mathbb{Z}_2 = S((O(2l_1) \times O(1))/SO(2l_1))$ on N_1,* *which commutes with the action of* G_2*, and a closed codimension one submanifold* A *of* N_1*, on which* $\mathbb{Z}_2$ *acts trivially.*

Let $E' = N(A, N_1)$ *equipped with the action of* G_2 *induced from the action on* N_1 *and the trivial action of* $\mathbb{Z}_2$*. Denote by* W *the standard real representation of* $SO(2l_1+1)$*. Then:*

(1) $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ *is orientable.*
(2) *The normal bundle of* $SO(2l_1+1)/S(O(2l_1) \times O(1)) \times A$ *in* $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ *is isomorphic to the tautological bundle over* $P_{\mathbb{R}}(E' \otimes W \oplus \{0\})$*.*

The lemma guarantees together with the discussion at the end of section 4 that one may remove $SO(2l_1+1)/S(O(2l_1)\times O(1))\times A$ from $SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N_1$ and replace it by A to get a torus manifold with G-action M such that $M^{SO(2l_1+1)} = A$. The blow up of M along A is $SO(2l_1+1)/S(O(2l_1)) \times_{\mathbb{Z}_2} N_1$.

Proof. The diagonal $\mathbb{Z}_2$-action on $SO(2l_1+1)/SO(2l_1) \times N_1$ is orientation preserving. Therefore $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ is orientable.

The normal bundle of $SO(2l_1+1)/S(O(2l_1)\times O(1))\times A$ in $SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N_1$ is given by $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N(A, N_1)$.

Consider the following commutative diagram

$$\begin{array}{ccc} SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N(A,N_1) & \xrightarrow{f} & P_{\mathbb{R}}(E'\otimes W) \times E' \otimes W \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ SO(2l_1+1)/S(O(2l_1)\times O(1)) \times A & \xrightarrow{g} & P_{\mathbb{R}}(E'\otimes W) \end{array}$$

where the vertical maps are the natural projections and f, g are given by

$$f([hSO(2l_1) : m]) = ([m \otimes he_1], m \otimes he_1)$$

and

$$g(hS(O(2l_1) \times O(1)), q) = [m_q \otimes he_1],$$

where $e_1 \in W - \{0\}$ is fixed such that for all $g' \in SO(2l_1)$, $g'e_1 = e_1$ and $m_q \neq 0$ some element of the fiber of E' over q.

The map f induces an isomorphism of the normal bundle of $SO(2l_1)/S(O(2l_1) \times O(1)) \times A$ in $SO(2l_1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ and the tautological bundle over $P_{\mathbb{R}}(E' \otimes W \oplus \{0\})$. ☐

Lemma 7.5. *If* $l_1 > 1$*, in the situation of Lemma 7.1, then* $\bigcap_{M_i \in \mathfrak{F}_1} M_i = M^{SO(2l_1)}$ *has at most two components. It has two components if and only if* $M = S^{2l_1} \times N_1$*.*

Proof. If $M = S^{2l_1} \times N_1$, then $\bigcap_{M_i \in \mathfrak{F}_1} M_i = \{N, S\} \times N_1$, where N, S are the north and the south pole of the sphere, respectively. Otherwise the blow up of M along $M^{SO(2l_1+1)}$ is given by $S^{2l_1} \times_{\mathbb{Z}_2} N_1$, which is a fiber bundle over $\mathbb{R}P^{2l_1}$. The characteristic submanifolds of $S^{2l_1} \times_{\mathbb{Z}_2} N_1$, which are permuted by $W(G_1)$, are given by the preimages of the following submanifolds of $\mathbb{R}P^{2l_1}$:

$$\mathbb{R}P_i^{2l_1-2} = \{(x_1 : x_2 : \dots : x_{2i-2} : 0 : 0 : x_{2i+1} : \dots : x_{2l_1+1}) \in \mathbb{R}P^{2l_1}\}, \quad i = 1, \dots, l_1.$$

These characteristic submanifolds are also given by the proper transforms $\tilde{M}_i$ of the characteristic submanifolds $M_i \in \mathfrak{F}_1$ of M. Because

$$\bigcap_{i=1}^{l_1} \mathbb{R}P_i^{2l_1-2} = \{(0 : 0 : \dots : 0 : 1)\},$$

it follows that

$$\bigcap_{M_i \in \mathfrak{F}_1} \tilde{M}_i = \tilde{N}_1 = \tilde{M}^{SO(2l_1)}.$$

Therefore, with Lemma 4.3 and Corollary 7.3,

$$\bigcap_{M_i\in\mathfrak{F}_1} M_i = N_1 = M^{SO(2l_1)}$$

follows. In particular $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected. □

Lemma 7.6. *If $l_1 = 1$, in the situation of Lemma 7.1, then the following statements are equivalent:*

- *$M^{SO(2)}$ has two components.*
- *$\#\mathfrak{F}_1 = 2$.*
- *$M = S^2 \times N_1$.*

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then $M^{SO(2)}$ is connected.

Proof. At first we prove that all components of $M^{SO(2)}$ are characteristic submanifolds of M belonging to $\mathfrak{F}_1$. By Lemma 7.1, N_1 is a characteristic submanifold of M and a component of $M^{SO(2)}$ such that $G_1N_1 = M$. Therefore, if $x \in M^{SO(2)}$, then there is a $g \in N_{G_1}SO(2)$ such that $g^{-1}x \in N_1$. This implies $x \in gN_1$. Because gN_1 is a characteristic submanifold belonging to $\mathfrak{F}_1$ and a component of $M^{SO(2)}$ it follows that $M^{SO(2)}$ is a union of characteristic submanifolds of M belonging to $\mathfrak{F}_1$.

Now assume that $\#\mathfrak{F}_1 = 1$. Then we have $M^{SO(2)} = N_1$. Therefore $M^{SO(2)}$ is connected.

Now assume that $M = SO(3)/SO(2) \times N_1$. Then it is clear that $M^{SO(2)}$ has two components.

Now assume that $M^{SO(2)}$ has two components. Because these components are characteristic submanifolds belonging to $\mathfrak{F}_1$ it follows that $\#\mathfrak{F}_1 = 2$.

Now assume that $\#\mathfrak{F}_1 = 2$. If there is no G_1-fixed point then it follows from Corollary 7.2 that $M = SO(3)/SO(2) \times N_1$. Assume that there is a G_1-fixed point in M. Then the blow up of M along M^{G_1} contains an orbit of type $SO(3)/S(O(2)\times O(1))$. Now Corollary 7.2 implies $\#\mathfrak{F}_1 = 1$. Therefore there is no G_1-fixed point if $\#\mathfrak{F}_1 = 2$. □

7.3. **Admissible pairs.** We are now in the position to state another classification theorem. To do so, we use the following definition.

Definition 7.7. Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$. Then a pair (N, A) with

- N a torus manifold with $G_2\times\mathbb{Z}_2$-action such that the $\mathbb{Z}_2$-action is orientation-reversing or trivial,
- $A \subset N$ the empty set or a closed $G_2 \times \mathbb{Z}_2$-invariant submanifold of codimension one, on which $\mathbb{Z}_2$ acts trivially, such that if $A \neq \emptyset$, then $\mathbb{Z}_2$ acts non-trivially on N,

is called *admissible for* $(\tilde{G}, G_1)$.

We say that two admissible pairs (N, A), (N', A') are equivalent if there is a $G_2 \times \mathbb{Z}_2$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$.

Theorem 7.8. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$. There is a one-to-one correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-actions such that G_1 is elementary and equivalence classes of admissible pairs for $(\tilde{G}, G_1)$.*

Proof. Let M be a torus manifold with $\tilde{G}$-action. If $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ has two components and $l_1 > 1$ or $\#\mathfrak{F}_1 = 2$ and $l_1 = 1$, then we assign to M the admissible pair $\Phi(M) = (N_1, \emptyset)$, where N_1 is a component of $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ or a characteristic submanifold belonging to $\mathfrak{F}_1$ in the case $l_1 = 1$. The action of $\mathbb{Z}_2$ is trivial in this case.

If $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected and $l_1>1$ or $\#\mathfrak{F}_1=1$ and $l_1=1$, then we assign to M the pair

$$\Phi(M)=\left(\bigcap_{M_i\in\mathfrak{F}_1} M_i, M^{SO(2l_1+1)}\right).$$

Because $\bigcap_{M_i\in\mathfrak{F}_1} M_i = M^{SO(2l_1)}$ there is a non-trivial action of

$$\mathbb{Z}_2 = S(O(2l_1)\times O(1))/SO(2l_1)$$

on $\bigcap_{M_i\in\mathfrak{F}_1} M_i$.

Now let (N,A) be a admissible pair for $(\tilde{G},G_1)$. If the $\mathbb{Z}_2$-action on N is trivial, we have $A=\emptyset$ and we assign to $(N,\emptyset)$ the torus manifold with $\tilde{G}$-action $\Psi((N,\emptyset)) = S^{2l_1}\times N$.

If the $\mathbb{Z}_2$-action on N is non-trivial, we assign to (N,A) the blow down $\Psi((N,A))$ of $SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N$ along $SO(2l_1+1)/S(O(2l_1)\times O(1))\times A$.

By Lemma 7.5, it is clear that this construction gives a one-to-one correspondence between torus manifolds with $\tilde{G}$-action such that $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ has two components and $l_1>1$ and admissible pairs with trivial $\mathbb{Z}_2$-action. With Lemma 7.6, we see that an analogous statement holds for $l_1=1$ and $\#\mathfrak{F}_1=2$.

Now let (N,A) be an admissible pair such that $\mathbb{Z}_2$ acts non-trivially on N. Then the discussion after Lemma 7.4 shows that $\Phi(\Psi((N,A)))$ is equivalent to (N,A).

If M is a torus manifold with $G_1\times G_2$-action such that G_1 is elementary and $N_1=\bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected the blow up of M along $M^{SO(2l_1+1)}$ is given by

$$SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N_1.$$

Therefore we find that $\Psi(\Phi(M))$ is equivariantly diffeomorphic to M. □

8. Classification

Here we use the results of the previous sections to state a classification of torus manifolds with G-action. We do not consider actions of groups, which have $SO(2l_1)$ as an elementary factor, because as explained in section 6 these factors may be replaced by $SU(l_1)\times S^1$. We get the classification by iterating the constructions given in Theorem 5.13 and Theorem 7.8.

We illustrate this iteration in the case that all elementary factors of G are isomorphic to $SU(l_i+1)$. Let $\tilde{G}=\prod_{i=1}^k G_i\times T^{l_0}$ and M a torus manifold with $\tilde{G}$-action such that all G_i are elementary and isomorphic to $SU(l_i+1)$.

In Theorem 5.13 we constructed a triple (ψ_1,N_1,A_1), which determines the $\tilde{G}$-equivariant diffeomorphism type of M. Here N_1 is a torus manifold with $\prod_{i=2}^k G_i\times T^{l_0}$-action. Therefore there is a triple (ψ_2,N_2,A_2) which determines the $\prod_{i=2}^k G_i\times T^{l_0}$-equivariant diffeomorphism type of N_1. Because $N_2\subset N_1$ such that $G_2N_2=N_1$ and A_1 is G_2-invariant we have $G_2(A_1\cap N_2)=A_1$. Therefore the G-equivariant diffeomorphism type of M is determined by

$$(\psi_1\times\psi_2, N_2, A_1\cap N_2, A_2).$$

Continuing in this manner leads to a triple

$$(\psi, N, (A_1,\dots,A_k)),$$

where $\psi\in \operatorname{Hom}\left(\prod_{i=1}^k S(U(l_i)\times U(1)), T^{l_0}\right)$, N is a $2l_0$-dimensional torus manifold and the A_i are codimension two submanifolds of N or empty.

The iteration becomes more complicated if there are more than one elementary factors of $\tilde{G}$ isomorphic to $SO(2l_i+1)$. To illustrate what happens here, we discuss the case $\tilde{G}=G_1\times G_2\times T^{l_0}$, where the G_i are elementary and isomorphic to $SO(2l_i+1)$.

Then, by Theorem 7.8, there is an admissible pair (N_1, B_1) for $(\tilde{G}, G_1)$ corresponding to M, where N_1 is a torus manifold with $G_2 \times T^{l_0} \times (\mathbb{Z}_2)_1$-action. By Lemmas 7.5 and 7.6, we have two cases

(1) $N_1^{SO(2l_2)}$ has two components.
(2) $N_1^{SO(2l_2)}$ is connected.

In the first case we have

$$N_1 = SO(2l_2 + 1)/SO(2l_2) \times N_2,$$

where N_2 is a $2l_0$-dimensional torus manifold. The action of $(\mathbb{Z}_2)_1$ on N_1 commutes with the action of $G_2 \times T^{l_0}$. Therefore the action of $(\mathbb{Z}_2)_1$ on N_1 splits as an product of an action on $SO(2l_2 + 1)/SO(2l_2)$ and an action on N_2. Because there is only one non-trivial action of $\mathbb{Z}_2$ on $SO(2l_2 + 1)/SO(2l_2)$ which commutes with the action of $SO(2l_2 + 1)$, the $G_2 \times T^{l_0} \times (\mathbb{Z}_2)_1$-equivariant diffeomorphism type of N_1 is completely determined by a pair (N_2, a_{12}), where N_2 is equipped with the action of $T^{l_0} \times (\mathbb{Z}_2)_1$ and $a_{12} \in \{0, 1\}$ is non-zero if and only if the $(\mathbb{Z}_2)_1$-action on $SO(2l_2 + 1)/SO(2l_2)$ is non-trivial.

In the second case the $G_2 \times T^{l_0}$-equivariant diffeomorphism type of N_1 is determined by a pair (N_2, B_2), where $N_2 = N_1^{SO(2l_2)}$. Because N_2 is $(\mathbb{Z}_2)_1$-invariant in this case, N_2 is a torus manifold with $T^{l_0} \times (\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$-action, where $(\mathbb{Z}_2)_2 = S(O(2l_2) \times O(1))/SO(2l_2)$. We put $a_{12} = 0$ in this case.

As in the case where there are only elementary factors isomorphic to $SU(l_i + 1)$, one sees that the $G_1 \times G_2 \times T^{l_0}$-equivariant diffeomorphism type of M is determined by

$$(N_2, (N_2 \cap B_1, B_2), a_{12}).$$

There are some relations between a_{12} and B_1. For example, if $a_{12} = 1$, then there are no $(\mathbb{Z}_2)_1$-fixed points in N_1. Therefore B_1 has to be empty.

If there are more than two elementary factors of $\tilde{G}$ isomorphic to $SO(2l_i + 1)$, we have to introduce more numbers a_{ij}. There are some relations between the a_{ij} coming from the fact that M is required to be orientable. This will be explained in the proof of Lemma 8.3.

8.1. **Admissible 5-tuples.** We use the following definition to make the above constructions more formal.

Definition 8.1. Let $\tilde{G} = \prod_{i=1}^k G_i \times G'$ with

$$G_i = \begin{cases} SU(l_i + 1) & \text{if } i \leq k_0 \\ SO(2l_i + 1) & \text{if } i > k_0 \end{cases}$$

and $k_0 \in \{0, \dots, k\}$. Then a 5-tuple

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k}, (a_{ij})_{k_0+1 \leq i < j \leq k})$$

with

(1) $\psi \in \operatorname{Hom}(\prod_{i=1}^{k_0} S(U(l_i) \times U(1)), Z(G'))$ and $\psi_i = \psi|_{S(U(l_i)\times U(1))}$,
(2) N a torus manifold with $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-action,
(3) $A_i \subset N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant closed submanifold of codimension two, on which $\operatorname{im} \psi_i$ acts trivially, such that if $A_i \neq \emptyset$, then $\ker \psi_i = SU(l_i)$,
(4) $B_i \subset N$ the empty set or a $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant closed submanifold of codimension one, on which $(\mathbb{Z}_2)_i$ acts trivially, such that if $B_i \neq \emptyset$, then the action of $(\mathbb{Z}_2)_i$ on N is non-trivial,
(5) $a_{ij} \in \{0, 1\}$ such that
    (a) if $a_{ij} = 1$, then

(i) the action of $(\mathbb{Z}_2)_j$ on N is trivial,
(ii) $a_{jk} = 0$ for $k > j$,
(iii) $B_i = \emptyset$,

(b) if the action of $(\mathbb{Z}_2)_i$ on N is non-trivial, then it is orientation preserving if and only if $\sum_{j>i} a_{ij}$ is odd,
(c) if the action of $(\mathbb{Z}_2)_i$ on N is trivial, then $\sum_{j>i} a_{ij}$ is odd or zero,

is called *admissible for* $(\tilde{G}, \prod_{i=1}^k G_i)$ if the A_i and B_i intersect pairwise transversely.

If G' is a torus we also say that a 5-tuple is admissible for $\tilde{G}$ instead of $(\tilde{G}, \prod_{i=1}^k G_i)$.

We say that two admissible 5-tuples

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k}, (a_{ij})_{k_0+1\leq i<j\leq k})$$

and

$$(\psi', N', (A'_i)_{i=1,\dots,k_0}, (B'_i)_{i=k_0+1,\dots,k}, (a'_{ij})_{k_0+1\leq i<j\leq k})$$

are equivalent if

- $\psi_i = \psi'_i$ if $l_i > 1$ and $\psi_i = \psi_i'^{\pm 1}$ if $l_i = 1$,
- $a_{ij} = a'_{ij}$,
- there is a $G' \times \prod_{i=k_0+1}^k (\mathbb{Z}_2)_i$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A_i) = A'_i$ and $\phi(B_i) = B'_i$.

Remark 8.2. By Lemma B.1, two submanifolds A_1, A_2 of N satisfying the condition (3) intersect transversely if and only if no component of A_1 is a component of A_2.

By Lemma B.4, two submanifolds A_1, B_1 of N satisfying the conditions (3) and (4), respectively, intersect always transversely.

By Lemma B.5, two submanifolds B_1, B_2 of N satisfying the condition (4) intersect transversely if and only if no component of B_1 is a component of B_2.

Lemma 8.3. *Let $\tilde{G}$ as above. Then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples*

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k}, (a_{ij})_{k_0+1\leq i<j\leq k})$$

for $(\tilde{G}, \prod_{i=1}^k G_i)$ and the equivalence classes of admissible 5-tuples

$$(\psi', N', (A'_i)_{i=1,\dots,k_0}, (B'_i)_{i=k_0+1,\dots,k-1}, (a'_{ij})_{k_0+1\leq i<j\leq k-1})$$

for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is elementary for the $G_k \times G'$-action on N'.

Proof. At first assume that $G_k = SU(l_k + 1)$. Let $(\psi, N, (A_i)_{i=1,\dots,k-1}, \emptyset, \emptyset)$ be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is elementary for the $G_k \times G'$-action on N.

Let (ψ_k, N_k, A_k) be the admissible triple for $(G_k \times G', G_k)$, which corresponds to N under the correspondence given in Theorem 5.13. Then N_k is a submanifold of N. By Lemma B.1, A_i, $i = 1, \dots, k-1$, intersects N_k transversely. Therefore $N_k \cap A_i$ has codimension 2 in N_k. Because $A_i = G_k(N_k \cap A_i)$, $N_k \cap A_i$ has no component, which is contained in A_k or $N_k \cap A_j$, $j \neq i$. Therefore by

$$(\psi \times \psi_k, N_k, (A_1 \cap N_k, \dots, A_{k-1} \cap N_k, A_k), \emptyset, \emptyset)$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$ is given.

Now let

$$(\psi \times \psi_k, N_k, (A_1, \dots, A_k), \emptyset, \emptyset)$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. Let $H_0 = G_k \times \operatorname{im} \psi_k$ and $H_1 = S(U(l_k) \times U(1)) \times \operatorname{im} \psi_k$. Then, by Lemma 5.11, the blow down N of $\tilde{N} = H_0 \times_{H_1} N_k$ along $H_0/H_1 \times A_k$ is a torus manifold with $G_k \times G'$-action. By Lemma 4.3, $F(H_0 \times_{H_1} A_i) = G_k F(A_i)$, $i < k$, are submanifolds of N satisfying the condition (3) of Definition 8.1.

Because $F(A_i)$ and $F(A_j)$, $i < j < k$, have no components in common, $G_kF(A_i)$ and $G_kF(A_j)$ intersect transversely. Therefore by

$$(\psi, N, (G_kF(A_1), \dots, G_kF(A_{k-1})), \emptyset, \emptyset)$$

an admissible triple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem 5.13 one sees that this construction leads to a one-to-one-correspondence.

Now assume that $G_k = SO(2l_k + 1)$. Let

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k-1}, (a_{ij})_{k_0+1\leq i<j\leq k-1}) \tag{8.1}$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is elementary for the $G_k \times G'$-action on N.

At first assume that, for the G_k-action on N, $N^{SO(2l_k)}$ is connected. Let (N_k, B_k) be the admissible pair for $(G_k \times G', G_k)$ which corresponds to N under the correspondence given in Theorem 7.8. Then N_k is a submanifold of N which is invariant under the action of $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$, where $(\mathbb{Z}_2)_k = S(O(2l_k) \times O(1))/SO(2l_k)$. For $i < k$, let $a_{ik} = 0$.

We claim that by

$$(\psi, N_k, (A_1 \cap N_k, \dots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \dots, B_{k-1} \cap N_k, B_k), (a_{ij})) \tag{8.2}$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ is given.

At first note that, for $i = 1, \dots, k-1$, the A_i and B_i intersect N_k transversely by Lemmas B.1 and B.4. Therefore $A_i \cap N_k$ and $B_i \cap N_k$ has codimension two or one, respectively, in N_k.

One sees as in the case $G_k = SU(l_k + 1)$ that the $N_k \cap A_i$ and $N_k \cap B_i$ intersect pairwise transversely.

Now we verify the condition (5) of Definition 8.1 for the 5-tuple (8.2). By Lemma 6.4, $(\mathbb{Z}_2)_i$, $i < k$, acts orientation preserving on N if and only if it acts orientation preserving on N_k. This proves (5b) because (8.1) is an admissible 5-tuple and $a_{ik} = 0$.

Because, by Lemma 7.1, $G_kN_k = N$, $(\mathbb{Z}_2)_i$, $i < k$, acts trivially on N_k if and only if it acts trivially on N. This proves (5c) and (5(a)i) because (5c) and (5(a)i) hold for the admissible 5-tuple (8.1) and $a_{ik} = 0$.

Because $a_{ik} = 0$, (5(a)ii) and (5(a)iii) are clear.

Now assume that $N^{SO(2l_k)}$ is non-connected. Then, by Lemmas 7.5 and 7.6, we have

$$N = SO(2l_k + 1)/SO(2l_k) \times N_k.$$

In this case the $(\mathbb{Z}_2)_i$-action, $i < k$, on N commutes with the action of $SO(2l_k+1)$. Therefore it splits in a product of an action on $SO(2l_k+1)/SO(2l_k)$ and an action on N_k. We put $a_{ik} = 1$ if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k + 1)/SO(2l_k)$ is non-trivial and $a_{ik} = 0$ otherwise. Because there is only one non-trivial action of $\mathbb{Z}_2$ on $SO(2l_k + 1)/SO(2l_k)$, which commutes with the action of $SO(2l_k + 1)$, we may recover the action of $(\mathbb{Z}_2)_i$ on N from the action on N_k and a_{ik}.

We identify $SO(2l_k)/SO(2l_k) \times N_k$ with N_k and equip it with the trivial action of $(\mathbb{Z}_2)_k = S(O(2l_k) \times O(1))/SO(2l_k)$. We claim that by

$$(\psi, N_k, (A_1 \cap N_k, \dots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \dots, B_{k-1} \cap N_k, \emptyset), (a_{ij})) \tag{8.3}$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ is given.

The conditions (3) and (4) of Definition 8.1 and the transversality condition are verified as in the previous cases.

Therefore we only have to verify condition (5). Because the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k + 1)/SO(2l_k)$ is orientation reversing, the $(\mathbb{Z}_2)_i$-action, $i < k$ on N_k has the same orientation behavior as the action on N if and only if the $(\mathbb{Z}_2)_i$-action on

$SO(2l_k+1)/SO(2l_k)$ is trivial. By the definition of a_{ik}, this is the case if and only if $a_{ik}=0$. Therefore (5b) follows because (8.1) is an admissible 5-tuple and $(\mathbb{Z}_2)_k$ acts trivially on N_k.

If the $(\mathbb{Z}_2)_i$-action on N_k is trivial and non-trivial on $SO(2l_k+1)/SO(2l_k)$, then the $(\mathbb{Z}_2)_i$-action on N is orientation reversing. Therefore $\sum_{j>i} a_{ij}$ is odd.

The $(\mathbb{Z}_2)_i$-actions on N_k and $SO(2l_k+1)/SO(2l_k)$ are trivial if and only if the $(\mathbb{Z}_2)_i$-action on N is trivial. Therefore $\sum_{j>i} a_{ij}$ is odd or trivial. This verifies (5c).

If there is a $j<i$ such that $a_{ji}=1$, then $(\mathbb{Z}_2)_i$ acts trivially on N because the admissible 5-tuple (8.1) satisfies (5(a)i). Therefore $a_{ik}=0$. This proves (5(a)ii).

If the $(\mathbb{Z}_2)_i$-action on $SO(2l_k+1)/SO(2l_k)$ is non-trivial the action on N has no fixed points. Therefore $B_i=\emptyset$. This proves (5(a)iii). The property (5(a)i) is clear.

Now let

$$(\psi, N_k, (A_1,\dots,A_{k_0}), (B_{k_0+1},\dots,B_k), (a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^k G_i)$. At first assume that $(\mathbb{Z}_2)_k$ acts non-trivially on N_k. Then the blow down N of $\tilde{N} = SO(2l_k+1)/SO(2l_k)\times_{(\mathbb{Z}_2)_k} N_k$ along $SO(2l_k+1)/SO(2l_k)\times_{(\mathbb{Z}_2)_k} B_k$ is a torus manifold with $G_k\times G'\times\prod_{i=k_0+1}^{k-1}(\mathbb{Z}_2)_i$-action. As in the case $G_k=SU(l_k+1)$ one sees that

$$(\psi, N, (G_kF(A_1),\dots,G_kF(A_{k_0})), (G_kF(B_{k_0+1}),\dots,G_{k-1}F(B_{k-1})), (a_{ij}))$$

is an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$.

If $(\mathbb{Z}_2)_k$ acts trivially on N_k, then put

$$N = SO(2l_k+1)/SO(2l_k)\times N_k.$$

Here $(\mathbb{Z}_2)_i$, $i<k$, acts by the product action of the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k+1)/SO(2l_k)$ and the action on N_k if $a_{ik}=1$. Otherwise $(\mathbb{Z}_2)_i$ acts by the product action of the trivial action on $SO(2l_k+1)/SO(2l_k)$ and the action on N_k. Now by

$$\begin{aligned}(\psi, N, &(SO(2l_k+1)/SO(2l_k)\times A_1),\dots,SO(2l_k+1)/SO(2l_k)\times A_{k_0}),\\ &(SO(2l_k+1)/SO(2l_k)\times B_{k_0+1},\dots,SO(2l_k+1)/SO(2l_k)\times B_{k-1}), (a_{ij}))\end{aligned}$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem 7.8 one sees that this construction leads to a one-to-one-correspondence. □

Let $\tilde{G} = \prod_i G_i \times T^{l_0}$ and

$$(\psi, M, (A_i), (B_i), (a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is an elementary factor of $\prod_{i\geq k} G_i\times T^{l_0}$ for the action on M. Furthermore, let

$$(\psi', N, (A_i'), (B_i'), (a_{ij}'))$$

be the admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ corresponding to $(\psi, M, (A_i), (B_i), (a_{ij}))$. Then the following lemma shows that G_i, $i>k$, is an elementary factor of $\prod_{i\geq k} G_i\times T^{l_0}$ for the action on M if and only if it is an elementary factor of $\prod_{i\geq k+1} G_i\times T^{l_0}$ for the action on N.

Lemma 8.4. *Let $\tilde{G} = G_1\times G'\times G''$, M a torus manifold with $\tilde{G}$-action and N a component of an intersection of characteristic submanifolds of M, which is $G_1\times G'$-invariant and contains a T-fixed point x such that G_1 acts non-trivially on N. Furthermore, assume that G'' is a product of elementary factors for the action on M.*

Then N is a torus manifold with $G_1 \times G' \times T^{l_0}$-action for some $l_0 \geq 0$ and G_1 is an elementary factor of $\tilde{G}$, with respect to the action on M, if and only if it is an elementary factor of $G_1 \times G' \times T^{l_0}$, with respect to the action on N.

Proof. Assume that G_1 is an elementary factor for one of the two actions on M and N. Then G_1 is isomorphic to a simple group or Spin(4). If G_1 is simple and not isomorphic to $SU(2)$ then the statement is clear.

Therefore there are two cases $G_1 = SU(2), \mathrm{Spin}(4)$.

If x is not fixed by G_1, then $G_1 = SU(2)$ is elementary for both actions on N and M by Lemma 3.1. Therefore we may assume that $x \in N^{G_1} \subset M^{G_1}$. Then there is a bijection

$$\mathfrak{F}_{xM} \rightarrow \mathfrak{F}_{xN} \amalg \mathfrak{F}_N^{\perp},$$

where

$$\mathfrak{F}_{xM} = \{\text{characteristic submanifolds of } M \text{ containing } x\},$$
$$\mathfrak{F}_{xN} = \{\text{characteristic submanifolds of } N \text{ containing } x\},$$
$$\mathfrak{F}_N^{\perp} = \{\text{characteristic submanifolds of } M \text{ containing } N\}.$$

This bijection is compatible with the actions of the Weyl-group of G_x.

At first assume that $G_1 = SU(2)$ is elementary for the action on M but not for the action on N. Then there is another simple factor $G_2 = SU(2)$ of $G_1 \times G' \times T^{l_0}$ such that $G_1 \times G_2$ is elementary for the action on N. At first assume that G_2 is elementary for the action on M.

Let $w_i \in W(G_i)$, $i = 1, 2$, be generators. Then there are two non-trivial $W(G_1 \times G_2)$-orbits $\mathfrak{F}_1, \mathfrak{F}_2$ in $\mathfrak{F}_{xM}$. We have:

- $\#\mathfrak{F}_i = 2$, $i = 1, 2$,
- w_i, $i = 1, 2$, acts non-trivially on $\mathfrak{F}_i$ and trivially on the other orbit.

But because, $G_1 \times G_2$ is elementary for the action on N, there is exactly one non-trivial $W(G_1 \times G_2)$-orbit $\mathfrak{F}_1'$ in $\mathfrak{F}_{xN}$. We have:

- $\#\mathfrak{F}_1' = 2$,
- w_i, $i = 1, 2$, acts non-trivially on $\mathfrak{F}_1'$.

This is a contradiction.

If G_2 is not elementary, then G_2 is a simple factor of an elementary factor. In this case the action of $W(G_1 \times G_2)$ on $\mathfrak{F}_{xM}$ behaves as in the first case. Therefore we also get a contradiction in this case.

Under the assumption that $G_1 = \mathrm{Spin}(4)$ is elementary for the action on M a similar argument shows that G_1 is elementary for the action on N.

Therefore G_1 is elementary for the action on N if it is elementary for the action on M.

If G_1 is elementary for the action on N but not elementary for the action on M, then it is a simple factor of an elementary factor $G_1' \neq G_1$ of $\tilde{G}$ or a product $G_2' \times G_3'$ of elementary factors G_2' and G_3' of $\tilde{G}$. But because G'' is a product of elementary factors, it contains all elementary factors of $\tilde{G}$ which have non-trivial intersection with G''. Because G_1 is not contained in G'', it follows that G_1', G_2' and G_3' are subgroups of $G_1 \times G'$. Therefore, by the above argument, G_1' or G_2' and G_3' are elementary for the action on N. Because elementary factors can not contain each other we get a contradiction to the assumption that G_1 is elementary for the action on N. □

Recall from section 3 that if M is a torus manifold with G-action, then we may assume that all elementary factors of G are isomorphic to $SU(l_i + 1)$, $SO(2l_i + 1)$ or $SO(2l_i)$. That means $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^{l_0}$. Because, as described in section 6, we may replace elementary factors isomorphic to

$SO(2l_i)$ by $SU(l_i)\times S^1$, the following theorem may be used to construct invariants of torus manifolds with $\tilde{G}$-action. By Theorem 6.3 these invariants determine the $\tilde{G}$-equivariant diffeomorphism type of simply connected torus manifolds with $\tilde{G}$-action.

Theorem 8.5. *Let* $\tilde{G}=\prod_{i=1}^k G_i\times T^{l_0}$ *with*

$$G_i=\begin{cases} SU(l_i+1) & \text{if } i\leq k_0\\ SO(2l_i+1) & \text{if } i>k_0\end{cases}$$

and $k_0\in\{0,\dots,k\}$. *Then there is a one-to-one correspondence between the equivalence classes of admissible* 5*-tuples for* $\tilde{G}$ *and the* $\tilde{G}$*-equivariant diffeomorphism classes of torus manifolds with* $\tilde{G}$*-action such that all* G_i *are elementary.*

Proof. This follows from Lemma 8.3 and Lemma 8.4 by induction. □

Using Lemma 2.8 and Theorem 5.16 we get the following result for quasitoric manifolds.

Theorem 8.6. *Let* $\tilde{G}=\prod_{i=1}^k G_i\times T^{l_0}$ *with* $G_i=SU(l_i+1)$. *Then there is a one-to-one correspondence between the equivalence classes of admissible* 5*-tuples for* $\tilde{G}$ *of the form*

$$(\psi,N,(A_i)_{1\leq i\leq k},\emptyset,\emptyset)$$

with N *quasitoric and* A_i*,* $1\leq i\leq k$*, connected and the* $\tilde{G}$*-equivariant diffeomorphism classes of quasitoric manifolds with* $\tilde{G}$*-action.*

Remark 8.7. Remark 2.9 and Theorem 5.15 lead to a similar result for torus manifolds with G-actions whose cohomologies are generated by their degree two parts.

Corollary 8.8. *Let* $\tilde{G}=\prod_{i=1}^{k_1} G_i\times T^{l_0}$ *with* G_i *elementary and* M *a torus manifold with* G*-action. Then* M/G *has dimension* $l_0+\#\{G_i;\ G_i=SO(2l_i)\}$.

Proof. At first we discuss the case, where all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i+1)$ or $SU(l_i+1)$, i.e. $\#\{G_i;\ G_i=SO(2l_i)\}=0$. By Lemma 4.7, replacing M by the blow up $\tilde{M}$ of M along the fixed points of G_1 does not change the orbit space. Therefore, by Corollaries 5.6 and 7.2, we have up to finite coverings

$$\begin{aligned}M/G&=(M/G_1)/(\prod_{i\geq 2}G_i\times T^{l_0})=(\tilde{M}/G_1)/(\prod_{i\geq 2}G_i\times T^{l_0})\\&=((H_0\times_{H_1}N_1)/G_1)/(\prod_{i\geq 2}G_i\times T^{l_0})=N_1/(\prod_{i\geq 2}G_i\times T^{l_0}),\end{aligned}$$

where N_1 is the $\prod_{i\geq 2}G_i\times T^{l_0}$-manifold from the admissible 5-tuple for $(\tilde{G},G_1)$ corresponding to M. Here H_0,H_1 are defined as in Lemma 5.3 if $G_1=SU(l_1+1)$. If $G_1=SO(2l_1+1)$, we have $H_0=SO(2l_1+1)$ and $H_1=S(O(2l_1)\times O(1))$.

By iterating this argument we find that $M/G=N/T^{l_0}$ up to finite coverings, where N is the T^{l_0}-manifold from the admissible 5-tuple for $\tilde{G}$ corresponding to M.

Now we study the case $l_0'=\#\{G_i;\ G_i=SO(2l_i)\}\neq 0$. As discussed in section 6, the orbits of the G-action on M do not change if we replace an elementary factor isomorphic to $SO(2l_i)$ by $SU(l_i)\times S^1$. Therefore this replacement does not change the dimension of the orbit space. But it increases l_0 by one, and decreases l_0' by one. Therefore the statement follows by induction on l_0'. □

8.2. **Applications.** Now we apply our classification results to special cases. We first discuss the case, where M is a torus manifold with G-action such that G is semi-simple and $H^*(M;\mathbb{Z})$ is generated by its degree two part.

Corollary 8.9. *If G is semi-simple and M is a torus manifold with G-action such that $H^*(M;\mathbb{Z})$ is generated by its degree two part, then*

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i+1)$$

and

$$M = \prod_{i=1}^{k} \mathbb{C}P^{l_i},$$

where each $SU(l_i+1)$ acts in the usual way on $\mathbb{C}P^{l_i}$ and trivially on $\mathbb{C}P^{l_j}$, $j \neq i$.

Proof. By Lemma 2.8 and Remark 2.9, all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i+1)$. Because G is semi-simple, there is only one admissible 5-tuple for $\tilde{G}$, namely $(\text{const}, \text{pt}, \emptyset, \emptyset, \emptyset)$. It corresponds to a product of complex projective spaces. □

Next we discuss torus manifolds M with G-action such that $\dim M/G \leq 1$. With Theorem 8.5, we recover the following two results of S. Kuroki [13, 11]:

Corollary 8.10. *Let M be a simply connected torus manifold with G-action such that M is a homogeneous G-manifold. Then M is a product of even-dimensional spheres and complex projective spaces.*

Proof. By Corollary 8.8, the center of G is zero-dimensional. Moreover, all elementary factors of G are isomorphic to $SU(l_i+1)$ or $SO(2l_i+1)$. Therefore the admissible 5-tuple corresponding to M is given by

$$(\text{const}, \text{pt}, \emptyset, \emptyset, (a_{ij})),$$

where the $a_{ij} \in \{0,1\}$ are unknown. In particular, no elementary factor of G has a fixed point in M. Therefore, by Corollaries 5.6 and 7.2, M splits into a direct product of complex projective spaces and even dimensional spheres. □

Corollary 8.11. *If the G-action on the simply connected torus manifold M has an orbit of codimension one, then M is the projectivication of a complex vector bundle or a sphere bundle over a product of complex projective spaces and even-dimensional spheres.*

Proof. By Corollary 8.8, we may assume that there is a covering group $\tilde{G} = S^1 \times \prod_i G_i$ of G with G_i elementary and $G_i = SU(l_i+1)$ or $G_i = SO(2l_i+1)$. We assume that the G_i are sorted in such a way that

- $G_i = SO(2l_i+1)$ and G_i has no fixed point in M if $i \leq k_0$,
- $G_i = SU(l_i+1)$ and G_i has no fixed point in M if $k_0+1 \leq i \leq k_1$,
- $G_i = SU(l_i+1), SO(2l_i+1)$ and G_i has fixed points in M if $i \geq k_1+1$,

where $k_0 \leq k_1$ are some constants.

By Corollaries 5.6 and 7.2, we know that M is of the form

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times H_{0k_0+1} \times_{H_1k_0+1} \left(H_{0k_0+2} \times_{H_{1k_0+2}} \left(\dots \left(H_{0k_1} \times_{H_{1k_1}} M'\right) \dots\right)\right),$$

where

$$H_{0i} = SU(l_i+1) \times \operatorname{im}\psi_i,$$
$$H_{1i} = S(U(l_i+1) \times U(1)) \times \operatorname{im}\psi_i,$$

for $i = k_0 + 1, \dots, k_1$, and M' is a torus manifold with $\tilde{G}'$-action, where $\tilde{G}' = \prod_{i \geq k_1+1} G_i \times S^1$.

Because the action of H_{1i} on H_{0j}, $j > i$, is trivial and the actions of the H_{1i} on M' commute, M may be written as

$$M = \prod_{i=1}^{k_0} S^{2l_i} \times \left(\prod_{i=k_0+1}^{k_1} H_{0i} \times_{\prod H_{1i}} M' \right).$$

Therefore M is a fiber bundle over a product of even dimensional spheres and complex projective spaces with fiber M'.

Let $(\psi, N', (A_i), (B_i), (a_{ij}))$ be the admissible 5-tuple for $\tilde{G}'$ corresponding to M'. Because $\dim N' = 2$ and all G_i, $i > k_1$, have fixed points in M', we have

$$N' = S^2, \qquad A_i \neq \emptyset, \qquad B_i \neq \emptyset.$$

Because the S^1-action on S^2 has only two fixed points, N and S, there are at most two elementary factors isomorphic to $SU(l_i + 1)$. The orientation reversing involutions of S^2 which commute with the S^1-action and have fixed points are given by "reflections" at S^1-orbits. Therefore there is at most one elementary factor isomorphic to $SO(2l_i + 1)$. If there is such a factor then there is at most one G_i isomorphic to $SU(l_i + 1)$ because N is mapped to S by such a reflection. Let

$$\phi_i : S(U(l_i) \times U(1)) \to U(1) \qquad \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix} \mapsto g \qquad (A \in U(l_i), g \in U(1)).$$

Then we have the following admissible 5-tuples:

$\tilde{G}'$	5-tuple	M'
S^1	$(\emptyset, S^2, \emptyset, \emptyset, \emptyset)$	S^2
$S^1 \times SU(l_1 + 1)$	$(\phi_1^{\pm 1}, S^2, \{N\}, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1+1}$
	$(\phi_1^{\pm 1}, S^2, \{N, S\}, \emptyset, \emptyset)$	S^{2l_1+2}
$S^1 \times SO(2l_1 + 1)$	$(\emptyset, S^2, \emptyset, S^1, \emptyset)$	S^{2l_1+2}
$S^1 \times SU(l_1 + 1) \times SU(l_2 + 1)$	$(\phi_1^{\pm 1}\phi_2^{\pm 1}, S^2, (\{N\}, \{S\}), \emptyset, \emptyset)$	$\mathbb{C}P^{l_1+l_2+1}$
$S^1 \times SU(l_1 + 1) \times SO(2l_2 + 1)$	$(\phi_1^{\pm 1}, S^2, \{N, S\}, S^1, \emptyset)$	$S^{2l_1+2l_2+2}$

Therefore the statement follows. ☐

Now we turn to the case, where M is a torus manifold with G-action such that G is semi-simple and has exactly two elementary factors G_1, G_2. We start with a discussion of the case, where $G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)$.

Corollary 8.12. *Let $\tilde{G} = G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)$ with G_1 and G_2 elementary of rank l_1, l_2, respectively, and M a torus manifold with G-action. Then M is one of the following:*

$$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \mathbb{C}P^{l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_2^{2l_2}, S^{2l_1+2l_2}.$$

Here S_1^l denotes the l-sphere together with the $\mathbb{Z}_2$-action generated by the antipodal map and S_2^l the l-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

Furthermore, the $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

Proof. First assume that $G_1, G_2 \neq SO(2l)$. Then we have the following possibilities for the admissible 5-tuple of M:

G_1	G_2	5-tuple	M
$SU(l_1+1)$	$SU(l_2+1)$	$(\text{const}, \text{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}$
$SU(l_1+1)$	$SO(2l_2+1)$	$(\text{const}, \text{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1} \times S^{2l_2}$
$SO(2l_1+1)$	$SO(2l_2+1)$	$(\emptyset, \text{pt}, \emptyset, \emptyset, a_{12}=0)$	$S^{2l_1} \times S^{2l_2}$
		$(\emptyset, \text{pt}, \emptyset, \emptyset, a_{12}=1)$	$S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}$

If $G_1 = SU(l_1+1)$ and $G_2 = SO(2l_2)$, then, by Corollary 3.6, there is one admissible triple for (G, G_1) namely $(\text{const}, S^{2l_2}, \emptyset)$. It corresponds to $\mathbb{C}P^{l_1} \times S^{2l_2}$.

Now assume that $G_1 = SO(2l_1+1)$ and $G_2 = SO(2l_2)$. Let (N, B) be the admissible pair for (G, G_1) corresponding to M. Then, by Corollary 3.6, we have $N = S^{2l_2}$. Up to equivariant diffeomorphism there are two orientation reversing involutions on S^{2l_2} which commute with the action of G_2, the anti-podal map and a reflection at an hyperplane in $\mathbb{R}^{2l_2+1}$. Therefore we have four possibilities for M:

$$S^{2l_1} \times S^{2l_2}, S^{2l_1+2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_2^{2l_2}.$$

□

For the discussion of the case $G_1 \times G_2 = SO(2l_1) \times SO(2l_2)$ we need the following lemma.

Lemma 8.13. *Let $\tilde{G} = SO(2l_1) \times S^1$ and M a simply connected torus manifold with G-action such that $SO(2l_1)$ is an elementary factor of $\tilde{G}$ and S^1 acts effectively on M and M^{S^1} has codimension two in M.*

Then M is equivariantly diffeomorphic to $\#_i(S^2 \times S^{2l_1})_i$ or S^{2l_1+2}.

Here the action of $\tilde{G}$ on S^{2l_1+2} is given by the restriction of the usual $SO(2l_1+3)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^2 \times S^{2l_1}$ is the product action of the usual action of S^1 and $SO(2l_1)$ on S^2 and S^{2l_1}, respectively. Moreover, the connected sums are equivariant.

Proof. As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S \times S^1$. Let (ψ, N, A) be the admissible triple corresponding to M. Then ψ is completely determined by the discussion in section 6 and $A = N^S = M^{SU(l_1)}$. Furthermore S and S^1 act effectively on N. All components of N^S and N^{S^1} have codimension two in N.

By Lemma 5.17, N is simply connected.

Denote by $\tilde{M}$ the blow up of M along A. Because all T-fixed points of M are contained in A we have $l_1 \# M^T = \# \tilde{M}^T$. On the other hand, $\tilde{M}$ is a fiber bundle with fiber N over $\mathbb{C}P^{l_1-1}$. Therefore we have $l_1 \# N^{S \times S^1} = \# \tilde{M}^T$.

From this $\# M^T = \# N^{S \times S^1}$ follows.

Because S and S^1 act both effectively on N such that their fixed point sets have codimension two, it follows from the classification of simply connected four-dimensional T^2-manifolds given in [20, p. 547,549] that the T-equivariant diffeomorphism type of N is determined by $\# M^T$ and that $\# M^T$ is even.

Therefore the $S \times S^1 \times SU(l_1)$-equivariant diffeomorphism type of M is uniquely determined by $\# M^T = \chi(M)$. It follows from Theorem 6.3 that the $SO(2l_1) \times S^1$-equivariant diffeomorphism type of M is uniquely determined by $\chi(M)$. Because

$$M_k = \begin{cases} \#_{i=1}^k (S^2 \times S^{2l_1})_i & \text{if } k \geq 1 \\ S^{2l_1+2} & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k+2$, the statement follows. □

Corollary 8.14. *Let $\tilde{G} = SO(2l_1) \times SO(2l_2)$ and M a simply connected torus manifold with G-action such that $SO(2l_1)$, $SO(2l_2)$ are elementary factors of $\tilde{G}$.*

Then M is equivariantly diffeomorphic to $\#_i(S^{2l_1} \times S^{2l_2})_i$ or $M = S^{2l_1+2l_2}$.

Here the action of $\tilde{G}$ on S^{2l_1+2} is given by the restriction of the usual $SO(2l_1 + 2l_2 + 1)$-action to $\tilde{G}$. The action of $\tilde{G}$ on $S^{2l_1} \times S^{2l_2}$ is the product action of the usual action of $SO(2l_1)$ and $SO(2l_2)$ on S^{2l_1} and S^{2l_2}, respectively. Moreover, the connected sums are equivariant.

Proof. As described in section 6, we may replace $\tilde{G}$ by $SU(l_1) \times S \times SO(2l_2)$. Let (ψ, N, A) be the admissible triple for $(SU(l_1) \times S \times SO(2l_2), SU(l_1))$ corresponding to M. Then ψ is completely determined by the discussion in section 6 and $A = N^S$. Furthermore, S acts effectively on N such that N^S has codimension two.

By Lemma 5.17, N is simply connected. Therefore, by Lemma 8.13, the equivariant diffeomorphism-type of N is uniquely determined by $\chi(N) \in 2\mathbb{Z}$. Because all other parts of the triple (ψ, N, A) are determined by the discussion in section 6 and the equivariant diffeomorphism type of N, it follows that the equivariant diffeomorphism type of M is determined by $\chi(N)$. Let T_2 be the maximal torus $T \cap SO(2l_2)$ of $SO(2l_2)$. Then as in the proof of Lemma 8.13 one sees that

$$\chi(M) = \#M^T = \#N^{S\times T_2} = \chi(N).$$

Therefore the equivariant diffeomorphism type of M is uniquely determined by $\chi(M) \in 2\mathbb{Z}$. Because

$$M_k = \begin{cases} \#_{i=1}^k (S^{2l_1} \times S^{2l_2})_i & \text{if } k \geq 1 \\ S^{2l_1+2l_2} & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k + 2$, the statement follows. □

At the end of this section we give a classification of four dimensional torus manifolds with G-action.

Corollary 8.15. *Let M be a four dimensional torus manifold with G-action, G a non-abelian Lie-group of rank two. Then M is one of the following*

$$\mathbb{C}P^2,\ \mathbb{C}P^1 \times \mathbb{C}P^1,\ S^4,\ S^2_1 \times_{\mathbb{Z}_2} S^2_1,\ S^2_1 \times_{\mathbb{Z}_2} S^2_2$$

or a S^2-bundle over $\mathbb{C}P^1$. Here S^2_1 denotes the two-sphere together with the $\mathbb{Z}_2$-action generated by the antipodal map and S^2_2 the two-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

Proof. Let $\tilde{G}$ be a covering group of G. Then there are the following possibilities using Convention 3.5:

$$\begin{aligned} \tilde{G} = SU(3),\ & SU(2) \times SU(2),\ SU(2) \times S^1, \\ & SU(2) \times SO(3),\ SO(3) \times SO(3),\ SO(3) \times S^1,\ \mathrm{Spin}(4). \end{aligned}$$

If $\tilde{G} = \mathrm{Spin}(4)$, we replace it by $SU(2) \times S^1$ as before.

Then we have the following admissible 5-tuples:

$\tilde{G}$	5-tuple	M
$SU(3)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^2$
$SU(2)\times SU(2)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^1\times \mathbb{C}P^1$
$SU(2)\times S^1$	$(\psi, S^2, \emptyset, \emptyset, \emptyset)$	S^2-bundle over $\mathbb{C}P^1$
	$(\psi, S^2, N, \emptyset, \emptyset)$	$\mathbb{C}P^2$
	$(\psi, S^2, \{N,S\}, \emptyset, \emptyset)$	S^4
$SU(2)\times SO(3)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^1\times S^2$
$SO(3)\times SO(3)$	$(\emptyset, \mathrm{pt}, \emptyset, \emptyset, a_{12}=1)$	$S_1^2\times_{\mathbb{Z}_2} S_1^2$
	$(\emptyset, \mathrm{pt}, \emptyset, \emptyset, a_{12}=0)$	$S^2\times S^2$
$SO(3)\times S^1$	$(\emptyset, S^2, \emptyset, \emptyset, \emptyset)$	$S^2\times S^2$
	$(\emptyset, S_1^2, \emptyset, \emptyset, \emptyset)$	$S_1^2\times_{\mathbb{Z}_2} S_1^2$
	$(\emptyset, S_2^2, \emptyset, \emptyset, \emptyset)$	$S_1^2\times_{\mathbb{Z}_2} S_2^2$
	$(\emptyset, S_2^2, \emptyset, S^1, \emptyset)$	S^4

Here ψ is a group homomorphism $S(U(1)\times U(1))\rightarrow S^1$. □

Appendix A. Lie-groups

Lemma A.1. *Let $l>1$. Then $S(U(l)\times U(1))$ is a maximal subgroup of $SU(l+1)$.*

Proof. Let H be a subgroup of $SU(l+1)$ with $S(U(l)\times U(1))\subset H\subsetneq SU(l+1)$.

Because $S(U(l)\times U(1))$ is a maximal connected subgroup of $SU(l+1)$ the identity component of H has to be $S(U(l)\times U(1))$. Therefore H is contained in the normalizer of $S(U(l)\times U(1))$. Because $l>1$,

$$N_{SU(l+1)}S(U(l)\times U(1))/S(U(l)\times U(1)) = (SU(l+1)/S(U(l)\times U(1)))^{S(U(l)\times U(1))} = \left(\mathbb{C}P^l\right)^{S(U(l)\times U(1))}$$

is just one point. Therefore $H=S(U(l)\times U(1))$ follows. □

Lemma A.2. *Let $\psi: S(U(l)\times U(1))\rightarrow S^1$ be a non-trivial group homomorphism and*

$$H_0 = SU(l+1)\times S^1,$$
$$H_1 = S(U(l)\times U(1))\times S^1,$$
$$H_2 = \{(g,\psi(g)), g\in S(U(l)\times U(1))\}.$$

Then H_1 is the only connected proper closed subgroup of H_0, which contains H_2 properly.

Proof. Let $H_2\subset H\subset H_0$ be a closed connected subgroup. Then we have

$$\operatorname{rank} H_0 \geq \operatorname{rank} H \geq \operatorname{rank} H_2 = \operatorname{rank} H_0 - 1.$$

At first assume that $\operatorname{rank} H = \operatorname{rank} H_0$. Then we have by [18, p. 297]

$$H = H'\times S^1,$$

where H' is a connected subgroup of maximal rank of $SU(l+1)$. Let $\pi_1: H_0\rightarrow SU(l+1)$ the projection to the first factor. Because $H'=\pi_1(H)\supset\pi_1(H_2)=S(U(l)\times U(1))$ and $S(U(l)\times U(1))$ is a maximal connected subgroup of $SU(l+1)$, we have by Lemma A.1 that $H=H_1$ or $H=H_0$.

Now assume that $\operatorname{rank} H=\operatorname{rank} H_2$. Then there is a non-trivial group homomorphism $H\rightarrow S^1$. Therefore locally H is a product $H'\times S^1$, where H' is a simple group which contains $SU(l)$ as a maximal rank subgroup. By [2, p. 219], we have

$$H' = E_7, E_8, G_2, SU(l).$$

If $H'=SU(l)$, then we have $H=H_2$. Therefore we have to show that the other cases do not occur.

l	$\dim H_0$	$\dim H' \times S^1$
8	81	$\dim E_7 \times S^1 = 134$
9	100	$\dim E_8 \times S^1 = 249$
3	16	$\dim G_2 \times S^1 = 15$

Therefore the first two cases do not occur. Because there is no G_2-representation of dimension less than seven, the third case does not occur. □

Lemma A.3. *Let T be a torus and $\psi_1, \psi_2 : S(U(l) \times U(1)) \to T$ be two group homomorphisms. Furthermore, let, for $i = 1, 2$,*

$$H_i = \{(g, \psi_i(g)) \in SU(l+1) \times T; g \in S(U(l) \times U(1))\}$$

be the graph of ψ_i.

(1) *If $l > 1$, then H_1 and H_2 are conjugated in $SU(l+1) \times T$ if and only if $\psi_1 = \psi_2$.*
(2) *If $l = 1$, then H_1 and H_2 are conjugated in $SU(l+1) \times T$ if and only if $\psi_1 = \psi_2^{\pm 1}$.*

Proof. At first assume that H_1 and H_2 are conjugated in $SU(l+1) \times T$. Let $g' \in SU(l+1) \times T$ such that

$$H_1 = g' H_2 g'^{-1}.$$

Because T is contained in the center of $SU(l+1) \times T$, we may assume that $g' = (g, 1) \in SU(l+1) \times \{1\}$. Let $\pi_1 : SU(l+1) \times T \to SU(l+1)$ be the projection on the first factor. Then:

$$S(U(l) \times U(1)) = \pi_1(H_1) = g\pi_1(H_2)g^{-1} = gS(U(l) \times U(1))g^{-1}.$$

By Lemma A.1, it follows that

$$g \in N_{SU(l+1)} S(U(l) \times U(1)) = \begin{cases} S(U(l) \times U(1)) & \text{if } l > 1 \\ N_{SU(2)} S(U(1) \times U(1)) & \text{if } l = 1. \end{cases}$$

Now for $h \in S(U(l) \times U(1))$ we have

$$(h, \psi_1(h)) = g'(g^{-1}hg, \psi_1(h))g'^{-1}.$$

Now $(g^{-1}hg, \psi_1(h))$ lies in H_2. Therefore we may write:

$$g'(g^{-1}hg, \psi_1(h))g'^{-1} = g'(g^{-1}hg, \psi_2(g^{-1}hg))g'^{-1} = (h, \psi_2(g^{-1}hg))$$

If $l > 1$ we have

$$\psi_2(g^{-1}hg) = \psi_2(g)^{-1}\psi_2(h)\psi_2(g) = \psi_2(h).$$

Otherwise we have

$$\psi_2(g^{-1}hg) = \psi_2(h^{\pm 1}) = \psi_2(h)^{\pm 1}.$$

The other implications are trivial. Therefore the statement follows. □

Lemma A.4. *Let $l \geq 1$. Spin$(2l)$ is a maximal connected subgroup of Spin$(2l+1)$. Its normalizer consists out of two components.*

Proof. By [2, p. 219], $\mathrm{Spin}(2l)$ is a maximal connected subgroup of $\mathrm{Spin}(2l+1)$.

$$N_{\mathrm{Spin}(2l+1)}\mathrm{Spin}(2l)/\mathrm{Spin}(2l) = (\mathrm{Spin}(2l+1)/\mathrm{Spin}(2l))^{\mathrm{Spin}(2l)} = \left(S^{2l}\right)^{\mathrm{Spin}(2l)}$$

consists out of two points. Therefore the second statement follows. □

Lemma A.5. *Let G be a Lie-group, which acts on the manifold M. Furthermore, let $N \subset M$ be a submanifold. If the intersection of Gx and N is transverse in x for all $x \in N$, then GN is open in M.*

Proof. We will show that $f : G \times N \to M$, $(h, x) \mapsto hx$ is a submersion. Because a submersion is an open map, it follows that $GN = f(G \times N)$ is open in M. For $g \in G$, let

$$\begin{aligned} l_g : G \times N &\to G \times N \\ (h, x) &\mapsto (gh, x) \end{aligned}$$

and

$$\begin{aligned} l'_g : M &\to M \\ x &\mapsto gx. \end{aligned}$$

Then we have for all $g \in G$

$$f = l'_g \circ f \circ l_{g^{-1}}.$$

Now for $(g, x) \in G \times N$ we have

$$D_{(g,x)} f = D_x l'_g \, D_{(e,x)} f \, D_{(g,x)} l_{g^{-1}}.$$

Because Gx and N intersect transversely in x, the differential $D_{(e,x)} f$ is surjective. Because l'_g, $l_{g^{-1}}$ are diffeomorphisms, it follows that $D_{(g,x)} f$ is surjective. Therefore f is a submersion. □

Appendix B. Generalities on torus manifolds

Lemma B.1. *Let M be a torus manifold and $M_1, \dots, M_k$ pairwise distinct characteristic submanifolds of M with $N = M_1 \cap \dots \cap M_k \neq \emptyset$. Then each M_i intersects transversely with $\bigcap_{j=1}^{i-1} M_j$. Therefore N is a submanifold of M with* $\operatorname{codim} N = 2k$ *and* $\dim\langle \lambda(M_1), \dots, \lambda(M_k)\rangle = k$. *Furthermore, N is the union of some components of $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$.*

Proof. We prove the lemma by induction on k. Let $k \geq 1$ and $x \in N$. Then we have

$$T_x M = \bigcap_{i=1}^{k} T_x M_i \oplus \bigoplus_j V_j,$$

where the V_j are one-dimensional complex $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$-representations. Since the M_i have codimension two in M, each $\lambda(M_i)$ acts non-trivially on exactly one V_{j_i}.

If $\operatorname{codim} \bigcap_{i=1}^{k} T_x M_i < 2k$, then there are i_1 and i_2, such that $V_{j_{i_1}} = V_{j_{i_2}}$. Therefore

$$T_x M_{i_1} = T_x M_{i_2} = T_x M^{\langle \lambda(M_{i_1}), \lambda(M_{i_2})\rangle}$$

has codimension two.

Since $\langle \lambda(M_{i_1}), \lambda(M_{i_2})\rangle$ has dimension two, it does not act almost effectively on M. This is a contradiction. Therefore $\bigcap_{i=1}^{k} T_x M_i$ has codimension $2k$. By induction hypothesis $\bigcap_{i=1}^{k-1} M_i$ is a submanifold of codimension $2k-2$ and $T_x \bigcap_{i=1}^{k-1} M_i = \bigcap_{i=1}^{k-1} T_x M_i$. Thus, M_k and $\bigcap_{i=1}^{k-1} M_i$ intersect transversely. Therefore N is a submanifold of M of codimension $2k$.

If $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$ has dimension smaller than k then the weights of the V_j are linear dependent. Therefore there is $(a_1, \dots, a_k) \in \mathbb{Z}^k - \{0\}$, such that

$$\mathbb{C} = V_1^{a_1} \otimes \dots \otimes V_k^{a_k},$$

where $\mathbb{C}$ denotes the trivial $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$-representation. This gives a contradiction because each $\lambda(M_i)$ acts non-trivially on exactly one V_j.

Because $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$ has dimension k, $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$ has dimension at most $2n-2k$. But N is contained in $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$ and has dimension $2n-2k$. Therefore it is the union of some components of $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$. □

Lemma B.2. *Let M be a torus manifold of dimension $2n$ and N a component of the intersection of $k(\leq n)$ characteristic submanifolds $M_1, \dots, M_k$ of M with $N^T \neq \emptyset$. Then N is a torus manifold. Moreover, the characteristic submanifolds of N are given by the components of intersections of characteristic submanifolds $M_i \neq M_1, \dots, M_k$ of M with N, which contain a T-fixed point.*

Proof. Let $M_i \neq M_1, \dots, M_k$ be a characteristic submanifold of M with $(M_i \cap N)^T \neq \emptyset$. Then, by Lemma B.1, each component of $M_i \cap N$ which contains a T-fixed point has codimension two in N. That means that they are characteristic submanifolds of N.

Now let $N_1 \subset N$ be a characteristic submanifold and $x \in N_1^T$. Then we have

$$T_xM = T_xN_1 \oplus V_0 \oplus N_x(N, M)$$

as T-representations with V_0 a one dimensional complex T-representation. Let M_i be the characteristic submanifold of M, which corresponds to V_0. Then N_1 is the component of the intersection $M_i \cap N$, which contains x. □

Lemma B.3. *Let M be a $2n$-dimensional torus manifold and T' a subtorus of T. If N is a component of $M^{T'}$, which contains a T-fixed point x, then N is a component of the intersection of some characteristic submanifolds of M.*

Proof. By Lemma B.1, the intersection of the characteristic submanifolds $M_1, \dots M_k$ is a union of some components of $M^{\langle \lambda(M_1), \dots, \lambda(M_k) \rangle}$.

Therefore we have to show that there are characteristic submanifolds $M_1, \dots, M_k$ of M such that

$$T_xN = T_x\left(M_1 \cap \dots \cap M_k\right).$$

There are n characteristic submanifolds $M_1, \dots, M_n$ which intersect transversely in x. Therefore we have

$$T_xM = N_x(M_1, M) \oplus \dots \oplus N_x(M_n, M).$$

We may assume that there is a $1 \leq k \leq n$ such that T' acts trivially on $N_x(M_i, M)$ for $i > k$ and non-trivially on $N_x(M_i, M)$ for $i \leq k$. Then we have

$$T_xN = (T_xM)^{T'} = N_x(M_{k+1}, M) \oplus \dots \oplus N_x(M_n, M) = T_x\left(M_1 \cap \dots \cap M_k\right).$$

□

Lemma B.4. *Let M be a torus manifold with $T^n \times \mathbb{Z}_2$-action, such that $\mathbb{Z}_2$ acts non-trivially on M. Furthermore, let $B \subset M$ be a submanifold of codimension one on which $\mathbb{Z}_2$ acts trivially and N the intersection of characteristic submanifolds $M_1, \dots, M_k$ of M. Then B and N intersect transversely.*

Proof. Let $x \in B \cap N$ then we have the $\langle \lambda(M_1), \dots, \lambda(M_k) \rangle \times \mathbb{Z}_2$-representation T_xM. It decomposes as the sum of the eigenspaces of the non-trivial element of $\mathbb{Z}_2$. Because B has codimension one the eigenspace to the eigenvalue -1 is one dimensional. Because the irreducible non-trivial torus representations are two-dimensional, we have

$$\begin{aligned} T_xN = (T_xM)^{\langle \lambda(M_1), \dots, \lambda(M_k) \rangle} &= T_xM^{\langle \lambda(M_1), \dots, \lambda(M_k) \rangle \times \mathbb{Z}_2} \oplus N_x(B, M)^{\langle \lambda(M_1), \dots, \lambda(M_k) \rangle} \\ &= T_xM^{\langle \lambda(M_1), \dots, \lambda(M_k) \rangle \times \mathbb{Z}_2} \oplus N_x(B, M). \end{aligned}$$

That means that the intersection is transverse. □

Lemma B.5. *Let M^{2n} be a $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$-manifold such that $(\mathbb{Z}_2)_i$ acts non-trivially on M. Furthermore, let $B_i \subset M$, $i = 1, 2$, be closed connected submanifolds of codimension one such that $(\mathbb{Z}_2)_i$ acts trivially on B_i. Then the following statements are equivalent:*

(1) *B_1, B_2 intersect transversely*

(2) $B_1 \neq B_2$
(3) $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ *acts effectively on* M *or* $B_1 \cap B_2 = \emptyset$

Proof. Denote by V_i the non-trivial real irreducible representation of $(\mathbb{Z}_2)_i$. Let $x \in B_1 \cap B_2$. Then for the $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$-representation T_xM there are two possibilities:

$$T_xM = \begin{cases} \mathbb{R}^{2n-1} \oplus V_1 \otimes V_2 \\ \mathbb{R}^{2n-2} \oplus V_1 \oplus V_2 \end{cases}$$

In the first case B_i, $i = 1, 2$, is the component of $M^{(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2}$ containing x and $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ acts non-effectively on M. In the second case $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ acts effectively on M and B_1, B_2 intersect transversely in x.

All conditions given in the lemma imply that we are in the second case or $B_1 \cap B_2 = \emptyset$. Therefore they are equivalent. □

Remark B.6. Lemmas B.1, B.4 also hold if we do not require that a characteristic manifold contains a T-fixed point.

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