Finiteness of p-Divisible Sets of Multiple Harmonic Sums

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Abstract. Let l be a positive integer and $\mathbf{s}=(s_1,\dots,s_l)$ be a sequence of positive integers. In this paper we shall study the arithmetic properties of multiple harmonic sum $H(\mathbf{s};n)$ which is the n-th partial sum of multiple zeta value series $\zeta(\mathbf{s})$. We conjecture that for every $\mathbf{s}$ and every prime p there are only finitely many p-integral partial sums $H(\mathbf{s};n)$. This generalizes a conjecture of Eswarathasan and Levine and Boyd for harmonic series. We provide a lot of evidence for this general conjecture as well as some heuristic argument to support it. This paper is a sequel to *Wolstenholme Type Theorem for multiple harmonic sums*, Intl. J. of Num. Thy. **4**(1) (2008) 73-106.

Résumé. Soit l un entier et $\mathbf{s}=(s_1,\dots,s_l)$ une séquence d'entiers positifs. Dans ce document, nous étudierons les propriétés arithmétique de sommes harmoniques multiples $H(\mathbf{s};n)$, qui est le n-me somme partielle de la valeur de la série multiple zeta $\zeta(\mathbf{s})$. On conjecture que pour tout $\mathbf{s}$ et de tous les premiers p, il n'y a que de nombreux finitely p-partie intégrante sommes $H(\mathbf{s},n)$. Ceci généralise une conjecture de Eswarathasan et Levine et Boyd pour la série harmonique. Nous fournissons beaucoup d'éléments de preuve pour cette conjecture générale ainsi que certaines heuristiques argument soutenir. Ce document fait suite à *Wolstenholme Type Theorem for multiple harmonic sums*, Intl. J. of Num. Thy. **4**(1) (2008) 73-106.

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1 Introduction

In recent years there is a revival of interest in the multiple zeta values defined by

$$\zeta(\mathbf{s}):=\zeta(s_1,\dots,s_l)=\sum_{0<k_1<\cdots<k_l} k_1^{-s_1}\cdots k_l^{-s_l}$$

for $\mathbf{s}=(s_1,\dots,s_l)\in\mathbb{N}^l$, where $\mathbb{N}$ is the set of positive integers (see, for e.g., [11, 13, 19]). In physics, however, not only these series but also their partial sums have significant meanings in applications (see [2, 3]). These partial sums are called the multiple harmonic sums (MHS for short) which generalize the notion of harmonic sums. In general it is defined as

$$H(\mathbf{s};n):=\sum_{1\le k_1<\cdots<k_l\le n} k_1^{-s_1}\cdots k_l^{-s_l},\quad n\in\mathbb{Z}_{\ge 0}. \tag{1}$$

where $|\mathbf{s}|:=s_1+\cdots+s_l$ is called the weight and $l=l(\mathbf{s})$ the length. By convention we set $H(\mathbf{s};r)=0$ for $r=0,\dots,l-1$, and $H(\emptyset;0)=1$. To save space, we denote by $\{e_1,\dots,e_t\}^d$ the string formed by repeating $(e_1,\dots,e_t)$ d times and $1^d=\{1\}^d$. We also write $H(\mathbf{s})$ for the set of all the partial sums $H(\mathbf{s};n)$ when no confusion is likely to arise .

This paper can be considered as a sequel to [21] whose main goal is to provide generalizations of Wolstenholme's Theorem for the MHS. Hoffman [12] obtained similar types of results independently. One of the results in [21] is the following generalization to homogeneous MHS:

Theorem 1.1. [21, Theorem 2.13]) *Let s and l be two positive integers. Let p be an odd prime such that $p \geq l+2$ and $p-1$ divides none of ks and $ks+1$ for $k=1,\dots,l$. Then*

$$H(\{s\}^l; p-1) \equiv \begin{cases} 0 & \pmod{p} \text{ if } ls \text{ is even,} \\ 0 & \pmod{p^2} \text{ if } ls \text{ is odd.} \end{cases}$$

In particular, the above is always true if $p \geq ls+3$.

One can also investigate the sums $H(\mathbf{s}; n)$ with fixed $\mathbf{s}$ but varying n. Such a study for harmonic series was initiated systematically by Eswarathasan and Levine [8] and Boyd [4], independently. It turns out that to obtain precise information one has to study Wolstenholme type congruences in some detail and so these two directions of research are interwoven into each other rather tightly. To state our main results and conjectures we define

$$H(\mathbf{s}; n) = \frac{a(\mathbf{s}; n)}{b(\mathbf{s}; n)}, \quad a(\mathbf{s}; n),\ b(\mathbf{s}; n) \in \mathbb{N}, \quad \gcd(a(\mathbf{s}; n), b(\mathbf{s}; n)) = 1.$$

For completeness, we set $a(\mathbf{s}; 0) = 0$ and $b(\mathbf{s}; 0) = 1$. Fixing a prime p we are interested in the p-divisibility of the integers $a(\mathbf{s}; n)$ and $b(\mathbf{s}; n)$ for varying n. Thus we put $H(\mathbf{s}; n)$ inside $\mathbb{Q}_p$, the fractional field of the p-adic integers and let v_p be the discrete valuation on $\mathbb{Q}_p$ such that $v_p(p) = 1$. In this general situation we're forced to change the notation used by the previous authors. For any $m \in \mathbb{N}$ and $\mathbf{s} \in \mathbb{N}^l$, put

$$\begin{aligned} I(\mathbf{s}|m) &:= \{n \in \mathbb{Z}_{\geq 0} : b(\mathbf{s}; n) \not\equiv 0 \pmod{m}\}, \\ J(\mathbf{s}|m) &:= \{n \in \mathbb{Z}_{\geq 0} : a(\mathbf{s}; n) \equiv 0 \pmod{m}\}. \end{aligned}$$

Note that $J(\mathbf{s}|m) \neq \emptyset$ since $0 \in J(\mathbf{s}|m)$ always. For any prime p we call $J(\mathbf{s}|p)$ the *p-divisible set* of the MHS $H(\mathbf{s})$ defined by Eq. (1).

In [4] Boyd presented a heuristic argument by modeling on simple branching processes to convince us that the p-divisible set of the harmonic series is finite for every prime p (this is also independently conjectured by Eswarathasan and Levine [8, Conjecture A]). Boyd also proves this conjecture for all primes less than 550 except for 83, 127 and 397. We now provide a generalization:

Conjecture 1.2. *For any $\mathbf{s}$ and any prime p the p-divisible set $J(\mathbf{s}|p)$ is finite.*

Although we are not able to prove this conjecture in general, we obtain a lot of partial results. The primary tool to prove these when $l(\mathbf{s}) \geq 2$ is our Criterion Theorem 2.1. Fixing an arbitrary prime p we define

$$G_0 = \{0\} \text{ and } G_t = \{n : p^{t-1} \leq n < p^t\} \text{ for } t \in \mathbb{N}.$$

Criterion Theorem. Let $l \geq 2$ be a positive integer and p be a prime such that $l \in G_{t_0}$. Let $\mathbf{s} = (s_1, \cdots, s_l) \in \mathbb{N}^l$ and put $m = \min\{s_i : 1 \leq i \leq l\}$. For $t \in \mathbb{N}$ set $f(\mathbf{s}, p; t) = \min\{-v_p\big(H(\mathbf{s}; n)\big) : n \in G_t\}$. If there is $\tau > t_0$ such that

$$f(\mathbf{s}, p; \tau) > (|\mathbf{s}| - m)(\tau - 1) - m,$$

then $J(\mathbf{s}|p)$ is finite.

We list only some results obtained by applying our Criterion Theorem below. More examples including those when $l = 1$ can be found in Sec. 2-4 and in the online supplement [20].

Theorem 1.3. *The p-divisible set $J(\mathbf{s}|p)$ is finite if*

(1). $\mathbf{s} = (1,1)$ *and* $p = 3, 7, 13, 31$, *or* $\mathbf{s} = (1,1,1)$ *and* $p = 3$, *or* $\mathbf{s} = (1,1,2)$ *and* $p = 7$.

(2). $\mathbf{s} = (4,3,5),\ (5,3,4)$ *and* $p = 17$.

(3). $\mathbf{s} = \{s\}^l,\ 1 \leq l \leq 20,\ s \geq 2,$ *and* $p = 2$.

(4). $\mathbf{s} = (s,t),\ s,t \leq 20,\ t \geq 2,$ *and* $p = 2, 3, 5$.

(5). $\mathbf{s} = (r,s,t),\ r,s,t \leq 10,\ t \geq 2,$ *and* $p = 2, 3, 5$.

(6). $\mathbf{s} = (q, r, s, t)$, $q, r, s, t \le 4$, $t \ge 2$, *and* $p = 2, 3, 5$.

Moreover, for $\mathbf{s}$ *in the last four cases we have* $J(\mathbf{s}|2) = \{0\}$.

Conjecture 1.4. *For every* $\mathbf{s}$ *the* 2*-divisible set* $J(\mathbf{s}|2) = \{0\}$.

In [8, Conjecture B] Eswarathasan and Levine state that there should be infinitely many primes p (so called harmonic primes) such that $J(1|p) = \{0, p-1, p^2-p, p^2-1\}$. Boyd [4] further suggest $1/e$ as the expected density of such primes. For any $\mathbf{s}$ we extend this notion to define the *reserved (divisibility) set* $RJ(\mathbf{s}; x)$ of polynomials in x with rational coefficients. For any prime $p \ge |\mathbf{s}| + 3$ we have $RJ(\mathbf{s}; p) \subseteq J(\mathbf{s}|p)$ and there are primes p (called *reserved primes* for $\mathbf{s}$) such that equality holds. We determine $RJ(\mathbf{s})$ for many types of $\mathbf{s}$ in Theorem 7.2. Furthermore, we argue by heuristics that the following conjecture should be true.

Conjecture 1.5. *If* $l(\mathbf{s}) = 1$ *and* $\mathbf{s} = s \ge 2$ *then the proportion of primes* p *with* $J(s|p) = RJ(s)$ *is* $1/\sqrt{e}$. *This proportion is equal to* $1/e$ *for all other* $\mathbf{s}$.

This conjecture is supported by very strong numerical and theoretical evidence which we gather in [20, Appendix II] and in Theorem 7.2. It also generalizes Boyd's density conjecture of the harmonic primes.

At the end of this paper we put forward some more conjectures of $J(\mathbf{s}|p)$ related to the distribution of irregular primes.

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2 A process to determine $J(\mathbf{s}|p)$

For any positive integer n let $n = p\tilde{n} + r$, where $\tilde{n}, r \in \mathbb{N}$ and $0 \le r \le p-1$. For any $\mathbf{s} \in \mathbb{N}^l$ define

$$H^*(\mathbf{s}; n) = \sum_{\substack{1 \le k_1 < \cdots < k_l \le n \\ p \nmid k_1 k_2 \cdots k_l}} \frac{1}{k_1^{s_1} \cdots k_l^{s_l}}.$$

Then by a straightforward computation using the shuffle trick we have: for any $s, l \in \mathbb{N}$

$$H(s; n) = H^*(s; n) + p^{-s} \cdot H(s; \tilde{n}), \tag{2}$$

$$H(\{s\}^l; n) = \sum_{k=0}^{l} p^{-ks} \cdot H(\{s\}^k; \tilde{n}) \cdot H^*(\{s\}^{l-k}; n). \tag{3}$$

where $H(\{s\}^0; m) = H^*(\{s\}^0; m) = 1$ for any integer m. We omit the proofs of these formulas whose main ingredient is contained in the proof of the main Criterion Theorem 2.1 below. Both of these formulas are generalizations of [8, (2.2)] for partial sums of harmonic series. They are the primary tools to study Conjecture 1.2 for homogeneous MHS.

For more general MHS we need a more complicated version of these formulas. Fixing an arbitrary prime p we define

$$G_0 = \{0\} \text{ and } G_t = \{n : p^{t-1} \le n < p^t\} \text{ for } t \in \mathbb{N}.$$

For any $\mathbf{s} \in \mathbb{N}^l$ we set $J_t(\mathbf{s}|p) = G_t \cap J(\mathbf{s}|p)$.

Theorem 2.1. (*Criterion Theorem*) *Let* $l \ge 2$ *be a positive integer and* p *be a prime such that* $l \in G_{t_0}$. *Let* $\mathbf{s} = (s_1, \cdots, s_l) \in \mathbb{N}^l$ *and put* $m = \min\{s_i : 1 \le i \le l\}$. *For* $t \in \mathbb{N}$ *set* $f(\mathbf{s}, p; t) = \min\{-v_p\big(H(\mathbf{s}; n)\big) : n \in G_t\}$. *If there is* $\tau > t_0$ *such that*

$$f(\mathbf{s}, p; \tau) > (|\mathbf{s}| - m)(\tau - 1) - m,$$

then $J(\mathbf{s}|p)$ *is finite.*

Proof. Let $n = p\tilde{n} + r \in G_{\tau+1}$. By definition we have

$$H(\mathbf{s}; n) = \sum_{\substack{\alpha+\beta=l \\ \alpha,\beta\geq 0}} \sum_{\substack{1\leq i_1<\cdots<i_\alpha\leq n \\ 1\leq j_1<\cdots<j_\beta\leq n \\ \{i_1,\dots,i_\alpha\}\cup\{j_1,\dots,j_\beta\}=\{1,\dots,n\}}} \sum_{\substack{1\leq K_1<\cdots<K_l\leq n \\ p\mid K_{i_a}, 1\leq a\leq\alpha \\ p\nmid K_{j_b}, 1\leq b\leq\beta}} \frac{1}{K_1^{s_1}\cdots K_l^{s_l}}. \tag{4}$$

Note that the terms corresponding to $\alpha = l, \beta = 0$ form the series $A = H(\mathbf{s}; \tilde{n})/p^{|\mathbf{s}|} \neq 0$ since $\tau > t_0$ so that $\tilde{n} \geq p^{t_0} > l$. For all other terms with $\beta \geq 1$ we have the natural bound

$$v_p(K_1^{s_1}\cdots K_l^{s_l}) \leq (|\mathbf{s}| - m)\tau$$

since $K_j < p^{\tau+1}$ for all j and one of them is prime to p. Set $B = H(\mathbf{s}; n) - A$ then obviously $v_p(B) \geq -(|\mathbf{s}| - m)\tau$. Since $\tilde{n} \in G_\tau$ by assumption of τ we know that

$$v_p(A) < -|\mathbf{s}| - (|\mathbf{s}| - m)(\tau - 1) + m = -(|\mathbf{s}| - m)\tau \leq v_p(B).$$

By induction it's easy to see that for all $t > \tau$ and $n \in G_t$ we have

$$v_p\big(H(\mathbf{s}; n)\big) < -|\mathbf{s}|(t - \tau) - (|\mathbf{s}| - m)(\tau - 1) + m = -|\mathbf{s}|(t - 1) + m\tau < 0.$$

This shows clearly that $J(\mathbf{s}|p)$ is finite. □

Corollary 2.2. *Let s and l be two positive integers, $l \geq 2$, and p be a prime. Suppose $l \in G_{t_0}$. Then the p-divisible set $J(\{s\}^l|p)$ is finite if there exists $\tau > t_0$ such that $f_l(\tau) := f(\{s\}^l, p; \tau) > (l-1)s\tau - s$.*

Proposition 2.3. *Let r, s, t be three positive integers. Let $\mathbf{s} = (r, s)$ with $1 \leq r \leq 10$ and $2 \leq s \leq 10$, or $\mathbf{s} = (r, s, t)$ with $2 \leq r, s, t \leq 5$. Then there's always some prime $p \geq |\mathbf{s}| + 3$ such that the p-divisible set $J(\mathbf{s}|p) = RJ(\mathbf{s}; p)$ is finite where $RJ(\mathbf{s}; p)$ is given in Theorem 7.2.*

Proof. The set $RJ(\mathbf{s}; p)$ will be defined for general $\mathbf{s}$ in Definition 7.1 and computed in Theorem 7.2. The proof of the proposition follows from the Criterion Theorem 2.1 by computation. To save space we put the details online [20]. □

3 Finiteness of $J(s|p)$

We now describe an approach to determine the p-divisible set $J(s|p)$ for any given positive integer s and odd prime p. This is essentially discovered by Eswarathasan and Levine [8] and by Boyd [4], independently. It follows quickly from Eq. (2) that

$$n \in I(s|p) \text{ if and only if } \tilde{n} \in J(s|p^s). \tag{5}$$

Therefore

$$n \in J(s|p) \text{ implies } \tilde{n} \in J(s|p^s). \tag{6}$$

It's also clear that

$$I(s|p) = pJ(s|p^s) + R, \quad R = \{0, 1, \cdots, p - 1\}.$$

Remark 3.1. The case when $s > 1$ is very different from that of $s = 1$ considered by previous authors in that the information of $I(s|p)$ is in general not enough to determine $J(s|p)$.

To get an equivalent condition of Eq. (6) we need a partial generalization of [8, Lemma 3.1]. Set the parity function $\mathfrak{p}(m) = 1$ if m is odd and $\mathfrak{p}(m) = 2$ if m is even.

Lemma 3.2. *Let p be an odd prime and s a positive integers. If $p - 1 \nmid s, s + 1$ then we have*

$$H^*(s; pn) \equiv 0 \pmod{p^{\mathfrak{p}(s-1)}}. \tag{7}$$

Proof. By definition

$$H^*(s;pn) = \sum_{\substack{1\le k\le n\\ (p,k)=1}} \frac{1}{k^s} = \sum_{m=0}^{n-1} \left(\sum_{mp<k<(m+1)p} \frac{1}{k^s} \right). \tag{8}$$

The lemma follows from the fact that each inner sum in the parentheses satisfies the congruence in Eq. (7) which can be proved by the same argument as that in the proof of [21, Lemma 2.2] (when n is odd the shorter proof suffices). It also follows from [16, Corollary 1]. □

Proposition 3.3. *Let $s, l \in \mathbb{N}$ and p be an odd prime such that $p \ge l+2$ and $p-1$ divides none of ks and $ks+1$ for $k = 1, \dots, l$. Then*

$$H^*(\{s\}^l; pn) \equiv 0 \pmod{p^{\mathfrak{p}(ls-1)}}.$$

The proof as well as the result itself is similar to that of Theorem 1.1 so we leave the details to the interested reader. The first step of induction is given as Lemma 3.2 above. In fact, the proposition itself reduces to Theorem 1.1 when $n = 1$.

Definition 3.4. For $n \in J(s|p^s)$ there is a unique integer $\psi_s(s|p;n) \in [0, p-1]$ such that

$$\psi_s(s|p;n) \equiv \frac{1}{p^s} H(s;n) \pmod{p}. \tag{9}$$

Lemma 3.5. *For $n = p\tilde{n} + r$, $0 \le r < p$, we have*

$$H^*(s;n) - H^*(s;p\tilde{n}) \equiv H(s;r) \pmod{p}. \tag{10}$$

Furthermore, if $\tilde{n} \in J(s|p^s)$ then

$$H(s;n) \equiv H(s;r) + \psi_s(s|p;\tilde{n}) \pmod{p} \tag{11}$$

Proof. This follows from Eq. (2) and Lemma 3.2. Also see the proof of [8, Lemma 3.2]. □

Theorem 3.6. *Let $n = p\tilde{n} + r$, $0 \le r < p$. Then $n \in J(s|p)$ if and only if*

$$\tilde{n} \in J(s|p^s) \text{ and } H(s;r) + \psi_s(\tilde{n}) \equiv 0 \pmod{p}. \tag{12}$$

Proof. If $n \in J(s|p)$ then Eq. (6) implies that $\tilde{n} \in J(s|p^s)$. In addition, congruence (12) follows immediately from Eq. (11). On the other hand, if Eq. (12) holds then Eq. (11) implies that $n \in J(s|p)$ and the proof is complete. □

We now use the above theorem to define a branching process by using the sets G_t which will compute $J(s|p)$ if it's finite.

Proposition 3.7. *Let s be a positive integer and p an odd prime. Then $J(s|p) = \cup_{t=0}^{\infty} J_t(s|p)$ where $J_t(s|p)$ can be determined recursively by*

$$J_{t+1}(s|p) = \{n = p\tilde{n} + r : \tilde{n} \in J_t(s|p^s), r \in R, v_p\big(H(s;r) + \psi_s(\tilde{n})\big) > 0\}$$

for $t \in \mathbb{N}$. Here, as before, $R = \{0, 1, \cdots, p-1\}$.

The next corollary follows naturally.

Corollary 3.8. *Let s be a positive integer and p an odd prime. Then $J(s|p)$ is finite if and only if $J_t(s|p^s) = \emptyset$ for some $t \in \mathbb{N}$.*

An easy computation according to Corollary 3.8 yields the following concrete result.

Proposition 3.9. *Let s be a positive integer. Then $J(s|p)$ is finite for primes $p = 2, 3, 5, 7$.*

Proof. (1) $p = 2$. We claim that $J(s|2) = \{0\}$. We can prove that 2 does not divide $H(s;n)$ by induction on n. This is clear for $n = 1$ and $n = 2$ because $H(s;1) = 1, H(s;2) = (1+2^s)/2^s$. Suppose $r \notin J(s|2)$ for all $r \le n$ and $n \in J(s|2)$. If n is odd then let $H(s;n-1) = \frac{a}{2b}$ where a is odd by inductive assumption. Then

$$H(s;n) = \frac{a}{2b} + \frac{1}{n^s} = \frac{Na + 2B}{\text{l.c.m}(2b, n^s)},$$

where $N = n^s/\gcd(n^s, b)$ and $B = b/\gcd(n^s, b)$. Hence $Na + 2B$ is odd because both N and a are odd, which is a contradiction. If $n = 2\tilde{n}$ then

$$H(s;2\tilde{n}) = \sum_{k=1}^{\tilde{n}} \left(\frac{1}{(2k-1)^s} + \frac{1}{(2k)^s} \right) \equiv \tilde{n} + \frac{1}{2^s} H(s;\tilde{n}) \pmod 2.$$

By inductive assumption $2 \nmid H(s;\tilde{n})$ which implies that $2 \nmid H(s;2\tilde{n})$. So n can not belong to $J(s|2)$ either if n is even. This shows that $J(s|2) = \{0\}$. In fact, it is not hard to see that for $n \in G_t$, $t \ge 1$ we have

$$v_2\big(H(s;n)\big) = -(t-1)s. \tag{13}$$

For $3 \le p \le 7$ Eswarathasan and Levine [8] have shown that $J(1|p)$ are finite. We also know that when $s \le 4$ then $J(s|p)$ are finite for these primes by explicit computation [20]. Assume $s \ge 4$. Then by Corollary 3.8 we only need to show that $J_1(s|p^s) = \emptyset$. We need [21, Corollary 2.7] which implies that if $p \ge 3$ is a regular prime then

$$H(s;p-1) \not\equiv 0 \pmod{p^s} \text{ for } s \ge 4. \tag{14}$$

(2) $p = 3$. Neither $H(s;1) = 1$ nor $H(s;2) = 1 + 1/2^s$ is divisible by 3^s. so $J_1(s|3^s) = \emptyset$.

(3) $p = 5$. Neither $H(s;1) = 1$ nor $H(s;2) = 1 + 1/2^s$ is divisible by 5^s. Now

$$6^s H(s;3) = 2^s + 3^s + 6^s \equiv 2^s + (-2)^s + 1 \equiv \begin{cases} 2 \cdot 4^n + 1 & \pmod 5 \quad \text{if } s = 2n, \\ 1 & \pmod 5 \quad \text{if } s \text{ is odd.} \end{cases}$$

So we always have $H(s;3) \not\equiv 0 \pmod 5$, i.e., $3 \notin J_1(s|5^s)$. Finally, Eq. (14) implies that $4 \notin J_1(s|5^s)$ for $s \ge 4$ because 5 is a regular prime. Hence $J_1(s|5^s) = \emptyset$.

(4) $p = 7$. Clearly $1, 2 \notin J_1(s|7^s)$ and $6^s H(s;3) = 2^s + 3^s + 6^s < 7^s$ when $s \ge 4$. Now

$$H(s;4) = H(s;6) - \frac{1}{5^s} - \frac{1}{6^s} \equiv (-1)^{s+1} \left(1 + \frac{1}{2^s} \right) \pmod 7.$$

Because $2^3 \equiv 1 \pmod 7$ we get

$$1 + \frac{1}{2^s} \equiv \begin{cases} 2 & \pmod 7 \quad \text{if } s \equiv 0 \pmod 3, \\ 3/2 & \pmod 7 \quad \text{if } s \equiv 1 \pmod 3, \\ 3 & \pmod 7 \quad \text{if } s \equiv -1 \pmod 3. \end{cases}$$

Therefore $4 \notin J_1(s|7^s)$. Similarly, $H(s;5) \equiv (-1)^{s+1} \not\equiv 0 \pmod 7$. Finally, it follows from Eq. (14) that $6 \notin J_1(s|7^s)$ for $s \ge 4$. These show that $J_1(s|7^s) = \emptyset$ for all $s \ge 4$. □

Remark 3.10. The case $p = 11$ is not so easy since $H(3;4) \equiv 0 \pmod{11}$ and moreover, for any positive integer e there is some $s < p^e(p-1)$ such that $H(s;4) \equiv 0 \pmod{11^{e+1}}$.

We also computed $J(s|p)$ for some other s and p (see [20]), which confirms the following

Proposition 3.11. *Let p be a prime such that $p \le 3001$. Then $J(s|p)$ is finite for $2 \le s \le 300$.*

4 Finiteness of $J(\{s\}^l|p)$

In order to apply Criterion Theorem 2.1 we set

$$f_l(t) := f(\{s\}^l, p; t) = \min\{-v_p(H(\{s\}^l; n)) : n \in G_t\} \quad \forall t \geq 1.$$

We first look at the case $s \geq 2$.

Lemma 4.1. *For all $s \geq 2$ we have $v_2(3^s + 1) = \mathfrak{p}(s-1)$ which is 1 if s is even and 2 if s is odd. In particular, we always have $3^s + 1 \not\equiv 0 \pmod{2^s}$.*

Proof. This is clear because

$$3^s + 1 = \begin{cases} 9^n + 1 \equiv 2 \pmod 8 & \text{if } s = 2n, \\ 3 \cdot 9^n + 1 \equiv 4 \pmod 8 & \text{if } s = 2n+1. \end{cases}$$

□

Proposition 4.2. *Let $s \geq 2$ and $l \leq 20$ be two positive integers. Then the 2-divisible set $J(\{s\}^l|2)$ is finite.*

Proof. When $l = 1$ this is included in Proposition 3.9. So we assume $s, l \geq 2$. Then

$$H(s; 2) = 1 + \frac{1}{2^s}, \quad H(s, s; 2) = \frac{1}{2^s}, \quad H(s; 3) = \frac{6^s + 3^s + 2^s}{6^s}.$$

Further, by Lemma 4.1 we know that

$$H(s, s; 3) = \frac{1}{2^s} + \left(1 + \frac{1}{2^s}\right)\frac{1}{3^s} = \frac{3^s + 1 + 2^s}{6^s}$$

has at least a factor 2 in the denominator. Therefore we can take $\tau = 2$ to get $f(\tau) \geq 1 > (2s - s)(\tau - 1) - s = 0$. So the condition in Corollary 2.2 is satisfied and consequently $J(s, s|2) = \{0\}$.

A detailed study of using Lemma 4.1 tells more. Let $t \geq 0$ and $n \in G_{t+2}$. Then by induction and Eq. (3) we can easily show that

$$v_2\big(H(s, s; n)\big) = \begin{cases} -(2t+1)s & \text{if } 2 \cdot 2^t \leq n < 3 \cdot 2^t, \\ \mathfrak{p}(s-1) - (2t+1)s & \text{if } 3 \cdot 2^t \leq n < 4 \cdot 2^t. \end{cases} \tag{15}$$

Putting $l = 3$ in the following equation

$$H(\{s\}^l; n) = H(\{s\}^l; n-1) + \frac{1}{n^s} H(\{s\}^{l-1}; n-1), \tag{16}$$

and applying induction on t we can show that

$$v_2\big(H(s, s, s; n)\big) = \begin{cases} \mathfrak{p}(s-1) - 3ts & \text{if } 4 \leq 2n/2^t < 5, \\ -3ts & \text{if } 5 \leq 2n/2^t < 6, \\ -(3t+1)s & \text{if } 6 \leq 2n/2^t < 8. \end{cases} \tag{17}$$

So we get $J(s, s, s|2) = \{0\}$ when $s \geq 2$.

When $l \geq 4$ we can utilize Eq. (16) again. However, even in the case $l = 4$ it is very complicated already. Nevertheless the idea is straightforward so we omit the details of the proof. Suppose $s = 2$ and $n \in G_{t+2}$ with $t \geq 1$ (note that $H(\{s\}^4; n) = 0$ for all $n \leq 3$). Then we have

$$v_2\big(H(\{2\}^4; n)\big) = \begin{cases} (1) & -2(4t-1) & \text{if } 32 \leq 16n/2^t < 48, \\ (2) & -8t & \text{if } 48 \leq 16n/2^t < 56, \\ (3) & -8t + \delta(t) & \text{if } 56 \leq 16n/2^t < 57, \\ (4) & -8t + 7 & \text{if } 57 \leq 16n/2^t < 58, \\ (5) & -2(4t-2) & \text{if } 58 \leq 16n/2^t < 60, \\ (6) & -2(4t-1) & \text{if } 60 \leq 16n/2^t < 64. \end{cases} \tag{18}$$

Here if $t = 1$ then (3)-(6) merge into (6); if $t = 2$ then $\delta(t) = 5$ and (3)-(5) merge into (3); if $t = 3$ then (3) and (4) merge into (3); if $t \geq 3$ then $\delta(t) = 6$. When $s = 3$ and $n \in G_{t+2}$ with $t \geq 1$ we have

$$v_2\big(H(\{3\}^4;n)\big) = \begin{cases} (1) \quad -3(4t-1) & \text{if } 8 \leq 4n/2^t < 12, \\ (2) \quad -12t & \text{if } 12 \leq 4n/2^t < 14, \\ (3) \quad -3(4t-1) & \text{if } 14 \leq 4n/2^t < 15, \\ (4) \quad -3(4t-1)+1 & \text{if } 15 \leq 4n/2^t < 16. \end{cases} \tag{19}$$

Here if $t = 1$ then (3) and (4) merge into (4). When $s \geq 4$ we have

$$v_2\big(H(\{s\}^4;n)\big) = \begin{cases} (1) \quad -s(4t-1) & \text{if } 4 \leq 2n/2^t < 6, \\ (2) \quad -4st & \text{if } 6 \leq 2n/2^t < 7, \\ (3) \quad -4st + 2\,\mathfrak{p}(s-1) & \text{if } 7 \leq 2n/2^t < 8. \end{cases} \tag{20}$$

Equations (18)-(20) imply that $J(\{s\}^4|2) = \{0\}$ for all $s \geq 2$.

Similar computation shows that when $l = 5$ and $n \in G_{t+2}$ with $t \geq 1$ we have

$$v_2\big(H(\{2\}^5;n)\big) = \begin{cases} (1) \quad -s(5t-3) + 2\,\mathfrak{p}(s-1) & \text{if } 16 \leq 8n/2^t < 17, \\ (2) \quad -s(5t-3)+3 & \text{if } 17 \leq 8n/2^t < 18, \\ (3) \quad -s(5t-3) & \text{if } 18 \leq 8n/2^t < 20, \\ (4) \quad -s(5t-2) & \text{if } 20 \leq 8n/2^t < 24, \\ (5) \quad -s(5t-1) & \text{if } 24 \leq 8n/2^t < 28, \\ (6) \quad -s(5t-1)+1 & \text{if } 28 \leq 8n/2^t < 32. \end{cases} \tag{21}$$

Here if $t = 1$ then (1)-(3) do not appear; if $s \geq 3$ then (1) and (2)) merge into (1). This implies that $J(\{s\}^5|2) = \{0\}$ for all $s \geq 2$.

As the length becomes longer (i.e. l gets bigger) there are more and more cases. The number of cases, denoted by $C(l)$, is independent of s when s is large enough and tends to increase with l though not always. We compute the following

$$\begin{aligned} &C(6) = 5, && C(7) = 7, && C(8) = 6, && C(9) = 8, && C(10) = 8, \\ &C(11) = 11, && C(12) = 10, && C(13) = 12, && C(15) = 15, && C(16) = 12, \\ &C(17) = 15, && C(18) = 14, && C(19) = 18, && C(20) = 15. \end{aligned}$$

After tedious verification we find $J(\{s\}^l|2) = \{0\}$ for all $l \leq 20$ and $s \geq 2$. □

For any given l by similar method we should be able to determine $J(\{s\}^l|2)$ for all $s \geq 2$. However, for odd primes p we can only extend this result to $J(\{s\}^l|p)$ for small l and small s with the aid of computers.

Proposition 4.3. *Let s and l be two positive integers. Suppose $2 \leq s \leq 10$ and $2 \leq l \leq 10$. Then the p-divisible set $J(\{s\}^l|p)$ is finite for the consecutive five primes immediately after $ls + 2$. Moreover there's always some prime p such that $J(\{s\}^l|p) = RJ(\{s\}^l;p)$ where*

$$RJ(\{s\}^l;p) = \begin{cases} \{0, p-1\} & \text{if } 2 \nmid s, \\ \{0, i + (p-1)/2, p-1 : 0 \leq i \leq l-1\} & \text{if } 2|s. \end{cases}$$

Proof. The set $RJ(\mathbf{s};p)$ will be defined for general $\mathbf{s}$ in Definition 7.1 and computed in Theorem 7.2. The proof of the proposition follows from Corollary 2.2 by computer computation. To save space we put the details online [20]. □

In the rest of this section we turn to the case $s = 1$. We may assume $l \geq 2$ since the harmonic series has been handled by [8] and [4]. According Corollary 2.2 if we can find τ large enough such that $f_l(\tau) \geq (l-1)(\tau-1)$ then $J(1^l|p)$ is finite.

Proposition 4.4. *1. The p-divisible set $J(1^2|p)$ is finite if $p = 3, 7, 13, 31$.*

2. Let $s, t \leq 20$ and $t \geq 2$. Then the set $J(s, t|p)$ is finite for $p = 2, 3, 5$.

3. Let $r, s, t \leq 10$ and $t \geq 2$. Then the set $J(r, s, t|p)$ is finite for $p = 2, 3, 5$.

4. Let $q, r, s, t \leq 4$ and $t \geq 2$. Then the set $J(q, r, s, t|p)$ is finite for $p = 2, 3, 5$.

Proof. We only need to find τ satisfying the condition of Corollary 2.2.

(1) For each τ in the following we have $f_2(\tau) = \tau - 1$.

$p = 3$. Take $\tau = 6$. Then computation reveals that $J(1, 1|3) = \{0, 5\}$. If $l = 3$ then we take $\tau = 10$. Then we have $f_3(\tau) \geq 2(\tau - 1)$. Note that in G_9 there is $n = 17770$ such that $v_3\big(H(1^3; n)\big) = -15$ so $f_3(9) = 15$. By Corollary 2.2 and simple computation we see that $J(1^3|3) = \{0, 8\}$.

$p = 7$. Take $\tau = 4$. Then $J(1, 1|7) = \{0, 4, 6, 7, 13\}$.

$p = 13$. Take $\tau = 4$. Then $J(1, 1|13) = \{0, 12, 13, 25\}$.

$p = 31$. Take $\tau = 4$. Then $J(1, 1|31) = \{0, 17, 22, 30, 31, 61\}$.

For the last three cases with $p = 2, 3, 5$ we put the result of computation online [20]. For example, we can take $\tau = 10$ and show that $J(1, 1, 1|3) = \{0, 8\}$. □

Remark 4.5. We could extend our results to larger l and some other primes p but it would be very time consuming with our slow PCs. However, even in the case $\mathbf{s} = (1, 1)$ similar process fails for $p = 2$. Computations suggest that $J(1, 1|2) = \{0\}$, $J(1, 1|5) = \{0, 4, 5, 9\}$, $J(1, 1|11) = \{0, 10, 11, 21\}$ and $J(1, 1|17) = \{0, 11, 13, 16, 17, 33\}$. We will analyze the situation for $p = 2$ in detail in the next section.

5 Sequences related to $J(s, 1|2)$

One may wonder what goes wrong in Proposition 4.4 if we let $\mathbf{s} = (1, 1)$ and $p = 2$. We will see that, amazingly, this problem is closely related to some pseudo-random process.

Only in this section we adopt the shorthand $H_1(n) := H(1; n)$ and $H_2(n) := H(1, 1; n)$. Let's start with the first few partial sums of $H_2(n)$ when $2 \leq n \leq 14$. Here $\sim$ means we only consider the fractional part of the numbers.

$$H_2(2) \sim \frac{1}{2}, \quad H_2(3) \sim 1, \quad H_2(4) \sim \frac{11}{24}, \quad H_2(5) \sim \frac{7}{8}, \quad H_2(6) \sim \frac{23}{90},$$
$$H_2(7) \sim \frac{109}{180}, \quad H_2(8) \sim \frac{9371}{10080}, \quad H_2(9) \sim \frac{467}{2016}, \quad H_2(10) \sim \frac{25933}{50400},$$
$$H_2(11) \sim \frac{25933}{50400}, \quad H_2(12) \sim \frac{39353}{50400}, \quad H_2(13) \sim \frac{13501}{415800}, \quad H_2(14) \sim \frac{4027}{14850}.$$

It looks like 2 never divides the numerator and moreover, the 2-powers in the denominators of $H_2(n)$ tend to increase with n, though not always. To proceed we need to know the 2-divisibility of $H_1^*(n)$.

Lemma 5.1. *Let n be a positive integer. Then*

$$H_1^*(n) \equiv \begin{cases} 0 \pmod 4 & \text{if } n \equiv 0, 3 \pmod 4, \\ 1 \pmod 4 & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$$

Proof. If n is even then obviously $H_1^*(n) = H_1^*(n-1)$. So we only need to consider $n \equiv 1, 3 \pmod 4$.

Set $\delta = 1$ if $n = 4l + 1$ and $\delta = 0$ if $n = 4l - 1$. Then

$$H_1^*(n) = \frac{\delta}{4l+1} + \sum_{i=1}^{2l} \frac{1}{2i-1} = \frac{\delta}{4l+1} + \sum_{i=1}^{l} \left(\frac{1}{2i-1} + \frac{1}{4l-2i+1} \right) \equiv \delta \pmod 4$$

as desired. □

Remark 5.2. By working more carefully we can obtain the following improvement of Lemma 5.1: if $n = 2^d m$ or $n = 2^d m - 1$ where m is odd and $d \geq 1$. Then $v_2\big(H_1^*(n)\big) = 2(d-1)$. However, the proof is complicated and it is not needed in the rest of the paper so we leave the proof of this general statement to the interested reader.

The following result is exactly the reason why Corollary 2.2 cannot be applied to $J(1,1|2)$.

Proposition 5.3. *For any $t \geq 2$, there is a unique $n_t \in G_t$ such that $v_2\big(H_2(n_t)\big) \geq 2 - t$ whereas for all $n_t \neq n \in G_t$ we have $v_2\big(H_2(n)\big) \leq 1 - t$. Therefore, for all positive integers $n \notin \{n_t\}_{t\geq 1}$ the numerator of $H_2(n)$ is an odd integer.*

Proof. Note that $G_1 = \{1\}$ and $G_2 = \{2,3\}$. Thus $n_2 = 3$ because $H_2(3) = 1$. Assume that $t \geq 3$ and each n_i has been found in G_i uniquely for $i \leq t$. Let $n = 2\tilde{n} + r \in G_{t+1}$ for $r = 0$ or 1. When $l = p = 2$ and $s = 1$ Eq. (3) becomes

$$H_2(n) = H_2^*(n) + \frac{1}{2}H_1(\tilde{n})H_1^*(n) + \frac{1}{4}H_2(\tilde{n}). \tag{22}$$

It's easy to show that $v_2\big(H_1(m)\big) = 1 - t$ for $m \in G_t$ by induction and the recursive relation $H_1(n) = H_1^*(n) + H_1(\tilde{n})/2$. If $\tilde{n} \neq n_t$ then we have $v_2\big(H_2(\tilde{n})\big) \leq 1 - t$ and hence

$$v_2\big(H_2(n)\big) = \min\{v_2\big(H_1^*(n)\big) - t, -1 - t\} = -1 - t < 1 - (t+1).$$

Suppose now $\tilde{n} = n_t$ and $n = 2n_t + r_t$. We consider four possible cases.

(i) If $v_2\big(H_2(n_t)\big) = 2 - t$ and $v_2\big(H_1^*(n)\big) \geq 1$ then $n \neq n_{t+1}$ because

$$v_2\big(H_2(n)\big) = \min\{v_2\big(H_1^*(n)\big) - t, -t\} = -t = 1 - (t+1).$$

(ii) If $v_2\big(H_2(n_t)\big) = 2 - t$ and $v_2\big(H_1^*(n)\big) = 0$ then

$$v_2\left(\frac{1}{2}H_1(n_t)H_1^*(n)\right) = v_2\left(\frac{1}{4}H_2(n_t)\right) = -t.$$

Hence $n = n_{t+1}$ because

$$v_2\big(H_2(n)\big) \geq 1 - t = 2 - (t+1).$$

(iii) If $v_2\big(H_2(n_t)\big) \geq 3-t$ and $v_2\big(H_1^*(n)\big) = 0$ then $n \neq n_{t+1}$ because $v_2\big(H_2(n)\big) = \min\{v_2\big(H_1^*(n)\big) - t, v_2\big(H_2(n_t)\big) - 2 - t\} = -t = 1 - (t+1)$.

(iv) If $v_2\big(H_2(n_t)\big) \geq 3 - t$ and $v_2\big(H_1^*(n)\big) \geq 1$ then $n = n_{t+1}$ because

$$v_2\big(H_2(n)\big) \geq \min\{v_2\big(H_1^*(n)\big) - t, v_2\big(H_2(n_t)\big) - 2 - t\} \geq 1 - t = 2 - (t+1).$$

Now if $n_t = 2l$ is even then by Lemma 5.1

(1) $2n_t + 1 \equiv 1 \pmod 4$ and $v_2\big(H_1^*(2n_t+1)\big) = 0$, and

(2) $2n_t \equiv 0 \pmod 4$ and $v_2\big(H_1^*(2n_t+1)\big) \geq 1$.

If $n_t = 2l + 1$ is odd then by Lemma 5.1

(3) $2n_t + 1 \equiv 3 \pmod 4$ and $v_2\big(H_1^*(2n_t+1)\big) \geq 1$ and

(4) $2n_t \equiv 2 \pmod 4$ and $v_2\big(H_1^*(2n_t+1)\big) = 0$.

Therefore, we have four situations to consider:

(a) n_t is even and $v_2\big(H_2(n_t)\big) = 2 - t$. Then $n_{t+1} = 2n_t + 1$ by (1) and (ii).

(b) n_t is even and $v_2\big(H_2(n_t)\big) \geq 3 - t$. Then $n_{t+1} = 2n_t$ by (2) and (iv).

(c) n_t is odd and $v_2\big(H_2(n_t)\big) \geq 3 - t$. Then $n_{t+1} = 2n_t + 1$ by (3) and (iv).

(d) n_t is odd and $v_2\big(H_2(n_t)\big) = 2 - t$. Then $n_{t+1} = 2n_t$ by (4) and (ii).

It follows that $n_{t+1} \in G_{t+1}$ is uniquely determined. This finishes the proof of the proposition by induction. ∎

Denote the dyadic valuation $v_2\big(H_2(n_t)\big)$ by $-w_t$. Then we have the following two interesting sequences:

$$\begin{aligned}\{n_t\}_{t\geq 2} = \{&3,6,13,27,54,109,219,439,879,1759,3518,7037,14075,28151,\\ &56303,112606,225212,450424,900848,1801696,3603393,\dots\} \quad (23)\\ \{w_t\}_{t\geq 2} = \{&0,1,3,4,3,3,5,7,9,10,9,10,12,14,13,13,15,17,19,19,\dots\} \quad (24)\end{aligned}$$

Set $r_1 = r_2 = 1$ and define $r_t = 0$ or 1 for $t \geq 3$ as determined in the proof of Proposition 5.3 such that $n_{t+1} = 2n_t + r_t$. Then clearly n_t can be written as

$$n_t = (r_1 r_2 \dots r_t)_2 \tag{25}$$

in binary system and apparently the sequence $\{n_t\}$ increases very fast. Further, the occurrence of $r_t = 0$ or $r_t = 1$ does not seem to have any predictable pattern and in fact we believe it is related to some pseudo-random process. By this we mean the following. First we of course conjecture that $\{w_t\}_{t\geq 3}$ is always bounded below by 1 which is equivalent to say $J(1,1|2) = \{0\}$. We also have proved that w_t is bounded above by $t-1$ and it is not hard to see that $w_t = t-1$ for infinitely many t's. It's also conceivable that w_t are near $t-1$ most of the time. However, we believe w_t could move very far away from $t-1$ for very large t. At the present stage, we could not even determine whether the difference between w_t and $t-1$ can be arbitrarily large.

Remark 5.4. We put the two sequences $\{n_t\}_{t\geq 2}$ and $\{w_t\}_{t\geq 2}$ in Sloane's online database of integer sequences as $A079403(n)$ and $A079404(n)$, respectively. Shortly after Benoit Cloitre emailed me a formula for the known terms of $\{n_t\}_{t\geq 2}$:

$$n_t = \lfloor 2^{t-1} c \rfloor \tag{26}$$

where $c = 1.718232....$ Indeed, it's easy to see that

$$n_1 = 1, n_t = 2^{t-1} \prod_{k=1}^{t-1} \left(1 + \frac{r_k}{2n_k}\right), \ \forall t \geq 2.$$

Further,

$$c = \lim_{t\to\infty} \prod_{k=1}^{t-1} \left(1 + \frac{r_k}{2n_k}\right)$$

exists by comparison test. From Eq. (25) we get

$$c = (r_1.r_2r_3r_4\dots)_2 = (1.101101111101111000001...)_2 = 1.718232...$$

Moreover, using binary system we see that the integral part of $2^{t-1}c$ is exactly n_t, as desired.

We can easily generalize Proposition 5.3 to the following.

Proposition 5.5. *For any $t \geq 2$, there is a unique $n_t \in G_t$ such that $v_2\big(H(s,1;n_t)\big) \geq -s(t-1)+1$ whereas for all $n_t \neq n \in G_t$ we have $v_2\big(H(s,1;n)\big) \leq -s(t-1)$. Therefore, for every positive integer $n \notin \{n_t\}_{t\geq 1}$ the numerator of $H(s,1;n)$ is not divisible by 2.*

Proof. We can assume that $s \geq 2$ because of Proposition 5.3. The key to the proof is Eq. (4) which yields that

$$H(s,1;n) = H^*(s,1;n) + U(s,1;n) + V(s,1;n) + \frac{1}{2^{s+1}} H(s,1;\tilde{n})$$

where

$$U(s,1;n) = \frac{1}{2^s} \sum_{1\leq 2k<l\leq n, 2\nmid l} \frac{1}{k^s l}, \quad V(s,1;n) = \frac{1}{2} \sum_{1\leq k<pl\leq n, 2\nmid k} \frac{1}{k^s l}.$$

Now it's easy to see that if $v_2\big(H(s,1;\tilde{n})\big) \leq -s(t-1)$ then $v_2\big(H(s,1;n)\big) \leq -st$.

When $t=2$ we find $n_2=2$ always because

$$H(s,1;2)=\frac{1}{2},\quad H(s,1;3)=\frac{1}{2}+\frac{2^s+1}{3\cdot 2^s}.$$

Assume that $t\ge 3$ and $\tilde{n}$ is the unique $n_t\in G_t$ such that $v_2\big(H(s,1;\tilde{n})\big)>-s(t-1)$. Then $v_2\big(H(s;\tilde{n})\big)=-s(t-1)$ by Eq. (13). So we can always uniquely choose r_t so that for $n=2\tilde{n}+r_t\in G_{t+1}$

$$v_2\big(U(s,1;n)\big)=v_2\big(U(s,1;2\tilde{n})+rH(s;\tilde{n})/2^s\big)=-st$$

if $v_2\big(H(s,1;\tilde{n})\big)=-s(t-1)+1$. If $v_2\big(H(s,1;\tilde{n})\big)>-s(t-1)+1$ then we can uniquely choose r_t so that

$$v_2\big(U(s,1;n)\big)\ge -st+1.$$

The upshot is for $n_t\in G_t$ there is a unique $n_{t+1}\in G_{t+1}$ satisfying the condition of the proposition. This finishes the proof. ☐

In general, we cannot apply Criterion Theorem to determine the finiteness of $J((\mathbf{s},1)|2)$ for any $\mathbf{s}\in\mathbb{N}^l$, because of the existence of similar sequences. Moreover we believe 2 never divides the numerator of any multiple harmonic sum.

Conjecture 5.6. *Let l be an arbitrary positive integer and $\mathbf{s}\in\mathbb{N}^l$. Then the 2-divisible set $J(\mathbf{s}|2)=\{0\}$.*

We have verified this conjecture for all $\mathbf{s}=(s,t)$ and $\mathbf{s}=(r,s,t)$ with $1\le r,s\le 10$ and $2\le t\le 10$, and for all $\mathbf{s}=(u,r,s,t)$ with $1\le u,r,s\le 4$ and $2\le t\le 4$. See [20]. The computation is very time-consuming, for example when $\mathbf{s}=(1,4,4,2)$ the Maple program runs more than 3.5 hours on my PC with Pentium 4 CPU 3.06GHZ and 512 MB RAM. The same program in GP Pari runs a little faster. We put the program at the end of our online supplement [20].

We believe that among all possible $\mathbf{s}$ and prime p the cases $(\mathbf{s},1)$ are the only ones that our Criterion Theorem fails (see [20]). Let me sketch a heuristic argument for this belief for the case $\mathbf{s}=1^l$ and $p\ge 3$.

By the recursive relation

$$H(1^l;n)=\sum_{j=0}^{l}p^{-j}\cdot H(1^j;\tilde{n})\cdot H^*(1^{l-j};n).$$

it is not hard to see that the size of τ we are looking for in the Criterion Theorem depends on the length of the sequences $\{n_t\}_{t>t_0}$ not satisfying the condition in the theorem, where $n_t\in G_t$ and $n_{t+1}=pn_t+r_t$ for some $0\le r_t<p$. If n_t is already found then the existence of n_{t+1} depends on $H^*(1;n)$ essentially, which we assume to distribute among $(p+1)/2$ values modulo p by the symmetric structure of $J_1(1|p)$ (see Sec. 6). So n_t produces two possible n_{t+1} or no n_{t+1} with the same probability $q=(p-1)/(2p)$, and it produces exactly one n_{t+1} with probability $1/p$.

Let's assume that a certain cell reproduces itself according a similar law as above, namely, it clones itself or dies in the next generation with the same probability q, and with probability $1/p$ it stays alive without reproduction. Let p_k be the probability that starting from k cells in the beginning the cells eventually all die out. We claim that $p_k=1$ for all k. Indeed, it is not too hard to see that we only need to show $p_1=1$. This follows from the Criticality Theorem for Galton-Watson branching process (see [10, Preface] or [1, p. 7, Theorem 1]) because the average offspring is $2q+1/p=1$.

6 The structure of $J_1(\{s\}^l|p)$

Set $J_t^0(\mathbf{s}|m)=\{0\}\cup J_t(\mathbf{s}|m)$ for any positive integers t, m and $\mathbf{s}\in\mathbb{N}^l$. The next result is easy but very useful in determining the structure of $J_1(s|p)$ since it tells us essentially that $J_1^0(s|p)$ is symmetric about $(p-1)/2$.

Proposition 6.1. *Let p be an odd prime and $s\in\mathbb{N}$. Let $r\in\{1,\dots,p-2\}$. Then $r\in J_1(s|p)$ if and only if $p-1-r\in J_1(s|p)$.*

Proof. If $p-1|s$ then $J_1(s|p)=\emptyset$ because $H(s;r)\equiv r \pmod p$ for all $r\in\{1,\dots,p-2\}$. If $p-1\nmid s$ then we have

$$H(s;r)=\sum_{k=1}^{r}\frac{1}{k^s}=\sum_{k=p-r}^{p-1}\frac{1}{(p-k)^s}\equiv(-1)^s\sum_{k=p-r}^{p-1}\frac{1}{k^s} \pmod p.$$

Subtracting $0\equiv(-1)^s\sum_{k=1}^{p-1}\frac{1}{k^s} \pmod p$ from the above we get the desired result. □

Remark 6.2. We feel prompted to mention that the symmetry of $J_1^0(s|p)$ is not enjoyed by $J_1^0(s|p^2)$. For instance, while $J_1^0(5|37)=\{0,6,9,12,18,24,27,30,36\}$ is symmetric about 18 the set $J_1^0(5|37^2)=\{0,6,36\}$ is not.

Now that we know $J_1^0(s|p)$ is symmetric we may wonder what happens to the center $(p-1)/2$. When s is odd the answer is related to the irregularity of primes.

Proposition 6.3. *Let p be an odd prime and s be a positive integer such that $p-1\nmid s$. Let n be the unique positive integer such that $s\equiv n \pmod{p-1}$ and $2\le n\le p-2$. If $n<p-3$ then we have*

$$H(s;(p-1)/2)\equiv\begin{cases}\frac{2-2^p}{p} \pmod p, & \text{if } n=1;\\ \frac{2-2^n}{n}B_{p-n} \pmod p, & \text{if } n>1 \text{ is odd;}\\ \frac{n(1-2^{n+1})}{2(n+1)}pB_{p-n-1} \pmod{p^2}, & \text{if } n \text{ is even.}\end{cases} \tag{27}$$

Therefore,

(a) If s is even then $(p-1)/2\in J_1(s|p)$.

(b) If s is odd and $(p,p-n)$ is an irregular pair then $(p-1)/2\in J_1(s|p)$.

(c) If s is odd and $(p-1)/2\in J_1(s|p)$ and $2^s\not\equiv 2 \pmod p$ then $(p,p-n)$ is an irregular pair.

In particular, if $s\ge 3$ is odd and $p>2^s-2$ then $(p-1)/2\in J_1(s|p)$ if and only if $(p,p-s)$ is an irregular pair.

Proof. The congruence (27) is essentially [17, Cor. 5.2(b)]. The rest follows immediately. □

Remark 6.4. Proposition 6.3 tells us that when $s\equiv 1 \pmod p$ then $(p-1)/2\in J(s|p)$ if and only if p is a Wieferich prime. There are only two known such primes: $p=1093$ and $p=3511$. If any others exist, they must be greater than 1.2510^{15} according to [15].

The above proposition says that if s is even and $p-1\nmid s$ then $(p-1)/2\in J_1(s|p)$. A natural question is when $(p-1)/2\in J_1(s|p^2)$? The answer is given below.

Corollary 6.5. *Suppose s is a positive integer such that $p-1\nmid s$. Suppose n is the integer between 2 and $p-1$ such that $n\equiv s \pmod{p-1}$ and $n<p-4$. For n even $H(s;(p-1)/2)\equiv 0 \pmod{p^2}$ if and only if either $(p,p-n-1)$ is an irregular pair or $p|2^{n+1}-1$. If n odd, $p|2^n-2$ and $(p,p-n)$ is an irregular pair then $H(s;(p-1)/2)\equiv 0 \pmod{p^2}$.*

Proof. This is clear for n even. If n is odd then the corollary follows from [17, Thm. 5.2(b)]. □

Remark 6.6. For every positive even integer s and every irregular prime $p>s+4$ up to 100,000, p^2 always exactly divides $H(s;(p-1)/2)$. Is this true in general? The answer is no. A calculation by Maple shows that for the 5952nd irregular pair $(p,p-n-1)=(130811,52324)$ we have $n=78486$ and $2^{n+1}\equiv 2 \pmod p$ and therefore $p^2|H(n+1;(p-1)/2)$ and $p^3|H(n;(p-1)/2)$ by [17, Thm. 5.2(a),(b)]. The peculiarity of this pair was already noticed in [6]. The next two such pairs are (599479,359568) (see [7]), and (2010401,1234960) (see [5]). Note that apparently this problem is *not* related to the problem of $2^p\equiv 2 \pmod{p^2}$.

Theorem 6.7. *Let s be a positive integer and $p>2ls+1$ be an odd prime. Then*

$$\{p-1,j+(p-1)/2:j=0,1,\cdots,l-1\}\subset J_1\big(\{2s\}^l|p\big).$$

Proof. Let $m = (p-1)/2$. By [21, Lemma 2.12] there are integers c_λ such that

$$l!H\big(\{s\}^l; n\big) = \sum_{\lambda\in P(l)} c_\lambda H_\lambda(s;n), \tag{28}$$

where $P(l)$ is the set of partitions of l, $H_{(\lambda_1,\dots,\lambda_r)}(s;n) = \prod_{j=1}^r H(\lambda_j s; n)$ and $c_{(l)} = (-1)^{l-1}(l-1)!$. Plugging in $n = m$ we get $m \in J_1\big(\{2s\}^l|p\big)$ by Proposition 6.3. By definition $H(\{s\}^l; q) = 0$ for $q = 1, \cdots, l-1$. When $q = 1$ this implies that $\sum_{\lambda\in P(l)} c_\lambda = 0$ by Eq. (28). Hence $m+1 \in J_1\big(\{2s\}^l|p\big)$. Similarly, because $1, 1/2^{2s}, \cdots, 1/(l-1)^{2s}$ are linearly independent when regarded as a function of s, we see that for all independent variables $x_1, \dots, x_j$, $j \le l-1$, we have

$$\sum_{\lambda=(\lambda_1,\dots,\lambda_r)\in P(d)} c_\lambda \prod_{j=1}^r (x_1^{\lambda_1} + \cdots x_j^{\lambda_j}) = 0.$$

The theorem now follows from setting $x_j = 1/(m+j)^{2s}$ for $j = 1, \cdots, l-1$. ☐

Corollary 6.8. *Let s be a positive integer and $p > 4s+1$ be an odd prime. Then*

(1) If s is even then $(p-1)/2, (p+1)/2 \in J_1(s,s|p)$.

(2) If s is odd and $(p, p-s)$ is an irregular pair then $(p-1)/2 \in J_1(s,s|p)$. Further, if s is odd, $2^s \not\equiv 2 \pmod p$, and $(p-1)/2 \in J_1(s,s|p)$, then $(p,p-s)$ is an irregular pair. In particular, if $s \ge 3$ is odd and $p > 2^s - 2$ then $(p-1)/2 \in J_1(s,s|p)$ if and only if $(p,p-s)$ is an irregular pair.

Proof. Let s and p be the integers satisfying the conditions of the corollary. When s is even the corollary follows from Theorem 6.7. If s is odd then by Proposition 6.3 and the shuffle relation we have

$$2H(s,s;(p-1)/2) \equiv H(s;(p-1)/2)^2 - H(2s;(p-1)/2). \tag{29}$$

The corollary follows immediately. ☐

7 Reserved set of MHS

In Conjecture B of [8] Eswarathasan and Levine state that there should be infinitely many primes p such that the divisible set $J(1|p) = \{0, p-1, p^2-p, p^2-1\}$. Boyd [4] further suggest $1/e$ as the expected density of such primes. The most important steps are to elucidate the structure of $J_1(1|p)$ and determine the relation between $J_t(1|p)$ and $J_{t+1}(1|p)$ for $t > 0$. We put forward some similar results and conjectures concerning the divisible sets of general MHS in this last section.

Definition 7.1. For any $\mathbf{s} \in \mathbb{N}^l$ there are finitely many function $f_0(x) = 0, f_1(x), \dots, f_r(x) \in \mathbb{Q}[x]$ such that for all primes $p \ge wt(\mathbf{s}) + 3$

$$f_0(p) < f_1(p) < \cdots < f_r(p) \text{ and } f_0(p), f_1(p), \dots, f_r(p) \in J(\mathbf{s}|p).$$

We call the largest r the *reserved (divisibility) number* of MHS $H(\mathbf{s})$, denoted by $\rho(\mathbf{s})$. We call the corresponding set $\{f_0(x), \dots, f_{\rho(\mathbf{s})}(x)\}$ the *reserved (divisibility) set* of $\zeta(\mathbf{s})$, denoted by $RJ(\mathbf{s}) = RJ(\mathbf{s}; x)$. Its t-th segment is $RJ_t(\mathbf{s}) = \{f(x) \in RJ(\mathbf{s}) : f(p) \le p^t - 1 \text{ for all prime } p\}$ for $t \ge 1$. Note that $0 \in RJ_t(\mathbf{s})$ for all $t \ge 0$. If $J(\mathbf{s}|p) = RJ(\mathbf{s}; p)$ for some prime p then is called a *reserved prime* for MHS $H(\mathbf{s})$.

For example, the reserved number of the harmonic series is 3, the reserved set is $RJ(1) = \{0, x-1, x^2-x, x^2-1\}$, and 5 is a reserved prime for the harmonic series because $J(1|5) = \{0, 4, 20, 24\}$. We summarize all known reserved sets in the following theorem.

Theorem 7.2. *Let $r, s, t, l \le 5$ be positive integers. Then*

(1). $RJ(1) = \{0, x-1, x^2-1, x^2-x\}$.

(2). *If $1 \ne l$ is odd then $RJ_{10}(1^l) = \{0, x-1\}$.*

(3). *If $s \ge 3$ is odd then $RJ_{10}(\{s\}^l) = \{0, x-1\}$.*

(4). *If s is even then* $RJ_8(\{s\}^l)=\{0, i+(x-1)/2, x-1 : 0\le i\le l-1\}$.

(5). *If $s\ge 3$ is odd then* $RJ(s,1)=\{0,x-1,x\}$.

(6). *If $s+t$ is odd then* $RJ(s,t)=\{0\}$.

(7). *If $s\ne t$, $s+t$ is even and $t\ne 1$ then* $RJ(s,t)=\{0,x-1\}$.

(8). *If $s\ne t$, $s+t$ is even and $t\ne 1$ then* $RJ(s,t,s,t)=\{0,x-1\}$.

(9). *If $r+s+t\ge 5$ is odd, $r,s,t\ge 2$, and $r\ne t$, then* $RJ(r,s,t)=\{0\}$.

(10). *If s is odd then* $RJ(r,s,r)=\{0,x-1\}$.

(11). *If $r+s+t\ge 6$ is even, $(r,s,t)\ne(4,3,5),(5,3,4)$, and $r,s,t\ge 2$ are not all the same, then* $RJ(r,s,t)=\{0\}$.

(12). $RJ_{10}(2,1,1)=RJ(1,1,2)=RJ(4,3,5)=RJ(5,3,4)=\{0,x-1\}$.

(13). *If $s,t\ge 0$ and $s+t$ is even then* $RJ_{10}\big(1^s,2,1^t\big)=\{0,x-1\}$.

(14). *If $s\ge 1$ then* $RJ_{10}\big(1^s,2,1^s,2,1^s\big)=\{0,x-1\}$.

(15). *If $s\ge 2$ is even then* $RJ_{10}\big(1^s,2,1^{s-1},2,1^{s+1}\big)=\{0,x-1\}$.

(16). *If $s\ge 2$ is even then* $RJ_{10}\big(1^{s+1},2,1^{s-1},2,1^{s}\big)=\{0,x-1\}$.

(17). *If l is even then* $RJ(1^l)=\{0,x-1,x,2x-1\}$.

(18). $RJ_{10}(1,2,1)=\{0,x-1,2x-1\}$.

Remark 7.3. We are sure that we can remove the subscript 10 in RJ_{10} when more powerful computational tools are available to us.

Proof. Even without the restriction of the bound 5, the inclusions $RJ(\mathbf{s};p)\subseteq J(\mathbf{s}|p)$ follow from Eq. (28), Theorem 1.1, [21, Theorems 2.14, 3.1, 3.5, 3.16, 3.18], and Theorem 6.7, except in case (7.2) and the last two cases.

For $\mathbf{s}=1^l$, $l\ge 2$, Theorem 1.1 implies $p-1\in J\big(1^l|p^{\mathfrak{p}(l-1)}\big)$ for all $l\ge 1$ and $p\ge l+3$. In fact, by [21, Theorems 2.8, 2.14] we have

$$H(1^l;p-1)\equiv\frac{H(l;p-1)}{(-1)^{l-1}l}\equiv\begin{cases}-\dfrac{pB_{p-l-1}}{l+1} & \pmod{p^2}\quad\text{if } 2|l,\\ -\dfrac{p^2(l+1)B_{p-l-2}}{2l+4} & \pmod{p^3}\quad\text{if } 2\nmid l.\end{cases}\tag{30}$$

So if $p\ge l+3$ then

$$H(1^l;p)=H(1^l;p-1)+\frac{H(1^{l-1};p-1)}{p}\equiv\begin{cases}\dfrac{-p(l+2)B_{p-l-1}}{2(l+1)} & \pmod{p^2}\quad\text{if } 2|l,\\ -\dfrac{B_{p-l}}{l} & \pmod{p}\quad\text{if } 2\nmid l.\end{cases}\tag{31}$$

Further, setting $h_i=H(1^i;p-1)$, $h_0=1$ and $h_{-1}=0$ we have

$$\begin{aligned}H(1^l;2p-1)&=\sum_{i=0}^{l}\sum_{1\le k_1<\dots<k_i\le p<k_{i+1}<\dots<k_l<2p}\frac{1}{k_1k_2\dots k_l}\\ &\equiv\sum_{i=0}^{l}\Big(h_i+\frac{h_{i-1}}{p}\Big)\Big(h_{l-i}+(-1)^{l-i}pH(l-i+1;p-1)\Big)\pmod{p^2}.\end{aligned}\tag{32}$$

Here we have used geometric series expansion inside $\mathbb{Q}_p$ such that for any $m<p$

$$\begin{aligned}\sum_{1\le i_1<\dots<i_m<p}\frac{1}{(p+i_1)\cdots(p+i_m)}&\equiv h_m-p\sum_{i=1}^{m}H\big((1^{i-1},2,1^{m-i});p-1\big)\pmod{p^3}\\ \equiv h_m+p((m+1)h_{m+1}-h_1h_m)&\equiv h_m+(-1)^m pH(m+1;p-1)\pmod{p^3}\end{aligned}$$

by Eq. (28). It follows from equations (28), (32) and [21, Theorem 3.1] that

$$H(1^l;2p-1)\equiv 2h_l+\frac{1}{p}\sum_{i=0}^{l-1}h_ih_{l-1-i}-(-1)^lH(l;p-1)\pmod{p^2}.$$

When l is even we have

$$H(1^l;2p-1)\equiv -\frac{(l+2)H(l;p-1)}{l}+\frac{2H(l-1;p-1)}{p(l-1)}\equiv -2pB_{p-l-1}\pmod{p^2}.$$

So it's divisible by p. This shows that $2x-1$ belongs to the reserved set $RJ(1^l)$ in the penultimate case of the theorem. When $l=2n+1$ is odd $h_l\equiv 0\pmod{p^2}$ and we get

$$H(1^l;2p-1)\equiv \frac{1}{p}\sum_{i=0}^{n}h_{2i}h_{2n-2i}\equiv \frac{-H(2n;p-1)}{np}\equiv \frac{-2B_{p-l}}{l}\pmod{p}$$

which is rarely congruent to 0. This explains why in case (2) we can't have $2x-1$ in the reserved set $RJ(1^l)$ when l is odd.

Finally let's turn to the last case of reserved set $RJ(1,2,1)$. We have for any prime $p\geq 7$

$$H(1,2,1;2p-1)=\sum_{1\leq l<m<n<2p}\frac{1}{lm^2n}=A+B+C+D,$$

where

$$\begin{aligned}
A&=\sum_{1\leq l<m<n\leq p}\frac{1}{lm^2n}=H(1,2,1;p-1)+\frac{1}{p}H(1,2;p-1),\\
B&=\sum_{1\leq l<m\leq p<n<2p}\frac{1}{lm^2n}=\Big(H(1,2;p-1)+\frac{1}{p^2}H(1;p-1)\Big)\sum_{1\leq k<p}\frac{1}{p+k},\\
C&=\sum_{1\leq l\leq p<m<n<2p}\frac{1}{lm^2n}=\Big(H(1;p-1)+\frac{1}{p}\Big)\sum_{1\leq m<n<p}\frac{1}{(p+m)^2(p+n)},\\
D&=\sum_{p<l<m<n<2p}\frac{1}{lm^2n}=\sum_{1\leq l<m<n<p}\frac{1}{(p+l)(p+m)^2(p+n)}.
\end{aligned}$$

Observe that $D\equiv H(1,2,1;p-1)\equiv 0\pmod{p}$ by [21, Corollary 3.6] and $H(1;p-1)\equiv 0\pmod{p^2}$ by Wolstenholme's Theorem. By geometric series expansion we get

$$\sum_{1\leq m<n<p}\frac{1}{(p+m)^2(p+n)}\equiv\sum_{1\leq m<n<p}\frac{(1-2p/m)(1-p/n)}{m^2n}\pmod{p^2}.$$

Hence

$$\begin{aligned}
H(1,2,1;2p-1)\equiv{}&\frac{1}{p}\big(H(1,2;p-1)+H(2,1;p-1)\big)\\
&-2H(3,1;p-1)-H(2,2;p-1)\pmod{p}.
\end{aligned}$$

By Theorem 1.1 and [21, Theorem 3.1] we have $H(2,2;p-1)\equiv H(3,1;p-1)\equiv 0\pmod{p}$. Further, from shuffle relation we have

$$H(1,2;p-1)+H(2,1;p-1)=H(1;p-1)H(2;p-1)-H(3;p-1)\equiv 0\pmod{p^2}$$

by Theorem 1.1. This shows that $H(1,2,1;2p-1)\equiv 0\pmod{p}$.

To prove the theorem we now only need to demonstrate that $RJ(\mathbf{s};p)=J(\mathbf{s}|p)$ for some $p\geq|\mathbf{s}|+3$ which can be done through a case by case computation. We put this part of verification online [20]. In fact, much more data are available in this supplement. ☐

Are there any other $\mathbf{s} \in \mathbb{N}^l$ ($l \leq 3$) besides those listed in Theorem 7.2 such that $RJ(\mathbf{s}) \neq \{0\}$? In view of the last conjecture of [21] we believe there are.

From Theorem 7.2 we see that to determine $RJ(\mathbf{s})$ we often only need to study $RJ_1(\mathbf{s})$ because for all non-homogeneous $\mathbf{s}$ not equal to $(2r-1,1)$ or $(1,2,1)$, the proportion of primes p such that $J_t(\mathbf{s}|p) = \emptyset$ for all $t \geq 2$ is supposed to be positive. This implies that $RJ(\mathbf{s}) = RJ_1(\mathbf{s})$ for all such $\mathbf{s}$. Precisely, we have the following

Conjecture 7.4. *Suppose* $\mathbf{s} \in \mathbb{N}^l$ *such that (i)* $\mathbf{s} = 1$*, or (ii)* $\mathbf{s} = (1,2,1)$*, or (iii)* $\mathbf{s} = (2r-1,1)$ *for some* $r \geq 1$*, or (iv)* $\mathbf{s} = 1^{2l}$ *for some* $l \geq 1$*. Then* $RJ(\mathbf{s}) = RJ_2(\mathbf{s})$*. For all other* $\mathbf{s}$ *we have* $RJ(\mathbf{s}) = RJ_1(\mathbf{s})$.

When $a \geq 2$ we assume that $H(\mathbf{s}'; ap-1)$ has random distribution modulo p^2 (the case $a=2$ and $\mathbf{s}' = (1,2,1)$ has to be dealt with separately, but that's not hard). Thus the chance that $p|H(\mathbf{s}; ap)$ is less than $1/p^3$ for large p. This implies that the probability of $J_2(\mathbf{s}|p) = \emptyset$ is roughly $(1-1/p^3)^{p^2} \to 1$ as $p \to \infty$.

Definition 7.5. Let $\mathbf{s} \in \mathbb{N}^l$. Define the *reserved density* of the MHS $H(\mathbf{s})$ by

$$\text{density}(RJ(\mathbf{s}); X) = \frac{\sharp\{\text{prime } p : |\mathbf{s}| + 2 < p < X, J(\mathbf{s}|p) = RJ(\mathbf{s})\}}{\sharp\{\text{prime } p : |\mathbf{s}| + 2 < p < X\}} \tag{33}$$

and the mth *reserved density* by

$$\text{density}(RJ^m(\mathbf{s}); X) = \frac{\sharp\{\text{prime } p : |\mathbf{s}| + 1 < p < X, \cup_{t=0}^{m} J_t(\mathbf{s}|p) = RJ_m(\mathbf{s})\}}{\sharp\{\text{prime } p : |\mathbf{s}| + 2 < p < X\}}. \tag{34}$$

Conjecture 7.6. *Let* $\mathbf{s} \in \mathbb{N}^l$*. Then*

$$\text{density}(RJ(\mathbf{s}); \infty) = \begin{cases} 1/\sqrt{e}, & \text{if } l = 1, \mathbf{s} \geq 2, \\ 1/e, & \text{if } l = \mathbf{s} = 1 \text{ or } l \geq 2. \end{cases}$$

Note that we always have $RJ(\mathbf{s}; p) \subseteq J(\mathbf{s}|p)$. We have put the data strongly supporting Conjecture 7.6 in [20, Appendix II]. In fact, we have only computed the first or the second reserved density because according to Conjecture 7.4 this is enough to determine the reserved density in whole.

We now provide a heuristic argument for Conjecture 7.6. Suppose $l = 1$ and $\mathbf{s} = s \geq 2$ first. Then by Proposition 6.1 we only consider $H(s; r)$ for $1 \leq r \leq (p-5)/2 + \mathfrak{p}(s-1)$ because for most p the midpoint $(p-1)/2 \in J_1(s|p)$ if and only if s is even (see Proposition 6.3). If we assume that when r varies the numbers $H(s; r)$ distribute randomly modulo p for any large fixed prime p then the probability that $J_1^0(s|p) = RJ_1(s; p)$ is $(1-1/p)^{(p-5)/2+\mathfrak{p}(s-1)} \to 1/\sqrt{e}$ as $p \to \infty$. By Conjecture 7.4 (we also have a heuristic argument for it in this case) we see that the probability that $J(s|p) = RJ(s; p)$ is $1/\sqrt{e}$.

Remark 7.7. Although we cannot prove the random distribution of $H(s; r)$ for $1 \leq r \leq (p-5)/2 + \mathfrak{p}(s-1)$ modulo large prime p , in a recent paper[9], Garaev et al. show that for any $\varepsilon > 0$, the set $\{H(s; r)) : r < p^{1/2+\varepsilon}\}$ is uniformly distributed modulo a sufficiently large p.

Now we assume $l \geq 2$. In general $J_1^0(\mathbf{s}|p)$ does not have any symmetry so we see that the probability that $J_1^0(\mathbf{s}|p) = RJ_1(\mathbf{s}; p)$ is $(1-1/p)^{p-\delta} \to 1/e$ as $p \to \infty$, where $\delta = \sharp RJ_1(\mathbf{s})$. When $\mathbf{s}$ does not belong to the cases (i)-(iv) in Conjecture 7.4 we see that the probability that $J(\mathbf{s}|p) = RJ(\mathbf{s}; p)$ is $1/e$ by Conjecture 7.4.

Finally let's deal with larger reserved sets when $l \geq 2$. By Theorem 7.2 we know that if $\mathbf{s} = 1^{2m}$ or $\mathbf{s} = (1,2,1)$ or $\mathbf{s} = (2r-1,1)$ for some $r \geq 1$ then $RJ(\mathbf{s}) = RJ_2(\mathbf{s})$. Let $\mathbf{s} = 1^{2m}$. Then for $p < n < 2p-1$ by definition $H(1^{2m}; n)$ is a sum of many product terms with v_p-value equal to either 0 or 1, i.e., the denominator has at most one p factor. Assuming random distribution of $pH(1^{2m}; n)$ modulo p^2 we see that when $p < n < 2p-1$ the probability that p divides $H(1^{2m}; n)$ is $1/p^2$. Thus the probability that none of $H(1^{2m}; n)$ ($p < n < 2p-1$) is multiple of p is equal to $(1-1/p^2)^{p-2}$. Similar heuristic argument implies that the probability that none of $H(1^{2m}; n)$ ($2p \leq n < 3p$) is multiple of p is equal to $(1-1/p^3)^p$, and so on. When $lp \leq n < (l+1)p$ and $2m \leq l \leq p$ the probability is equal to $(1-1/p^{2m+1})^p$. In general for $p^t \le n < p^{t+1}$ $(t \ge 1)$ we can break it into p subintervals and estimate inside each of the subintervals. It is easy to conclude that the probability that none of $H(1^{2m}; n)$ $(p^t \le n < p^{t+1})$ is multiple of p is at least $(1-1/p^{2m+2t-1})^{p^{t+1}}$. Now

$$\sum_{t=1}^{\infty} \frac{1}{p^{2m+t-2}} = \frac{1}{p^{2m-2}(p-1)} \to 0 \quad \text{as } p \to \infty.$$

We see that the probability that $J(1^{2m}|p) = RJ(1^{2m}; p)$ is

$$(1-1/p)^{p-2} \to \frac{1}{e} \quad \text{as } p \to \infty.$$

We omit the arguments for $\mathbf{s} = (1,2,1)$ and $(2r-1,1)$ which are similar.

We conclude our paper by some conjectures which concern distributions of irregular primes in disguised forms.

Conjecture 7.8. *Let r, s and t be positive integers. Then*

(1). *If $s > 1$ is odd then $J_1(s|p) = \{(p-1)/2, p-1\}$ for infinitely many primes p.*

(2). *If $s > 1$ is odd then $J_1(s,s|p) = \{(p-1)/2, p-1\}$ for infinitely many primes p.*

(3). *Let s,t be positive integers. Suppose $s+t$ is odd. Then $J_1(s,t|p) = \{p-1\}$ for infinitely many primes p.*

(4). *Let r,s,t be positive integers such that $r+s+t$ is odd and $r \neq t$. Then $J_1(r,s,t|p) = \{p-1\}$ for infinitely many primes p.*

Note that by various results of this paper and [21] an affirmative answer to any part of our Conjecture 7.8 would imply that there are infinitely many irregular pairs $(p, p-w)$ for any fixed odd number w (≥ 5 in case (7.8)). Therefore, even if the sets of primes in Conjecture 7.8 are expected to be infinite they are extremely sparse; very likely they have zero density.

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