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THE GREEN'S FUNCTIONS OF THE BOUNDARIES AT INFINITY OF THE HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. The work is motivated by a result of Manin in [12], which relates the Arakelov Green function on a compact Riemann surface to configurations of geodesics in a 3-dimensional hyperbolic handlebody with Schottky uniformization, having the Riemann surface as conformal boundary at infinity. A natural question is to what extent the result of Manin can be generalized to cases where, instead of dealing with a single Riemann surface, one has several Riemann surfaces whose union is the boundary of a hyperbolic 3-manifold, uniformized no longer by a Schottky group, but by a Fuchsian, quasi-Fuchsian, or more general Kleinian group. We have considered this question in this work and obtained several partial results that contribute towards constructing an analog of Manin's result in this more general context.

INTRODUCTION

Results. Lets consider that S is a compact Riemann surface. Then we can define a Green's function with respect to a metric and a divisor on S. Suppose that M is a hyperbolic 3 - Manifold with infinite volume and having compact Riemann surfaces $S_1, S_2, ..., S_n$ as its conformal boundaries at infinity. Also choose a geometry on M which bears a metric of constant negative curvature. The main new results of this paper express the Green's functions on each S_i in terms of the length of some certain geodesics in M. Manin has done this provided that M has one boundary component (i.e. n=1) and is uniformized by a Schottky group in [12]. In this work we will generalize Manin's results in general case, M has more than one boundary components and is uniformized by a Fuchsian, quasi-Fuchsian or Kleinian group.

In this introduction we state the main definitions, motivation and the plan of doing the work. We start by introducing the definition of a Green's function on a compact Riemann surface for a divisor and with respect to a normalized volume form on it.

Green's functions on Riemann surfaces. Let S be a compact Riemann surface and $A = \sum_x m_x(x)$ (with m_x integer number) be a divisor on S. we show the support of A by $|A|$. Also lets consider that $d\mu$ is a positive real - analytic 2 - form on S. By a Green's function on S for A with respect to $d\mu$ we mean a real analytic function

$$g_{\mu,A} = g_A : S \setminus |A| \longrightarrow \mathbb{R}$$

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satisfying the following conditions:
(i) *Laplace equation:*

$$\partial\bar{\partial}g_A = \pi i(deg(A)d\mu - \delta_A).$$

Where $deg(A) = \sum_x m_x$, and δ_A is the standard δ - current $\varphi \mapsto \sum_x m_x\varphi(x)$.

(ii) *Singularities:* Let z be a complex coordinate in a neighborhood of the point x. Then $g_A - m_x \log|z|$ is locally real analytic.

A function satisfying these two conditions is uniquely determined up to an additive constant. And the third condition is
(iii) *Normalization:*

$$\int_S g_A d\mu = 0.$$

Which eliminates the remaining ambiguous constant. g_A is additive on A and for $x \neq y$, $g_x(y)$ is symmetric, i.e. $g_x(y) = g_y(x)$.

Lets consider that $B = \sum_y n_y(y)$ is another divisor on S that is prime to the divisor A. That means, $|A| \cap |B| = \emptyset$. And Put

$$g_\mu(A,B) := \sum_y n_y g_{\mu,A}(y). \tag{0.0.1}$$

g_A is additive and $g_x(y)$ is symmetric, then $g_\mu(A,B)$ is symmetric and biadditive in A, B.

In general, the function g_μ depends on the metric $d\mu$, but in the case that both of the divisors are of the degree zero, from the condition (i) we see that $g_{\mu,A,B}$ depends only on A, B. Notice that, as a particular case of the general Kahler formalism, to choose $d\mu$ is the same as to choose a real analytic Riemannian metric on S compatible with the complex structure. This means that $g_\mu(A,B) = g(A,B)$ are conformal invariants when both divisors are of degree zero. Also in the case that the divisor A is principal, i.e. A is the divisor of a meromorphic function like f_A, then

$$g(A,B) = \log \prod_{y\in|B|} |f_A(y)|^{n_y} = Re\int_{\gamma_B} \frac{df_A}{f_A} \tag{0.0.2}$$

Where the curve γ_B is a 1 - chain with boundary B. The divisors of degree zero on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ are principal. Then this formula can be directly applied to divisors of degree zero on Riemann sphere.

It is well known that the Green's function of the degree zero divisors on a Riemann surface of arbitrary genus can be expressed exactly via the differential of the third kind ω_A with pure imaginary periods and residues m_x at x when the divisor is $A = \sum m_x(x)$(see, [5], [11]). Then the generalization of the previous formula for arbitrary divisors A, B of degree zero is

$$g(A,B) = Re \int_{\gamma_B} \omega_A. \tag{0.0.3}$$

In general, when the degree of the divisors are not restricted to zero, the basic Green's function $g_{\mu,(x)}(y)$ can be expressed explicitly via theta functions (as in [6]) in the case when μ is the *Arakelov metric* constructed with the help of an orthonormal basis of the differentials of the first kind on the Riemann surface.

Motivations. The result of Manin in [12] which relates the Arakelov Green function on a compact Riemann surface to configurations of geodesics in a 3-dimensional hyperbolic handlebody with Schottky uniformization, having the Riemann surface as conformal boundary at infinity, was extremely innovative and influential and had a wide range of consequences in the arithmetic context of Arakelov geometry as well as and in other contexts, ranging from p-adic geometry, real hyperbolic 3-manifolds, the holography principle and AdS/CFT correspondence in string theory, and noncommutative geometry. A natural question is to what extent the result of Manin can be generalized to cases where, instead of dealing with a single Riemann surface, one has several Riemann surfaces whose union is the boundary of a hyperbolic 3-manifold, uniformized no longer by a Schottky group, but by a Fuchsian, quasi-Fuchsian, or more general Kleinian group. Such a generalization is not only interesting because it is a very natural question to pass from Kleinian-Schottky groups to more general Kleinian groups, but also for its potential applications to Arakelov geometry, to the case of curves defined over number fields with several Archimedean places, while Manin's result was formulated for the case of arithmetic curves defined over the rationales.

Plan. We have focused on the formula in Manin's work, that expresses the Arakelov Green's function on a compact Riemann surface in terms of a basis of holomorphic differentials of the first kind and of differentials of the third kind. In the case of Schottky uniformization, when the limit set has Hausdorff dimension strictly smaller than one, one can construct such differentials in terms of averages over the Schottky group. While the same type of formula no longer holds in the Fuchsian or quasi-Fuchsian case, we use the canonical covering map relating Fuchsian and Schottky uniformization and the coding of limit sets for the Fuchsian and Schottky case, to express the Green function in the Fuchsian or quasi-Fuchsian case in terms of the one in the Schottky case. The approach we follow for the more general Kleinian case is via a decomposition of the uniformizing group as a free product of quasi-Fuchsian and Schottky groups and applying the results we obtained for these cases individually. The paper consists of three chapters devoted to the cases that the hyperbolic 3-manifold M have 1, 2 and $n < \infty$ many boundary components at infinity respectively. First chapter pays to the one boundary component case and includes four subchapter devoted to the Foundations and the genera 0,1 and ≥ 2 respectively. Also this chapter contains fundamental definitions and basic computations that are bases for the computations in the next chapters. The results of this chapter are a summary of the reference [12]. All hyperbolic 3-manifolds with two boundary components with the same genera are uniformized by Quasi-fuchsian groups; the second chapter is devoted to this manifolds. In third chapter we consider the general case i.e. the case that M is uniformized by some Kleinian groups and have $n < \infty$ boundary components. Finally, as a remark for the chapter three we show that for manifolds with ∞ many boundary components at infinity the situation is like the previous.

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1. One boundary component case

In this chapter, first we gather some fundamental definitions and some useful notations that will be used in all of the paper. Next we bring the computation of Green's function for Riemann sphere $\mathbb{P}^1(\mathbb{C})$, the boundary at infinity of the hyperbolic space. All 3-manifolds with one boundary component that is a compact Riemann surface with genus ≥ 1, are uniformized by Schottky groups. The rest of the chapter is devoted to these manifolds. The results of this chapter are a summary of the reference [12].

1.1. **Foundations.** Consider the hyperbolic space H^3 with the upper half space model and coordinate (z, y) that comes from $\mathbb{C}\times\mathbb{R}_+$ equipped by the hyperbolic distance function corresponding to the metric

$$ds^2 = \frac{|dz|^2 + dy^2}{y^2} \tag{1.1.1}$$

of constant curvature -1. The geodesics in this model are vertical half - lines $Z = constant$ and also vertical half - circles orthogonal to the plane at infinity $\mathbb{C}$ (i.e. for $y = 0$).

If we consider the end points of the geodesics (including ∞ for the ends of the vertical half-lines), then we can consider the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (or S^2 in the unit boll model) as the boundary at infinity of H^3.

1.1.1. *Notations.* Lets show the geodesic joining a to b in $H^3 \cup \hat{\mathbb{C}}$ by $\{a, b\}$; by $a * \gamma$ the point on the geodesic γ closest to the point a (i.e, the intersection point of γ and a geodesic passing through a and orthogonal to γ); by $d_u(a, b)$ the distance from u to the geodesic $\{a, b\}$ and by $ordist(a, b)$ the oriented distance between two points lying on an oriented geodesic in H^3 (Figure 1). Also lets show by $\varphi_u(a, b)$ the angle (at u) between the semi - geodesics joining u to a and u to b, and for a, b in $\hat{\mathbb{C}}$ by $\psi_\gamma(a, b)$ the oriented angle between the semi - geodesics joining $a * \gamma$ to a and $b * \gamma$ to b. In order to calculate $\psi_\gamma(a, b)$ we must first make the parallel translation along γ identifying the normal spaces to γ at $a * \gamma$ and at $b * \gamma$. The orientation of a normal space is defined by projecting it along oriented γ to its initial end into the tangent space to $\hat{\mathbb{C}}$, which is canonically oriented by the complex structure.

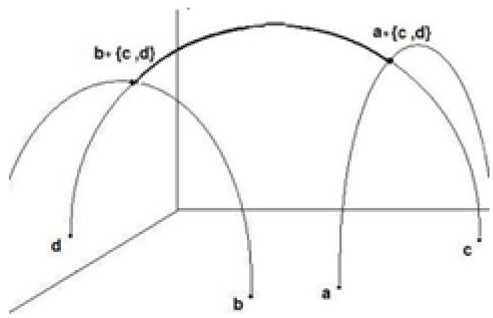

FIGURE 1.

1.2. **Genus zero case.** In this case we consider that the hyperbolic manifold M is the hyperbolic space H^3 with Riemann-sphere as its boundary at infinity. For this case we have

Proposition 1.2.1. *For* a, b, c *and* d *in* $\mathbb{P}^1(\mathbb{C})$, *denote by* $w_{(a)-(b)}$ *a meromorphic function on* $\mathbb{P}^1(\mathbb{C})$ *with the divisor* $(a)-(b)$. *Then we have*

$$\log \left| \frac{w_{(a)-(b)}(c)}{w_{(a)-(b)}(d)} \right| = -ordist(a * \{c,d\}, b * \{c,d\}) \tag{1.2.1}$$

and

$$\arg \frac{w_{(a)-(b)}(c)}{w_{(a)-(b)}(d)} = -\psi_{\{c,d\}}(a,b) \tag{1.2.2}$$

Proof. Mobius transformations preserve hyperbolic distance and angel. Then both sides of (1.2.1) and (1.2.2) are invariant under these transformations. Hence it suffices to consider the case when $(a,b,c,d) = (z,1,0,\infty)$ in $\mathbb{P}^1(\mathbb{C})$. Then the geodesic $\{a,b\} = \{0,\infty\}$ is the y semi-axis in $(z,y) = (z_1 + iz_2, y)$ coordinate for H^3 and the geodesics joining the points $a = z$ and $b = 1$ normally to this semi-axis are half circles passing from these points with center in 0 and normal to $\mathbb{C}$ (Figure 2). Then we have $a * \{c,d\} = z * \{0,\infty\} =$

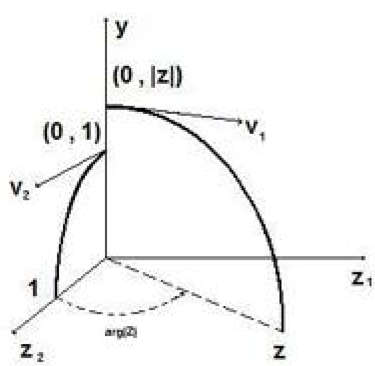

FIGURE 2.

$(0,|z|)$, $b * \{c,d\} = 1 * \{0,\infty\} = (0,1)$ and

$$ordist(a * \{c,d\}, b * \{c,d\}) = ordist((0,|z|),(0,1)) = -\log|z|.$$

Also from the properties of cross - ratio we have

$$\frac{w_{(a)-(b)}(c)}{w_{(a)-(b)}(d)} = \frac{w_{(z)-(1)}(0)}{w_{(z)-(1)}(\infty)} = z.$$

This gives (1.2.1) directly. To see (1.2.2), we know that angles in H^3 can be calculated using the Euclidean metric $|dz|^2 + dy^2$ and also the parallel transport along y semi-axis coincides with the Euclidean one. As it's shown in figure 2 the vectors v_1 and v_2 tangent to the geodesics from the points z and 1, are normal to $\{0, \infty\}$. Hence if we transport (parallel) the vector v_2 in the point $(0, 1)$ to the point $(0, |z|)$ then both vectors are in an Euclidean plane normal to y semi-axis. Then the angel from the vector v_1 to the vector v_2 can be calculated via there angels from z_1 axis. Then we have

$$\psi_{\{0,\infty\}}(z, 1) = arg(v_2) - arg(v_1) = -arg \frac{w_{(z)-(1)}(0)}{w_{(z)-(1)}(\infty)}.$$

This proves (1.2.2). □

The part $w_{(a)-(b)}(c)/w_{(a)-(b)}(d)$ in the proposition above is the classical cross - ratio of four points on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, for which it is convenient to have a special notation

$$\langle a, b, c, d\rangle := \frac{w_{(a)-(b)}(c)}{w_{(a)-(b)}(d)}. \tag{1.2.3}$$

Theorem 1.2.2. *Let a, b, c and d be in $\mathbb{P}^1(\mathbb{C})$. Then we have*

$$\begin{aligned} g((a) - (b), (c) - (d)) &= -ordist(a * \{c, d\}, b * \{c, d\}) \\ &= \log |\langle a, b, c, d\rangle|. \end{aligned} \tag{1.2.4}$$

Proof. From the formula (0.0.2) for the Green's function in the case that the divisors are principal, we have

$$\begin{aligned} g((a) - (b), (c) - (d)) &= \log |w_{(a)-(b)}(c)|^1 \log |w_{(a)-(b)}(d)|^{-1} \\ &= \log \left| \frac{w_{(a)-(b)}(c)}{w_{(a)-(b)}(d)} \right| . \end{aligned}$$

Then from proposition (1.2.1) and notation (1.2.3) we have (1.2.4). □

1.3. **Genus one case.** In this case the group that uniformize the hyperbolic 3 - manifold with its boundary at infinity is a cyclic group, and there is a nice explicit formula for the basic Green's function on the boundary at infinity of 3-manifold; but because we do not use it here we do not want to point to it here. But one can find it in [12](or for a physical point of view in [14]). Of course the process in the next part can be used for this case too. We have pointed some notes about this case in section 1.4.1.

1.4. **Genus>1 case and Schottky Groups.** Consider the complex projective linear transformations group $\mathrm{PGL}(2,\mathbb{C})$. *(i) Kleinian groups.* A subgroup G of $\mathrm{PGL}(2,\mathbb{C})$ is called a Kleinian group if G acts on H^3 properly discontinuously. For a Kleinian group G and a point p in H^3 lets denote by $G(p)$, the orbit of the point P under the action of G. Since G acts on H^3 Properly discontinuously, $G(p)$ has accumulation points only on $\mathbb{P}^1(\mathbb{C})$. They are independent of the choice of the reference point p and are called the *limit set* of G, which is denoted by $\Lambda(G)$. Equivalently, it is the closure of the set of all fixed points of the elements of G other than the identity. $\Lambda(G)$ is the minimal non - empty closed G - invariant set. The complement of the limit set $\mathbb{P}^1(\mathbb{C}) \setminus \Lambda(G)$ is denoted by $\Omega(G)$ and is called the *region of discontinuity* of G.

The Kleinian group G is called of the *first kind* if $\Lambda(G) = \mathbb{P}^1(\mathbb{C})$ and of the *second kind* if $\Omega(G) \neq \emptyset$. In the case that G is of the second kind G acts on $\Omega(G)$ properly discontinuously. Therefor in this case $\Omega(G)$ is the maximal open G - invariant subset of $\mathbb{P}^1(\mathbb{C})$ where G acts properly discontinuously. The quotient space $\Omega(G)/G$ has the complex structure induced from that of $\Omega(G)$. Thus $\Omega(G)/G$ is a countable union of Riemann surfaces lying at infinity of the complete hyperbolic 3 - manifold H^3/G. For a torsion free Kleinian group G, a manifold $(H^3 \cup \Omega(G))/G$ possibly with boundary is denoted by M_G and is called a *Kleinian manifold*. The interior H^3/G of M_G which admits the hyperbolic structure is denoted by N_G.

(ii) Loxodromic, Parabolic and Elliptic elements. An element g in $\mathrm{PGL}(2,\mathbb{C})$ is called loxodromic if it has exactly two different fixed points in $\mathbb{P}^1(\mathbb{C})$. These points are denoted by $z^+(g)$ and $z^-(g)$ and are called attracting one and repelling one. For any $z_0 \neq z^{\pm}(g)$ and $h \in \mathrm{PGL}(2,\mathbb{C})$, we have $z^{\pm}(g) = \lim_{n\to\pm\infty} g^n,\ z^{\pm}(g) = z^{\mp}(g^{-1}),\ z^{\pm}(hgh^{-1}) = hz^{\pm}(g)$. If we denote by $q(g)$ the eigenvalue of g on the complex tangent space to $z^+(g)$ then there is a local coordinate for $\mathbb{P}^1(\mathbb{C})$ that in this coordinate g is represented by $\begin{pmatrix} q(g) & 0 \\ 0 & 1 \end{pmatrix}$. For this reason $q(g)$ is called the multiplier of g, and we have $|q(g)| < 1$, $q(g) = q(g^{-1}) = q(hgh^{-1})$. Also by definition g is called a parabolic element if it has precisely one fixed point in $\mathbb{P}^1(\mathbb{C})$, and elliptic if it has fixed points in H^3. In fact an elliptic element fixes all points of a geodesic in H^3 joining two fixed points in $\mathbb{P}^1(\mathbb{C})$. For a loxodromic or elliptic element g the geodesic joining two fixed points in $\mathbb{P}^1(\mathbb{C})$ is called axis of g. An element is elliptic if and only if it has finite order, then elliptic elements cause a singularity in N_G. For this reason we consider Kleinian groups without elliptic elements. This means that the group is torsion free or equivalently acts freely on H^3.

(iii) Schottky groups. A finitely generated, free and purely loxodromic Kleinian group is called a Schottky group. Purely loxodromic means, all elements except the identity are loxodromic. For such a group Γ the number of a minimal set of generators is called the genus of Γ. A marking for a Schottky group Γ of genus p by definition is a family of $2p$ open connected domains $D_1, ..., D_{2p}$ in $\mathbb{P}^1(\mathbb{C})$ and a family of generators $g_1, ..., g_p \in \Gamma$ with the following properties. For $i = 1, ..., p$

a) The boundary C_i of D_i is a Jordan curve homeomorphic to S^1 and closures of D_i are pairwise disjoint.
b) $g_i(C_i) \subseteq C_{p+i}$ and $g_i(D_i) \subset \mathbb{P}^1(\mathbb{C}) \setminus D_{p+i}$.

We say that a marking is classical, if all C_i are circles. It is known that every Schottky group admits a marking. In fact each Schottky group admit infinitely many marking but there are some Schottky groups for which no classical marking exists.

(iv) Γ *- invariant sets of the Schottky groups and there quotient spaces.* A Schottky group Γ is a Kleinian group, then it acts on H^3 properly discontinuously. This action is free too. Then the quotient space $N_\Gamma = H^3/\Gamma$ has a complete hyperbolic 3-manifold structure this means that it is a non-compact Riemann space of constant curvature -1. Topologically, If Γ is of genus p then $N_\Gamma = H^3/\Gamma$ is the interior of a handlebody of genus p.

Now lets consider the marking $\{ D_1, ..., D_{2p} \ ; \ g_1, ..., g_p \}$ for the Schottky group Γ of genus p and put

$$X_\Gamma := \mathbb{P}^1(\mathbb{C}) \setminus \bigcup_{i=1}^{p} (D_i \cup \bar{D}_{p+i}), \qquad \Omega(\Gamma) := \bigcup_{g \in \Gamma} g(X_\Gamma). \tag{1.4.1}$$

The set $\Omega(\Gamma)$ is the region of discontinuity of Γ and X_Γ is a fundamental domain for the action of Γ on $\mathbb{P}^1(\mathbb{C})$. Then Γ acts on $\Omega(\Gamma)$ freely and properly discontinuously. So the quotient space $S_\Gamma = \Omega(\Gamma)/\Gamma$ is a complex Riemann surface of genus p. All compact Riemann surfaces can be obtained in this way (see [8]) and every compact Riemann surface admits infinitely many different Schottky covers.

The Cayley graph of a Schottky group Γ of genus p is an infinite tree with multiplicity of $2p$ at each vertex. Γ is free, then this tree is without loops and each path between two points is unique. Now, by the definition of a marking for Γ, each generator g_i takes X_Γ to the inside of C_{p+i} and again the generator g_j takes this part to the inside of the second copy of C_{p+j} inside C_{p+i}. This means that for each element g in Γ we can associate a path in the Cayley graph that the points of X_Γ moves along that path on a tubular neighborhood. This express the action of Γ on the set $\Omega(\Gamma)$ (Figure 3 illustrating this for the case $p = 2$). We can consider the Riemann surface S_Γ as the boundary at infinity of the manifold N_Γ by identifying the points of S_Γ with the points of the set of equivalence classes of unbounded ends of oriented geodesics in N_Γ modulo the relation "distance=0".

By the definition of the limit set for a Kleinian group, the complement

$$\Lambda(\Gamma) := \mathbb{P}^1(\mathbb{C}) \setminus \Omega(\Gamma) \tag{1.4.2}$$

is the limit set of Γ. For the cyclic Schottky groups (of genus 1) the limit set $\Lambda(\Gamma)$ consists of two points which can be chosen as $0, \infty$, but for genus≥ 2 this set is an infinite Cantor set (see [18]).

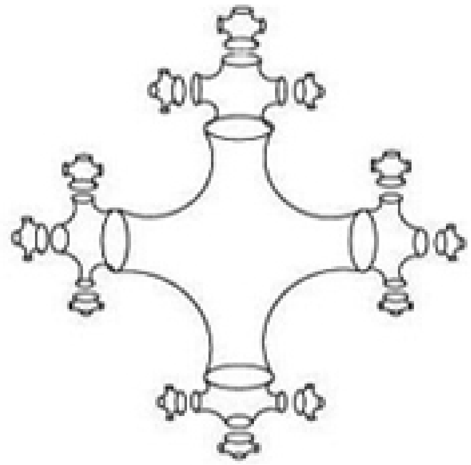

FIGURE 3.

Lets consider the irreducible left - infinite words like $\bar{h} = ...h_i^{\varepsilon_i}...h_0^{\varepsilon_0}$. Where $h_i \in \{g_1, ..., g_p\}$, $\varepsilon_i = \pm 1$, and z_0 is a point in the fundamental domain X_Γ. And put

$$z^+(\bar{h}) = \lim_{i \to +\infty} h_i^{\varepsilon_i}...h_0^{\varepsilon_0}(z_0) \tag{1.4.3}$$

This is a well defined point of $\Lambda(\Gamma)$ and is independent of the point z_0. Since Γ is a free group, the map $\bar{h} \mapsto z^+(\bar{h})$ establishes a bijection as following

Irreducible left - infinite words in g_i

$\updownarrow$

Ends of the Cayley graph of Γ, $\{g_i\}$

$\updownarrow$

Points of $\Lambda(\Gamma)$.

Denote by $a(\Gamma)$ the Hausdorff dimension of the set $\Lambda(\Gamma)$. It can be characterized as the convergence abscissa of any Poincare series

$$\sum_{g \in \Gamma} \left| \frac{dg(z)}{dz} \right|^s. \tag{1.4.4}$$

Where z is any coordinate function on $\mathbb{P}^1(\mathbb{C})$ with a zero and a pole in $\Omega(\Gamma)$. For $s \in \mathbb{C}$ that $\mathrm{Re}(s) > a(\Gamma)$, this series converges uniformly on compact subsets of $\Omega(\Gamma)$. Generally we have $0 < a(\Gamma) < 2$ (See [3]). In the next section we will consider $a(\Gamma) < 1$ to have the convergency of some infinite products defining some Γ - automorphic functions. This class of Schottky groups was characterized by Bowen [3] in following way: $a(\Gamma) < 1$ if and only if Γ admits a rectifiable invariant quasi-circle (which then contains $\Lambda(\Gamma)$).

Choose marking for Γ of genus p and denote by a_i the image of C_i in S_Γ with induced orientation. Also for $i = 1, ..., p$ choose the points x_i in C_i and denote by b_i the images in S_Γ of oriented pathes from x_i to $g_i(x_i)$ lying in X_Γ. These images are obviously closed paths and we can choose them in such a way that they don't intersect. If we denote the classes of these pathes in 1- homology group $H_1(S_\Gamma, \mathbb{Z})$ by the same notations, then the set $\{a_i, b_j\}$ form a canonical basis of this group i.e. for all $i = 1, ..., p$ we have

$$(a_i, a_j) = (b_i, b_j) = 0\ ,\ (a_i, b_j) = \delta_{ij}. \tag{1.4.5}$$

Moreover the kernel of the map $H_1(S_\Gamma, \mathbb{Z}) \longrightarrow H_1(M_\Gamma, \mathbb{Z})$ which is induced by the inclusion $S_\Gamma \hookrightarrow M_\Gamma$ is generated by the classes a_i.

1.4.1. *Differentials of the first kind.* Lets consider the cyclic Schottky group $\Gamma =< qz >$, for $q \in \mathbb{C}^*$, $|q| < 1$. In this case $\Lambda(\Gamma) = \{0, \infty\}$ and consequently $\Omega(\Gamma) = \mathbb{C}^*$. Then a differential of the first kind on $\Omega(\Gamma) = \mathbb{C}^*$ can be written as

$$\omega = d\log z = d\log \frac{w_{(0)-(\infty)}(z)}{w_{(0)-(\infty)}(z_0)} = d\log\langle 0, \infty, z, z_0\rangle. \tag{1.4.6}$$

where z_0 is any point $\neq 0, \infty$. And for $d\log q^n z = d\log z$, this differential determines a differential of the first kind on S_Γ. In general case for a Schottky group Γ of genus p we can make a differential of the first kind ω_g for any $g \in \Gamma$ on $\Omega(\Gamma)$ and S_Γ by an appropriate averaging of this formula. Lets consider a marking for Γ and Denote by $C(|g)$ a set of representatives of $\Gamma/(g^{\mathbb{Z}})$; by $C(h|g)$ a similar set for $(h^{\mathbb{Z}}) \setminus \Gamma/(g^{\mathbb{Z}})$; and by $S(g)$ the conjugacy class of g in Γ. Then for any $z_0 \in \Omega(\Gamma)$ we have

Proposition 1.4.1. *(a) If $a(\Gamma) < 1$, the following series converges absolutely for $z \in \Omega(\Gamma)$ and determines (the lift to S_Γ of) a differential of the first kind on S_Γ:*

$$\omega_g = \sum_{h \in C(|g)} d\log\langle h(z^+(g)), h(z^-(g)), z, z_0\rangle. \tag{1.4.7}$$

This differential does not depend on z_0, and depends on g additively. Also If the class of g is primitive (i.e. non - divisible in $H := \Gamma/[\Gamma, \Gamma]$), ω_g can be rewritten as following

$$\omega_g = \sum_{h \in S(g)} d\log\langle z^+(h), z^-(h), z, z_0\rangle. \tag{1.4.8}$$

(b) If g_i form a part of the marking of Γ, and a_i are the homology classes described before, we have

$$\int_{a_i} \omega_{g_j} = 2\pi i \delta_{ij}. \tag{1.4.9}$$

It follows that the map g mod $[\Gamma, \Gamma] \mapsto \omega_g$ embeds H as a sublattice in the space of all differentials of the first kind.

(c) Denote by $\{b_j\}$ the complementary set of homology classes in $H_1(S_\Gamma, \mathbb{Z})$ as in before. Then we have for $i \neq j$, with an appropriate choice of logarithm branches:

$$\tau_{ij} := \int_{b_i} \omega_{g_j} = \sum_{h \in C(g_i|g_j)} \log\langle z^+(g_i), z^-(g_i), h(z^+(g_j)), h(z^-(g_j))\rangle \tag{1.4.10}$$

And

$$\tau_{ii} = \log q(g_i) + \sum_{h \in C_0(g_i|g_i)} \log\langle z^+(g_i), z^-(g_i), h(z^+(g_i)), h(z^-(g_i))\rangle \tag{1.4.11}$$

where $C_0(g_i|g_i)$ *is* $C(g_i|g_i)$ *without the identity class.*

Proof. For the proofs, see [12] and [13]. Notice that our notation here slightly differs from [12]; in particular, τ_{ij} here corresponds to $2\pi i\tau_{ij}$ of [12]. □

1.4.2. *Differentials of the third kind and Green's functions.* Lets consider the points a and b in X_Γ the fundamental domain of Γ and put $\nu_{(a)-(b)} = \sum_{h\in\Gamma} d\log\langle a, b, h(z), h(z_0)\rangle$. Then assuming $a(\Gamma) < 1$, we see that this series absolutely converges and because it's Γ - automorphic it gives us a differential of the third kind on S_Γ with residues ± 1 at the images of a, b in S_Γ. Moreover, since both points a, b are out of the circles C_i, its a_i - periods vanish. Now, if we consider the linear combination $\nu_{(a)-(b)} - \sum_{j=1}^{p} X_j(a,b)\omega_{g_j}$ with real coefficients X_j, then it will have pure imaginary a_i - periods and If we find real coefficients X_j so that the real part of b_i - periods of the form

$$\omega_{(a)-(b)} = \nu_{(a)-(b)} - \sum_{j=1}^{p} X_j(a,b)\omega_{g_j} \tag{1.4.12}$$

vanish, we will be able to use this differential in order to calculate conformally invariant Green's functions. The set of the equations for calculating the coefficients $X_j(a,b)$ are as following

$$\sum_{j=1}^{p} X_j(a,b) Re\tau_{ij} = Re\int_{b_i} \nu_{(a)-(b)} = \sum_{h\in S(g_i)} \log|\langle a, b, z^+(h), z^-(h)\rangle| \tag{1.4.13}$$

for $i = 1, ..., p$. The parts $Re\tau_{ij}$ are calculated by means of formula (1.4.10) and (1.4.11), and b_i - periods of $\nu_{(a)-(b)}$ are given in [12].

Now if we denote the points a, b, c, and d in S_Γ and their images in the fundamental domain X_Γ by the same notations, then we have

$$Re\int_d^c \nu_{(a)-(b)} = \sum_{h\in\Gamma} \log|\langle a, b, h(c), h(d)\rangle|, \quad Re\int_d^c \omega_{g_j} = \sum_{h\in S(g_j)} \log|\langle z^+(h), z^-(h), c, d\rangle|$$

And finally we have the Green's function on S_Γ as following

$$\begin{aligned} g((a)-(b),(c)-(d)) &= Re\int_d^c \omega_{(a)-(b)} \\ &= \sum_{h\in\Gamma} \log|\langle a, b, h(c), h(d)\rangle| \\ &\quad - \sum_{j=1}^{p} X_j(a,b) \sum_{h\in S(g_j)} \log|\langle z^+(h), z^-(h), c, d\rangle|\,. \end{aligned} \tag{1.4.14}$$

2. Two boundary component case

All hyperbolic Riemann surfaces are uniformized by Fuchsian groups. Also Fuchsian groups act on hyperbolic space by Poincare extension and uniformize some hyperbolic 3-manifolds with two boundaries at infinity with the same genera. But they do not uniformize all such manifolds; In fact they are uniformized by an extension of Fuchsian groups that are called Quasi-Fuchsian groups. In this chapter first we intend to extend the method in previous chapter to fuchsian groups and then for Quasi-fuchsian groups.

Consider hyperbolic plane H^2 with the upper half plane model and coordinate $z = x + iy \in \mathbb{C}$ with $y > 0$, equipped by the hyperbolic distance function corresponding to the metric

$$ds^2 = \frac{|dz|^2}{y^2} \tag{2.0.15}$$

of the constant curvature -1. The geodesics in this model are vertical half-lines $x = \textit{constant}$ and vertical half-circles orthogonal to the line at infinity $\mathbb{R}$ (i.e, for $y = 0$).

If we consider the end points of geodesics (including ∞ for the end of vertical half-lines other than the end in $\mathbb{R}$) then we can consider circle $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (or $\mathbb{P}^1(\mathbb{R}) = S^1$ in unit disc model) as boundary at infinity of H^2.

2.1. **Fuchsian Groups.** Each subgroup F of $PGL(2, \mathbb{R})$, the general projective linear group over $\mathbb{R}$, that acts freely and properly discontinuously on H^2 is called a Fuchsian group. Similar to the Kleinian groups the limit set $\Lambda(F)$ and the region of discontinuity $\Omega(F)$ are defined but in $\hat{\mathbb{R}}$, the boundary at infinity of H^2. And the properties are similar to them. Also similar to the loxodromic elements, a hyperbolic element is an element $h \in PGL(2, \mathbb{R})$ with two fixed points $z^{\pm}(h)$ in $\hat{\mathbb{R}}$. For a Fuchsian group F the quotient space H^2/F is a hyperbolic Riemann surface possibly with boundary $\Omega(F)/F$ (If $\Omega(F) \neq \emptyset$).

Let F be a Fuchsian group such that $S = H^2/F$ is a compact Riemann surface with the genus $p > 1$. Then F is a purely hyperbolic group of finite order $2p$ (for example see [9]). Lets denote by f_i the generators of F and by $Fix(f_i) = \{z^{\pm}(f_i)\}$ the fixed point set of f_i. Also consider that P is the fundamental polygon of F with $a_1, b_1, a'_1, b'_1, \ldots, b'_p$ as it's sides, such that $f_i(a_i) = a'_i$ and $f_{p+i}(b_i) = b'_i$ for $i = 1, \ldots, p$. We represent P as following:

$$P = a_1(x_1)b_1(y_1)a'_1(x'_1)b'_1(y'_1)\ldots a'_p(x'_p)b'_p(y'_p) \tag{2.1.1}$$

Where x_i, x'_i, y_i and y'_i are the intersection points of a_i and b_i, a'_i and b'_i and so on. Now we have

$$\pi_1(S) \simeq F = \langle \{f_i\ ; i = 1, \ldots, 2p\}; \prod_1^p [f_i, f_{p+i}] = I \rangle$$

And also (See for example [9]).

$$H_1(S,\mathbb{Z}) = \frac{\pi_1(S)}{\langle\{aba^{-1}b^{-1}|a,b\in\pi_1(S)\}\rangle} = \mathbb{Z}\oplus\ldots\oplus\mathbb{Z} \qquad (2p\ times)$$

If we denote by $\overline{a}_i$ and $\overline{b}_i$ the images of a_i and b_i in S, then $\{\overline{a}_i, \overline{b}_j\}$ generate $H_1(S,\mathbb{Z})$.

2.2. **Extension of Fuchsian group on H^3.** F acts similarly, freely and properly discontinuously on lower half plane $-H^2$ of $\mathbb{C}$ too, and $-H^2$ is F-invariant. Lets denote by F the Poincare extension of F on H^3 too (see [15] or [18]). By this extension F can be considered as a Kleinian group. And we have

$$N_F = \frac{H^3}{F} \cong \frac{H^2}{F}\times(0,1) = S\times(0,1) \qquad and \qquad F=\pi_1(N_F)=\pi_1(S)$$

(See [15]). And N_F has a hyperbolic structure, [16].

2.3. **F invariants.** We know that $\Lambda(F)$ the limit point set of F is $\hat{\mathbb{R}}$ (S^1 in unit disc model), [3]. Then as a Kleinian group $\Lambda(F)=\hat{\mathbb{R}}\subset\hat{\mathbb{C}}$ and the region of discontinuity of F considering as a Kleinian group is

$$\Omega(F) = \hat{\mathbb{C}}\backslash\Lambda(F) = \hat{\mathbb{C}}\backslash\hat{\mathbb{R}} = -H^2\cup H^2 \tag{2.3.1}$$

And also the Kleinian manifold is

$$M_F = \frac{H^3\cup(-H^2\cup H^2)}{F} = S\times[0,1]. \tag{2.3.2}$$

Now because S is compact then $\partial S=\emptyset$. Consequently N_F is a hyperbolic 3 - manifold with two compact boundary component at infinity $S_0 = H^2/F = S\times\{0\}$ and $S_1 = -H^2/F = S\times\{1\}$ with the same genus p.

2.4. **Coding of the points of $\Lambda(F)$ and the Geodesics.** For convenience and having a simple intuition lets consider the unit disc model for H^2 in this section. We can code points of $\Lambda(F) = S^1$ and geodesics with beginning and end points on $\Lambda(F) = S^1$ as following:

Lets mark semicircles including sides of P by $c_1, ..., c_{4p}$ in the counter clockwise direction around S^1 and put $g_1 = f_1, g_2 = f_{p+1}, g_3 = f_1^{-1}, g_4 = f_{p+1}^{-1}, g_5 = f_2, g_6 = f_{p+2}, g_7 = f_2^{-1}, g_8 = f_{p+2}^{-1}$, and so on. In general for $k = 0, ..., p-1$, $g_{4k+1} = f_{k+1}, g_{4k+2} = f_{p+k+1}, g_{4k+3} = f_{k+1}^{-1}, g_{4k+4} = f_{p+k+1}^{-1}$. And label end points of c_i on S^1 by P_i and Q_{i+1} (with $Q_{4p+1} = Q_1$) with P_i occurring before Q_{i+1} in the counter clockwise direction, (See figure 4). And define

$$f_F : S^1 \to S^1$$

$$f_F(x) = g_i(x) \quad if \quad x\in[P_i, P_{i+1})$$

Then f_F is a well defined map and is called a Markov map related to the Fuchsian group F. We have the following lemma

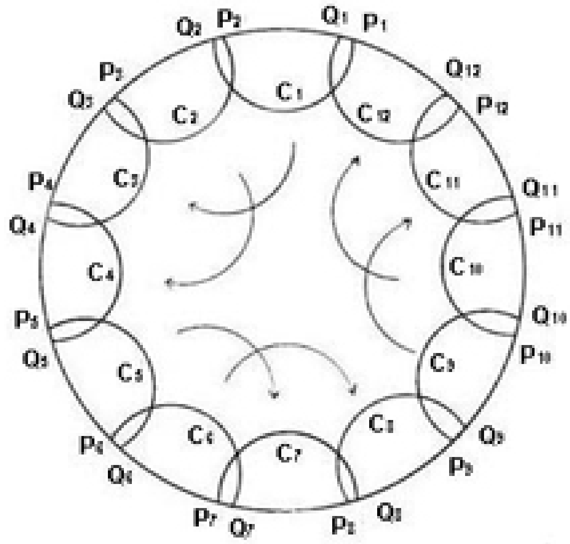

FIGURE 4.

Lemma 2.4.1. *The map f_F and the group F are orbit equivalent on S^1, namely except for the pairs $(Q_i, g_{i-1}Q_i)$, for $i = 1, 2, ..., 4p$; for each x,y in S^1, $x = f(y)$ for some f in F if and only if there exists nonnegative integers n, m such that $f_F^n(x) = f_F^m(y)$.*

Proof. (See [4]). □

Now label each arc $[P_i, P_{i+1})$ by g_i, and for each element x in S^1 put $x = (..., x_2, x_1, x_0)$. Where the component x_n is the label of the segment to which $f_F^n(x)$ belongs. We know that for each generator f_i of F, the fixed point $Z^+(f_i)$ is in the circle $a_i' \cup (-a_i')$ and $Z^-(f_i)$ is in $a_i \cup (-a_i)$ then from the definition of Markov map we have

$$Z^+(f_i) = (..., f_i^{-1}, f_i^{-1}, f_i^{-1}) \qquad and \qquad Z^-(f_i) = (..., f_i, f_i, f_i) \tag{2.4.1}$$

Next, if $\{x, y\}$ be a geodesic with the beginning point $x \in S^1$ and end point $y = (..., y_2, y_1, y_0) \in S^1$ then put

$$\{x, y\} = (..., x_2, x_1, x_0, y_0^{-1}, y_1^{-1}, y_2^{-1}, ...). \tag{2.4.2}$$

Now for each $h \in F$ let $\gamma_h = \{z^+(h), z^-(h)\}$ show the geodesic arc between $z^+(h)$ and $z^-(h)$ (the axis of h) and $\overline{\gamma}_h$ the image of γ_h in N_F. Then we have the following geodesics in N_F

- Closed geodesics: a geodesic in N_F is closed if and only if it's the projection of the axis of a hyperbolic element in F,
- Images of the geodesics with beginning and end points in $\Lambda(F)$,
- Geodesics with beginning points in ∂N_F that are limit cycle to $\overline{\gamma}_{f_i}$, for a generator f_i of F,
- Geodesics with beginning and end points in S_0 or S_1.,
- Geodesics with beginning point in S_0 and end point in S_1 and vis versa.

2.5. **Schottky groups associated to the Fuchsian groups.** For the Riemann surface S and associated Fuchsian group F as above let N be the smallest normal subgroup of F including f_i for $i = p+1, ..., 2p$. Then it's obvious that the factor group F/N is a free group generated by p generators. If we consider the covering $\tilde{S} \to S$ associated to N, then from normality of N the group of deck transformations of this cover is F/N and according to the classical Koebe uniformization theorem [10](see also [7])there is a planar region $\Omega \subset \hat{\mathbb{C}}$ that is a region of discontinuity of a Schottky group $\Gamma =< \gamma_1, \gamma_2, ..., \gamma_p >$ and $F/N \simeq \Gamma$. $\tilde{S}$ is covering isomorphic to $\Omega = \Omega(\Gamma)$ and for coverings $J : H^2 \longrightarrow \Omega(\Gamma)$, $\pi_\Gamma : \Omega(\Gamma) \longrightarrow \Omega(\Gamma)/\Gamma$ and $\pi_F : H^2 \longrightarrow H^2/F$ we have $\pi_\Gamma \circ J = \pi_F$ i.e. the following diagram is commutative

$$\begin{array}{ccc} H^2 & \stackrel{J}{\longrightarrow} & \Omega(\Gamma) \\ \pi_F \searrow & & \swarrow \pi_\Gamma \\ & S & \end{array}$$

Also we have $J \circ f_i = \gamma_i \circ J$ for $i = 1, ..., p$ and $J \circ f = J$ for each f in N and each mappings is complex-analytic covering and Γ is uniquely determined to within conjugation in $PGL(2, \mathbb{C})$ [19].

For an extension of J to a set some more than H^2 that we need it in the later computations, lets denote by F_0 the free subgroup of F generated by $f_1, ..., f_p$. We know that $\Lambda(F_0)$ is a subset of $\Lambda(F)$. $\Lambda(F_0)$ is F_0 invariant and is out of the closer of the fundamental domain of F_0, then all points of $\Lambda(F_0)$ are in only the circles C_i that are related to the generators of F_0 and their inverses. This shows that the coding of each point in $\Lambda(F_0)$ includes only the generators of F_0 and their inverses. Also since each generator of F like f is an isometry of H^2, it's a composition of two reflections, then $f(x)$ is not in the isometry circle of f^{-1} for each point x in S^1. Then by the definition of the Markov map f_F all the codings of the points of S^1 are irreducible. This means that each point x in $\Lambda(F_0)$ can be coded by the irreducible formal combination $x = (..., f_{i_2}^{\epsilon_{i_2}}, f_{i_1}^{\epsilon_{i_1}}, f_{i_0}^{\epsilon_{i_0}})$ for $i_j \le p$. This Fuchsian coding is suitable for expressing geodesics in N_F or S_F when the beginning and end points of the geodesics are in $\Lambda(F_0)$ and also can be used for extending the map J on $\Lambda(F_0)$ and maybe on some more.

Now we want to extend J to $H^2 \cup \Lambda(F_0)$ (onto $\overline{\Omega(\Gamma)} = \hat{\mathbb{C}}$). For this, we know that each element of $\Lambda(\Gamma)$ can be coded by the infinite words $z = ...\gamma_{i_2}^{\epsilon_{i_2}} \gamma_{i_1}^{\epsilon_{i_1}} \gamma_{i_0}^{\epsilon_{i_0}}(z_0)$ for a constant z_0 in $\Omega(\Gamma)$ and $\epsilon_i = \pm 1$. Similarly, since F_0 is free and purely loxodromic group then it's a Schottky group too and we can use Schottky coding for it too. For conveniences in some proofs we will use Schottky codings for the points of $\Lambda(F_0)$. Then each point x in $\Lambda(F_0)$ can be coded by the infinite words $x = ...f_{i_2}^{\epsilon_{i_2}} f_{i_1}^{\epsilon_{i_1}} f_{i_0}^{\epsilon_{i_0}}(x_0)$ for $i_j \le p$ and a constant point x_0 in $\Omega(F_0) \subset \hat{\mathbb{C}}$ (also $H^2 \subset \Omega(F_0)$). Then we can consider x_0 in $\Omega(F_0) \cap H^2$ such that $z_0 = J(x_0)$. Also from (2.4.1) and using the properties of a loxodromic element and invariance of the limit set under F (or F_0) it is not hard to show that on $\Lambda(F_0)$ we have the following relation between Fuchsian and Schottky coding

$$\begin{aligned} ...f_{i_k}^{\epsilon_{i_k}}...f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}f_{i_k}^{\epsilon_{i_k}}...f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0) &= Z^+(f_{i_k}^{\epsilon_{i_k}}...f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}) \\ &= (..., f_{i_0}^{-\epsilon_{i_0}}, ..., f_{i_{k-1}}^{-\epsilon_{i_{k-1}}}, f_{i_k}^{-\epsilon_{i_k}}, f_{i_0}^{-\epsilon_{i_0}}, ..., f_{i_{k-1}}^{-\epsilon_{i_{k-1}}}, f_{i_k}^{-\epsilon_{i_k}}) \end{aligned} \tag{2.5.1}$$

And also this is true for repelling points too because of the relation $Z^-(f) = Z^+(f^{-1})$ for each f in F . Now, lets put

$$J(...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0)) = ...\gamma_{i_2}^{\epsilon_{i_2}}\gamma_{i_1}^{\epsilon_{i_1}}\gamma_{i_0}^{\epsilon_{i_0}}(z_0) \tag{2.5.2}$$

in other words, by the definition of infinite words, the continuity of J and the relation $J \circ f(x) = \gamma \circ J(x)$ on H^2

$$\begin{aligned} J(...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0)) &= J(\lim_{j\to\infty} f_{i_j}^{\epsilon_{i_j}}...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0)) \\ &= \lim_{j\to\infty} Jf_{i_j}^{\epsilon_{i_j}}...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0) \\ &= \lim_{j\to\infty} \gamma_{i_j}^{\epsilon_{i_j}}...\gamma_{i_2}^{\epsilon_{i_2}}\gamma_{i_1}^{\epsilon_{i_1}}\gamma_{i_0}^{\epsilon_{i_0}}(J(x_0)) \\ &= ...\gamma_{i_2}^{\epsilon_{i_2}}\gamma_{i_1}^{\epsilon_{i_1}}\gamma_{i_0}^{\epsilon_{i_0}}(z_0) \end{aligned} \tag{2.5.3}$$

Then the map $J : \Lambda(F_0) \to \Lambda(\Gamma)$ is well defined and onto. Also from (2.5.1) we see that we can explain J and consequently the functions (spatially Green's function) on S_0 by the Fuchsian coding, when these functions are expressed via this map.

For each point x in $\Lambda(F_0)$ and element f in F_0 equivalent to γ (i.e. $f = f_{i_k}^{\epsilon_{i_k}}...f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}} \sim \gamma = \gamma_{i_k}^{\epsilon_{i_k}}...\gamma_{i_1}^{\epsilon_{i_1}}\gamma_{i_0}^{\epsilon_{i_0}}$, where $i_j \leq p$ and f_i is replaced by γ_i and vice versa) we have

$$J \circ f(...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0)) = \gamma \circ J(...f_{i_2}^{\epsilon_{i_2}}f_{i_1}^{\epsilon_{i_1}}f_{i_0}^{\epsilon_{i_0}}(x_0))$$

Then $J \circ f = \gamma \circ J$ and $J(z^{\pm}(f)) = z^{\pm}(\gamma)$.

2.6. **Automorphic Functions on S_0.** Let F_0 be the free group generated by $f_1, ..., f_p$ and $D = \Sigma m_x(x)$ be a divisor with support $|D|$ in H^2. Put $D_0 := \Sigma m_x(Jx)$ and again lets denote by w_A a meromorphic function on $\mathbb{P}^1(\mathbb{C})$ with the divisor A, and for an element x_0 in $H^2 \setminus \bigcup_{f\in F_0} f(|D|)$ define:

$$W_{D,x_0}(x) = \prod_{f\in F_0} \frac{w_{D_0}(Jf(x))}{w_{D_0}(Jf(x_0))}. \tag{2.6.1}$$

Theorem 2.6.1. *For the Schottky group Γ associated to the Fuchsian group F, if $a(\Gamma) < 1$, then the product (2.6.1) converges absolutely and uniformly on any compact subset of H^2, after deleting a finite number of factors that may have a pole or zero on this subset.*

Proof. Let K be a compact subset of H^2. Since F acts on H^2 properly discontinuously the set $K_{F_0} = \{f \in F_0 | f(K) \cap |D| \neq \emptyset\}$ is finite. If we delete the set K_{F_0} from the index set of the product (2.6.1) then for each $f \in F_0 \setminus K_{F_0}$ when $f(x)$ and $f(x_0)$ lie outside a fixed compact neighborhood of $|D|$ we have

$$\left| \frac{w_{D_0}(Jf(x))}{w_{D_0}(Jf(x_0))} - 1 \right| \leq c|Jf(x) - Jf(x_0)| = c|\alpha(z) - \alpha(z_0)|$$
$$\leq \frac{c}{2}|z - z_0| \left| \frac{d\alpha(z)}{dz} + \frac{d\alpha(z_0)}{dz} \right|$$

Where c is a constant, $z = J(x)$, $z_0 = J(x_0)$ and $f \sim \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (and $ad - bc = 1$). The last inequality comes from the equality $\frac{1}{|cz+d|^2} = |\frac{d\alpha(z)}{dz}|$. Now since $a(\Gamma) < 1$ the series

$$\sum_{\gamma \in \Gamma} \left| \frac{d\gamma(z)}{dz} \right| \tag{2.6.2}$$

converges uniformly on the compact subsets of $\Omega(\Gamma)$. Then the product (2.6.1) is convergent. □

In $W_{D,x_0}(x)$, if we change the point x_0 to x_1, then we have $W_{D,x_0}(x) = C_{x_0}^{x_1} W_{D,x_1}(x)$. Where $C_{x_0}^{x_1}$ is a nonzero complex number that depends on the points x_0 and x_1 and $C_{x_0}^{x_1} C_{x_1}^{x_0} = 1$. Also, for $f \in F$ and $x \in H^2$ we have

$$W_{D,x_0}(fx) = \prod_{h \in F_0} \frac{w_{D_0}(Jh(x_0)))}{w_{D_0}(Jhf^{-1}(x_0))} \prod_{h \in F_0} \frac{w_{D_0}(Jh(x))}{w_{D_0}(Jh(x_0))} = \mu_D(f) W_{D,x_0}(x).$$

$\mu_D(f)$ is a nonzero complex number multiplicative on D and f and also independent of x_0. We can see this as following

$$\mu_D^{x_0}(f) W_{D,x_0}(x) = W_{D,x_0}(fx) = C_{x_0}^{x_1} \mu_D^{x_1}(f) C_{x_1}^{x_0} W_{D,x_0}(x) = \mu_D^{x_1}(f) W_{D,x_0}(x)$$

Then $\mu_D^{x_0}(f) = \mu_D^{x_1}(f)$. When Γ is a cyclic group $\mu_D(f) = 1$. This shows that W_{D,x_0} is not F automorphic function on H^2 in general.

Theorem 2.6.2. *a) Lets denote by $C(f|g)$ a set of the representatives of $(f^n) \backslash F_0/(g^n)$. Then for $f \neq g$ mod$[F_0, F_0]$ in F_0 we have*

$$\mu_{(g(x_1))-(x_1)}(f) = \prod_{h \in F_0} \frac{w_{(Jg(x_1))-(Jx_1)}(J(h(x_0)))}{w_{(Jg(x_1))-(Jx_1)}(J(h \circ f^{-1}(x_0)))}$$
$$= \prod_{h \in C(f|g)} \frac{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^+(g)))}{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^-(g)))} \tag{2.6.3}$$

And by defining $Q(f) := \langle J(z^+(f)), J(z^-(f)), J(f(x_1)), J(x_1) \rangle$ we have

$$\mu_{(f(x_1))-(x_1)}(f) = Q(f) \prod_{h \in C_0(f|f)} \frac{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^+(f)))}{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^-(f)))} \tag{2.6.4}$$

Where $C_0(f|f)$ is the set $C(f|f)$ without the identity class.
b) Lets denote by $C(|f)$ a set of representatives of $F_0/(f^n)$ and by $S(f)$ the conjugacy

class of f in F_0. Then for some x_1 in H^2 such that $z_1 = J(x_1)$ stays in $\Omega(\Gamma) \setminus \Gamma_\infty$, we have

$$W_{(f(x_1))-(x_1),x_0}(x) = \prod_{h \in C(|f)} \frac{w_{(Jh(z^+(f)))-(Jh(z^-(f)))}(J(x))}{w_{(Jh(z^+(f)))-(Jh(z^-(f)))}(J(x_0))}$$
$$= \prod_{h \in S(f)} \frac{w_{(J(z^+(h)))-(J(z^-(h)))}(J(x))}{w_{(J(z^+(h)))-(J(z^-(h)))}(J(x_0))}. \tag{2.6.5}$$

And this is independent of x_1.

Proof. Lets put $J(g(x_1)) = a$, $J(x_1) = b$ and $w_{(a)-(b)}(x) = \frac{a-x}{b-x}$ then we have

$$\mu_{(g(x_1))-(x_1)}(f) = \prod_{h \in F_0} \frac{a - Jh(x_0)}{b - Jh(x_0)} \Big/ \frac{a - Jhf^{-1}(x_0)}{b - Jhf^{-1}(x_0)}$$
$$= \prod_{h \in C(f|g)} \prod_{m=-\infty}^{m=\infty} \prod_{n=-\infty}^{n=\infty} \frac{a - J(f^{-m}hg^n(x_0))}{a - J(f^{-m}hg^{n-1}(x_0))} \frac{b - J(f^{-m}hg^{n-1}(x_0))}{b - J(f^{-m}hg^n(x_0))}$$
$$= \prod_{h \in C(f|g)} \prod_{m=-\infty}^{m=\infty} \prod_{n=-\infty}^{n=\infty} \frac{A_n}{A_{n-1}} \frac{B_{n-1}}{B_n}$$
$$= \prod_{h \in C(f|g)} \prod_{m=-\infty}^{m=\infty} \frac{A_\infty}{A_{-\infty}} \frac{B_{-\infty}}{B_\infty} \qquad \left(\prod_{-N}^{N} \frac{A_n}{A_{n-1}} \frac{B_{n-1}}{B_n} = \frac{A_N}{A_{-N-1}} \frac{B_{-N-1}}{B_N} \right)$$
$$= \prod_{h \in C(f|g)} \prod_{m=-\infty}^{m=\infty} \frac{a - J(f^{-m}h(z^+(g))}{a - J(f^{-m}h(z^-(g)))} \frac{b - J(f^{-m}h(z^-(g)))}{b - J(f^{-m}h(z^+(g)))}$$
$$= \prod_{h \in C(f|g)} \prod_{m=-\infty}^{m=\infty} \frac{J(f^{m+1}(x_1)) - J(h(z^+(g))}{J(f^{m+1}(x_1)) - J(h(z^-(g)))} \frac{J(f^m(x_1)) - J(h(z^-(g)))}{J(f^m(x_1)) - J(h(z^+(g)))} \tag{2.6.6}$$
$$= \prod_{h \in C(f|g)} \frac{J(z^+(f)) - J(h(z^+(g))}{J(z^-(f)) - J(h(z^+(g)))} \frac{J(z^-(f)) - J(h(z^-(g)))}{J(z^+(f)) - J(h(z^-(g)))}$$
$$= \prod_{h \in C(f|g)} \frac{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^+(g)))}{w_{(J(z^+(f)))-(J(z^-(f)))}(Jh(z^-(g)))}$$

The part (2.6.6) comes from the equation $J \circ g = \gamma \circ J$ and the invariance of the Cross-Ratio on the action of Mobius transformations. The second part of a) and b) can be proved similarly. For b) we should consider that $h(z^\pm(g)) = z^\pm(hgh^{-1})$. $\square$

In above theorem in the case that the divisor is $(a) - (b)$ we have

$$\mu_{(a)-(b)}(f) = \prod_{h \in S(f)} \frac{w_{(J(a))-(J(b))}(J(z^+(h)))}{w_{(J(a))-(J(b))}(J(z^-(h)))}. \tag{2.6.7}$$

By previous theorem $W_{(f(x_1))-(x_1),x_0}(x)$ is a meromorphic function without any poles and zeroes in H^2 then it is a holomorphic function on H^2. Also as we see in the expression, it is independent of the point x_1.

2.7. **Differentials of the first kind on S_0.** For each f in F_0 lets put

$$\omega_f = d\log W_{(f(x_1))-(x_1),x_0}(x).$$

However W_{D,x_0} is not F- automorphic function on H^2 in general but we have

$$d\log W_{D,x_0}(fx) = d\log W_{D,x_0}(x).$$

And this shows that the differential $d\log W_{D,x_0}(x)$ is F automorphic. Then ω_f is a differential of the first kind on S_0.

According to the classical theorem of cuts we can choose $\Omega(\Gamma)$ the region of discontinuity of the Schottky group Γ with the marking $\{D_1, ..., D_{2p}; \gamma_1, \gamma_2, ..., \gamma_p\}$ with $C_i = \partial D_i$ such that $\bar{a}_i$ the image of a_i for $i = 1, ..., p$ in S_0 be coincident with the image of C_{p+i}. And these together with $\bar{b}_i$ the image of b_i for $i = 1, ..., p$ in S_0 make a canonical base for $H_1(S_0, \mathbb{Z})$ i.e.

$$(\bar{a}_i, \bar{a}_j) = (\bar{b}_i, \bar{b}_j) = 0 \qquad \textit{and} \qquad (\bar{a}_i, \bar{b}_j) = \delta_{ij}. \tag{2.7.1}$$

Proposition 2.7.1. *a) $\{\omega_{f_i}\}_1^p$ is a Riemann's basis for the space of differentials of the first kind on S_0 by choosing the previous base for $H_1(S_0, \mathbb{Z})$ i.e.*

$$\frac{1}{2\pi i}\int_{\bar{a}_j} \omega_{f_i} = \delta_{ij}. \tag{2.7.2}$$

b) If we denote by $C(f|g)$ a set of the representatives of $(f^n)\backslash F_0/(g^n)$ and by $C_0(f|g)$, the set $C(f|g)$ without the identity class, Then for $i \neq j$ we have

$$\tau_{ij} := \int_{\bar{b}_j} \omega_{f_i} = \sum_{h\in C(f_j|f_i)} \log\langle J(z^+(f_j)), J(z^-(f_j)), J(h(z^+(f_i)), J(h(z^-(f_i))\rangle \tag{2.7.3}$$

and for $i = j$ by defining $Q(f_i) := \langle J(z^+(f_i)), J(z^-(f_i)), J(f_i(x_1)), J(x_1)\rangle$ we have

$$\tau_{ii} = \log Q(f_i) + \sum_{h\in C_0(f_i|f_i)} \log\langle J(z^+(f_i)), J(z^-(f_i)), J(h(z^+(f_i)), J(h(z^-(f_i))\rangle. \tag{2.7.4}$$

Proof. We have

$$W_{(f(x_1))-(x_1),x_0}(x) = \prod_{\alpha\in\Gamma} \frac{w_{(\gamma(z_1))-(z_1)}(\alpha(z))}{w_{(\gamma(z_1))-(z_1)}(\alpha(z_0))} \tag{2.7.5}$$

where $z = J(x), z_0 = J(x_0),\ z_1 = J(x_1)$ and $f \sim \gamma$. If we show this equality for $f_i,\ i = 1, 2, 3, ..., p$ by

$$W_{(f_i(x_1))-(x_1),x_0}(x) = \overline{W}_{(\gamma_i(z_1))-(z_1),z_o}(z) \tag{2.7.6}$$

then for a) we have

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\bar{a}_j}\omega_{f_i} &= \frac{1}{2\pi i}\int_{\bar{a}_j} d\log W_{(f_i(x_1))-(x_1),x_0}(x)\\
&= \frac{1}{2\pi i}\int_{a_j}\frac{dJ}{dx}\,\frac{d\log W_{(f_i(x_1))-(x_1),x_0}(x)}{dz}dx\\
&= \frac{1}{2\pi i}\int_{J(a_j)}\frac{d\log \overline{W}_{(\gamma_i(z_1))-(z_1),z_o}(z)}{dz}dz\\
&= \frac{1}{2\pi i}\int_{C_{p+j}} d\log \overline{W}_{(\gamma_i(z_1))-(z_1),z_o}(z)\\
&= \frac{1}{2\pi i}\sum_{\alpha\in C(|\gamma_i)}\int_{C_{p+j}} d\log w_{(\alpha(z^+(\gamma_i)))-(\alpha(z^-(\gamma_i)))}(z)\\
&= \sum_{\alpha\in C(|\gamma_i)}\begin{cases}1 & if\ \alpha(z^+(\gamma_i))\in D_{p+j},\alpha(z^-(\gamma_i))\notin D_{p+j}\\ -1 & if\ \alpha(z^+(\gamma_i))\notin D_{p+j},\alpha(z^-(\gamma_i))\in D_{p+j}\\ 0 & otherwise\end{cases}
\end{aligned}
$$

The last equality comes from this fact that if $i=j$ since $z^+(\gamma_i))\in D_{p+i}$ and $z^-(\gamma_i)\in D_i$, only for $\alpha = id$ the first alternative and for all $\alpha\neq id$ the third one is valid. If $i\neq j$ only the third alternative is right for all α. We can see these from the Figure 1.

For the first part of b). We have shown by the point x_j the intersection of a_j and b_j in the representation of P, the fundamental domain of F. Then $\bar{b}_j$ is the image of the part of b_i that is between the points x_j and $f_j(x_j)$ in the fundamental domain. Now, from (2.6.3) we have

$$
\begin{aligned}
\int_{\bar{b}_j}\omega_{f_i} &= \int_{x_j}^{f_j(x_j)} d\log W_{(f_i(x_1))-(x_1),x_0}(x)\\
&= \log\frac{W_{(f_i(x_1))-(x_1),x_0}(f_j(x_j))}{W_{(f_i(x_1))-(x_1),x_0}(x_j)}\\
&= \log\left(\prod_{h\in F_0}\frac{w_{(Jf_i(x_1))-(Jx_1)}(Jhf_j(x_j))}{w_{(Jf_i(x_1))-(Jx_1)}(Jh(x_0))}\Big/\prod_{h\in F_0}\frac{w_{(Jf_i(x_1))-(Jx_1)}(Jh(x_j))}{w_{(Jf_i(x_1))-(Jx_1)}(Jh(x_0))}\right)\\
&= \log\mu_{(f_i(x_1))-(x_1)}(f_j)\\
&= \sum_{h\in C(f_j|f_i)}\log\frac{w_{(J(z^+(f_j)))-(J(z^-(f_j)))}(Jh(z^+(f_i)))}{w_{(J(z^+(f_j)))-(J(z^-(f_j)))}(Jh(z^-(f_i)))}\\
&= \sum_{h\in C(f_j|f_i)}\log\langle J(z^+(f_j)),J(z^-(f_j)),J(h(z^+(f_i)),J(h(z^-(f_i))\rangle \qquad (2.7.7)
\end{aligned}
$$

Where the last equality comes from the definition in chapter one. Similarly we can reach to the second part of b) using (2.6.4). □

2.8. **Differentials of the third kind and Green's functions.** For $a, b, c, d \in S_0$ lets denote by the same words the corresponding points in $P \subset H^2$ the fundamental domain of F (then Ja, Jb, Jc and Jd are in $\hat{\mathbb{C}} \setminus \cup_{i=1}^{p} (D_i \cup \bar{D}_{p+i})$ the fundamental domain of Γ) and put $\nu_{(a)-(b)} = d \log W_{(a)-(b),x_0}(x)$. Then $\nu_{(a)-(b)}$ is a differential of third kind on S_0 with the residues 1 and -1 at the images of a and b. Also, because the points a and b are in the fundamental domain then they are out of the circles C_i. Then $\bar{a}_i$ - periods of $\nu_{(a)-(b)}$ are zero, and from (2.6.7) the $\bar{b}_k$ - periods are

$$\begin{aligned}
\int_{\bar{b}_j} \nu_{(a)-(b)} &= \int_{x_j}^{f_j(x_j)} d \log W_{(a)-(b),x_0}(x) \\
&= \log \frac{W_{(a)-(b),x_0}(f_j(x_j))}{W_{(a)-(b),x_0}(x_j)} \\
&= \log \left(\prod_{h \in F_0} \frac{w_{(J(a))-(J(b))}(Jhf_j(x_j))}{w_{(J(a))-(J(b))}(Jh(x_0))} \Big/ \prod_{h \in F_0} \frac{w_{(J(a))-(J(b))}(Jh(x_j))}{w_{(J(a))-(J(b))}(Jh(x_0))} \right) \\
&= \log \mu_{(a)-(b)}(f_j) \\
&= \log \prod_{h \in S(f_j)} \frac{w_{(J(a))-(J(b))}(J(z^+(h)))}{w_{(J(a))-(J(b))}(J(z^-(h)))} \\
&= \sum_{h \in S(f_j)} \log \langle J(a), J(b), J(z^+(h)), J(z^-(h)) \rangle. \qquad (2.8.1)
\end{aligned}$$

Then we have

$$Re \int_{\bar{b}_j} \nu_{(a)-(b)} = \log |\mu_{(a)-(b)}(f_j)| = \sum_{h \in S(f_j)} \log |\langle J(a), J(b), J(z^+(h)), J(z^-(h)) \rangle|.$$

Now, we can reach to a differential of the third kind with pure imaginary periods by defining

$$\omega_{(a)-(b)} = \nu_{(a)-(b)} - \sum_{i=1}^{p} X_i(a,b) \omega_{f_i} \qquad (2.8.2)$$

Where the real coefficients $X_i(a,b)$ are such that the set of the equations

$$\sum_{j=1}^{p} X_j(a,b) Re \tau_{ij} = Re \int_{\bar{b}_i} \nu_{(a)-(b)} = \sum_{h \in S(f_i)} \log |\langle Ja, Jb, J(z^+(h)), J(z^-(h)) \rangle|$$

for $i = 1, ..., p$ are satisfied. In fact the coefficients $X_i(a,b)$ kill the real part of the $\bar{b}_i$ - periods of $\nu_{(a)-(b)}$. Notice that the new differential form $\omega_{(a)-(b)}$ probably have nonzero $\bar{a}_i$ - periods, but since the coefficients $X_i(a,b)$ are real and $\int_{\bar{b}_i} \omega_{f_i}$ are pure imaginary then they are pure imaginary too.

Finally, if we denote the points a, b, c, and d in S_0 and the images of these points in the fundamental domain of F by the same notations, then we have

$$\begin{aligned}
\int_d^c \nu_{(a)-(b)} &= \int_d^c d\log W_{(a)-(b),x_0}(x) \\
&= \log \frac{W_{(a)-(b),x_0}(c)}{W_{(a)-(b),x_0}(d)} \\
&= \log\left(\prod_{h\in F_0} \frac{w_{(J(a))-(J(b))}(Jh(c))}{w_{(J(a))-(J(b))}(Jh(x_0))} \Big/ \prod_{h\in F_0} \frac{w_{(J(a))-(J(b))}(Jh(d))}{w_{(J(a))-(J(b))}(Jh(x_0))}\right) \\
&= \log \prod_{h\in F_0} \frac{w_{(J(a))-(J(b))}(Jh(c))}{w_{(J(a))-(J(b))}(Jh(d))} \\
&= \sum_{h\in F_0} \log\langle J(a), J(b), J(h(c)), J(h(d))\rangle. \qquad (2.8.3)
\end{aligned}$$

And by (2.6.5)

$$\begin{aligned}
\int_d^c \omega_{f_i} &= \int_d^c d\log W_{(f_i(x_1))-(x_1),x_0}(x) \\
&= \log \frac{W_{(f_i(x_1))-(x_1),x_0}(c)}{W_{(f_i(x_1))-(x_1),x_0}(d)} \\
&= \log \frac{\prod_{h\in S(f_i)} \frac{w_{(J(z^+(h)))-(J(z^-(h)))}(J(c))}{w_{(J(z^+(h)))-(J(z^-(h)))}(J(x_0))}}{\prod_{h\in S(f_i)} \frac{w_{(J(z^+(h)))-(J(z^-(h)))}(J(d))}{w_{(J(z^+(h)))-(J(z^-(h)))}(J(x_0))}} \\
&= \log \prod_{h\in S(f_i)} \frac{w_{(J(z^+(h)))-(J(z^-(h)))}(J(c))}{w_{(J(z^+(h)))-(J(z^-(h)))}(J(d))} \\
&= \sum_{h\in S(f_i)} \log\langle J(z^+(h)), J(z^-(h)), J(c), J(d)\rangle. \qquad (2.8.4)
\end{aligned}$$

Then the Green's function on S_0 can be computed as following

$$\begin{aligned}
g((a)-(b),(c)-(d)) &= Re\int_d^c \omega_{(a)-(b)} \\
&= Re\int_d^c \nu_{(a)-(b)} - \sum_{i=1}^p X_i(a,b) Re\int_d^c \omega_{f_i} \\
&= \sum_{h\in F_0} \log \mid \langle J(a), J(b), J(h(c)), J(h(d))\rangle \mid - \\
&\quad \sum_{i=1}^p X_i(a,b) \sum_{h\in S(f_i)} \log \mid \langle J(z^+(h)), J(z^-(h)), J(c), J(d)\rangle \mid \qquad (2.8.5)
\end{aligned}$$

Because of the commutativity of the diagram

$$\begin{array}{ccc} H^2 & \xrightarrow{J} & \Omega(\Gamma) \\ \pi_F \searrow & & \swarrow \pi_\Gamma \\ & S_0 & \end{array}$$

the image of the point $x \in H^2$ and $J(x) \in \Omega(\Gamma)$ are the same in S_0. Then the above formula for the Green's function on S_0 gives an expression via the points on S_0. In fact for the points a, b in $\Omega(F)$ the images of the geodesics $\{a, b\}$ and $\{J(a), J(b)\}$ in the Kleinian manifold M_F are the same. We can see this by using the following covering spaces

$$H^3 \cup \Omega(F) = H^3 \cup H^2 \cup (-H^2) \xrightarrow{J} \frac{H^3}{N} \cup \frac{H^2}{N} \cup \frac{-H^2}{N} \xrightarrow{\pi_\Gamma} M_F. \tag{2.8.6}$$

For all parts $Aut(\tilde{X}) \simeq \Gamma$, and again we have $\pi_\Gamma \circ J = \pi_F$. Also since J is extended on $\Lambda(F_0)$ in a natural way then for each $f \in F_0$ the image of the geodesics $\{Z^+(f), Z^-(f)\}$ and $\{J(Z^+(f)), J(Z^-(f))\}$ in N_F are the same.

One can reach to a formula for the Green's function on S_1 similarly by replacing $-H^2$ and $-P$ instead of H^2 and P.

Definition 2.8.1. A finitely generated, torsion free Kleinian group Q is called Quasi - Fuchsian if the limit set $\Lambda(Q)$ be a Jordan curve and each of the two simply connected components of $\Omega(Q)$ be Q-invariant.

Proposition 2.8.2. *Given a Quasi - Fuchsian group Q, there exist a Fuchsian group F and a quasiconformal diffeomorphism between Kleinian manifolds M_Q and M_F.*

Proof. (See [1]). □

Lets denote by D_1 and D_2 the simply connected components of $\Omega(Q)$ for Quasi-Fuchsian group Q. Then there are two Fuchsian groups F_1 and F_2 related to Q such that D_i/Q is homeomorphic to H^2/F_i. Then each F_i is isomorphic to Q. Also since $Q|_{\Lambda(Q)}$ is topologically conjugate to $F_i|_{\hat{\mathbb{R}}}$ (or $F_i|_{S^1}$ in unit disk model) then naturally there is a Markov map $f_Q : \Lambda(Q) \to \Lambda(Q)$ like Fuchsian case. Then all the statements for Fuchsian groups can be extended to the Quasi - Fuchsian groups. In this case because the Jordan curve $\Lambda(Q)$ is generally not smooth nor even rectifiable ([2] p.263) then the hausdorff dimension of the subgroup Q_0 of Q (like F_0 for Fuchsian case F) may be not less than 1 in general. We have the following theorem

Theorem 2.8.3. *For a finitely generated Kleinian group G the following conditions are equivalent*

(1) $M_G = (H^3 \cup \Omega(G))/G$ is diffeomorphic to $S \times [0, 1]$ where S is a component of ∂M_G;
(2) G is Quasi - Fuchsian;
(3) $\Omega(G)$ has an invariant component that is a Jordan domain;
(4) $\Omega(G)$ has two invariant components;

Proof. (See [15], page 125). □

Also two homeomorphic Riemann surfaces can be made uniform simultaneously by a single Quasi - Fuchsian group (theorem of Bers on simultaneous uniformization). Then we have the computations for all hyperbolic 3 - manifolds with two compact Riemann surfaces with the same genus as it's boundary components.

3. n Boundary components case

Already, we have computed the Green's function for the boundary components of a hyperbolic 3-manifolds with two boundary at infinity with the same genera. In this chapter we want to do the problem for the most general case that the Green's function can be defined i.e. for manifolds with $n < \infty$ boundary components that are compact Riemann surfaces probably with different genera. The case with two boundary is different genera are included in this case too. Such manifolds are uniformized by some Kleinian groups and for the first, we will try to find a decomposition for such Kleinian groups. This, help us to find invariants of these groups that are necessary for computing the automorphic functions that will be used for computing the differentials and the Green's functions. In all of this section we consider G to be a finitely generated and torsion free Kleinian group of the second kind (i.e. $\Omega(G) \neq \varnothing$).

Theorem 3.0.4. ***(The Ahlfors finiteness theorem)*** *Let G be a finitely generated and torsion free Kleinian group. Then $\partial M_G = \Omega(G)/G$ is a finite union of analytically finite Riemann surfaces(closed Riemann surface from which a finite number of points are removed).*

Proof. (See [1]). □

For the rest of the section we consider

$$\partial M_G = \frac{\Omega(G)}{G} = S_1 \cup S_2 \cup ... \cup S_{n+1} \tag{3.0.7}$$

such that for each i, S_i is a compact Riemann surface of genus p_i and without cusped point. Then we should consider that G doesn't have any parabolic elements. Also consider that the component S_{n+1} is the one that for $i = 1, ..., n$, $p_{n+1} \geq p_i$. In this case G is a function group (A finitely generated non - elementary Kleinian group which has an invariant component in its region of discontinuity) and there is an invariant component Ω_0 in $\Omega(G)$.

Proposition 3.0.5. *Let G be a function group with an invariant component Ω_0 in $\Omega(G)$. Then $\Lambda(G) = \partial\Omega_0$. And hence all the other components of $\Omega(G)$ are simply connected.*

Proof. (See [15]). □

We consider two cases for Ω_0 in the proposition above:
Case1: The component Ω_0 is simply connected. Then in this case G is a B - group (a function group with simply connected invariant component). And we know that a B-group with a compact boundary component say S_k is a Quasi - Fuchsian group (See [17] page 411).
Case2: The component Ω_0 is not simply connected. In this case we have the following theorem.

Theorem 3.0.6. *Let G be a function group with invariant component Ω_0 that is not a simply connected component. Then G has a decomposition into a free product of subgroups as following*

$$G = \mathfrak{B}_1 * \cdots * \mathfrak{B}_r * \mathfrak{A}_1 * \cdots * \mathfrak{A}_s * < p_1 > * \cdots * < p_t > * < f_1 > * \cdots * < f_u >$$

Where each $\mathfrak{B}_i$ is a B - group , $\mathfrak{A}_i$ is a free abelian group of rank two with two parabolic generators, $< p_i >$ is the cyclic group generated by the parabolic transformation p_i and $< f_i >$ is the cyclic group generated by the loxodromic transformation f_i. Furthermore, each parabolic transformation in G is conjugate in G to one a listed subgroup. G has finite - sided fundamental polyhedron if and only if all the groups $\mathfrak{B}_i$ do.

Proof. [17] page 420 (see also [15]). □

This theorem shows that in our special case we have

$$G = \mathfrak{B}_1 * \cdots * \mathfrak{B}_r * < f_1 > * \cdots * < f_u > . \tag{3.0.8}$$

For B - groups if one boundary component be compact then it's a Quasi - Fuchsian group. And $< f_1 > * \cdots * < f_u >$ is a free and purely loxodromic group and consequently it's a Schottky group [15]. Hence we have

$$G = Q_1 * \cdots * Q_n * \Gamma \tag{3.0.9}$$

Where Q_i is a Quasi - Fuchsian group and Γ is a schottky group. Then by Van Kampen theorem we have

$$M_G = M_{Q_1} \# M_{Q_2} \# \cdots \# M_{Q_n} \# M_\Gamma \tag{3.0.10}$$

and if $\partial M_{Q_i} = S_{Q_i} \cup S'_{Q_i}$ then

$$S_{n+1} = S'_{Q_1} \# S'_{Q_2} \# \cdots \# S'_{Q_n} \# S_\Gamma \quad and \quad S_{Q_i} = S_i. \tag{3.0.11}$$

We will denote by p_0 the genus of Schottky group Γ in the decomposition of the group G. We intend to analyze the structure of the region of discontinuity of G. For the first, we consider the case that we don't have the Schottky group Γ in the decomposition of G. Lets denote by D_i and D'_i the simply connected components of the Quasi - Fuchsian group Q_i.

Lemma 3.0.7. *For each $g \in G$, $g(D_i) \cap D_i = D_i$ or $g(D_i) \cap D_i = \emptyset$.*

Lemma 3.0.8. *For each $i = 1, ..., n$, the sets $\mathcal{D}_i = \bigcup_{g \in G} g(D_i)$ are G - invariant. Then $\bigcap_1^n \mathcal{D}_i = \emptyset$.*

From proposition (3.0.5) all other components than Ω_0 of $\Omega(G)$ are simply connected. Now, since $S_1 \cap \cdots \cap S_n = \emptyset$ and from lemmas (3.0.7), (3.0.8) and the decomposition of ∂M_G we see that $\{D_1, D_2, ..., D_n, \Omega_0\}$ is a complete system of representatives of the equivalent classes (Ω_1 and Ω_2 are equivalent or conjugate if and only if $stab_G(\Omega_1)$ is conjugate in G to $stab_G(\Omega_2)$; and since $z^{\pm}(hgh^{-1}) = hz^{\pm}(g)$ then Ω_1 and Ω_2 are equivalent or conjugate if and only if there is a g in G such that $\Omega_1 = g(\Omega_2)$) of the components of $\Omega(G)$. Since Ω_0 is G - invariant the class of Ω_0 has one member. Then we have

Proposition 3.0.9. *If we consider that the class of D_i is for $S_{Q_i} = S_i$, then*

$$\Omega_0 = \hat{\mathbb{C}} \setminus (\bigcup_{i=1}^{n} \bigcup_{g\in G} g(\bar{D}_i)) = \bigcap_{i=1}^{n} \bigcap_{g\in G} g(D_i').$$

Proof. $\Omega(G)$ contains only the G - invariant sets $\mathcal{D}_i$ and Ω_0. □

By lemmas (3.0.7), (3.0.8) and proposition (3.0.9) we have

$$\Lambda(G) = \bigcup_{i=1}^{n} \bigcup_{g\in G} g(\partial(D_i)) = \bigcup_{i=1}^{n} \bigcup_{g\in G} g(\Lambda(Q_i)). \tag{3.0.12}$$

(Figure 5 for the case that $n = 3$ and there is no Schottky group). Then we have

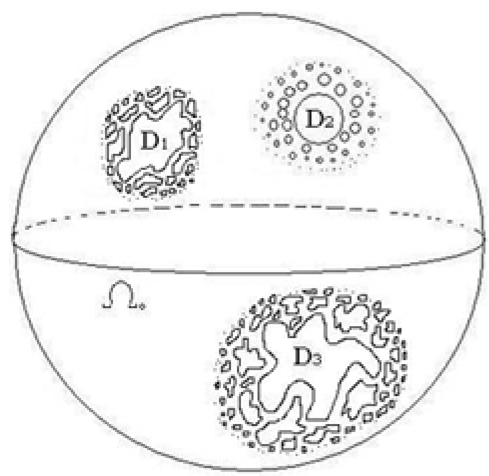

FIGURE 5.

$$S_1 = D_1/Q_1, \cdots, S_n = D_n/Q_n \qquad and \qquad S_{n+1} = \Omega_0/G. \tag{3.0.13}$$

3.1. **Fundamental domain of the group G.** Let $K_i = F_i \cup F_i'$ be a fundamental domain for the group Q_i such that $F_i \subset D_i$ and $F_i' \subset D_i'$. Then a fundamental domain for the group G is (Figure 6)

$$K = (\bigcup_{i=1}^{n} F_i) \cup (\bigcap_{i=1}^{n} F_i'). \tag{3.1.1}$$

In this case we have $p_{n+1} = p_1 + p_2 + \cdots + p_n$.

Now, lets consider the general case that there is a Schottky group $\Gamma_0 =< \gamma_{01}, ..., \gamma_{0p_0} >$ (of order p_0) in the decomposition of the group G. Again we know that all other components of $\Omega(G)$ than Ω_0 is simply connected. Then ∂M_{Γ_0} can not be glued to the components other than S_{n+1} in ∂M_G. Also this means that in this case we have

$$\Omega_0 = (\hat{\mathbb{C}} \setminus (\bigcup_{i=1}^{n} \bigcup_{g\in G} g(\bar{D}_i))) \setminus \bigcup_{g\in G} g(\Lambda(\Gamma_0)) = \bigcap_{g\in G} (\bigcap_{i=1}^{n} g(D_i') \setminus g(\Lambda(\Gamma_0)))$$

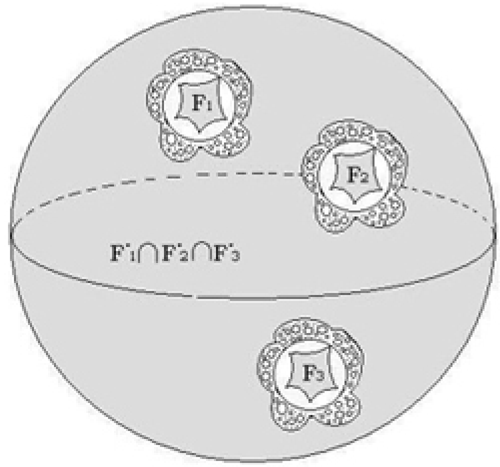

FIGURE 6.

And

$$\Lambda(G)=\bigcup_{i=1}^{n}\bigcup_{g\in G} g(\Lambda(Q_i)\cup\Lambda(\Gamma_0)). \tag{3.1.2}$$

In this case if $\{A_1,...,A_{2p_0};\gamma_{01},...,\gamma_{0p_0}\}$ be a marking for the Schottky group Γ_0, then the fundamental domain is

$$K=(\bigcup_{i=1}^{n}F_i)\cup((\bigcap_{i=1}^{n}F_i')\backslash\bigcup_{i=1}^{p_0}(A_i\cup\bar{A}_{p_0+i})). \tag{3.1.3}$$

And also for the genus of the component S_{n+1} we have $p_{n+1}=p_0+p_1+\cdots+p_n$.

3.2. **Green's Function on S_i.** For each $i=1,...,n$ lets consider

$$Q_i=\langle\{q_{ij};j=1,...,2p_i\};\prod_{j=1}^{p_i}[q_{ij},q_{ip_i+j}]=I\rangle. \tag{3.2.1}$$

Then we can compute Green's function on the components S_i for $i=1,...,n$ as like as the previous section.

3.3. **Schottky group associated to the group G.** From the definition of Q_i we know that Q_i/N_i is a free group generated by p_i generators. We have the following lemma

Lemma 3.3.1. *Lets denote by N the smallest normal subgroup of the group $Q_1*\cdots*Q_n$ including the generators q_{ij} for $i=1,...,n$ and $j=p_i+1,...,2p_i$ and N_i be the smallest normal subgroup of the group Q_i including the generators q_{ij} for $j=p_i+1,...,2p_i$. Then we have*

$$\frac{Q_1*\cdots*Q_n}{N}\cong\frac{Q_1}{N_1}*\cdots*\frac{Q_n}{N_n}. \tag{3.3.1}$$

Proof. In the case $n=2$ the map

$$\varphi : Q_1 * Q_2 \longrightarrow \tfrac{Q_1}{N_1} * \tfrac{Q_2}{N_2}$$
$$x_1 ... x_k \longmapsto \bar{x}_1 ... \bar{x}_k$$

is an isomorphism. The general case is true by induction on n. □

Now if N be the smallest normal subgroup of the group G including the generators q_{ij} for $i = 1, ..., n$ and $j = p_i + 1, ..., 2p_i$, Then by the previous lemma one can see that G/N is isomorphic to the group

$$\Gamma_0 * \frac{Q_1}{N_1} * \cdots * \frac{Q_n}{N_n} = \Gamma_0 * \Gamma_1 * \cdots * \Gamma_n.$$

Then G/N is a free group of order p_{n+1}.

Now lets consider the covering space map $\Omega_0 \longrightarrow S_{n+1}$. Since N is a normal subgroup of G then the covering space $\Omega_0/N \longrightarrow S_{n+1}$ is regular and is between $\Omega_0 \longrightarrow S_{n+1}$ with the covering transformations group $Aut(\Omega_0/N) = G/N$ that is a free group of order p_{n+1}. Then similar to the previous section we have a Schottky group $\Gamma = \langle \gamma_{i1}, \gamma_{i2}, ..., \gamma_{ip_i} \quad | \quad i = 0, ..., n \quad \rangle$ (we can arrange the generators of the group Γ in this way because of the isometry) and a complex - analytic covering mapping $J : \Omega_0 \longrightarrow \Omega(\Gamma)$ such that the following diagram is commutative

$$\begin{array}{ccc} \Omega_0 & \xrightarrow{J} & \Omega(\Gamma) \\ \pi_G \searrow & & \swarrow \pi_\Gamma \\ & S_{n+1} & \end{array}$$

and for $i = 1, ..., n$ and $j \leq p_i$, $J \circ q_{ij} = \gamma_{ij} \circ J$ and for $j > p_i$, $J \circ q_{ij} = J$ and $J \circ \gamma_{0j} = \gamma_{0j} \circ J$ (we have considered the same notations for the generators of Γ_i and Γ). As a matter of fact there is a Fuchsian group F and there are normal subgroups H_1 and H_2 of F such that the sequence

$$H^2 \xrightarrow{J'} \frac{H^2}{H_1} \cong \Omega_0 \xrightarrow{J} \frac{H^2}{H_2} \cong \frac{\Omega_0}{N} \longrightarrow S_{n+1} \tag{3.3.2}$$

is the composition of analytic covering maps. Then we have the composition of covering maps

$$H^2 \xrightarrow{J \circ J'} \frac{H^2}{H_2} \cong \frac{\Omega_0}{N} \longrightarrow S_{n+1} \tag{3.3.3}$$

and since $F/H_2 = Aut(\Omega_0/N) = G/N$ is a free group, like the previous section we have $\Omega_0/N \cong H^2/H_2 \cong \Omega(\Gamma)$ for some Schottky group Γ. In fact, for the first, because of the isometries $F/H_1 = Aut(H^2/H_1 \cong \Omega_0) = G$ we can choose the Fuchsian group

$$F = \langle \{f_{ij} \ ; i = 0, \ldots, n \quad j = 1, ..., 2p_i\}; \prod_{i=0}^{n} \prod_{j=1}^{p_i} [f_{ij}, f_{i,p_i+j}] = I \quad \rangle \tag{3.3.4}$$

such that the equations $g_{ij} \circ J' = J' \circ f_{ij}$ for $i = 1, ..., n$ and $j = 1, ..., p_i$ are satisfied. Where $g_{ij}(= q_{ij} \quad or \quad \gamma_{0j})$ are the generators of G. Then like the previous section we can choose uniquely up to conjugation in $\mathrm{PGL}(2,\mathbb{C})$ the Schottky group Γ that satisfies the equations

$$(J\circ J')\circ f_{ij}=\gamma_{ij}\circ(J\circ J') \quad and \quad (J\circ J')\circ f_{ip_i+j}=J\circ J' \tag{3.3.5}$$

Then since J' is onto, the relations $J\circ g_{ij}=\gamma_{ij}\circ J$ and $J\circ g_{ip_i+j}=J$ are satisfied automatically for $i=1,...,n$ and $j=1,...,p_i$. This means that the following diagram is commutative in all loops

$$\begin{array}{ccccc} H^2 & \xrightarrow{J'} & \Omega_0 & \xrightarrow{J} & \Omega(\Gamma) \\ f_{ij}\updownarrow & & g_{ij}\updownarrow & & \gamma_{ij}\updownarrow \\ H^2 & \xrightarrow{J'} & \Omega_0 & \xrightarrow{J} & \Omega(\Gamma) \end{array}$$

This gives a proof for the existing the of covering space $\Omega(\Gamma)$ and Schottky group Γ satisfying the properties that we need and also shows the relations between the Fuchsian group F , Kleinian group G and the Schottky group Γ associated to S_{n+1}.

Now lets put $G_0=\Gamma_0*Q_{10}*Q_{20}*\cdots*Q_{n0}$. Where Q_{i0} is the free subgroup of Q_i generated by the elements $q_{i1},...,q_{ip_i}$. For the extension of J to $\Lambda(G_0)$ i.e. defining the map $J:\Omega_0\cup\Lambda(G_0)\longrightarrow\hat{\mathbb{C}}$, we know that each Q_{i0} is free and purely loxodromic then it is a Schottky group. Like the previous section we can code an element of $\Lambda(Q_{i0})$ by Schottky coding $x=\cdots q_{ij_2}^{\epsilon_{j_2}}q_{ij_1}^{\epsilon_{j_1}}q_{ij_0}^{\epsilon_{j_0}}(x_0)$ for $j_k\le p_i$ and a point x_0 in $\Omega_0\subset D_i'$. For each element g in G_0 lets denote by γ the element in Γ corresponding to g, such that $J\circ g=\gamma\circ J$. Now put

$$J_0:\bigcup_{g\in G}g(\Lambda(\Gamma_0))\longrightarrow\Lambda(\Gamma)$$

$$g(\cdots\gamma_{i_2}^{\epsilon_2}\gamma_{i_1}^{\epsilon_1}\gamma_{i_0}^{\epsilon_0}(x_0))\mapsto\gamma(\cdots\gamma_{0i_2}^{\epsilon_2}\gamma_{0i_1}^{\epsilon_1}\gamma_{0i_0}^{\epsilon_0}(z_0))$$

Where $z_0=J(x_0)$. And for $i=1,...,n$ put

$$J_i:\bigcup_{g\in G}g(\Lambda(Q_{i0}))\longrightarrow\Lambda(\Gamma)$$

$$g(\cdots q_{ij_2}^{\epsilon_{j_2}}q_{ij_1}^{\epsilon_{j_1}}q_{ij_0}^{\epsilon_{j_0}}(x_0))\mapsto\gamma(\cdots\gamma_{ij_2}^{\epsilon_{j_2}}\gamma_{ij_1}^{\epsilon_{j_1}}\gamma_{ij_0}^{\epsilon_{j_0}}(z_0))$$

And finally lets define

$$J:\Lambda(G_0)\longrightarrow\Lambda(\Gamma)$$

$$J(x)=\begin{cases} J_0(x) & if\ x\in\bigcup_{g\in G}g(\Lambda(\Gamma_0)) \\ \\ J_i(x) & if\ x\in\bigcup_{g\in G}g(\Lambda(Q_{i0})) \end{cases}.$$

Then like the previous section we can show that for each element g in G_0 and γ the element of Γ corresponding to g and each x in $\Lambda(G_0)$ we have $J\circ g(x)=\gamma\circ J(x)$ and $J(z^{\pm}(g))=z^{\pm}(\gamma)$. Also like the previous chapter we can consider the Fuchsian coding for the points of $\Lambda(G_0)$ and from (2.5.1) we can explain J and consequently the Green's function on S_{n+1} via the Fuchsian coding.

3.4. **Green's function of S_{n+1}.** If we consider the condition $a(\Gamma) < 1$ and use the subgroup $G_0 = \Gamma_0 * Q_{10} * Q_{20} * \cdots * Q_{n0}$ instead of F_0 in the previous section then one can bring all of the definitions like before with some changes, and compute the Green's function on S_{n+1} and the other parts in the same way. In this case we should notice that if $\{a_{ij}, b_{ij} | i = 0, ..., n, j = 1, ..., p_i\}$ be a set that makes a base for $H_1(S_{n+1}, \mathbb{Z})$ then each member of it has a class of images in Ω_0. But we can consider that the representatives that are used in the computations are in the fundamental domain. As its shown in figure 6. In this case these images are in $D'_i \cap \Omega_0$ for $i = 1, ..., n$ and also in $\Omega(\Gamma_0) \cap \Omega_0$ for $i = 0$. Then the representatives for S_i and S_{n+1} are coincide if we uniformize S_i by D'_i, i.e. identifying both components of ∂M_{Q_i}. Final formula for the Green's function on S_{n+1} is as following

$$g_{s_{n+1}}((a)-(b),(c)-(d)) = \sum_{f \in G_0} \log | \langle J(a), J(b), J(f(c)), J(f(d)) \rangle |$$

$$- \sum_{\substack{i=0,\dots,n \\ j=1,\dots,p_i}} \overline{X}_{ij}(a,b) \sum_{f \in S(f_{ij})} \log | \langle J(z^+(f)), J(z^-(f)), J(c), J(d) \rangle | \tag{3.4.1}$$

Where f_{ij} is equal to q_{ij} for $i = 1, ..., n$ and to γ_{0j} for $i = 0$, and $\overline{X}_{ij}(a,b)$ are the multipliers for $g_{s_{n+1}}$.

3.5. **Remark (Infinitely many boundary component case).** When we have infinitely many boundary components in the boundary of N_G that are Riemann surfaces without puncture, according to the Ahlfors finiteness theorem, G is an infinitely generated Kleinian group. And hence one of the boundary components of M_G is with infinity genus and is not a compact Riemann surface then for this component the Green's function is not defined. But for the other components we can bring the computations similar to the previous condition using some subgroups of G that are isomorphic to the Kleinian groups in the previous sections.

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