KÄHLER-EINSTEIN METRICS ON STRICTLY PSEUDOCONVEX DOMAINS

CRAIG VAN COEVERING

Abstract. Extending the results of S. Y. Cheng and S.-T. Yau it is shown that a strictly pseudoconvex domain $M \subset X$ in a complex manifold carries a complete Kähler-Einstein metric if and only if its canonical bundle is positive, i.e. admits an Hermitian connection with positive curvature. We consider the restricted case in which the CR structure on ∂M is normal. In this case M must be a domain in a resolution of the Sasaki cone over ∂M. We give a condition on a normal CR manifold which it cannot satisfy if it is a CR infinity of a Kähler-Einstein manifold. We are able to mostly determine those normal CR 3-manifolds which can be CR infinities.

We give many examples of Kähler-Einstein strictly pseudoconvex manifolds on bundles and resolutions. In particular, the tubular neighborhood of the zero section of every negative holomorphic vector bundle on a compact complex manifold whose total space satisfies $c_1 < 0$ admits a complete Kähler-Einstein metric.

Contents

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1. Introduction

S. Y. Cheng and S.-T. Yau proved in [16] that a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a complete negative scalar curvature Kähler-Einstein metric. Their arguments also extended to other types of domains, such as a pseudoconvex domain which is the intersection of pseudoconvex domains with C^2 boundary. Many cases of domains in arbitrary complex manifolds are already dealt with in [16], and in [32]. In [32] N. Mok and S.-T. Yau proved the existence of a Kähler-Einstein metric on strictly pseudoconvex domains under some hypotheses. These include, in particular, domains in Stein manifolds and domains which admit a negative Ricci curvature metric. This article considers the existence of a complete negative scalar curvature Kähler-Einstein metric on a strictly pseudoconvex domain of an arbitrary complex manifold.

Theorem 1.1. *Suppose M is a strictly pseudoconvex domain in M'. Then M admits an unique complete Kähler-Einstein metric of negative scalar curvature if and only if $\mathbf{K}_M$ is positive, i.e. admits an Hermitian metric with positive curvature.*

Strict pseudoconvexity means that the boundary $S := \partial M$ has a positive CR structure. We consider the case in which the CR structure on S is in addition normal, that is, admits transverse vector field preserving it. We prove that any strictly pseudoconvex manifold with a normal CR structure on S must be a domain in a resolution of the Sasaki cone $C(S)$ of the natural Sasaki structure on S. This severely restricts the strictly pseudoconvex manifolds with a normal CR structure on the boundary. In particular, a domain in a Stein manifold must be in $\mathbb{C}^n$ and have a boundary diffeomorphic to $\mathbb{S}^{2n-1}$ with CR structure a deformation of the standard one. We consider the following:

Problem 1.2. *Which positive normal CR manifold (S, D, J) is the conformal boundary of Kähler-Einstein manifold?*

There are far too many CR structures on even simple manifolds for this to be a tractable problem in general. This is true even for $S = \mathbb{S}^{2n-1}$. It is shown in [9, 10] that there are inequivalent families of CR structures on all odd dimensional spheres with the number of deformation classes growing doubly exponentially with the dimension. In particular, $\mathbb{S}^5$ has 68 inequivalent deformation classes. More interesting is that these CR structures have associated Sasaki-Einstein metrics. But in the present context, we prove here that they cannot be the conformal boundary of a Kähler-Einstein manifold. We prove that any simply connected normal CR manifold (S, D, J) satisfying the topological condition for a compatible Sasaki-Einstein metric, $c_1(D) = 0$ and $c_1^B > 0$, are negative examples of 1.2. Thus not surprisingly, those (S, D, J) which can be the boundary at infinity of a complete Ricci-flat manifold as considered in [43] are excluded. We are able to mostly answer Problem 1.2 in dimension 3, just leaving open some cases of finite quotients of $\mathbb{S}^3$.

We give many examples in which Theorem 1.1 applies. This gives many examples with normal CR structures at infinity and otherwise. An easy case is that of negative holomorphic bundles over a compact complex manifold.

Corollary 1.3. *Let $\pi : \mathbf{E} \to N$ be a negative holomorphic bundle over a compact complex manifold N. If $-c_1(M) - c_1(\mathbf{E}) > 0$ on N, then the disk subbundles $D \subset \mathbf{E}$ admit unique complete Kähler-Einstein metrics of negative scalar curvature.*

We also construct some examples on resolutions of hypersurface singularities and on some familiar resolutions of quotient singularities.

2. Background

Let S be a real $2n-1$-dimensional manifold. A *CR structure* on S is a pair (D, J) consisting of a distribution $D \subset TS$ of real $2n-2$-dimensional hyperplanes and an almost complex structure J on D such that, if $D^{1,0} \subset D \otimes \mathbb{C} \subset TS \otimes \mathbb{C}$ denotes the type $(1,0)$-vectors, the *formal integrability* condition holds:

$$[D^{1,0}, D^{1,0}] \subseteq D^{1,0}. \tag{1}$$

The *Levi form* $\mathbf{L}^D : D \times D \to TS/D$ is defined by $\mathbf{L}^D(X,Y) = -[X,Y] \mod D$, for $X, Y \in \Gamma(D)$.

It is easy to check that (1) is equivalent to both $[X, JY] + [JX, Y] \in \Gamma(D)$ and the vanishing of the Nijenhuis tensor

$$\mathcal{N}_J(X,Y) = [JX, JY] - [X,Y] - J\big([X, JY] - [JX, Y]\big) = 0, \quad X, Y \in \Gamma(D). \tag{2}$$

Note that the former condition implies $\mathbf{L}^D$ is J-invariant, i.e. $\mathbf{L}^D(J\cdot, J\cdot) = \mathbf{L}^D(\cdot,\cdot)$.

We will always assume S is orientable, so TS/D is a trivial real line bundle. Then there is a 1-form η with $D = \ker\eta$, and we may identify $\mathbf{L}^D = d\eta|_D$. Note that $\mathbf{L}^D$ is only defined up to a choice of orientation of TS/D and a positive conformal factor.

Definition 2.1. *The CR structure (D, J) is* strictly pseudoconvex *if, for a given orientation of TS/D, the Levi form $\mathbf{L}^D$ is positive, i.e. $\mathbf{L}^D(X, JX) > 0$ for any non-zero $X \in D$.*

Note that formal integrability does not in general imply integrability, that is that (S, D, J) is a real hypersurface in a complex manifold. The analogue of the Newlander-Nirenberg theorem only holds for analytic CR manifolds [36, 35].

Definition 2.2. *A* finite manifold *is a pair (M, M') of complex manifolds with M an open relatively compact submanifold of M' with smooth non-empty boundary ∂M.*

Let ϕ be a defining function of $M \subset M'$. That is, ϕ is C^∞ in a neighborhood of $\bar{M}$, $M = \{\phi < 0\}$, and $d\phi \neq 0$ on ∂M.

Let J denote the complex structure of M'. The real $2n-1$-dimensional manifold $S := \partial M$ has the CR structure (D, J) where $D := TS \cap JTS$ and J is restricted to D.

Define a 1-form on S

$$\eta := d^c\phi|_S, \tag{3}$$

where $d^c := \frac{\sqrt{-1}}{2}(\bar\partial - \partial)$. Then it is easy to see that $D = \ker\eta$, and the Levi form is $\mathbf{L}^D = d\eta|_D$, which is a $(1,1)$-form on D as follows from the comments after (2).

Definition 2.3. *We say that the pair (M, M') is* strictly pseudoconvex *or that M is a strictly pseudoconvex domain, if the induced CR-structure (D, J) is.*

One can check that by altering ϕ, for instance considering $e^{A\phi}\phi$ for a constant $A > 0$, one may assume that ϕ is strictly plurisubharmonic on a neighborhood of $S = \partial M$. That is, $\sqrt{-1}\partial\bar\partial\phi$ is a positive $(1,1)$-form.

A strictly pseudoconvex domain $M \subset M'$ is a particular type of 1-convex manifold. A complex manifold X is said to be 1-convex if there is a Stein space Y, a proper holomorphic surjective mapping $\pi : X \to Y$ satisfying $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, and a finite set $A \subset Y$ such that if $E = \pi^{-1}(A)$ the map $\pi : X \setminus E \to Y \setminus A$ is a biholomorphism. Then Y is called the Remmert reduction of X and E is called the exceptional set of X. Note that E is the maximal compact analytic subvariety of X, i.e. the union of all compact analytic subvarieties of dimension ≥ 1.

Let $\xi \in \Gamma(TS)$ be a vector field on S so that

$$TS = D \oplus \mathbb{R}\xi. \tag{4}$$

We extend J to a $(1,1)$ tensor Φ on S by

$$\Phi(X) = JX, \text{ for } X \in D, \text{ and } \Phi\xi = 0. \tag{5}$$

Definition 2.4. *The CR structure (D, J) on S is* normal *if there is a ξ satisfying (4) whose flow preserves (D, J). Or in other words $\mathcal{L}_\xi \Phi = 0$.*

Suppose $\eta(\xi) > 0$ for an oriented contact form η. So we may assume by changing by a conformal factor that η is the unique 1-form with $\ker \eta = D$ and $\eta(\xi) = 1$, i.e. ξ is the Reeb vector field of η. Then (D, J) is normal if and only if

$$\mathcal{N}_\Phi = \xi \otimes d\eta. \tag{6}$$

In fact, it is easy to check that (6) is equivalent to (2) and $\mathcal{L}_\xi \Phi = 0$. And these two conditions are equivalent to the integrability of the almost complex structure on the *cone* $C(S) := \mathbb{R}_+ \times S$

$$I(X) = \Phi(X) - \eta(X) r\partial_r, \quad I(r\partial_r) = \xi, \tag{7}$$

where r is the radial coordinate on $\mathbb{R}_+$ and $X \in TS$.

If the CR structure (D, J) is positive and normal, then with ξ positively oriented as above, we have a natural metric

$$g(X, Y) := \frac{1}{2} d\eta(X, \Phi Y) + \eta(X)\eta(Y), \quad X, Y \in TS. \tag{8}$$

In this case S has a special type of metric contact structure, known as a *Sasaki structure*, which we denote by (g, ξ, η, Φ). See [8] for more details.

We denote by $\mathfrak{CR}(S, D, J)$ the automorphism group of the CR manifold (S, D, J) and its Lie algebra by $\mathfrak{cr}(S, D, J)$. R. Schoen [40] proved the following result.

Theorem 2.5. *The CR automorphism group of a strictly pseudoconvex CR manifold (M, D, J) is proper unless M is either $\mathbb{S}^{2n-1}$ or the Heisenberg group $\mathcal{H}_{2n-1}$ with the standard CR structures.*

Corollary 2.6. *If (M, D, J) is a compact strictly pseudoconvex CR manifold, then the CR automorphism group $\mathfrak{CR}(S, D, J)$ is compact unless $M = \mathbb{S}^{2n-1}$ with the standard CR structure, in which case the CR automorphism group is* $\mathrm{PSU}(1, n)$.

It will be useful to consider the space of compatible Sasaki structures on a normal strictly pseudoconvex CR manifold (S, D, J). See [11, 12] for more details.

Definition 2.7. *Let (S, D, J) be a strictly pseudoconvex CR structure. A vector field $X \in \mathfrak{cr}(S, D, J)$ is* positive *if $\eta(X) > 0$ for an oriented contact form η. We denote by $\mathfrak{cr}^+(S, D, J)$ the space of all positive elements of $\mathfrak{cr}(S, D, J)$.*

It is not difficult to see that $\mathfrak{cr}^+(S,D,J)$ is isomorphic the space of Sasaki structures compatible with (D,J), $\mathfrak{cr}^+(S,D,J)$ is an open convex cone in $\mathfrak{cr}(S,D,J)$, and is invariant under the adjoint action of $\mathfrak{CR}(S,D,J)$.

Definition 2.8. *Let (S,D,J) be a normal strictly pseudoconvex CR manifold. The* Sasaki cone $\kappa(S,D,J)$ *is the moduli space of Sasaki structures compatible with (D,J) on S. We have*

$$\kappa(S,D,J)=\mathfrak{cr}^+(S,D,J)/\mathfrak{CR}(S,D,J). \tag{9}$$

Choose a maximal torus $T_k\subset\mathfrak{CR}(S,D,J)$ of rank k, $1\leq k\leq n$, with Lie algebra $\mathfrak{t}_k$. Then if $\mathfrak{t}_k^+$ denotes the subspace of positive elements, we have

$$\kappa(S,D,J)=\mathfrak{t}_k^+/\mathcal{W}, \tag{10}$$

where $\mathcal{W}$ is the Weyl group of $T_k\subset K\subseteq\mathfrak{CR}(S,D,J)$ for a maximal compact subgroup K. Of course, $K=\mathfrak{CR}(D,J)$ unless $\mathfrak{CR}(D,J)=\mathrm{PSU}(1,n)$. Let $Z_k\subset\mathfrak{t}_k$ be the lattice of *integral* elements, that is $Z_k=\{\xi\in\mathfrak{t}_k:\exp(2\pi\xi)=1\}$. And define $Z_k^+=Z_k\cap\mathfrak{t}_k^+$. Then every $\xi\in Z_k^+$ defines a quasi-regular Sasaki structure, that is all the orbits of the Reeb vector field ξ close to give a locally free $\mathrm{U}(1)$-action. The $\mathrm{U}(1)$-action on S extends to a locally free holomorphic $\mathbb{C}^*$-action on $C(S)$. And $C(S)$ is biholomorphic to the total space minus the zero section $\mathbf{L}^\times$ of a negative holomorphic orbibundle over a Kähler orbifold W (cf. [8]).

3. The Kähler-Einstein metric

3.1. **The approximate metric.** Let $M\subset M'$ be a smooth strictly pseudoconvex domain in a Kähler manifold (M',g_0). And let ϕ be a plurisubharmonic defining function which is strictly plurisubharmonic on a neighborhood of ∂M. Then $h=-\log(-\phi)$ is strictly plurisubharmonic near ∂M, and dd^ch is the Kähler form of a metric near ∂M which in coordinates is

$$h_{i\bar{j}}=\frac{\phi_{i\bar{j}}}{-\phi}+\frac{\phi_i\phi_{\bar{j}}}{\phi^2}. \tag{11}$$

Computation gives

$$h^{i\bar{j}}=(-\phi)\left(\phi^{i\bar{j}}+\frac{\phi^i\phi^{\bar{j}}}{\phi-|d\phi|^2}\right), \tag{12}$$

where $\phi^{i\bar{j}}=(\phi_{i\bar{j}})^{-1}$, $\phi^i=\sum\phi^{i\bar{j}}\phi_{\bar{j}}$, and $|d\phi|^2=\phi^{i\bar{j}}\phi_i\phi_{\bar{j}}$.

It is also easy to see that

$$h^{i\bar{j}}h_ih_{\bar{j}}=\frac{|d\phi|^2}{|d\phi|^2-\phi}\leq 1. \tag{13}$$

Thus since $h(x)\to\infty$ as $x\to\partial M$, the metric $h_{i\bar{j}}$ is complete toward ∂M. Therefore, *a fortiori* the metric $g_{i\bar{j}}=(g_0)_{i\bar{j}}+h_{i\bar{j}}$ with Kähler form

$$\omega=\omega_0+dd^ch, \tag{14}$$

is a complete Kähler metric on M.

3.2. **Existence of the metric.** We will consider the existence of a complete Kähler-Einstein metric on M, that is a Kähler metric g with

$$\mathrm{Ric}_g = -\lambda g, \quad \lambda > 0. \tag{15}$$

For convenience we will set $\lambda = n+1$. If g is a complete Kähler metric on M with Kähler form ω, suppose we have $F \in C^\infty$ with

$$(n+1)\omega + \mathrm{Ricci}(\omega) = dd^c F. \tag{16}$$

Then a solution to the Monge-Ampère equation

$$(\omega + dd^c u)^n = e^{(n+1)u+F}\omega^n \tag{17}$$

provides a Kähler metric $\omega' = \omega + dd^c u$ satisfying (15). Equation (17) on non-compact manifolds was extensively studied by S.-Y Cheng and S.-T. Yau [16]. See also [42]. There it was proved that (17) has a unique solution if $F \in C^{3,\alpha}(M)$ and (M,g) has bounded geometry.

We use this method to find a complete solution to (15) where $M \subset M'$ is a strictly pseudoconvex domain in a Kähler manifold M'. With ϕ a defining function of M and ω_0 a Kähler form on M' we consider the complete metric with Kähler form

$$\omega = \omega_0 - dd^c \log(-\phi). \tag{18}$$

If a line bundle $\mathbf{L}$ is given by a system of charts and transition functions $(U_\alpha, g_{\alpha\beta})$, then an Hermitian metric on $\mathbf{L}$ is given by a system $\{h_\alpha\}$ of smooth positive functions on $\{U_\alpha\}$ which satisfy $h_\alpha = |g_{\beta\alpha}|^2 h_\beta$ on $U_\alpha \cap U_\beta$. In particular, we will use that any other Hermitian metric h' on $\mathbf{L}$ is of the form $h' = e^f h$ for $f \in C^\infty$.

An holomorphic line bundle $\mathbf{L}$ is *positive* if it has an Hermitian metric h such that the curvature of the associated Chern connection, $\Theta_{\mathbf{L}} = -\partial\bar{\partial}\log h$, satisfies $\frac{\sqrt{-1}}{2\pi}\Theta_{\mathbf{L}} > 0$, i.e. is a positive $(1,1)$-form.

The following theorem is mostly due to S. Y. Cheng and S. T. Yau [16].

Theorem 3.1. *Let (M, M') be a strictly pseudoconvex finite manifold. Then M admits a complete Kähler-Einstein metric of negative scalar curvature if and only if $\mathbf{K}_M$ is positive.*

Proof. Let h be a positive Hermitian metric on $\mathbf{K}_M$ and let h' be any connection on $\mathbf{K}_{M'}$. Choose $\varsigma \in C^\infty(\mathbb{R})$ with $\varsigma(x) = 1$ for $x \geq 1$ and $\varsigma(x) = 0$ for $x \leq 1/2$. Set $\varsigma_R(x) := \varsigma(\frac{x}{R})$. Consider the metric $\tilde{h} = \varsigma_R(-\phi)h + (1-\varsigma_R(-\phi))h'$, which has positive curvature on $\{-\phi > R\} \subset M$. Choose $R > 0$ sufficiently small that this set contains the maximal compact analytic subset E of M, and choose a plurisubharmonic function ψ on M' which is strictly plurisubharmonic away from E. Then $e^{-A\psi}\tilde{h}$ has positive curvature on a neighborhood of $M \subset M'$ for $A \gg 0$.

Suppose h is a metric on $\mathbf{K}_{M'}$ with $\omega_0 = \frac{\sqrt{-1}}{(n+1)}\Theta_h$ positive on a neighborhood of M. Then the volume form $\frac{1}{n!}\omega_0^n$ defines an Hermitian metric on $\mathbf{K}_{M'}$ by

$$\|\Omega\|^2 := \left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} \frac{\Omega \wedge \bar{\Omega}}{\omega_0^n}, \tag{19}$$

for $(n,0)$-form Ω. Then $\|\cdot\|^2 = e^f h$, for some $f \in C^\infty(M')$. So we have

$$(n+1)\omega_0 + \mathrm{Ricci}(\omega_0) = dd^c f. \tag{20}$$

We define

$$F = \log\left[\frac{e^f(-\phi)^{-(n+1)}\omega_0^n}{(\omega_0 - dd^c\log(-\phi))^n}\right]. \tag{21}$$

Then for the metric $\omega = \omega_0 - dd^c\log(-\phi)$ we have that F satisfies (16). It is easy to see that $F \in C^\infty(\overline{M})$. In fact one checks that

$$\frac{e^f(-\phi)^{-(n+1)}\omega_0^n}{(\omega_0 - dd^c\log(-\phi))^n}\Big|_{\partial M} = \frac{e^f\omega_0^n}{|d\phi|^2(dd^c\phi)^n}\Big|_{\partial M}. \tag{22}$$

Then the proof in [16] shows that (17) has a unique solution $u \in C^\infty(M)$. The proof follows from an application of the generalized maximum principle to formulae of [45] to obtain the necessary *a priori* estimates.

The converse is clear. Since if g_0 is Kähler-Einstein, then the curvature of (19) satisfies $\frac{\sqrt{-1}}{2\pi}\Theta = -\frac{1}{2\pi}\operatorname{Ricci}(g_0) > 0$. □

3.3. **uniqueness.** The following uniqueness result is due to S.-T. Yau and follows from a more general Schwartz lemma.

Proposition 3.2 ([32]). *Let (M_1, g_1) and (M_2, g_2) be complete complete Kähler-Einstein manifolds of negative scalar curvature, normalized to have equal Einstein constants, λ. If $\sigma : M_1 \to M_2$ is a biholomorphism, then $\sigma^* g_2 = g_1$.*

Proof. Let μ_{M_i}, $i = 1, 2$ be the respective volume forms, and define $f = \sigma^*\mu_{M_2}/\mu_{M_1}$. Then we have

$$\begin{aligned}\Delta \log f &= n\lambda - g_1^{i\bar{j}}\sigma^*\operatorname{Ricci}(g_2)_{i\bar{j}}\\ &= n\lambda - \lambda g_1^{i\bar{j}}(\sigma^* g_2)_{i\bar{j}}\end{aligned} \tag{23}$$

The arithmetic-geometric inequality applied to the second term on the right of (23) gives

$$\Delta \log f \geq n\lambda - \lambda n f^{1/n}. \tag{24}$$

From which we have

$$\Delta f \geq n\lambda f - n\lambda f^{\frac{n+1}{n}}, \tag{25}$$

and it follows from the maximum principle as in [42, Lemma 1.1] that $\sup f \leq 1$. Applying the same argument to σ^{-1} gives $\sigma^*\mu_{M_2} = \mu_{M_1}$, from which we have $\sigma^*\operatorname{Ricci}(g_2) = \operatorname{Ricci}(g_1)$ and $\sigma^* g_2 = g_1$. □

Let $\mathfrak{Hol}(M)$ denote the group of biholomorophisms of M, $\mathfrak{Isom}(M, g)$ the group of isometries of (M, g), and $\mathfrak{hol}(M)$, $\mathfrak{isom}(M, g)$ their respective Lie algebras. We also have the following easy converse to Proposition 3.2.

Proposition 3.3. *Let $\sigma : M \to M$ be an isometry of Kähler-Einstein strictly pseudoconvex manifold. Then σ is a biholomorphism up to conjugation, i.e. $\sigma^* J = \pm J$ where J is the complex structure of M. Thus*

$$\mathfrak{isom}(M, g) = \mathfrak{hol}(M). \tag{26}$$

Furthermore, if $X \in \mathfrak{isom}(M, g)$, then $JX \notin \mathfrak{isom}(M, g)$.

Proof. First note that since (M,g) has curvature asymptotic to constant -2 holomorphic bisectional curvature it must be irreducible as a Kähler manifold. There are two, J and $\sigma^* J$, parallel complex structures on M. Since (M,J) is irreducible, either $\sigma^* J = \pm J$, or the holonomy group $\mathrm{Hol}(g_1) \subseteq Sp(\frac{n}{2})$. In other words, in the second case one can show the existence of three parallel complex structures J_1, J_2, J_3 satisfying the quaternionic identities. But in this case $\mathrm{Ric}_g = 0$, and (26) follows.

Suppose that $X, JX \in \mathfrak{isom}(M,g,J)$. The following argument is due to S. Kobayashi [28, Ch. III,§1]. Define

$$A_X = \mathcal{L}_X - \nabla_X.$$

Since $X \in \mathfrak{hol}(M)$ and (M,g) is Kähler, we have

$$JA_X = A_X J = A_{JX}. \tag{27}$$

We have

$$g(A_{JX}Y, Z) + g(Y, A_{JX}Z) = 0 \tag{28}$$

and from (27)

$$g(A_X JY, Z) - g(JY, A_X Z) = 0. \tag{29}$$

It follows that A_X is symmetric. But since it is also skew-symmetric, we have $A_X = -\nabla X = 0$. This implies $\mathrm{Ric}_g(X,X) = 0$, a contradiction. □

3.4. **Boundary behavior.** We consider the boundary behavior of the metric $g_{i\bar{j}} = (g_0)_{i\bar{j}} + h_{i\bar{j}}$ and the Einstein metric $g'_{i\bar{j}}$ of Theorem 3.1, where $h = -\log(-\phi)$ for a defining function ϕ and g_0 is a Kähler metric on $\bar{M}$. First a straight forward calculation as in [16] gives the Christoffel symbols Γ^k_{ij} and the curvature $R_{i\bar{j}l\bar{k}}$ of $h_{i\bar{j}}$ near the boundary of M.

$$\Gamma^k_{ij} = \phi_{ij\bar{l}}\phi^{k\bar{l}} + \frac{\phi_i\delta^k_j + \phi_j\delta^k_i}{-\phi} + \frac{1}{\phi - |d\phi|^2}\big(\phi_{ij\bar{l}}\phi^{\bar{l}}\phi^k - \phi_{ij}\phi^k\big) \tag{30}$$

$$R_{i\bar{j}l\bar{k}} = -(g_{i\bar{j}}g_{l\bar{k}} + g_{i\bar{k}}g_{l\bar{j}}) + \frac{1}{\phi}\big(R^{\phi}_{i\bar{j}l\bar{k}} + \frac{1}{\phi - |d\phi|^2}(\phi_{,il}\phi_{,\bar{j}\bar{k}})\big) \tag{31}$$

Here R^ϕ denotes the curvature and $\phi_{,ij}$ the covariant derivative with respect to $\phi_{i\bar{j}}$.

The optimal regularity and asymptotic behavior of the solution $u \in C^\infty(M)$ to (17) of Theorem 3.1 was given in [31] for $M \subset \mathbb{C}^n$. The proof works with only minor modifications to an arbitrary strictly pseudoconvex $M \subset M'$ with the initial metric $g_{i\bar{j}}$. An essential step is to find a defining function ϕ_0 so that F defined in (21) vanishes to high order on ∂M. The following was first proved by C. Fefferman [19] for $M \subset \mathbb{C}^n$.

Lemma 3.4. *There exists a defining function ϕ_0 of $M \subset M'$ so that F given in (21) satisfies*

$$F = O(\phi_0^{n+1}) \tag{32}$$

Proof. We seek $\beta \in C^\infty(\bar{M})$ so that $\phi' = e^\beta \phi$ so that

$$\frac{e^f(-\phi')^{-(n+1)}\omega_0^n}{(\omega_0 - dd^c\log(-\phi'))^n} \to 1 \text{ on } \partial M. \tag{33}$$

But since

$$\frac{(\omega_0 - dd^c\log(-\phi))^n}{(\omega_0 - dd^c\log(-\phi'))^n} = \frac{(\omega_0 - dd^c\log(-\phi))^n}{(\omega_0 - dd^c\log(-\phi) - dd^c\beta)^n} \to 1 \text{ on } \partial M, \tag{34}$$

we may take $\beta = \frac{F}{n+1}$, and (33) is satisfied. Then the inductive argument in the proof of [31] goes through with the operator

$$\beta \to \frac{(\omega_0 - dd^c\log(-\phi) + dd^c\beta)^n e^{-f}(-e^{-\beta}\phi)^{n+1}}{\omega_0^n} \tag{35}$$

substituting that used there. □

The results of [31] on the asymptotic behavior of the solution $u \in C^\infty(M)$ to (17) with defining function ϕ_0 are valid in this situation. One can define Hölder spaces $C^{k,\alpha}(M)$ with respect to the metric $(g_0)_{i\bar{j}} + h_{i\bar{j}}$. Then if F given in (21) vanishes to order $0 < r < n+1$, we have

$$u \in \bigcap_k \phi^r C^{k,\alpha}(M). \tag{36}$$

Moreover, there is an asymptotic expansion of u. There are $\alpha_j \in C^\infty(\bar{M}),\ j \geq 1$, such that for $N \in \mathbb{N}$

$$u - \sum_{j=1}^{N} \alpha_j \phi_0^{(n+1)j}(\log(-\phi_0))^j \in C^{(n+1)(N+1)-1,\alpha}(\bar{M}), \tag{37}$$

and vanishes to order $(n+1)(N+1)-1$.

Proposition 3.5. *Let $g_{i\bar{j}}$ be either the metric $(g_0)_{i\bar{j}} + h_{i\bar{j}}$ or $(g_0)_{i\bar{j}} + h_{i\bar{j}} + u_{i\bar{j}}$ solving (15), i.e. the Kähler-Einstein metric, on a strictly pseudoconvex M, and let $r = \mathrm{dist}(o,x)$ be the distance from a fixed point $o \in M$, then the curvature of $g_{i\bar{j}}$ satisfies*

$$R_{i\bar{j}l\bar{k}} = -(g_{i\bar{j}}g_{l\bar{k}} + g_{i\bar{k}}g_{l\bar{j}}) + O(e^{-2r}). \tag{38}$$

Thus metric g is asymptotically of constant holomorphic sectional curvature -2 and are asymptotically complex hyperbolic*(ACH).*

If g is the Kähler-Einstein metric on a strictly pseudoconvex $M \subset M$ and ϕ is any defining function, then $(-\phi)g \to \mathbf{L}^D(\cdot, J\cdot)$ on $D \subset T\partial M$, where the Levi form $\mathbf{L}^D$ is of course only defined up to a conformal factor.

3.5. **Comments on the theorem and the $\partial\bar{\partial}$-lemma.** One could also consider the weaker condition that $-c_1(\tilde{M})$ is represented by a positive $(1,1)$-form. This is *a priori* weaker assumption as the $\partial\bar{\partial}$-lemma does not generally hold on a 1-convex manifold. It remains whether this weaker assumption is a sufficient condition for Theorem 3.1. The following is easy.

Lemma 3.6. *Let X be a complex manifold. Then X satisfies the $\partial\bar{\partial}$-lemma if and only if for any holomorphic line bundle $\mathbf{L}$ any $\omega \in c_1(\mathbf{L})$ is represented by $\frac{\sqrt{-1}}{2\pi}\Theta_h$ for some Hermitian metric h.*

We make the following

Conjecture 3.7. *A strictly pseudoconvex domain M admits a complete Kähler-Einstein metric if and only if there is a Kähler form $\omega \in -c_1(M)$.*

We will give some results that make the conjecture plausible. At least the following results will show that constructing a counterexample to Conjecture 3.7 would be very difficult. We consider some examples of 1-convex surfaces due to M. Colţoiu [17] on which the $\partial\bar{\partial}$-lemma does not hold, but nevertheless Theorem 3.1 applies.

Let X be an 1-convex manifold with exceptional set E. We denote by $B(X) \subset \operatorname{Pic} X$ the subgroup of line bundles $\mathbf{L}$ which are topologically trivial on X and holomorphically trivial in a neighborhood of E.

Proposition 3.8 ([17]). *For a 1-convex manifold with exceptional set E there is a group isomorphism*

$$B(X) \xrightarrow{\sim} H^1(E,\mathbb{Z})/\operatorname{Im}\bigl[H^1(X,\mathbb{Z}) \to H^1(E,\mathbb{Z})\bigr].$$

Let C_1 and C_2 be smooth curves in $\mathbb{C}P^2$ intersecting transversely of degrees $d_1 \geq 3$ and $d_2 > d_1$ respectively. Let $\pi : Y \to \mathbb{C}P^2$ be the blow up at each of the $d_1 d_2$ points $p_1, \ldots, p_{d_1 d_2}$ of intersection $C_1 \cdot C_2$. If $\hat{C}_i, i = 1,2$ are their strict transforms, then as divisors

$$\hat{C}_i = \pi^* d_i H - \sum_{j=1}^{d_1 d_2} E_j, \quad i = 1,2, \tag{39}$$

where E_j , $j = 1, \ldots, d_1 d_2$ are the exceptional divisors. Then $\hat{C}_1^2 = d_1^2 - d_1 d_2 < 0$ and $\hat{C}_2^2 = d_2^2 - d_1 d_2 > 0$. The first inequality, by a theorem of Grauert, implies that $\hat{C}_1$ is exceptional, and the second inequality implies that $X = Y \setminus \hat{C}_2$ is 1-convex. And it turns out that $\hat{C}_1$ is the entire exceptional set. See [17] for details.

In addition it is shown that $\pi_1(X) = 1$ and by the genus formula we have $g(\hat{C}_1) = \frac{(d_1-1)(d_1-2)}{2}$. Thus

$$B(X) = \mathbb{Z}^{2g}. \tag{40}$$

Since $K_Y = \pi^*(-3H) + \sum_{j=1}^{d_1 d_2} E_j$, we have

$$D = K_Y + 2\hat{C}_2 = \pi^*(2d_2 - 3)H - \sum_{j=1}^{d_1 d_2} E_j. \tag{41}$$

One can show using the Nakai-Moishezon criterion that $D > 0$. Clearly, $D^2 > 0$. We need to show that a curve $C \subset \mathbb{C}P^2$ with $\deg C = d$ does not intersect $d(2d_2 - 3)$ of the points $p_1, \ldots, p_{d_1 d_2}$ when counted with the multiplicity of C at each point. But by Bézout's theorem

$$\sum_j i(C, C_1, q_j) = d d_1, \tag{42}$$

where the sum is over the points of intersection of C with C_1. We have

$$i(C, C_1, q_j) \geq \mu_{q_j}(C)\mu_{q_j}(C_1) = \mu_{q_j}(C), \tag{43}$$

where $\mu_{q_j}(C)$, $\mu_{q_j}(C_1)$ denote the multiplicity of C, respectively C_1, at q_j. And

$$\sum_j \mu_{q_j}(C) \leq d d_1 < d(2d_2 - 3) \tag{44}$$

shows that C cannot intersect $d(2d_2 - 3)$ of the points $p_1, \ldots, p_{d_1 d_2}$ when counted with the multiplicity.

For any relatively compact strictly pseudoconvex domain $M \subset X$ we have $\mathbf{K}_M > 0$.

Let $\mathcal{E} \subset \mathcal{A}^{1,1}(X)$ be the space of smooth exact $(1,1)$-forms. The $\partial\bar{\partial}$-lemma hold on a manifold X precisely when the map

$$\Psi : C^{\infty}(X) \xrightarrow{\sqrt{-1}\partial\bar{\partial}} \mathcal{E} \tag{45}$$

is surjective. As observed in [27] $B(X)$ provides a nontrivial cokernel of (45). In fact, let $\mathbf{L} \in B(X)$, and let h be any Hermitian metric on $\mathbf{L}$. Then $\beta_{\mathbf{L}} = \frac{\sqrt{-1}}{2\pi}\Theta_h$ is a real $(1,1)$-form which is exact because $\mathbf{L}$ is topologically trivial. But suppose $\beta_{\mathbf{L}} = \sqrt{-1}\partial\bar{\partial}f$. Then the metric $h' = e^{2\pi f}h$ has curvature $\frac{\sqrt{-1}}{2\pi}\Theta_{h'} = \beta_{\mathbf{L}} + \sqrt{-1}\bar{\partial}\partial f = 0$. Thus the Chern connection of h' is flat, and since X is simply connected, there is a parallel section $\sigma \in \Gamma(\mathbf{L})$. Since $\bar{\partial}\sigma = \nabla^{0,1}\sigma = 0$, σ is holomorphic and $\mathbf{L}$ is trivial, a contradiction. This defines an injective map

$$B(X) \otimes \mathbb{Q} \hookrightarrow \mathcal{E}/C^{\infty}(X). \tag{46}$$

If $M \subset X$ is a sufficiently large relatively compact strictly pseudoconvex domain, then $\pi_1(M) = 1$. And from Proposition 3.8 and the above we have the

Proposition 3.9. *There exist infinitely many, topologically distinct, 1-convex surfaces which contain strictly pseudoconvex domains which do not satisfy the $\partial\bar{\partial}$-lemma but nevertheless satisfy Theorem 3.1.*

The above arguments lead to the following more general result which is perhaps worth mentioning.

Proposition 3.10. *Let X be a complex manifold with $H_1(X,\mathbb{R}) = 0$. Then X satisfies the $\partial\bar{\partial}$-lemma if and only if $H^1(X,\mathcal{O}) = 0$.*

Proof. Denote by $\operatorname{Pic}^o X$ the subgroup of $\operatorname{Pic} X$ of topologically trivial line bundles. Then we have

$$\operatorname{Pic}^o X = H^1(X,\mathcal{O})/\operatorname{Im}\big[H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O})\big]. \tag{47}$$

Choose $p \in \mathbb{Z}_+$ so that $p \cdot H_1(X,\mathbb{Z}) = 0$. Let $\mathbf{L} \in \operatorname{Pic}^o X$ be an element with $\mathbf{L}^p$ non-trivial. For any Hermitian metric h, $\omega_{\mathbf{L}} = \frac{\sqrt{-1}}{2\pi}\Theta_h \in \mathcal{E}$. If $\omega_{\mathbf{L}} = \sqrt{-1}\partial\bar{\partial}f$, then there exists a metric h' with a flat connection on $\mathbf{L}$. But a flat connection corresponds to an $\alpha_{\mathbf{L}} \in \operatorname{Hom}(\pi_1(X), S^1) = \operatorname{Hom}(H_1(X,\mathbb{Z}), S^1)$. We have $\alpha_{\mathbf{L}}^p = \alpha_{\mathbf{L}^p} = 1$, which implies that $\mathbf{L}^n$ is trivial. □

The following shows that a counterexample to Conjecture 3.7 would have to have singular exceptional set.

Proposition 3.11. *Let (M, M') be a strictly pseudoconvex finite manifold with Kähler form $\omega \in -c_1(M)$. If the exceptional set $E \subset M$ is smooth, possibly not connected, then $\mathbf{K}_{M'}$ admits an Hermitian metric which is positive in a neighborhood $\tilde{M}$ of $\bar{M}$.*

Proof. Let h an Hermitian metric on $\mathbf{K}_{M'}$. Since each connected component E_i of E is obviously Kähler, there exists an $f \in C^{\infty}(M')$ so that the metric $h' = e^f h$ satisfies

$$\begin{aligned} \omega|_{E_i} &= -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h'|_{E_i} \\ &= \big(-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f\big)|_{E_i}. \end{aligned} \tag{48}$$

We may assume that M' is 1-convex, and let $\pi : M' \to Y$ be the Remmert reduction. If ψ is the pull-back by π of a strictly plurisubharmonic function on Y, then we have

$$\sqrt{-1}\partial\bar{\partial}\psi(x) \geq 0, \text{for } x \in M' \tag{49}$$

$$\sqrt{-1}\partial\bar{\partial}\psi(x) > 0, \text{for } x \in M' \setminus E \tag{50}$$

$$\sqrt{-1}\partial\bar{\partial}\psi(x) > 0, \text{on } \mathbf{N}_{M'/E} \text{ for } x \in E. \tag{51}$$

(52)

Then it is easy to see from (48), (50), (51), and (52) that for $A > 0$ sufficiently large the metric $e^{-A\psi}h'$ has curvature

$$-\sqrt{-1}\partial\bar{\partial}\log h' + A\sqrt{-1}\partial\bar{\partial}\psi > 0, \quad \text{on } \tilde{M}, \tag{53}$$

where $\tilde{M}$ is a relatively compact neighborhood of $\bar{M}$. □

4. Normal CR infinity

4.1. **Consequences of a normal CR infinity.** Assuming that the CR boundary $S = \partial M$ of a strictly pseudoconvex manifold is normal has strong consequences on M. First we mention an embedding result for 1-convex manifolds in [18] which was generalized to complex spaces in [41].

Theorem 4.1. *Let X be a 1-convex complex space. Then X is embeddable in $\mathbb{C}P^M \times \mathbb{C}^N$ if and only if there is a positive holomorphic line bundle on X.*

Given a positive line bundle $\mathbf{L}$ on X it shown that there is an $N_0 \in \mathbb{N}$ so that for $k \geq N_0$ there finitely many sections $s_0, \dots, s_p \in H^0(X, \mathcal{O}(\mathbf{L}^k))$ so that $\{z \in X : s_0(z) = \cdots = s_p(z) = 0\}$ is empty and the map $\Psi : X \to \mathbb{C}P^p$ restricts to an embedding on a neighborhood U of the exceptional set E of the Remmert reduction $\pi : X \to Y$. This is combined with the embedding $\Upsilon : Y \to \mathbb{C}^n$ of the Stein space Y (cf. [33]) gives an embedding

$$\Psi \times \Upsilon : X \to \mathbb{C}P^p \times \mathbb{C}^n. \tag{54}$$

This gives us another necessary and sufficient condition for a strictly pseudoconvex domain M to admit a Kähler-Einstein metric.

Corollary 4.2 (to Theorem 3.1). *A strictly pseudoconvex domain M admits a Kähler-Einstein metric if and only if for some $k \geq 1$ there are finitely many sections $s_0, \dots, s_M \in H^0(M, \mathcal{O}(\mathbf{K}_M^k))$ inducing an embedding of a neighborhood U of the exceptional set of M in $\mathbb{C}P^M$.*

Proposition 4.3. *Suppose M is a strictly pseudoconvex manifold such that the induced CR structure on $S = \partial M$ is normal. Then the Remmert reduction of M is $\hat{M} = C(S)_{r<1}$, where $C(S)_{r<1} = \{(x, r) \in C(S) : r < 1\}$ is the domain in the Sasaki cone of S, with its induced Sasaki structure.*

Remark 4.4. Note that $C(S) \cup \{o\}$, with the vertex, has a unique structure of a normal Stein variety [43], and also an affine variety [44]. We will consider $C(S)$ as such with the addition of the vertex.

In other words, $M \subset X$ is a domain in a resolution $\pi : X \to C(S)$ of the cone $C(S)$ with exceptional fiber $E = \pi^{-1}(o)$ over $o \in C(S)$.

Proof. We may suppose that $M \subset X$ with X a 1-convex manifold with Remmert reduction $\pi : X \to Y$. Thus π maps M to the strictly pseudoconvex domain $N \subset Y$. We first prove the following.

Lemma 4.5. *The action of $\mathfrak{CR}(S, D, J)$ extends to a holomorphic action on N.*

Proof of Lemma. Since Y has finitely many isolated singular points, it has finite embedding dimension and there there is an embedding $\iota : Y \to \mathbb{C}^N$ (cf. [33]). Let $\psi \in \mathfrak{CR}(S, D, J)$, and define $f_j^\psi = \psi^*(z_j \circ \iota),\ j = 1, \dots, N$. We have $\bar{\partial}_b f_j^\psi = 0$, i.e. f_j^ψ is annihilated by $D^{0,1} \subset D \otimes \mathbb{C} \subset TS \otimes \mathbb{C}$, so by the extension theorem of J. Kohn and H. Rossi [29] the f_j^ψ extend to holomorphic functions on $\bar{M}$. There are holomorphic functions $g_j^\phi : \bar{N} \to \mathbb{C}^N$ with $g_j^\phi \circ \pi = f_j^\psi$. Denote $F^\psi := (f_1^\psi, \dots, f_N^\psi)$ and $G^\psi = (g_1^\psi, \dots, g_N^\psi)$. So $F^\psi : \bar{M} \to \mathbb{C}^N$ and $G^\psi : \bar{N} \to \mathbb{C}^N$ with $G^\psi \circ \pi = F^\psi$.

We have $\operatorname{Im} F^\psi|_S = \operatorname{Im} \iota|_S$. If $h \in \mathcal{O}_{\mathbb{C}^N}(U)$ is any function defined with $U \cap \operatorname{Im} \iota|_S \neq \emptyset$ with U connected and vanishing on $\operatorname{Im} \iota \subset \mathbb{C}^N$, then $h \circ F^\psi$ vanishes on $S \cap (F^\psi)^{-1}(U)$ so vanishes identically. Therefore there is a neighborhood $V \subset X$ of S with $F^\psi(V) \subset \operatorname{Im} \iota$. Now cover $F^\psi(\bar{M})$ with finitely many neighborhoods $U_\alpha,\ \alpha = 1, \dots, m$, for which $\operatorname{Im} \iota \cap U_\alpha = \{h_1^\alpha = \cdots = h_{k_\alpha}^\alpha = 0\}$ for defining functions $h_i^\alpha \in \mathcal{O}(U_\alpha)$. Let $V_\alpha = (F^\psi)^{-1}(U_\alpha)$. If $V_\alpha \cap V \neq \emptyset$, then $h_i^\alpha \circ F^\psi$ vanish on V_α. Thus $F^\psi(V \cup V_\alpha) \subset \operatorname{Im} \iota$. Continuing this argument shows that $F^\psi(\bar{M}) \subset \operatorname{Im} \iota$.

Since ι maps Y biholomorphically onto its image, we can define $\mu^\psi : \bar{N} \to Y$ by $\iota^{-1} \circ G^\psi$. Let ϕ be a plurisubharmonic defining function of $N \subset Y$, i.e. $N = \{\phi < 0\}$. Then $(\pi \circ \mu^\psi)^*\phi$ is plurisubharmonic and takes the value 0 on $S = \partial N$, so $(\pi \circ \mu^\psi)^*\phi(x) < 0$ for $x \in M$ by the maximum principle. Therefore we have $\mu^\psi : \bar{N} \to \bar{N}$.

Suppose $\psi_1, \psi_2 \in \mathfrak{CR}(S, D, J)$. Since $\iota \circ \mu^{\psi_1} \circ \mu^{\psi_2} \circ \pi - \iota \circ \mu^{\psi_1 \circ \psi_2} \circ \pi$ vanishes on S it must vanish identically. Therefore $\mu^{\psi_1 \circ \psi_2} = \mu^{\psi_1} \circ \mu^{\psi_2}$, so $\mathfrak{CR}(S, D, J)$ acts on $\bar{N}$ by biholomorphisms. $\square$

Suppose $\xi \in\in \mathfrak{t}_k^+ \subseteq \mathfrak{cr}^+(S, D, J)$. Then we can replace ξ with an integral element in $Z_k^+ \subset \mathfrak{t}_k^+$. This ξ generates an U(1)-action on $\bar{N}$. If ϕ is a defining function for $M \subset X$, $0 < \eta(\xi) = -\frac{1}{2} d\phi(J\xi)$, so $J\xi$ points inward at S. If $\epsilon < 0$, then

$$(\epsilon, 0] \times S \ni (t, x) \longrightarrow \exp(-tJ\xi)x \tag{55}$$

is a diffeomorphism onto a neighborhood of S in $\bar{N}$. We consider a complex structure on $\mathbb{R} \times S$ which is that of (7) in the coordinate $t = \log r$ of $\mathbb{R}$; that is,

$$I(X) := \Phi(X) - \eta(X)\partial_t, \quad I(\partial_t) = \xi, \tag{56}$$

for $X \in TS$. It is not difficult to see that (55) is a biholomorphism between $((\epsilon, 0] \times S, I)$ and (V, J) where V is a neighborhood of S in $\bar{N}$.

Since $C(S)_{r<1}$ and N are both normal Stein spaces Hartogs' theorem implies that an holomorphic function on $((\epsilon, 0) \times S, I)$ extends to $C(S)$ and likewise for holomorphic functions on $V \setminus S \subset N$. Therefore $\mathcal{O}(C(S)_{r<1}) \cong \mathcal{O}(N)$, and we have a biholomorphism $C(S)_{r<1} \cong N$. $\square$

Corollary 4.6. *If M is a strictly pseudoconvex Stein domain with the CR structure on $S = \partial M$ normal, then $S = \mathbb{S}^{2n-1}$, the $2n - 1$ sphere with a transversal deformation of the standard CR structure.*

See Section 4.2 for an explanation of "transversal deformation."

Proof. Since M is a domain in the resolution $\pi : X \to C(S)$, we must have $\pi^{-1}(o) = E =$. Thus $M \subset C(S)$ and $C(S)$ is nonsingular. Suppose $\xi \in Z_k^+ \subset \mathfrak{t}_k^+$. We have an action $\iota : \mathrm{U}(1) \to \mathrm{Aut}\big(T_o C(S)\big)$, with weights $(w_1, \dots, w_n) \in \mathbb{Z}^n_{>0}$. One can get an equivariant coordinate system $(U, z_1, \dots, z_n)$, i.e. $\iota(u)(z_1, \dots, z_n) = (u^{w_1} z_1, \dots, u^{w_n} z_n)$, by a simple averaging argument. Since $\exp(tJ\xi)$ maps M into U for large enough $t > 0$, we have $U \cong C(S) \cong \mathbb{C}^n$. We have $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ with the standard CR structure (D, J) and ξ is a CR Reeb vector field. For $z \in S$ there is a unique $t \in \mathbb{R}$ with $(e^{w_1 t} z_1, \dots, e^{w_n t} z_n) \in \mathbb{S}^{2n-1}$, and if we define $\psi(z) = t$ Then if we set $r' = e^{-\psi} r$, $\mathbb{S}^{2n-1} = \{r' = 1\}$. Note that $r' \partial_{r'} = r \partial_r$, so the Euler vector field and also the Reeb vector field is unchanged. And the contact forms are related by

$$\eta' = 2d^c \log r' = 2d^c \log r - 2d^c \psi = \eta - 2d^c \psi. \tag{57}$$

□

Proposition 4.7. *Let $M \subset X$ be a strictly pseudoconvex domain in one of the 1-convex surfaces of Section 3.5 so that $\pi_1(M) = 1$. Then the CR structure on $S = \partial M$ is not normal.*

Proof. Suppose otherwise, then the Remmert reduction is $\pi : M \to \hat{M} = \{r < 1\} \subset C(S)$. Let $\mathbf{L} \in B(M) = \mathbb{Z}^{2g}$ be a non-trivial element. By definition $\mathbf{L}$ is holomorphically trivial in a neighborhood of the exceptional curve E. So there is an $\hat{\mathbf{L}}$ on $\hat{M}$ with $\pi^* \hat{\mathbf{L}} = \mathbf{L}$. But obviously, $H^2(\hat{M}, \mathbb{Z}) = 0$, so $\operatorname{Pic} \hat{M} = 0$ implying that $\mathbf{L}$ is holomorphically trivial.

This proposition also simply follows from [17, Prop. 1] which give as a condition for $B(M) = 0$ that $\pi^* H^2(\hat{M}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is injective. □

This proposition has an obvious generalization.

Corollary 4.8. *Let $M \subset X$ be a strictly pseudoconvex domain. If $B(M) \neq 0$, then the CR structure on ∂M is not normal.*

Definition 4.9. *A strictly pseudoconvex finite manifold M is called a* normal *Kähler-Einstein manifold if it has a complete Kähler-Einstein metric and its conformal CR infinity is normal.*

It is well known [38], see also [44], that the Kähler cone $C(S) \cup \{o\}$ over a Sasaki manifold S is an affine variety. The homomorphism $\iota : \mathrm{U}(1) \to \mathfrak{CR}(S, D, J)$ with $\iota_* \frac{\partial}{\partial \theta} = \xi \in \mathfrak{cr}^+(D, J)$ extends to $\iota : \mathbb{C}^* \to \mathfrak{Hol}(C(S) \cup \{o\})$, an algebraic action. The following result is evident when this is combined with Theorem 4.1.

Proposition 4.10. *A normal Kähler-Einstein manifold is a domain in a quasi-projective variety.*

Proof. We have that $M \subset X$ where X is a resolution of $C(S) \cup \{o\}$ with $S = \partial M$. We compactify X to $\hat{X}$ as follows. Choose $\xi \in Z_k^+ \subset \mathfrak{t}_k^+$. Then $C(S)$ is biholomorphic to $\mathbf{L}^\times$ where $\mathbf{L} \to W$ is an holomorphic orbibundle over W. Now define an orbifold $\hat{X}$ by replacing $C(S) \subset X$ with the total space of $\pi : \mathbf{L}^{-1} \to W$, so $\hat{X}$ is X with the divisor W with positive normal bundle $\mathbf{L}^{-1} = [W]|_W$ added. (The divisor $W \subset \hat{X}$ is only $\mathbb{Q}$-Cartier, but the following arguments work in the orbifold setting. See [8].) Let $\sigma \in H^0(\hat{X}, \mathcal{O}(W))$ be a section vanishing on W. Let h be a metric on $[W]$ with $\frac{\sqrt{-1}}{2\pi} \Theta_h|_W > 0$. In a neighborhood of W identified with a neighborhood of the zero section of $\pi : \mathbf{L}^{-1} \to W$ define $\tilde{h} = e^{-\pi^* h|z|^2} h$, where z is the fiber coordinate, and extend $\tilde{h}$ to all of $[W]$. Then $-\log \tilde{h}|\sigma|^2$ is strictly plurisubharmonic near W. Since $-\log \tilde{h}|\sigma|^2 \to \infty$ at W, we can modify it away from W to a plurisubharmonic function f on X. If we set $q = -f - \log \tilde{h}|\sigma|^2$, then $\hat{h} = e^q \tilde{h}$ is a metric on $[W]$ with $\frac{\sqrt{-1}}{2\pi}\Theta_{\hat{h}} \geq 0$ and $\frac{\sqrt{-1}}{2\pi}\Theta_{\hat{h}} > 0$ in a neighborhood N of W. Then for sufficiently large $k > 0$, $\mathbf{F} = \mathbf{K}_{\hat{X}} \otimes [kW]$ admits a metric with positive curvature. By the Baily embedding theorem [1] for sufficiently large p, the sections of $\mathbf{F}^p$ define an embedding $\psi_{\mathbf{F}^p} : \hat{X} \to \mathbb{C}P^N$. □

Theorem 4.11. *Let M be a normal Kähler-Einstein manifold with CR infinity (S, D, J). Then the action of the connected component $\mathfrak{CR}_0(S, D, J)$ extends to a holomorphic action on M. Thus there is an injection*

$$\iota : \mathfrak{CR}_0(S, D, J) \hookrightarrow \mathfrak{Hol}(M) = \mathfrak{Isom}(M, g, J) \tag{58}$$

Proof. Let $\mathbf{F}^p = \mathbf{K}^p_{\hat{X}} \otimes [kpW]$ be the very ample bundle as above. Choose any $\xi \in Z^+_k \subset \mathfrak{t}^+_k$, then as above the action of $\mathbb{C}^*$ on $C(S)$ is algebraic. Let $\sigma \in H^0(\hat{X}, \mathcal{O}(\mathbf{F}^p))$, so $\sigma|_{C(S)} \in H^0(C(S), \mathcal{O}(\mathbf{K}^p_{C(S)}))$. If $\alpha : C(S) \times \mathbb{C}^* \to C(S)$ is the projection, then we have a section $\tilde{\sigma}$ of $\alpha^* \mathbf{K}^p_{C(S)}$ $\tilde{\sigma} = \iota(z)^* \sigma|_{C(S)}$, $z \in \mathbb{C}^*$. And $\tilde{\sigma}$ is a rational section of $\alpha^* \mathbf{K}^p_{\hat{X}}$, where $\alpha : \hat{X} \times \mathbb{C}^* \to \hat{X}$ is again the projection. We know that $\tilde{\sigma}$ is regular except perhaps on some $E_i \times \mathbb{C}^*$ where $E_i \subset E$ is an exceptional divisor of X or on $W \times \mathbb{C}^*$. But since $\tilde{\sigma}|_{X \times \{1\}}$ is smooth, it is easy to see that $\tilde{\sigma}$ is regular along $E \times \mathbb{C}^*$. It is easy to see that $\tilde{\sigma}$ has at most a pole of order kp along $W \times \mathbb{C}^*$. So $\iota(z)^* \sigma \in H^0(\hat{X}, \mathcal{O}(\mathbf{F}^p))$ for $z \in \mathbb{C}^*$.

As above we have the embedding

$$\psi_{\mathbf{F}^p} : \hat{X} \to \mathbb{C}P^N = \mathbb{P}(H^0(\hat{X}, \mathcal{O}(\mathbf{F}^p))^*). \tag{59}$$

If we denote the action of $\mathbb{C}^*$ on $\mathbb{C}P^N$ by $\iota(z)$, then for $x \in \hat{X} \setminus E$ $\iota(z)(\psi_{\mathbf{F}^p}(x) = \psi_{\mathbf{F}^p}(\iota(z)(x))$. So $\mathbb{C}^*$ preserves $\psi_{\mathbf{F}^p}(\hat{X} \setminus E) \subset \mathbb{C}P^N$. Since E is nowhere dense, it is easy to see that $\psi_{\mathbf{F}^p}(\hat{X}) \subset \mathbb{C}P^N$ is preserved by the action of $\mathbb{C}^*$ on $\mathbb{C}P^N$.

For each $\xi \in Z^+_k$ the U(1) action extends to X. These actions generate $T_k \subset \mathfrak{CR}(S, D, J)$. Since all maximal tori are conjugate, Z^+_k generates $\mathfrak{t}_k$ for all maximal tori. So the action of every maximal torus $T_k \subset \mathfrak{CR}(S, D, J)$ extends to M. When $\mathfrak{CR}(S, D, J)$ is compact every element of $\mathfrak{CR}_0(S, D, J)$ is contained in a maximal torus so the result follows. Otherwise, (S, D, J) is the round sphere $\mathfrak{CR}(\mathbb{S}^{2n-1}, D, J) = \mathrm{PSU}(1, n)$ and $M = \mathcal{H}^n_{\mathbb{C}}$ the complex hyperbolic space, and the theorem follows. □

4.2. **Transversal deformations.** Every normal Kähler-Einstein manifold M, with fixed CR Reeb vector field, has a natural infinite dimensional space of deformations parameterized by basic functions on $S = \partial M$ with sufficiently small 2nd derivatives. By basic we mean invariant under the Reeb action of ξ. These correspond to *transversal deformations* of the CR structure on S.

As above $S = \{r = 1\} \subset C(S)$, so we may take $r^2 - 1$ as the defining function of M. The CR distribution $D = \ker \eta$, where $\eta = 2d^c \log r$. Here η is the unique 1-form with $\ker \eta = D$ and $\xi \lrcorner \eta = 1$.

If $\psi \in C^\infty_B(S)$ is a basic function, which we may take as a function on $C(S)$, then set $r' = e^\psi r$. Then $r' = 1$ defines the boundary S' of a domain in $C(S)$ with defining function $r'^2 - 1$. One can check that S' is naturally diffeomorphic to S, one has $\xi \lrcorner \eta' = 1$, and this alters the CR structure on S by

$$\eta' = \eta + 2d^c\psi \tag{60}$$

$$\Phi' = \Phi - \xi \otimes \eta' \circ \Phi. \tag{61}$$

If $\eta' \wedge (d\eta')^{n-1}$ is nowhere zero, this defines another normal strictly pseudoconvex CR structure on S. This is equivalent to the new Levi form $\mathbf{L}^{D'} = d\eta'$ being positive.

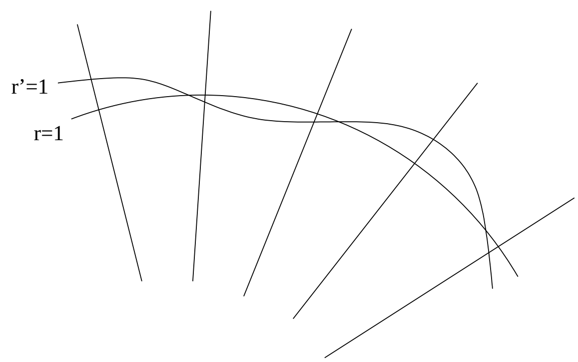

FIGURE 1. Transversal deformation

Equations (60) and (61) define the deformed CR structure on S. Alternatively, one may fix the complex structure on $C(S)$ deform the domain via $r' = e^{\psi}r$. Figure 1 shows the respective domains $C(S)_{r<1}$ and $C(S)_{r'<1}$.

Another type of deformation will be of interest in the final section which we call *transversal deformations of the second kind*. Let $\alpha \in H^1_B(S)$, then we have the CR structure on S

$$\eta' = \eta + \alpha, \tag{62}$$

with Φ given again by (61). If $\alpha = df$, with $f \in C^\infty_B(S)$, then the new CR structure is just a gauge transformation of (D, J) in the Reeb direction.

4.3. **possible CR infinities.** We consider the problem of which strictly pseudoconvex CR manifolds S are CR infinities of some Kähler-Einstein manifold.

Problem 4.12. *Which strictly pseudoconvex CR manifolds S are conformal CR infinities of complete Kähler-Einstein manifolds.*

By Theorem 3.1 this is equivalent to whether there exists a complex manifold M with $\partial M = S$ and $\mathbf{K}_M$ positive. One necessary condition is that S is embeddable, meaning there is a smooth embedding $S \hookrightarrow \mathbb{C}^N$ with the CR structure on S induced from the complex structure of $\mathbb{C}^N$. Conversely by a theorem of Harvey and Lawson [25] $S = \partial M$ for some M if S is embeddable. It is known [7] that all strictly pseudoconvex (S, D, J) are embeddable in dimension $2n - 1 \geq 5$. But generic perturbations of the standard CR structure on $\mathbb{S}^3$ are nonembeddable.

We consider the more restricted Problem 1.2, considering only normal CR structures. We can prove a negative result in case S has a normal CR structure (D, J). We will need a definition.

Definition 4.13. *We say that a normal CR manifold* (S, D, J) *has* property S-E *if* $c_1(D, J) = 0$ *and for* $\xi \in \mathfrak{cr}^+(D, J)$ *the transversal first Chern class* $c_1^B > 0$*, i.e. represented by a positive* $(1, 1)$*-form.*

If this property holds for some $\xi \in \mathfrak{cr}^+(D, J)$, then it holds for all of $\mathfrak{cr}^+(D, J)$, so it is intrinsic to (D, J). If $\xi' \in \mathfrak{cr}^+(D, J)$, then by conjugation with an element of $\mathfrak{CR}(D, J)$ we may suppose $\xi, \xi' \in \mathfrak{t}^+$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of $\mathfrak{CR}(D, J)$. The respective contact forms satisfy $\eta' = f\eta$ where $f = (\eta(\xi'))^{-1}$, so

$$(d\eta')^{n-1}|_D = f^{n-1}(d\eta)^{n-1}|_D. \tag{63}$$

Recall that $\frac{1}{2\pi}\operatorname{Ricci}^T(\frac{1}{2}d\eta) \in c_1^B$, where c_1^B is with respect to the foliation generated by ξ. Thus from (63) with respect to the foliation of ξ'

$$c_1^B \ni \frac{1}{2\pi}\operatorname{Ricci}^T(\frac{1}{2}d\eta') = \frac{1}{2\pi}\operatorname{Ricci}^T(\frac{1}{2}d\eta) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log f^{n-1}. \tag{64}$$

Here the Ricci forms are computed on (D, J).

This is precisely the topological condition for the associated Sasaki structure (S, g, η, ξ, Φ) to admit a possible transversal deformation to a Sasaki-Einstein structure. Sasaki manifolds satisfying this condition have been studied extensively (cf. [22, 22, 9, 10, 21],) mainly in order to construct new Einstein manifolds.

In terms of the Kähler cone $C(S)$ property S-E is equivalent to the existence of an holomorphic $(n, 0)$-form Ω with $\mathcal{L}_\xi\Omega = a\sqrt{-1}\Omega$, with $a > 0$, and satisfying

$$\left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}}\Omega\wedge\bar{\Omega} = e^h\frac{1}{n!}\omega^n, \tag{65}$$

where ω is the natural Kähler metric on $C(S)$ for the Sasaki manifold $(S, \frac{n}{a}\xi, D, J)$ and h is invariant under the action generated by ξ and $r\partial_r$. See [21]. If S is not simply connected then one may have to take Ω to be multivalued, i.e. it defines a section of $\mathbf{K}_{C(S)}^{\otimes p}$ for some $p \in \mathbb{N}$. We have $e^h\omega^n$ defining an Hermitian metric on $\mathbf{K}_{C(S)}$ via (65)

$$\|\Psi\|^2 = \left(\frac{i}{2}\right)^n (-1)^{\frac{n(n-1)}{2}} e^{-h}\frac{\Psi\wedge\bar{\Psi}}{\omega^n}, \tag{66}$$

for $(n, 0)$-form Ψ. It is easy to see that the associated connection ∇^h on $\mathbf{K}_{C(S)}$ is flat. And we have the surjective homomorphism to the holonomy group $\pi_1(S) \to \operatorname{Hol}(\nabla^h) \subset \mathrm{U}(1)$, whose image is a finite group $\mathbb{Z}_p$. If $p = 1$, then the singularity $o \in C(S) \cup \{o\}$ is *Gorenstein.* And if $p > 1$, $o \in C(S) \cup \{o\}$ is $\mathbb{Q}$-*Gorenstein.*

Theorem 4.14. *Let* (S, D, J) *be a normal strictly pseudoconvex CR manifold with property S-E such that the singularity* $C(S) \cup \{o\}$ *is Gorenstein, e.g.* $\pi_1(S) = e$*. Then* (S, D, J) *is the CR infinity of a complete Kähler-Einstein manifold* M *only if* $S = \mathbb{S}^{2n-1}$*, with* (D, J) *a transversal deformation of the standard CR structure and* $M \subset \mathbb{C}^n$*.*

Remark 4.15. One can show if (S, D, J) has property S-E, then $\pi_1(S)$ must be finite. This follows from the existence of a Sasaki structure with positive Ricci curvature and an application of Meyer's theorem. In most applications one will have (S, D, J) simply connected.

Proof. By Proposition 4.3 M is a resolution $\pi : M \to C(S)_{r<1}$. We have a holomorphic n-form Ω on $C(S)$ satisfying (65). It is a result of [30] and [13] that $o \in C(S)$ is a rational singularity if and only if it has a small neighborhood U with

$$\int_U \Omega \wedge \bar{\Omega} < \infty. \tag{67}$$

And this easily follows from (65). Moreover, it is a consequence of (67) that for any resolution $\pi : M \to C(S)_{r<1}$ the form Ω extends to a holomorphic form on M (cf. [30]). We recall a definition.

Definition 4.16. *Let X be a normal $\mathbb{Q}$-Gorenstein variety. Then X has* canonical singularities *if for every resolution $\pi : \hat{X} \to X$ one has*

$$K_{\hat{X}} = \pi^* K_X + \sum_i a_i E_i, \tag{68}$$

with $a_i \geq 0$ for each exceptional divisor E_i. The equality in (68) means linear equivalence.

It is sufficient to check (68) for one resolution, and we have each $a_i \geq 0$. By Corollary 4.2 for some $q \geq 1$ $\mathbf{K}_M^q$ is ample in a neighborhood of the exceptional set E, and we have $qK_M = q\sum_i a_i E_i$. If $\sigma \in H^0(M, \mathcal{O}(\mathbf{K}_M^q))$, then $f = \frac{\sigma}{\Omega^q}$ is a meromorphic function which is holomorphic on $C(S)_{r<1} \setminus \{o\}$. By the Riemann extension theorem f extends holomorphically to $\tilde{f}$ on $C(S)_{r<1}$. So $f = \pi^* \tilde{f}$, and $\sigma|_E$ is a constant multiple of $\Omega^q|_E$. Therefore, we must have $E = \emptyset$, $M = C(S)_{r<1}$. Since M is smooth we must have $M \subset \mathbb{C}^n$ and the rest follows as in the proof of Corollary 4.6. □

Remark 4.17. Theorem 4.14 is a global result. If (S, D, J) satisfies the assumptions of the theorem and in addition is Sasaki-Einstein, then the Ansatz in Section 5.3.2, due to E. Calabi, constructs an incomplete Kähler-Einstein metric with CR infinity (S, D, J).

Note also that the examples in 5.3.1 and 5.3.2 show that Gorenstein assumption in the theorem in necessary.

5. Examples

We consider some cases in which Theorem 3.1 is easily applicable. The following easy result will be helpful in some of the cases that follow.

Proposition 5.1. *Suppose X is a projective manifold and $W \subset X$ is a smooth divisor (X may have orbifold singularities along W) with $\mathbf{K}_X \otimes [kW] > 0$, for some $k \geq 1$, and $[W]|_W > 0$. Then $X \setminus W$ is 1-convex and $\mathbf{K}_{X \setminus W} > 0$.*

5.1. **Negative bundles.**

Let $\pi : \mathbf{E} \to N$ be an holomorphic bundle with a Hermitian metric h. Recall, that $\mathbf{E}$ has a unique Chern connection ∇ which is compatible with h and $\nabla^{0,1} = \bar{\partial}$. In a holomorphic local frame $(e_1, \dots, e_r)$ the connection form is $\theta = \partial h h^{-1}$. We have the curvature $\Theta \in \Omega^{1,1}(\mathrm{Hom}(\mathbf{E}, \mathbf{E})$ given by

$$\Theta_i^j = d\theta_i^j + \theta_k^j \wedge \theta_i^k. \tag{69}$$

Definition 5.2. *A connection on* $\mathbf{E}$ *has* positive (resp. negative) curvature *if for each* $x \in N$ *and for all nonzero* $v \in \mathbf{E}_x$ $\sqrt{-1}h(\Theta_x v, v)$ *is a positive (resp. negative)* $(1,1)$*-form.*

A holomorphic bundle $\mathbf{E}$ *is* positive (resp. negative) *if it admits a metric whose Chern connection has positive (resp. negative) curvature.*

This condition was called weakly positive by P. Griffiths [24], as it is not strong enough to ensure the properties of a positive line bundle such as Kodaira vanishing. Although, a weaker condition than in 5.2 was called weakly positive by H. Grauert [23].

Define a smooth function $r^2 := h(v,v)$ on the total space of $\mathbf{E}$.

Proposition 5.3. *Suppose the metric* h *has negative curvature. Then the disk bundles* $\{r^2 < c\}$*, for* $c > 0$*, are strictly pseudoconvex. In fact,* $dd^c r^2$ *is a positive* $(1,1)$*-form outside the zero section.*

Proof. Choose a local holomorphic frame $(e_1, \dots, e_r)$, with fiber coordinates $(w_1, \dots, w_r)$, so that $\theta = \partial h h^{-1} = 0$ at $x \in N$. Then at $x \in N$ we have

$$\Theta_x = d\theta = \bar{\partial}(\partial h h^{-1}) = (\bar{\partial}\partial h)h^{-1}. \tag{70}$$

While

$$\partial\bar{\partial} r^2 = \partial\bar{\partial} h_{i\bar{j}} w_i \bar{w}_{\bar{j}} + dw_i \wedge \bar{\partial} h_{i\bar{j}} \bar{w}_{\bar{j}} + \partial h_{i\bar{j}} w_i \wedge d\bar{w}_{\bar{j}} + h_{i\bar{j}} dw_i \wedge d\bar{w}_{\bar{j}}. \tag{71}$$

The two middle terms on the right of (71) vanish. Thus (70) and (71) show that at $(w_r, \dots, w_r) \in \mathbf{E}_x$ we have $\partial\bar{\partial} r^2 > 0$. □

If $\mathbf{E}$ is an arbitrary holomorphic vector bundle and $\mathbf{L}$ is a negative line bundle on N, then the curvature of $\mathbf{E} \otimes \mathbf{L}^\mu$ is

$$\Theta_{\mathbf{E}\otimes\mathbf{L}^\mu} = \Theta_{\mathbf{E}} + \mu\Theta_{\mathbf{L}}. \tag{72}$$

Thus for sufficiently large $\mu \gg 0$ $\mathbf{E} \otimes \mathbf{L}^\mu$ is negative.

Associated to a vector bundle $\pi : \mathbf{E} \to N$ is the bundle of projective spaces $\tilde{\pi} : \mathbb{P}(\mathbf{E}) \to N$ with fibers $\tilde{\pi}^{-1}(x) = \mathbb{P}(\mathbf{E}_x)$. Let $\rho : \mathbf{L} \to \mathbb{P}(\mathbf{E})$ be the universal bundle of lines in $\mathbf{E}$. The Hermitian metric h on $\mathbf{E}$ defines a natural metric on $\mathbf{L}$. We will compute the curvature $\Theta_{\mathbf{L}}$ of this metric on $\mathbf{L}$ in terms of the curvature $\Theta_{\mathbf{E}}$ of $\mathbf{E}$. This was proved in [24]. We prove it here as it is important to what follows.

Let $(e_1, \dots, e_r)$ be a local holomorphic frame of $\mathbf{E}$, and let $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r \setminus \{0\}$ be fiber coordinates. Then denote $\langle \xi, \xi \rangle := \sum_{i,j} \xi_i h_{i\bar{j}} \bar{\xi}_{\bar{j}}$. We have

$$\begin{aligned} \Theta_{\mathbf{L}} = \bar{\partial}\partial \log\langle \xi, \xi \rangle = & \frac{\partial\xi_i h_{i\bar{j}} \bar{\partial}\bar{\xi}_{\bar{j}} - \partial\xi_i \bar{\partial} h_{i\bar{j}} \bar{\xi}_{\bar{j}} - \xi_i \partial h_{i\bar{j}} \bar{\partial}\bar{\xi}_{\bar{j}} + \xi_i \bar{\partial}\partial h_{i\bar{j}} \bar{\xi}_{\bar{j}}}{\langle \xi, \xi \rangle} \\ & + \frac{(\partial\xi_i h_{i\bar{j}} \bar{\xi}_{\bar{j}} + \xi_i \partial h_{i\bar{j}} \bar{\xi}_{\bar{j}}) \wedge (\xi_i h_{i\bar{j}} \bar{\partial}\bar{\xi}_{\bar{j}} + \xi_i \bar{\partial} h_{i\bar{j}} \bar{\xi}_{\bar{j}})}{\langle \xi, \xi \rangle^2}. \end{aligned} \tag{73}$$

Rearranging terms in (73) we get

$$\Theta_{\mathbf{L}} = \frac{\langle \Theta_{\mathbf{E}} \xi, \xi \rangle}{\langle \xi, \xi \rangle} - P(x, \xi) + Q(x, \xi), \tag{74}$$

where

$$P(x, \xi) = \frac{\langle \partial\xi, \partial\xi \rangle}{\langle \xi, \xi \rangle} - \frac{\langle \partial\xi, \xi \rangle \wedge \langle \xi, \partial\xi \rangle}{\langle \xi, \xi \rangle^2} \tag{75}$$

is the Fubini-Study metric on the fibers and
(76)
$$Q(x,\xi) = \frac{-\langle\theta\xi,\theta\xi\rangle - 2\sqrt{-1}\operatorname{Im}\langle\partial\xi,\theta\xi\rangle}{\langle\xi,\xi\rangle} + \frac{2\sqrt{-1}\operatorname{Im}(\langle\partial\xi,\xi\rangle\langle\xi,\theta\xi\rangle) + \langle\theta\xi,\xi\rangle\wedge\overline{\langle\theta\xi,\xi\rangle}}{\langle\xi,\xi\rangle^2}.$$
One can choose a frame $(e_1,\dots,e_r)$ so that the connection form θ vanishes at $x_0\in N$. Then $Q(x_0,\xi)=0$, so we have the following.

Proposition 5.4. *If* $\mathbf{E}$ *is a negative vector bundle, then* $\rho:\mathbf{L}\to\mathbb{P}(\mathbf{E})$ *is negative.*

Let $\pi:\mathbf{E}\to N$ be a rank r bundle with associated projective bundle $\tilde{\pi}:\mathbb{P}(\mathbf{E})\to N$ and tautological line bundle $\rho:\mathbf{L}\to\mathbb{P}(\mathbf{E})$, then the canonical bundle of $\mathbb{P}(\mathbf{E})$ is given by
$$\mathbf{K}_{\mathbb{P}(\mathbf{E})} = \tilde{\pi}^*\mathbf{K}_N\otimes\tilde{\pi}^*\det(\mathbf{E})^{-1}\otimes\mathbf{L}^r. \tag{77}$$
Consider the compactification $X=\mathbb{P}(\mathbf{E}\oplus\mathbb{C})$ of $\mathbf{E}$. So we have
$$\mathbf{K}_X = \tilde{\pi}^*\mathbf{K}_N\otimes\tilde{\pi}^*\det(\mathbf{E})^{-1}\otimes\mathbf{L}^{r+1}. \tag{78}$$
Let $D_\infty\subset X$ be the divisor at infinity, that is $D_\infty=\{[v:0]\in\mathbb{P}(\mathbf{E}\oplus\mathbb{C})\}=\mathbb{P}(\mathbf{E})$. Clearly, $[D_\infty]$ and $\mathbf{L}^{-1}$ restrict to the hyperplane bundle on each fiber $\tilde{\pi}^{-1}(x)=\mathbb{P}(\mathbf{E}\oplus\mathbb{C})_x$. Thus $[D_\infty]\otimes\mathbf{L}=\tilde{\pi}^*(\mathbf{F})$ for a line bundle $\mathbf{F}$ on N. One can check that the normal bundle $\mathcal{N}_{D_\infty}\cong[D_\infty]|_{D_\infty}\cong\mathbf{L}^{-1}|_{D_\infty}$, thus $\tilde{\pi}^*(\mathbf{F})|_{D_\infty}$ is trivial. The projection $D_\infty=\mathbb{P}(\mathbf{E})\to N$ induces an injection on the Picard group, therefore $[D_\infty]=\mathbf{L}^{-1}$.

In particular, suppose $\pi:\mathbf{E}\to N$ is a negative bundle. Further, suppose that if M' denotes the total space of $\mathbf{E}$, $c_1(M')<0$. This can be seen to be equivalent to $c_1(N)+c_1(\mathbf{E})<0$. For if $-\varpi\in c_1(N)+c_1(\mathbf{E})$ with $-\varpi$ negative, then $-\pi^*\varpi-\sqrt{-1}\partial\bar{\partial}r^2$ is a negative form in $c_1(M')$ where $r^2=h(v,v)$. By (78), Proposition 5.4, and the above comments we have the following.

Proposition 5.5. *Let* $\pi:\mathbf{E}\to N$ *be a negative bundle of rank* r *with* $c_1(N)+c_1(\mathbf{E})<0$. *The canonical bundle of* $X=\mathbb{P}(\mathbf{E}\oplus\mathbb{C})$ *satisfies*
$$\mathbf{K}_X\otimes[kD_\infty]>0,\quad \text{for } k>r+1. \tag{79}$$

Proposition 5.1 then gives the following existence result for Einstein metrics on negative bundles.

Corollary 5.6. *Let* M' *be the total space of a negative holomorphic bundle* $\pi:\mathbf{E}\to N$ *such that* $c_1(M')<0$, *equivalently* $c_1(N)+c_1(\mathbf{E})<0$, *then the strictly pseudoconvex tubular neighborhoods* $M_c=\{v\in\mathbf{E}:r^2=h(v,v)<c\}$ *of the zero section admit unique complete Kähler-Einstein metrics, with normal CR infinity the sphere bundle* $S_c\subset\mathbf{E}$.

5.2. **Resolutions weighted homogeneous hypersurfaces.** One can easily construct examples by taking weighted blow-ups of simple weighted homogeneous hypersurface singularities.

A polynomial $f\in\mathbb{C}[z_0\dots,z_n]$ is *weighted homogeneous* with weights $\mathbf{w}=(w_0,\dots,w_n)\in\mathbb{Z}_+^{n+1}$ and degree d if
$$f(u^{w_0}z_0,\dots,u^{w_n}z_n)=u^d f(z_0,\dots,z_n),\quad u,z_0,\dots,z_n\in\mathbb{C}. \tag{80}$$
Here we assume that $\gcd(w_0,\dots,w_n)=1$. We assume that $X=\{z\in\mathbb{C}^{n+1}: f(z)=0\}$ is smooth away from $o\in\mathbb{C}^n$. Then it is well known, see [9], that $S=X\cap\mathbb{S}^{2n+1}$ has a natural Sasaki structure with Reeb vector field generating

the action $(z_0,\dots,z_n) \to (u^{w_0}z_0,\dots,u^{w_n}z_n)$ and the CR structure of S satisfies property S-E precisely when $|\mathbf{w}| = \sum w_i > d$, *loc. cit.*. The codimension of the singular set of X is ≥ 2, so X is normal. And X is easily seen to be Gorenstein with holomorphic form given on $X \setminus \{o\}$ by adjunction by

$$\Omega = \frac{(-1)^{i+1}}{\partial f/\partial z_i} dz_0 \wedge \cdots \wedge \hat{dz_i} \wedge \cdots \wedge dz_n, \tag{81}$$

where $\frac{\partial f}{\partial z_i} \neq 0$ for $i = 1,\dots,n$. Therefore we have the following.

Proposition 5.7. *A weighted homogeneous hypersurface $X = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ with an isolated singularity has a resolution $\hat{X}$ with $\mathbf{K}_{\hat{X}} > 0$ only if $\sum w_i \leq d = \deg(f)$.*

We have $C(S)_{\leq 1} = X \cap \mathbb{B}^{n+1}$. And by Proposition 4.3 any strictly pseudoconvex domain with CR infinity S must be the domain $\pi^{-1}(X \cap \mathbb{B}^{n+1})$ in a resolution $\pi : \hat{X} \to X$. Examples of such resolutions are easy to find by taking blow-ups or more generally weighted blow-ups. The weight $\mathbf{w} = (w_0,\dots,w_n)$ defines a grading $\mathbb{C}[z_0,\dots,z_n] = \oplus_{k\geq 0}\mathbb{C}[z_0,\dots,z_n]_k$. We define $\deg(f) = \max\{j : f \in \oplus_{k\geq j}\mathbb{C}[z_0,\dots,z_n]_k\}$. The *weighted blow-up* $\varpi : B^{\mathbf{w}}\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is constructed similarly to the usual but with the weighted grading, and the exceptional fiber $E = \varpi^{-1}(o) = \mathbb{P}(w_0,\dots,w_n)$, the weighted projective space. And, of course, one obtains the usual blow-up with $\mathbf{w} = (1,\dots,1)$. If $X' \subset B^{\mathbf{w}}\mathbb{C}^{n+1}$ is the strict transform, then we have the adjunction formula for the canonical bundle

$$\mathbf{K}_{X'} = \varpi^*(\mathbf{K}_X) + (|\mathbf{w}| - \deg(f) - 1)X' \cap E, \tag{82}$$

provided X' does not contain a divisor singular along a singular set of $B^{\mathbf{w}}\mathbb{C}^{n+1}$. See [39] for more details.

5.2.1. *Example 1.* Consider the hypersurface

$$X = \{z_0^d + \cdots + z_{n-1}^d + z_n^k = 0\} \subset \mathbb{C}^{n+1}, \tag{83}$$

with $k \geq d \geq n+1$. We consider a series of blow-ups of X. Blowing up gives $\pi : X_1 \to X$ where X_1 is the strict transform of X in $\pi : \hat{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$, the blow-up of $\mathbb{C}^{n+1}$ at the origin. Then X_1 is covered with affine neighborhoods $U_i, i = 0,\dots,n$. Take for example $U_0 \subset \mathbb{C}^{n+1}$ which has coordinates $y_0,\dots,y_n$ and π is given by $z_0 = y_0, z_1 = y_0y_1,\dots,z_n = y_0y_n$. Thus if $f = z_0^d + \cdots + z_{n-1}^d + z_n^k$, then $\pi^* f = y_0^d(1 + y_1^d + \cdots + y_{n-1}^d + y_0^{k-d}y_n^k)$. So if $g = 1 + y_1^d + \cdots + y_{n-1}^d + y_0^{k-d}y_n^k$, then $X_1 \cap U_0 = \{g = 0\} \subset \mathbb{C}^{n+1}$. It is elementary to check that this is a non-singular hypersurface and similarly for $X_1 \cap U_i, i = 1,\dots,n-1$. We have $X_1 \cap U_n = \{g = 0\} \subset \mathbb{C}^{n+1}$ where $g = y_0^d + \cdots + y_{n-1}^d + y_n^{k-d}$, and this hypersurface has a singular point at the origin unless $k - d = 0$ or 1. Repeating the procedure we get a resolution $\pi : \hat{X} \to X$, $\hat{X} = X_{\lfloor \frac{k}{d} \rfloor}$, if $k \equiv 0$ or $1 \mod d$.

Denote by E_i the strict transform of the exceptional set of the $i-th$ blow-up. Then if follows from (82) that

$$\mathbf{K}_{\hat{X}} = \sum_{i=1}^{\lfloor \frac{k}{d} \rfloor} i(n-d)E_i. \tag{84}$$

In order to prove that $\mathbf{K}_{\hat{X}} > 0$ we will compactify $\hat{X}$ and employ a lemma of H. Grauert. Let $s = \mathrm{lcm}(d,k)$ and set $a = \frac{s}{d}$ and $b = \frac{s}{k}$. Then we have $\mathbb{C}^{n+1} \subset$

$\mathbb{C}P^{n+1}_{a,\dots,a,b,1}$, where $\mathbb{C}P^{n+1}_{a,\dots,a,b,1}$ is the weighted projective space, and $f = z_0^d + \cdots + z_{n-1}^d + z_n^k$ is weighted homogeneous with respect to these weights. Let $Y = \{f = 0\} \subset \mathbb{C}P^{n+1}_{a,\dots,a,b,1}$. If $z_0, \dots, z_{n+1}$ are homogeneous coordinates on $\mathbb{C}P^{n+1}_{a,\dots,a,b,1}$, then we have added $\{z_{n+1} = 0\}$ to $\mathbb{C}^{n+1}$. Let $E_\infty = Y \cap \{z_{n+1} = 0\}$. Let $\hat{Y}$ be the above resolution of Y given by resolving $X \subset Y$. We will prove that

$$\mathbf{F} = \sum_{i=1}^{\lfloor \frac{k}{d} \rfloor} i(n-d)[E_i] + t[E_\infty] > 0 \tag{85}$$

on $\hat{Y}$ for $t \in \mathbb{N}$ sufficiently large. Note that $\mathbf{F}$ is not a Cartier divisor unless $\operatorname{lcm}(a,b)|t$. We will use the following due to H. Grauert [23].

Lemma 5.8. *A line bundle $\mathbf{L}$ on a compact complex space X is positive if and only if for every irreducible compact nowhere discrete analytic subspace $Z \subset X$ there is an holomorphic section σ of $\mathbf{L^k}|_Z$, for some k, with a zero on Z but not vanishing entirely.*

Let $H = \{f = 0\} \subset \mathbb{C}^{n+1}$ be an hypersurface which is tangent to the line $\mathbb{C}(0,\dots,0,1)$ at $(0,\dots,0) \in \mathbb{C}^{n+1}$ to at least order $\lfloor \frac{k}{d} \rfloor - 1$. If $D = H \cap X$, then one can check that $\pi^* D = D' + \sum_{i=1}^{\lfloor \frac{k}{d} \rfloor} iE_i$, where D' is the strict transform of D. Assume that f is algebraic, so it extends to a rational function on Y with a pole along E_∞. If $t > 0$ is sufficiently large, then $\pi^* f^{(n-d)}$ gives an holomorphic section of $\mathbf{F}$ with $(\pi^* f^{(n-d)}) = D' + qE_\infty$, for some $q > 0$. It is not very difficult to check that, for various such D, the condition of Lemma 5.8 is satisfied for A not contained in E_∞.

Suppose $A \subset E_\infty$. The above argument gives a section σ of $\mathbf{F}^k$ with $(\sigma) = kD' + kqE_\infty$. If kq is sufficiently divisible by a and b, then there are many rational functions $\frac{g}{z_{n+1}^{kq}}$ with g not vanishing along E_∞. We have $(\sigma) \sim kD' + (g)$, and various choices of D and g give the required section of $\mathbf{F}^k|_A$.

5.2.2. *Example 2.* Let

$$X = \{z_0^d + z_1^{2d} + \cdots + z_{n-1}^{2d} + z_n^k = 0\} \subset \mathbb{C}^{n+1}, \tag{86}$$

with $k \geq 2d \geq n+2$. Then X can be resolved similar to Example 1 but by taking blow-ups with weight $\mathbf{w} = (2,1,\dots,1)$. One can repeatedly blowing up the unique singular with this weight $\lfloor \frac{k}{2d} \rfloor$ times. And if $k \equiv 0$ or $1 \mod 2d$, this ends in a smooth resolution $\pi : \hat{X} \to X$, $X_{\lfloor \frac{k}{2d} \rfloor}$. One can check using arguments as in Example 1 that $\mathbf{K}_{\hat{X}} > 0$.

5.2.3. *Example 3.* Let

$$X = \{z_0^{2d} + z_1^{3d} + z_2^{6d} + \cdots + z_{n-1}^{6d} + z_n^k\}, \tag{87}$$

with $k \geq 6d \geq n+4$. Again X can repeatedly blowing up with weight $\mathbf{w} = (3,2,1,\dots,1)$ the unique singular point at each step $\lfloor \frac{k}{6d} \rfloor$ times. If $k \equiv 0$ or 1 mod $6d$, this ends in a smooth resolution $\pi : \hat{X} \to X$, $X_{\lfloor \frac{k}{6d} \rfloor}$. And again similar arguments show that $\mathbf{K}_{\hat{X}} > 0$.

5.3. **Locally strongly ACH Kähler-Einstein manifolds.** We will consider examples of ACH Kähler manifolds where the metric converges to that of the complex hyperbolic space $\mathcal{H}^n_{\mathbb{C}}$. More precisely, suppose $\Gamma \subset \mathrm{PSU}(1,n)$ is a finite group. By making a conjugation, we may assume that $\Gamma \subset \mathrm{U}(n) \subset \mathrm{PSU}(1,n)$. Assume that Γ acts freely away from $o \in \mathbb{B}^n = \mathcal{H}^n_{\mathbb{C}}$, where $\mathbb{B}^n \subset \mathbb{C}^n$ is the unit ball.

Definition 5.9. *The Kähler manifold (M,g) is* locally strongly ACH*, of order $\alpha > 0$, if there is a compact set $K \subset M$, a ball $B \subset \mathcal{H}^n_{\mathbb{C}}$, and a biholomorphism $\psi : M \setminus K \to \mathcal{H}^n_{\mathbb{C}} \setminus B/\Gamma$, such that if g_0 is the hyperbolic metric of $\mathcal{H}^n_{\mathbb{C}} \setminus B/\Gamma$,*

$$|\psi_* g - g_0|_{g_0} = O\big(e^{-\alpha r}\big), \tag{88}$$

where $r = \mathrm{dist}(o,x)$ is the distance from a fixed point of $\mathcal{H}^n_{\mathbb{C}}/\Gamma$.

Remark 5.10. One generally also has to assume an analogous condition to (88) on some derivatives of g for most analytical purposes, i.e.

$$|\nabla^k\big(\psi_* g - g_0\big)|_{g_0} = O\big(e^{-\alpha r}\big), \tag{89}$$

where ∇^k is the k-th covariant derivative of g_0.

Also, one may consider the weaker condition that ψ is merely a diffeomorphism and the complex structure J of M converges to J_0 of $\mathcal{H}^n_{\mathbb{C}}/\Gamma$ as in (88). See [26, 6] for some interesting rigidity results for such strongly ACH manifolds. Especially considering Proposition 5.11, it is an interesting question whether there are similar rigidity results for *locally* strongly ACH manifolds.

Proposition 5.11. *Let (M,g) be a Kähler manifold with Kähler form $\omega \in -\frac{2\pi}{n+1}c_1(M)$, and assume that $M \setminus K$ is biholomorphic to $\mathbb{B}^n \setminus B/\Gamma$, with Γ as above. Then there is a locally strongly ACH Kähler-Einstein metric g on M which is of order α for all $\alpha < 2n+2$, with convergence including all derivatives. That is (89) holds for all $k \geq 0$.*

Proof. By Proposition 4.3 M is a domain in a resolution $\pi : X \to \mathbb{C}^n/\Gamma$. Since this a resolution of a rational singularity the $\partial\bar{\partial}$-lemma holds. Thus $\mathbf{K}_X$ admits an Hermitian metric h so that $\omega = \frac{\sqrt{-1}}{n+1}\Theta_h$ is a Kähler form.

If $k = |\Gamma|$, then $(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k} \in \Gamma\big(\mathbf{K}^k_{\mathbb{C}^n/\Gamma}\big)$ and $\sigma = \pi^*(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}$ is a meromorphic section of $\mathbf{K}^k_X$, which incidentally must have poles on the exceptional divisors. We have a metric on $\mathbf{K}^k_X$, also denoted by h, with $\omega = \frac{\sqrt{-1}}{k(n+1)}\Theta_h$. Let $r^2 = \sum_j |z_j|^2$, and for $0 < \epsilon < \frac{1}{2}$ define a cut-off function $0 \leq \rho(r) \leq 1$

$$\rho(r) = \begin{cases} 1, & r < \epsilon \\ 0, & r > 2\epsilon \end{cases}$$

Let $\tilde{h}$ be the metric on $\mathbf{K}^k_{X\setminus E}$, where E is the exceptional set, with $\tilde{h}|\sigma|^2 = 1$. Define the metric

$$\hat{h} := e^{-Cr^2}\big(\rho(r)h + (1-\rho(r))\tilde{h}\big), \tag{90}$$

which for $C > 0$ sufficiently large $\omega_0 := \frac{\sqrt{-1}}{k(n+1)}\Theta_{\hat{h}}$ is positive. Since ω_0^n also defines an Hermitian metric on $\mathbf{K}^k_X$, we can define $f \in C^\infty(X)$ by

$$e^{kf}\hat{h} = \frac{1}{\big(\omega_0^n\big)^k}, \tag{91}$$

where $f = c + Cr^2$ for $r > 2\epsilon$. In particular, we have (20). The defining function $\phi = e^{\frac{f}{n+1}}\phi_0$ where $\phi_0 = \left[\frac{C}{n+1}\right]^{\frac{n}{n+1}}\left(r^2 - 1\right)$ has $-dd^c \log(-\phi) > 0$ in a neighborhood of $\partial\mathbb{B}^n$. We may modify ϕ on $r < \varepsilon < 1$ so that $-dd^c \log(-\phi) \geq 0$ on M. Then on $r > \max(2\epsilon, \varepsilon)$ (21) becomes

$$F = \log\left[\frac{(-\phi_0)^{-(n+1)}\omega_0^n}{(-dd^c \log(-\phi_0))^n}\right] = 0, \tag{92}$$

as is not difficult to check. Therefore, when we solve (17) to get the Kähler-Einstein metric ω' on $r > \max(2\epsilon, \varepsilon)$ we have

$$\begin{aligned} \omega' &= \omega_0 - dd^c \log(-\phi) + dd^c u \\ &= -dd^c \log(-\phi_0) + dd^c u \\ &= -dd^c \log(1 - r^2) + dd^c u, \end{aligned} \tag{93}$$

where $u \in C^\infty(M)$ satisfies both (36) and (37). The conclusion follows from the observation that if $r = \operatorname{dist}(o, x)$ is the distance with respect to the Bergman metric and ϕ is any defining function, then $c(-\phi) \leq e^{-2r} \leq C(-\phi)$ for $C > c > 0$. □

5.3.1. *Resolutions of Hirzebruch-Jung singularities.* Let $p > q > 0$ be relatively prime integers and consider the finite group $\Gamma \subset U(2)$ generated by

$$\begin{bmatrix} e^{\frac{2\pi i q}{p}} & 0 \\ 0 & e^{\frac{2\pi i}{p}} \end{bmatrix}. \tag{94}$$

Then $\mathbb{C}^2/\Gamma$ has an isolated orbifold singularity at the origin, and its minimal resolution given by a Hirzebruch-Jung string is well known. See [2, Ch. II, §5] and [37, §1.6] for a description in terms of toric geometry. This minimal resolution $\pi : X \to \mathbb{C}^2/\Gamma$ has the following properties:

(i) The exceptional divisor $E = \pi^{-1}(0) = \cup_{i=1}^k C_i$, where each C_i is an embedded $\mathbb{C}P^1$.
(ii) $C_i^2 = -e_i \leq -2$, $C_i \cdot C_j = 1$ for $|i - j| = 1$, and $C_i \cdot C_j = 0$ for $|i - j| > 1$.

The integers e_i are given by the continued fraction expansion

$$\frac{p}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{e_3 - \cfrac{1}{\ddots - \cfrac{1}{e_k}}}} \tag{95}$$

In other words, the e_i are determined by the Euclidean algorithm where we define $q_i, -1 \leq i \leq k$, inductively

$$q_{-1} := p, \quad q_0 := q, \quad q_{i-1} = e_{i+1}q_i - q_{i+1}, \quad \text{with } 0 \leq q_{i+1} < q_i. \tag{96}$$

Note that the resolution $\pi : X \to \mathbb{C}^2/\Gamma$ is the unique minimal toric resolution. And one can retrieve the toric diagram, i.e. the stabilizers of the C_i, from the e_i as follows. Suppose C_i has stabilizer $(m_i, n_i) \in \mathbb{Z}^2$, with m_i and n_i coprime, then we

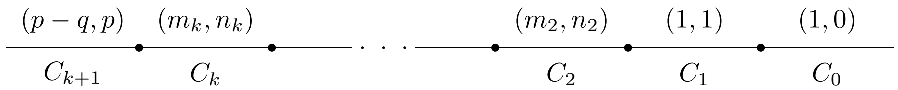

FIGURE 2. resolution of $\mathbb{C}^2/\Gamma$

have

$$\frac{n_i}{n_i-m_i}=e_1-\cfrac{1}{e_2-\cfrac{1}{e_3-\cfrac{1}{\ddots-\cfrac{1}{e_{i-1}}}}}. \tag{97}$$

And from (97) we obtain (m_i,n_i) unambiguously since $m_i>0$. If we denote by C_0 and C_{k+1} the non-compact curves with stabilizers $(1,0)$ and $(p-q,p)$ respectively corresponding to the axis of $\mathbb{C}^2/\Gamma$, then the arrangement of curves is given in Figure 5.3.1.

We have $c_1(X)<0$ if and only if $e_i\geq 3$ for $0\leq i\leq k$, in which case Proposition 5.11 gives a locally strongly ACH Kähler-Einstein metric on the domain $M_a=\{r<a\}\subset X$, where $r^2=|z_1|^2+|z_2|^2$, which is invariant under $T^2\subset \mathrm{U}(2)$.

It is known from D. Calderbank and M. Singer [15] that the domains M_a admit ASD (Self-dual Weyl curvature vanishes, $W_+=0$.) Hermitian Einstein metrics with negative scalar curvature. These metrics are toric but not Kähler. And furthermore, these metrics have the same CR-infinity as the Kähler-Einstein metrics given here. It would be interesting to know how these metrics are related in say the moduli of Einstein structures on M_a.

5.3.2. *Resolutions of diagonal quotients* $\mathbb{C}^n/\Gamma$*.* Let $\Gamma=\mathbb{Z}_k$ be the group acting on $\mathbb{C}^n$ with generator $(z_1,\ldots,z_n)\mapsto(\zeta\cdot z_1,\ldots,\zeta\cdot z_n)$, $\zeta=e^{\frac{2\pi i}{k}}$. Then the singularity $\mathbb{C}^n/\Gamma$ has as resolution $\pi: X\to\mathbb{C}^n/\Gamma$ the total space of $\mathbf{L}^k\to\mathbb{C}P^{n-1}$, with $\mathbf{L}$ the tautological line bundle. Then $c_1(X)<0$, if and only if $k>n$, and in this case Proposition 5.11 gives a locally strongly ACH Kähler-Einstein metric on the domain $M_a=\{r<a\}\subset X$, where $r^2=\sum_j|z_j|^2$, which is invariant under $\mathrm{U}(n)$.

The metrics in this example are of cohomogeneity one, so one expects to find an explicit formula for the Kähler potential of ω. There is a simple formula for these metrics due to A. Futaki [20]. The 2-form $dd^c\log r^2$ is basic with respect to the $\mathbb{C}^*$-action on $\mathbb{C}^n/\Gamma$ and restricts to the Fubini Study metric on the quotient $\mathbb{C}P^{n-1}$. Define

$$\omega^T=kdd^c\log r^2. \tag{98}$$

Then the transversal Ricci form, i.e. that of ω^T on $\mathbb{C}P^{n-1}$, satisfies

$$\mathrm{Ricci}^T=\frac{n}{k}\omega^T. \tag{99}$$

Let $t=\log r^{2k}$. The technique of E. Calabi [14] is to consider metrics of the form

$$\omega=\omega^T+\sqrt{-1}dd^cF(t). \tag{100}$$

It turns out to be easier to work in a *momentum coordinate* along the fiber. So set

$$\tau = F'(t), \tag{101}$$

$$\phi(\tau) = F''(t). \tag{102}$$

Then (100) becomes

$$\begin{aligned} \omega &= (1+\tau)\omega^T + \phi(\tau) dt \wedge d^c t \\ &= (1+\tau)\omega^T + \phi(\tau)^{-1} d\tau \wedge d^c \tau. \end{aligned} \tag{103}$$

Then its Ricci form is computed in [20] to be

$$\mathrm{Ricci}(\omega) = \mathrm{Ricci}^T - dd^c \log\big((1+\tau)^{n-1}\phi(\tau)\big), \tag{104}$$

and the Einstein equation $\mathrm{Ricci}(\omega) = -\lambda\omega$ is satisfied with $\lambda = 2 - \frac{n}{k}$ if

$$\phi(\tau) = \frac{1}{k}(1+\tau) - \frac{(n-2k)}{k(n+1)}(1+\tau)^2 - \frac{(2n+k)}{n(n+1)}\frac{1}{(1+\tau)^{n-1}}. \tag{105}$$

One retrieves the complex coordinate expression (100) by integrating, for fixed τ_0,

$$t = \int_{\tau_0}^{\tau(t)} \frac{dx}{\phi(x)}. \tag{106}$$

And one also obtains

$$F(t) = \int_{\tau_0}^{\tau(t)} \frac{x dx}{\phi(x)}. \tag{107}$$

Since (105) grows quadratically (106) shows that the range of t is finite. In fact, $F'(t)$ maps $(-\infty, c)$ to $(0,\infty)$. And one can show that it is a complete metric defined on $\{r < e^{\frac{c}{2k}}\} \subset X$. The differing radii arise from the ambiguity in the integral (106).

It would be interesting to obtain a closed formula for the Kähler potential of the metric (100). There is such a formula for the Ricci-flat metric in case $k = n$ on X due to E. Calabi [14, 4.14]. Since the explicit formula involves integrals it is easier to see the strongly ACH nature of the metric from Proposition 5.11.

5.4. **Normal CR infinities in dimension 3.** Using the classification of normal CR structures on 3-manifolds in [4] and [5] we are able to mostly classify those normal CR 3-manifolds which bound Kähler-Einstein manifolds and the unique Kähler-Einstein surfaces which thus arise.

The classification of normal CR structures on 3-manifolds follows from a classification of Sasaki structures on 3-manifolds which in turn follows from a classification [3] of Vaisman metrics (or locally comformally Kähler metrics with a parallel Lee forms) on compact surfaces. The Riemannian product of a Sasaki manifold with a circle is a Vaisman manifold.

Theorem 5.12 ([4])**.** *If (S, g, ξ) is a Sasaki 3-manifold then it is one of the follówning.*

(i) S is a Seifert $\mathbb{S}^1$-bundle over a Riemann surface of genus $g > 1$, ξ generates the $\mathbb{S}^1$-action, and the Vaisman manifold $S \times \mathbb{S}^1$ is a properly elliptic surface admitting two holomorphic circle actions.

(ii) S is a Seifert $\mathbb{S}^1$-bundle over an elliptic curve, ξ generates the $\mathbb{S}^1$-action, and the Vaisman manifold $S \times \mathbb{S}^1$ is a Kodaira surface admitting two holomorphic circle actions.

(iii) *S is a finite quotient of $\mathbb{S}^3$, with holomorphic coordinates $(z_1, Z_2) = (x_1 + iy_1, x_2 + iy_2)$ we have $\xi = a(x_1\partial_{y_1} - y_1\partial_{x_1}) + b(x_2\partial_{y_2} - y_2\partial_{x_2})$ with $a \geq b > 0$, and the Vaisman manifold $S \times \mathbb{S}^1 = \mathbb{C}^2 \setminus \{(0,0)\}/G$ is a Hopf surface of class 1 where G is generated by the contraction $g(z_1, z_2) = (e^{-a}z_1, e^{-b}z_2)$.*

The classification of normal CR manifolds (S, D, J) is more complicated as as it involves identifying and distinguishing the underlying CR structures of the above Sasaki structures. This is solved for (i) and (ii) by showing there are no other CR Reeb vector fields, so other Sasaki structures are deformations as in Section 4.2.

An above $\mathbb{S}^1$-Seifert bundle is an $\mathbb{S}^1$ subbundle of a negative orbifold bundle $\mathbf{L}$ over a Riemann surface N. Suppose it has multiple fibers over $p_1, \dots, p_k \in N$ of multiplicities $m_1, \dots, m_k$. We can classify the multiple fibers by $(m_j; q_j)$, $q_j < m_j$, $j = 1, \dots, k,$, where there is a neighborhood U of p_j and locally N is the quotient of $U \times \mathbb{S}^1$ by the $\mathbb{Z}_{m_j}$-action generated by $(z, w) \mapsto (e^{2\pi i/m_j}z, e^{2\pi i q_j/m_j}w)$.

Proposition 5.13. *Suppose (S, D, J) is a normal CR 3-manifold.*

(i) *If $b_1(S) > 0$, that is cases (i) and (ii) in Theorem 5.12, then (S, D, J) is the boundary of an unique Kähler-Einstein manifold if and only if for each $(m_j; q_j)$ $j = 1, \dots, k$, the e_i produced by the Euclidean algorithm of Section 5.3.1 satisfy $e_i \geq 3$.*

(ii) *If $b_1(S) = 0$, case (iii), then we have a biholomorphism $C(S) \cong \mathbb{C}^2/\Gamma$, with $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ finite and acting freely on $\mathbb{C}^2 \setminus \{(0,0)\}$. Then $(0,0) \in \mathbb{C}^2/\Gamma$ is a rational singularity, thus the exceptional set of the minimal resolution $\pi : X \to \mathbb{C}^2/\Gamma$ is a tree of rational curves C_i, $i = 1, \dots, k$. This is the unique Kähler-Einstein manifold with boundary (S, D, J) if and only if $C_i^2 \leq -3$, $i = 1, \dots, k$.*

Proof. For case (i) S is a Seifert $\mathbb{S}^1$ subbundle of a negative orbifold bundle $\mathbf{L}$ over a Riemann surface N of genus $g > 0$. Either we have a regular Sasaki structure and $\mathbf{L}$ is smooth, or there are multiple fibers and corresponding singularities $(m_j; q_j)$ $j = 1, \dots, k$. Each of these can be resolved as in Section 5.3.1, altogether giving the unique minimal resolution $\pi : X \to C(S)$. It is clear that $e_i = -C_i^2 \geq 3$ is necessary from the adjunction formula $2g(C_i) - 2 = -2 = K_X \cdot C_i + C_i^2$. If C_0 denotes the zero section of $\mathbf{L}$, then

$$0 \leq 2g(C_0) - 2 = K_X \cdot C_0 + C_0^2. \tag{108}$$

By Grauert's criterion for an exceptional curve [23, p. 367] $C_0^2 < 0$, so $K_X \cdot C_0 > 0$. Therefore by assumption $K_X \cdot C > 0$ for each irreducible exceptional curve. And if $E = \cup_i C_i$ denotes the exceptional set, the argument in the proof of the lemma on p. 347 of [23] shows that $\mathbf{K}_X|_E > 0$.

We have $C(S) = \mathbf{L}^\times$, minus the zero section. Then, after a possible homothetic change of ξ, the radial r on $C(S)$ is given by $r^2 = h(v, v)$ for an Hermitian metric h on $\mathbf{L}$. Let $\omega \in \Gamma(\mathbf{K}_N)$ be an holomorphic section which vanishes on $x_1, \dots, x_{2g-2}$. Define the $(1,0)$-form on the total space of $\mathbf{L}$

$$\beta = J^*\eta + \sqrt{-1}\eta = \frac{dr}{r} + \sqrt{-1}\eta = \partial \log r^2. \tag{109}$$

Then the $(2,0)$-form $\Omega := \beta \wedge \pi^*\omega$ satisfies $\bar{\partial}\Omega = 0$ and has a pole of order 1 on the zero section. Strictly speaking, we need to take this local construction to the power $\ell = \mathrm{lcm}(m_1, \dots, m_k)$ to get a true meromorphic section $\Omega^\ell \in \Gamma(\mathbf{K}_X^\ell)$. So Ω^ℓ is a meromorphic section with zeros along $\pi^*(x_i)$, $i = 1, \dots 2g - 2$, and poles of varying orders on the exceptional curves C_i. Then the arguments in Example 1 above or in the proof of Satz 4 in [23, p. 367] show that $\mathbf{K}_X > 0$.

In case (ii), $C(S) \cong \mathbb{C}^2/\Gamma$ follows from Theorem 5.12.iii. It is well known that $\mathbb{C}^2/\Gamma$ has only rational singularities, and the properties of the exceptional curves follow from well known properties of rational surface singularities (cf. [2]). If the exceptional curves satisfy $C_i^2 \leq -3$ the arguments in part (i) show what $\mathbf{K}_X > 0$. □

In part (ii) the groups $\Gamma \subset \mathrm{GL}(2,\mathbb{C})$ are classified [34]. This follows from

$$1 \to \mathbb{C}^* \to \mathrm{GL}(2,\mathbb{C}) \to \mathrm{PGL}(2,\mathbb{C}) \to 1, \tag{110}$$

and the fact that the finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ are the polyhedral groups.

We already know which cyclic groups $\mathbb{Z}_p = \Gamma \subset \mathrm{GL}(2,\mathbb{C})$ have the require resolution from Section 5.3.1. And all the groups $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ are ruled out by Theorem 4.14.

References

[1] W. L. Baily. On the imbedding of V-manifolds in projective space. *Amer. J. Math.*, 79:403–430, 1957.

[2] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[3] Florin Alexandru Belgun. On the metric structure of non-Kähler complex surfaces. *Math. Ann.*, 317(1):1–40, 2000.

[4] Florin Alexandru Belgun. Normal CR structures on compact 3-manifolds. *Math. Z.*, 238(3):441–460, 2001.

[5] Florin Alexandru Belgun. Normal CR structures on S^3. *Math. Z.*, 244(1):125–151, 2003.

[6] Hassan Boualem and Marc Herzlich. Rigidity at infinity for even-dimensional asymptotically complex hyperbolic spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(2):461–469, 2002.

[7] L. Boutet de Monvel. Intégration des équations de Cauchy-Riemann induites formelles. In *Séminaire Goulaouic-Lions-Schwartz 1974–1975; Équations aux derivées partielles linéaires et non linéaires*, pages Exp. No. 9, 14. Centre Math., École Polytech., Paris, 1975.

[8] Charles P. Boyer and Krzysztof Galicki. *Sasakian geometry*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.

[9] Charles P. Boyer, Krzysztof Galicki, and János Kollár. Einstein metrics on spheres. *Ann. of Math. (2)*, 162(1):557–580, 2005.

[10] Charles P. Boyer, Krzysztof Galicki, János Kollár, and Evan Thomas. Einstein metrics on exotic spheres in dimensions 7, 11, and 15. *Experiment. Math.*, 14(1):59–64, 2005.

[11] Charles P. Boyer, Krzysztof Galicki, and Santiago R. Simanca. Canonical Sasakian metrics. *Comm. Math. Phys.*, 279(3):705–733, 2008.

[12] Charles P. Boyer, Krzysztof Galicki, and Santiago R. Simanca. The Sasaki cone and extremal Sasakian metrics. In *Riemannian topology and geometric structures on manifolds*, volume 271 of *Progr. Math.*, pages 263–290. Birkhäuser Boston, Boston, MA, 2009.

[13] D. Burns. On rational singularities in dimensions > 2. *Math. Ann.*, 211:237–244, 1974.

[14] E. Calabi. Métriques kählériennes et fibrés holomorphes. *Ann. Sci. École Norm. Sup. (4)*, 12(2):269–294, 1979.

[15] David M. J. Calderbank and Michael A. Singer. Einstein metrics and complex singularities. *Invent. Math.*, 156(2):405–443, 2004.

[16] Shiu Yuen Cheng and Shing Tung Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[17] Mihnea Colţoiu. On the Oka-Grauert principle for 1-convex manifolds. *Math. Ann.*, 310(3):561–569, 1998.

[18] Shun Ichi Eto, Hideaki Kazama, and Kiyoshi Watanabe. On strongly q-pseudoconvex spaces with positive vector bundles. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 28:135–146, 1974.

[19] Charles L. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.

[20] Akito Futaki. Momentum construction on Ricci-flat Kähler cones. preprint, arXiv: math.DG/0703138.

[21] Akito Futaki, Hajime Ono, and Guofang Wang. Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds. *J. Differential Geom.*, 83(3):585–635, 2009.

[22] Jerome P. Gauntlett, Dario Martelli, James Sparks, and Daniel Waldram. Sasaki-Einstein metrics on $S^2 \times S^3$. *Adv. Theor. Math. Phys.*, 8(4):711–734, 2004.

[23] Hans Grauert. Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.*, 146:331–368, 1962.

[24] Phillip A. Griffiths. Hermitian differential geometry and the theory of positive and ample holomorphic vector bundles. *J. Math. Mech.*, 14:117–140, 1965.

[25] F. Reese Harvey and H. Blaine Lawson, Jr. On boundaries of complex analytic varieties. I. *Ann. of Math. (2)*, 102(2):223–290, 1975.

[26] Marc Herzlich. Scalar curvature and rigidity of odd-dimensional complex hyperbolic spaces. *Math. Ann.*, 312(4):641–657, 1998.

[27] Hideaki Kazama and Shigeharu Takayama. On the $\partial\overline{\partial}$-equation over pseudoconvex Kähler manifolds. *Manuscripta Math.*, 102(1):25–39, 2000.

[28] Shoshichi Kobayashi. *Transformation groups in differential geometry.* Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70.

[29] J. J. Kohn and Hugo Rossi. On the extension of holomorphic functions from the boundary of a complex manifold. *Ann. of Math. (2)*, 81:451–472, 1965.

[30] Henry B. Laufer. On rational singularities. *Amer. J. Math.*, 94:597–608, 1972.

[31] John M. Lee and Richard Melrose. Boundary behaviour of the complex Monge-Ampère equation. *Acta Math.*, 148:159–192, 1982.

[32] Ngaiming Mok and Shing-Tung Yau. Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions. In *The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980)*, volume 39 of *Proc. Sympos. Pure Math.*, pages 41–59. Amer. Math. Soc., Providence, RI, 1983.

[33] Raghavan Narasimhan. Imbedding of holomorphically complete complex spaces. *Amer. J. Math.*, 82:917–934, 1960.

[34] K. A. Nguyen, M. van der Put, and J. Top. Algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$. *Indag. Math. (N.S.)*, 19(2):287–297, 2008.

[35] L. Nirenberg. A certain problem of Hans Lewy. *Russian Math. Surveys*, 29(2):251–262, 1974.

[36] Louis Nirenberg. *Lectures on linear partial differential equations.* American Mathematical Society, Providence, R.I., 1973. Expository Lectures from the CBMS Regional Conference held at the Texas Technological University, Lubbock, Tex., May 22–26, 1972, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 17.

[37] Tadao Oda. *Convex bodies and algebraic geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.

[38] Liviu Ornea and Misha Verbitsky. Embeddings of compact Sasakian manifolds. *Math. Res. Lett.*, 14(4):703–710, 2007.

[39] Miles Reid. Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.

[40] R. Schoen. On the conformal and CR automorphism groups. *Geom. Funct. Anal.*, 5(2):464–481, 1995.

[41] Vo Van Tan. On the embedding problem for 1-convex spaces. *Trans. Amer. Math. Soc.*, 256:185–197, 1979.

[42] G. Tian and S.-T. Yau. Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 574–628. World Sci. Publishing, Singapore, 1987.

[43] C. van Coevering. Ricci-flat Kähler metrics on crepant resolutions of Kähler cones. *Math. Annalen*, 347(3):581–611, 2010.

[44] Craig van Coevering. Examples of asymptotically conical Ricci-flat kähler manifolds. arXiv:math.DG/0812.4745 v.2, to appear in Math. Zeitschrift, 2008.

[45] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.

Department of Mathematics, University of Science and Technology China, Hefei, Anhui Pronvince 230026, P.R. China

E-mail address: `craigvan@ustc.edu.cn`