K-trivial structures on Fano complete intersections

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It is proven that any structure of a fiber space into varieties of Kodaira dimension zero on a generic Fano complete intersection of index 1 and dimension M in $\mathbb{P}^{M+k}$ for $M \geq 2k+1$ is a pencil of hyperplane sections. We describe K-trivial structures on varieties with a pencil of Fano complete intersections.
Bibliography: 9 items.

1. Formulation of the main result. Fix $k \geq 2$, $M \geq 2k+1$ and a set of integers $(d_1, \ldots, d_k) \in \mathbb{Z}_+^k$ satisfying the conditions $d_k \geq \ldots \geq d_1 \geq 2$ and

$$d_1 + \ldots + d_k = M + k.$$

Assume in addition that if $M = 2k+2$, then $k \geq 4$, and if $M = 2k+1$, then $k \geq 5$ and $(d_1, \ldots, d_k) \neq (2, \ldots, 2, k+3)$.

The symbol $\mathbb{P}$ denotes the complex projective space $\mathbb{P}^{M+k}$. Consider a smooth complete intersection of codimension k in $\mathbb{P}$

$$V = \{f_1 = \ldots = f_k = 0\} \subset \mathbb{P},$$

where $f_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i))$ are homogeneous polynomials of degree d_i. The aim of this note is to give a complete description of structures of non-maximal Kodaira dimension on a generic complete intersection V, from which one immediately derives a description of K-trivial structures on varieties with a pencil of complete intersections.

Let $\beta: W \to S$ be a morphism of projective varieties with connected fibers (a fiber space). By the *relative Kodaira dimension* $\kappa(W/S)$ of the fiber space W/S we mean the Kodaira dimension of a fiber of general position $\beta^{-1}(s)$, $s \in S$. By a *structure* of a fiber space of the relative Kodaira dimension $\kappa \in \{-\infty, 0, 1, \ldots, M\}$ on the variety V we mean an arbitrary birational map $\chi: V \dashrightarrow W$, where $\beta: W \to S$ is a fiber space of the relative Kodaira dimension κ. It is well known [1], that the generic complete intersection V is birationally superrigid, in particular, on V there are no structures of negative relative Kodaira dimension. The main result of this note is

Theorem 1. *Let $\chi: V \dashrightarrow W$ be a structure of a fiber space of non-maximal relative Kodaira dimension on a sufficiently general complete intersection $V \subset \mathbb{P}$, that is, the inequality*

$$\kappa(W/S) + \dim S < \dim W = M$$

holds. Then $\kappa(W/S)=0$, $S=\mathbb{P}^1$ and there exists a uniquely determined linear subspace $\Lambda\subset\mathbb{P}$ of codimension two such that the following diagram of maps is commutative

$$\begin{array}{ccccc} & V & \stackrel{\chi}{\dashrightarrow} & W & \\ \pi & \downarrow & & \downarrow & \beta \\ & \mathbb{P}^1 & = & S & , \end{array}$$

where $\pi=\pi_\Lambda|_V\colon V\dashrightarrow\mathbb{P}^1$ is the restriction onto V of the linear projection $\pi_\Lambda\colon\mathbb{P}\dashrightarrow\mathbb{P}^1$ from the subspace Λ.

Obviously, the restriction onto V of the projection π_Λ from a subspace of codimension two is a K-trivial structure. Theorem 1 immediately implies a complete description of such structures on varieties with a pencil of Fano complete intersections. Recall the construction of such varieties [2].

Let $\Pi\stackrel{\mu}{\to}\mathbb{P}^1$ be a locally trivial bundle with the fiber $\mathbb{P}$. Consider a smooth subvariety $X\subset\Pi$ of codimension k, such that every fiber of the projection $\mu_X=\mu|_X\colon X\to\mathbb{P}^1$ is a (possibly singular) Fano complete intersection of the type $d_1\cdot\ldots\cdot d_k$ in $\mu^{-1}(t)\cong\mathbb{P}$. The variety X is assumed to be generic in the sense of regularity conditions [2,Theorem 5]; besides, the fiber of general position $\mu_X^{-1}(t)$ is assumed to be generic in the sense of Proposition 1, which is formulated and proved below.

Let $\sigma\colon\mathbb{P}^1\to\mathrm{Gr}(M+k-2,\Pi)$ be an arbitrary map, associating to a point $t\in\mathbb{P}^1$ a linear subspace of codimension two $\sigma(t)\in\mathrm{Gr}(M+k-2,\mu^{-1}(t))$ in the fiber $\mu^{-1}(t)\cong\mathbb{P}$. By the symbol Δ_σ we denote the ruled surface, consisting of the hyperplanes $H\subset\mu^{-1}(t)$, containing the subspace $\sigma(t)$, by the symbol

$$\pi_\sigma\colon\Pi\dashrightarrow\Delta_\sigma$$

the fiber-wise projection, $\pi_\sigma|_{\mu^{-1}(t)}$ is the projection from the subspace $\sigma(t)$ onto $\mathbb{P}^1$. Theorem 1 implies directly

Theorem 2. *Assume that a Fano fiber space $X/\mathbb{P}^1$ satisfies the conditions*
(i) $K_X^2+2H_F\notin\mathrm{Int}\,A^2_+X$,
(ii) $-K_X\notin A^1_{\mathrm{mov}}X$,
where $H_F=(-K_X\cdot F)$ is the class of a hyperplane section of the fiber of the projection μ_X in A^2X. Then for any structure $\chi\colon X\dashrightarrow W$ of a fiber space of the relative Kodaira dimension zero we get $\dim S=2$ and, moreover, there are a section $\sigma\colon\mathbb{P}^1\to\mathrm{Gr}(M+k-2,\Pi)$ and a birational map $\gamma\colon\Delta_\sigma\dashrightarrow S$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & X & \stackrel{\chi}{\dashrightarrow} & W & \\ \pi_\sigma & \downarrow & & \downarrow & \beta \\ & \Delta_\sigma & \stackrel{\gamma}{\dashrightarrow} & S. & \end{array}$$

One obtains Theorem 2 from Theorem 1 in a trivial way, see the proof of Theorem 5 in [2].

2. A summary of known results. It was observed in XIX century that the problems of description of structures of a rationally connected fiber space (or structures of negative relative Kodaira dimension) and of structures of relative Kodaira dimension zero are parallel to each other. If for a given rationally connected variety the first problem admits a complete solution, then the second problem can be solved by the same methods (but not the other way round!). In the modern period of algebraic geometry the first paper describing the structures of relative Kodaira dimension zero was [3]. In [4] a description of K-trivial structures on generic Fano hypersurfaces of index 1 (an analog of Theorem 1 for these varieties) was derived from the results of [5]. See also [6,7]. In [1] a sketch of the proof of the following fact was given.

Theorem 3. *In the assumptions of Theorem 1 every structure* $\chi\colon V \dashrightarrow W$ *of relative Kodaira dimension zero on* V *is a pencil:* $\dim S = 1$.

Theorem 3 is weaker than Theorem 1. In fact, in [1, Sec. 0.4] a proof of the following fact was sketched: in the assumptions of Theorem 1, let Σ be the $(\beta \circ \chi)$-preimage on V of an arbitrary movable linear system on the base S, $\Sigma \subset |-nK_V|$ for some $n \geq 1$. Then there exists an irreducible subvariety of codimension two $B \subset V$ such that

$$\operatorname{mult}_B \Sigma = n, \tag{1}$$

in particular, the self-intersection of this movable linear system $Z = (D_1 \circ D_2)$, $D_i \in \Sigma$, is $Z = n^2 B$ (whence Theorem 3 immediately follows). Since the subvariety B is numerically equivalent to a section of the variety V by a linear subspace of codimension two, in [1, Sec. 0.4] it was conjectured that Σ is composed from some pencil of hyperplane sections of V (this is equivalent to the claim of Theorem 1). In [8] a more detailed proof of Theorem 3 was given, using the cone techniques [5,9]. However, in the argument, given in [8], there was a gap in the proof of the lemma (the only lemma in that paper), generalizing the cone technique. (Namely, to describe the intersection of the base of a cone and a curve lying on the cone, a set of free linear systems on the cone was considered. Taking a generic divisor in each system, it was concluded that the zero-dimensional intersection of all those divisors with the base of the cone consists of distinct points, that is, each point in the intersection is of multiplicity one. However, such a conclusion is not obvious and requires a proof. Usually such claims are deduced from the Bertini theorem, but in the situation under consideration the Bertini theorem can not be used without a special justification. The Bertini theorem claims that the singularities of a generic divisor of a movable linear system are concentrated on its base set. In [8] a set of movable linear systems is considered, however, the divisors in those systems are chosen not independently of each other, more precisely, once a generic divisor in one of them has been chosen, the choice of the remaining divisors is uniquely determined. Therefore, the Bertini theorem does not apply or, in order to apply it, one needs a special argument.) Finally, in [2] the argument of [8] was replaced by another one and the proof of Theorem 3 was completed (see [2, Proposition 3.5]). As an immediate corollary, the following fact was obtained.

Theorem 4. *In the assumptions of Theorem 2 for any structure* $\chi\colon X \dashrightarrow W$ *of relative Kodaira dimension zero we get* $\dim S = 2$ *and the structure* χ *is compatible with the projection* $X \to \mathbb{P}^1$. (See [2, Theorem 5].)

Theorem 4 is a weaker version of Theorem 2. In this note, we make the concluding step in the description of structures of zero Kodaira dimension on Fano complete intersections.

3. The structure of the proof of Theorem 1. The variety V is assumed to be generic in the following sense. Firstly, V satisfies the regularity condition [1] at every point, see [1, Sec. 1.2]. Secondly, the following claim holds.

Proposition 1. *A sufficiently general complete intersection V satisfies the following condition: for any linear subspace $\Lambda \subset \mathbb{P}$ of codimension $k+1$ the intersection $V \cap \Lambda$ is an irreducible reduced variety of dimension $M-k-1$.*

Proof is given in Sec. 5. By the Lefschetz theorem the claim of the proposition holds for $M \geq 2k+3$ for any smooth complete intersection V. Only two cases need to be considered: $M = 2k+2$ and $M = 2k+1$.

We assume that the complete intersection V satisfies the property of Proposition 1.

Let $\chi\colon V \dashrightarrow W$ be a structure of a fiber space of non-maximal relative Kodaira dimension, Σ the strict transform on V with respect to χ of the pull back on W of some movable linear system on the base S. Then $\Sigma \subset |-nK_V|$, $n \geq 1$, the pair $(V, \frac{1}{n}\Sigma)$ is not terminal, the system Σ is composed from a pencil and there is an irreducible subvariety $B \subset V$ of codimension two, such that the inequality (1) holds [1, Proposition 3.5]. Let $b \in B$ be a point of general position, $\Delta = T_b B \subset \mathbb{P}$ the tangent space.

Proposition 2. *Let $\mu_\Delta\colon \mathbb{P} \dashrightarrow \mathbb{P}^{k+1}$ be the linear projection from Δ,*

$$\mu = \mu_\Delta|_V\colon V \dashrightarrow \mathbb{P}^{k+1}$$

its restriction onto V. Then the fibers of the rational map μ are irreducible and reduced, whereas the linear system Σ is the pull back via μ of a movable linear system Γ on $\mathbb{P}^{k+1}$.

Proof is given in Sec. 4.

Proof of Theorem 1. By Proposition 1, the set $V \cap \Delta$ has codimension $k+2$ in V. Considering the dimensions, we conclude that $\mu(B) = \bar{B}$ is an irreducible subvariety of codimension two ($\bar{B}$ can not be a divisor, since in that case the linear system Γ, and therefore also Σ, would have had a fixed component), so that $B = \mu^{-1}(\bar{B})$. Therefore,

$$\operatorname{mult}_{\bar{B}} \Gamma = n,$$

where Γ is a linear system of hypersurfaces of degree n. This is possible in one case only, when $\bar{B} \subset \mathbb{P}^{k+1}$ is a linear subspace of codimension two, and the system Γ is composed from the pencil of hyperplanes, containing $\bar{B}$. But then $B = \Lambda \cap V$, where $\Lambda = \mu_\Delta^{-1}(\bar{B})$ is a linear subspace of codimension two in $\mathbb{P}$, and the system Σ is composed from the pencil of sections of V by hyperplanes containing Λ. Q.E.D. for Theorem 1.

As we noted above, Theorem 2 follows immediately from Theorem 1.

4. Linear projections and cones. Let us prove Proposition 2. Proposition 1 implies that the fibers of μ are irreducible and reduced. To prove the main claim that $\Sigma = \mu^*\Gamma$, let us consider a point of general position $p \in V$. Let $D \in \Sigma$ be the divisor, containing that point. Now we get

Proposition 3. *The following inclusion holds*

$$T_p\mu^{-1}(\mu(p)) \subset T_p D.$$

Proposition 2 follows immediately from Proposition 3: as the point p is generic, we get that $\mu(D) \subset \mathbb{P}^{k+1}$ is a divisor (that is, $\mu(D) \neq \mathbb{P}^{k+1}$), so that $\Sigma = \mu^*\Gamma$ for some linear system Γ, which is what we need. Q.E.D.

Proof of Proposition 3. Set $P = \langle \Delta, p \rangle$ to be the fiber of the linear projection μ_Δ, so that $\mu^{-1}(\mu(p)) = P \cap V$. Since the point p is generic and the fiber $\mu^{-1}(\mu(p))$ is non-singular at this point, we get

$$\operatorname{codim}_{\mathbb{P}}(P \cap T_p V) = \operatorname{codim}_{\mathbb{P}} P + k,$$

that is, the linear subspaces P and $T_p V \subset \mathbb{P}$ are in general position.

Assume now that the point of general position $b \in B$ was chosen in the following way: on the variety B we considered an arbitrary family of irreducible k-dimensional subvarieties $\{Y_u, u \in U\}$, sweeping out B, in that family we chose a variety $Y = Y_u$ of general position, and the point b is a point of general position on Y. In particular, $T_b Y$ is a generic linear subspace of dimension k in $\Delta = T_b B \subset \mathbb{P}$. In particular,

$$\dim(\langle T_b Y, p \rangle \cap T_p V) = 1, \tag{2}$$

that is, the linear subspaces $\langle T_b Y, p \rangle$ $T_p V$ are in general position.

By the symbol $[b, p]$ we denote the line in $\mathbb{P}$, joining these two points, by the symbol (b, p) the set $[b, p] \setminus \{b, p\}$. Take a point $x \in (b, p)$. Set $C(Y, x)$ to be the cone with the vertex x and the base Y.

Proposition 4. *For sufficiently general Y, b, p, x the following claims are true:*
(i) *the point $z \in C(Y, x)$ is a singularity of that cone, if either $z = x$, or $z \in [y, x]$, where $y \in \operatorname{Sing} Y$,*
(ii) *the closed algebraic set $R(Y, x)$, which is the union of all irreducible components of the intersection $C(Y, x) \cap V$, containing the point p, is an irreducible curve, non-singular at the point p,*
(iii) *the curve $R(Y, x)$ intersects the subvariety Y outside the closed subset $\operatorname{Sing} Y$ of singular points of this variety.*

Proof of the claim (i) is given in [2, Sec. 3.3.1] (in addition to the arguments, given in [2], one needs to note that V can not be contained in the variety of secant lines $\operatorname{Sec} Y$ of the variety Y: even if $M = 2k + 1$, the variety V is not covered by lines). Furthermore, $[b, p] \cap Y = \{b\}$, so that the point $p \in C(Y, x)$ is non-singular. Obviously,

$$T_p C(Y, x) = \langle T_b Y, p \rangle,$$

whence, taking into account (2), it follows that the varieties $C(Y, x)$ and V intersect transversally at the point p, which proves (ii). Finally, the claim (iii) is proved by the arguments of [2, Sec. 3.3.1], taking into account that when Y, $b \in Y$ and $p \in V$ vary, the points $x \in (b, p)$ fill out an open subset of the projective space $\mathbb{P}$ (for instance, by the surjectivity of the map μ). Q.E.D. for Proposition 4.

Now we argue as in [2, Sec. 3.3]: the curve $R(Y,x)$ meets Y at the points, which are non-singular both on Y and on the cone $C(Y,x)$. Therefore, we get that the intersection number $(R(Y,x)\cdot Y)_{C(Y,x)}$ is well defined.

Lemma 1. *The following equality holds:* $(R(Y,x)\cdot Y)_{C(Y,x)} = \deg R(Y,x)$.

Proof is given in [9, §1].

Now let us come back to the divisor $D\in\Sigma$, containing the point p.

Lemma 2. *The following inclusion holds:* $R(Y,x)\subset D$.

Proof. Obviously, $(D\cdot R(Y,x)) = n\deg R(Y,x)$. On the other hand, as it was shown in [2, Sec. 3.3.2],

$$\sum_{y\in D\cap R(Y,x)\cap Y}(D\cdot R(Y,x))_y \geq (R(Y,x)\cdot Y)_{C(Y,x)}\,\mathrm{mult}_Y D.$$

Taking into account that $\mathrm{mult}_Y D = \mathrm{mult}_B D = n$ and the equality of Lemma 1, we obtain the claim of Lemma 2, since

$$p\in D\cap R(Y,x)$$

and $p\notin Y$. Q.E.D. for the lemma.

Lemma 2 implies the inclusion

$$\langle T_bY, p\rangle\cap T_pV\subset T_pD,$$

whence, since the subspace $T_bY\subset\Delta$ is generic, we get the inclusion

$$\langle\Delta, p\rangle\cap T_pV\subset T_pD,$$

which completes the proof of Proposition 3.

5. Complete intersections of general position. Let us prove Proposition 1. As we noted in Sec. 3, we have to consider the two cases: when $M=2k+2$ and $M=2k+1$. Consider first the following general problem. Let $X\subset\mathbb{P}^N$ be an irreducible subvariety of dimension $l\geq 2$. By the symbol $\mathcal{P}_d=\mathcal{P}_{d,N}$ we denote the space of homogeneous polynomials of degree d on $\mathbb{P}^N$. Let $U_d(X)\subset\mathcal{P}_d$ be the open set, consisting of such polynomials $f\in\mathcal{P}_d$, that $\{f|_X=0\}$ is an irreducible reduced subvariety of dimension $(l-1)$. Respectively, let $R_d(X)=\mathcal{P}_d\setminus U_d(X)$ be the set of "incorrect" polynomials. The problem is to estimate from below the codimension of the closed set $R_d(X)$ in the space $\mathcal{P}_d$.

Lemma 3. *The following estimate holds:*

$$\mathrm{codim}(R_d(X)\subset\mathcal{P}_d)\geq\binom{d+l-2}{d}-l+1.$$

Proof. Let $\gamma\colon\mathbb{P}^N \dashrightarrow \mathbb{P}^{l-1}$ be the linear projection from a $(N-l)$-plane of general position, $\gamma_X=\gamma|_X\colon X\dashrightarrow\mathbb{P}^{l-1}$ its restriction onto X (the set of points where γ_X is not defined is zero-dimensional). Obviously, the map γ_X is surjective, all its fibers are one-dimensional and the fiber $\gamma_X^{-1}(z)$ over a point of general position $z\in\mathbb{P}^{l-1}$ is an irreducible curve. The set $\Delta\subset\mathbb{P}^{l-1}$, consisting of such points z, that the fiber $\gamma_X^{-1}(z)$ is reducible or non-reduced, is a proper closed subset of $\mathbb{P}^{l-1}$ (at most a divisor). Therefore, for any irreducible divisor D on $\mathbb{P}^{l-1}$, such that $D \not\subset \Delta$, its inverse image

$$\gamma_X^{-1}(D) = \gamma^{-1}(D) \cap X$$

is irreducible and reduced. In other words, for any irreducible polynomial f on $\mathbb{P}^{l-1}$, such that $\{f = 0\} \not\subset \Delta$, we get $\gamma^* f \in U_d(X)$. The set $\gamma^* \mathcal{P}_{d,l-1}$ is a linear subspace of the space $\mathcal{P}_d$ (the same polynomials considered as polynomials in a larger number of variables). Let $R_{d,l-1} \subset \mathcal{P}_{d,l-1}$ be the closed subset of reducible polynomials. From what was said, it follows that

$$\operatorname{codim}(R_d(X) \subset \mathcal{P}_d) \geq \operatorname{codim}(R_{d,l-1} \subset \mathcal{P}_{d,l-1}).$$

The last codimension is easy to compute: the irreducible component of maximal dimension of the set $R_{d,l-1}$ consists of the polynomials of the form $f = f^\sharp h$, where $h \in \mathcal{P}_{1,l-1}$ is a linear form. Q.E.D. for Lemma 3.

Let us come back to the proof of Proposition 1. Assume that $M = 2k + 2$. Assume also that the complete intersection V is generic in the following sense: the variety

$$V^\sharp = \{f_1 = \ldots = f_{k-1} = 0\} \subset \mathbb{P} \tag{3}$$

is smooth. Obviously, $\dim V^\sharp = 2k + 3$, so that by the Lefschetz theorem the intersection $V^\sharp \cap \Lambda$ is irreducible, reduced and has dimension $k + 2$ for any linear subspace $\Lambda \subset \mathbb{P}$ of codimension $k + 1$. Fix such a subspace. Obviously,

$$V \cap \Lambda = \{f_k|_{V^\sharp \cap \Lambda} = 0\}.$$

Let $R_\Lambda \subset \mathcal{P}_{d_k, M+k}$ be the closed subset of polynomials f_k of degree d_k, for which $V \cap \Lambda$ is not an irreducible reduced subvariety of dimension $k + 1$. By Lemma 3,

$$\operatorname{codim} R_\Lambda \geq \binom{d_k + k}{d_k} - k - 1. \tag{4}$$

The equality $d_1 + \ldots + d_k = M + k$ implies that $d_k \geq 4$. Elementary computations show that the right hand part of the inequality (4) is strictly higher than the dimension of the projective Grassmanian of $(2k + 1)$-planes in $\mathbb{P}$ for $k \geq 4$. This proves Proposition 1 for $M = 2k + 2$.

Assume now that $M = 2k+1$. In this case the arguments are completely similar to those given above, but we need two steps. First, we consider the complete intersection

$$V^+ = \{f_1 = \ldots = f_{k-2} = 0\} \subset \mathbb{P},$$

which is assumed to be smooth. By the Lefschetz theorem the intersection $V^+ \cap \Lambda$ is irreducible and reduced. Now the arguments similar to those given above for $M = 2k + 2$, show that for a generic polynomial f_{k-1} of degree $d_{k-1} \geq 3$ the closed set

$$\{f_{k-1}|_{V^+ \cap \Lambda} = 0\}$$

is irreducible and reduced for any subspace Λ. Now we consider the variety $V^\sharp$, defined by the formula (3), and argue as in the case $M=2k+2$ and complete the proof. We omit the details of elementary computations. Q.E.D. for Proposition 1.

Remark 1. The additional (compared to [1]) restrictions for the parameters k, d_1,... d_k are needed precisely for the reason that for the excluded values the proof of Proposition 1 does not work. However, there are no doubts that both the claim of Proposition 1 and, the more so, Theorem 1 are true for those values as well. Here is the list of excluded families:

$$2\cdot 5 \quad \text{and} \quad 3\cdot 4 \quad \text{in} \quad \mathbb{P}^7,$$
$$2\cdot 6, \quad 3\cdot 5 \quad \text{and} \quad 4\cdot 4 \quad \text{in} \quad \mathbb{P}^8,$$
$$2\cdot 2\cdot 6, \quad 2\cdot 3\cdot 5, \quad 2\cdot 4\cdot 4 \quad \text{and} \quad 3\cdot 3\cdot 4 \quad \text{in} \quad \mathbb{P}^{10},$$
$$2\cdot 2\cdot 7, \quad 2\cdot 3\cdot 6, \quad 2\cdot 4\cdot 5, \quad 3\cdot 3\cdot 5 \quad \text{and} \quad 3\cdot 4\cdot 4 \quad \text{in} \quad \mathbb{P}^{11},$$
$$2\cdot 2\cdot 3\cdot 6, \quad 2\cdot 2\cdot 4\cdot 5, \quad 2\cdot 3\cdot 3\cdot 5 \quad \text{and} \quad 2\cdot 3\cdot 4\cdot 4 \quad \text{in} \quad \mathbb{P}^{13},$$

and the infinite series $2\cdot\ldots\cdot 2\cdot(k+3)$ in $\mathbb{P}^{3k+1}$, $k\geq 2$.

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