arXiv:1109.4084v1 [math.AT] 19 Sep 2011

CENTERS AND HOMOTOPY CENTERS IN ENRICHED MONOIDAL CATEGORIES.

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Abstract. We consider a theory of centers and homotopy centers of monoids in monoidal categories which themselves are enriched in duoidal categories. The duoidal categories (introduced by Aguillar and Mahajan under the name 2-monoidal categories) are categories with two monoidal structures which are related by some, not necessary invertible, coherence morphisms. Centers of monoids in this sense include many examples which are not 'classical.' In particular, the 2-category of categories is an example of a center in our sense. Examples of homotopy center (analogue of the classical Hochschild complex) include the **Gray**-category **Gray** of 2-categories, 2-functors and pseudonatural transformations and Tamarkin's homotopy 2-category of dg-categories, dg-functors and coherent dg-transformations.

Contents

1991 *Mathematics Subject Classification.* Primary 18D10,18D20, 18D50, secondary 55U40, 55P48.
Key words and phrases. Monoidal categories, center, Hochschild complex, operads, Deligne's conjecture.
The first author acknowledges the financial support of Scott Russel Johnson Memorial Foundation, Max Plank Institut für Mathematik and Australian Research Council (grant No. DP1095346). The second author was supported by the grant GA ČR 201/08/0397 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

1. Introduction

This paper grew up from our attempts to comprehend a construction by D. Tamarkin in [36] which answers a question: what do dg-categories, dg-functors, and coherent up to all higher homotopies dg-transformations, form? In the process we discovered that the most natural language which allows easy development of such a construction is a generalization to the enriched categorical context of the classical Hochschild complex theory for algebras. Enrichment, however, should be understood in a more general sense. In this paper we therefore want to set up some basic definitions and constructions of the proposed theory of enrichment and the corresponding theory of centers and the Hochschild complexes. In the sequel [9] we will consider homotopical aspects of the theory of the Hochschild complexes. The higher dimensional generalization of our theory will be addressed in yet another paper.

Classically, the Hochschild complex of an associative algebra can be understood as its derived or homotopical center. Our theory generalizes this point of view by extending the notions of center and homotopy center to a much larger class of monoids. Of course, the classical center construction and the Hochschild complex are special cases of the center in our sense. But, perhaps, the most striking feature of our theory is that the 2-category of categories is an example of our center construction as well. An example of a homotopy center is then the symmetric monoidal closed category **Gray** of 2-categories, 2-functors and pseudonatural transformations [22]. Tamarkin's homotopy 2-category of dg-categories, dg-functors and their coherent natural transformations is also an example of the homotopy center. In some philosophical sense, we have here a new understanding of the center as a universal method for (higher dimensional) enrichment. Other nontrivial examples of duoidal categories and centers are presented in the lecture notes of Ross Street concerning invariants of 3-dimensional manifolds [34].

Let us now provide more detail about where we enrich. Classically, we can enrich over any monoidal category $\mathscr{D}$. However, monoidal $\mathscr{D}$-enriched categories make sense only if $\mathscr{D}$ has some degree of commutativity, more precisely, we need $\mathscr{D}$ to be braided. It was observed by Forcey in [20] that we can slightly weaken this requirement. It is enough for $\mathscr{D}$ to be 2-fold monoidal in his sense. Even Forcey's conditions can be weaken. It is enough for $\mathscr{D}$ to be 2-monoidal in Aguillar-Mahajan sense [1]. In our paper we call such a $\mathscr{D}$ a *duoidal category*.[1] In a duoidal category we have two tensor products with the corresponding unit objects making $\mathscr{D}$ a monoidal category in two different ways. In addition, we require that these two tensor products are related by a not necessary invertible middle interchange law and that the unit objects also satisfy some interesting coherence relations.

[1] This terminology was proposed by Ross Street and we found it very convenient. The terminology of [1] suffers from the existence of a similarly sounding terminology of Balteanu, Fiedorowicz, Schwänzl, and Vogt [3], and Forcey [20].

Let $\mathscr{D}$ be such a duoidal category. To incorporate the theory of the Hochschild complex, we also assume that $\mathscr{D}$ itself is enriched over a base closed symmetric monoidal category V. In this situation one can consider a monoidal category $\mathcal{K}$ enriched in $\mathscr{D}$ (and the underlying category of $\mathcal{K}$ is a monoidal V-category). Then we could define a monoid in $\mathcal{K}$ in a usual way as an object equipped with a unit and an associative multiplication. But, unexpectedly, such a monoid notion is in general not correct – it amounts to a monoid in the underlying monoidal category. We introduce a new notion of a monoid in $\mathcal{K}$ by adding more unitary operations which makes all theory nontrivial (all these operations coincide if $\mathscr{D}$ is a braided monoidal category, so their multitude is not visible in the classical theory). Then for a monoid $\mathtt{M}$ in $\mathcal{K}$ we define a cosimplicial object in $\mathscr{D}$ which is an analogue of the classical Hochschild cosimplicial complex of an algebra. If we fix a cosimplicial object δ in V, we can take a kind of geometric realization of the cosimplicial Hochschild complex. This is an object $CH_\delta(\mathtt{M},\mathtt{M})$ from $\mathscr{D}$ and this is our definition of the δ-center of a monoid. If δ is the constant object $\delta_n = I$ (the unit of V), then the δ-center is called the center. Notice that the δ-center of a monoid lives in $\mathscr{D}$, not in $\mathcal{K}$. When $V, \mathscr{D}$ and $\mathcal{K}$ have compatible model structures, we also define a homotopy center $CH(\mathtt{M},\mathtt{M})$ of $\mathtt{M}$ by using an appropriate cofibrant and contractible δ and a fibrant replacement of $\mathtt{M}$.

In the classical theory we know that the center of a monoid is a commutative monoid. We have an analogue of this statement in our settings: the center of a monoid in $\mathcal{K}$ is a duoid (double monoid in the terminology of [1]) in $\mathscr{D}$. The homotopy version of this statement is the following: there is a canonical action of a contractible 2-operad on the homotopy center of a monoid in $\mathcal{K}$. This is an analogue of Deligne's conjecture for the classical Hochschild complex. Tamarkin's main result from [36] is a special case of this Deligne's conjecture applied to a particular monoid in a monoidal category $\mathfrak{J}(O, \mathfrak{Chain})$ constructed in Section 10. The classical Deligne's conjecture follows from this statement by a theorem from [5] if $\mathscr{D}$ is a symmetric monoidal category. In this paper we set up a version of the theory of 2-operads which allows a precise formulation of such a statement. A proof of this form of Deligne's conjecture will be given in [9].

Finally, let us say a few words about possible further directions. One interesting and almost obvious possibility is to replace duoidal categories by n-oidal categories. An n-oidal category is a category with n monoidal structures related by interchange morphisms and various coherence morphisms between unit objects which satisfy some coherence relations [1]. Many results of our paper admit more or less obvious generalization to the n-oidal case. In particular, we can consider n-oids in n-oidal categories and centers and homotopy centers of n-oids.

Conjecture ($(n{+}1)$-oidal Deligne's conjecture)**.** *There is a canonical action of a contractible $(n+1)$-operad on the homotopy center of an n-oid $\mathtt{N}$ which lifts the $(n+1)$-oid structure on the center of $\mathtt{N}$.*

Analogously to Tamarkin's theorem, this conjecture answers a question: what do n-categories enriched in a symmetric monoidal model category V form? This conjecture should imply also the n-dimensional form of the classical Deligne conjecture [28] via the results of [6, 5]. We hope to address the proof of these conjectures in the near future.

Another very interesting direction is a construction of the so-called semistrict n-categories. In the theory of higher dimensional categories it is highly desirable to have some sort of a minimal model of the theory of weak n-categories. Many important statements in higher category theory, like the equivalences amongst almost all definitions of weak n-categories, or the Grothendieck hypothesis on algebraic models of n-homotopy types [16], will follow naturally once we have at hands a well developed theory of semistrict n-categories. So far, however, a good notion of semistrict n-category is known only for $n \leq 3$. For $n = 2$, it is the category of strict 2-categories. For $n = 3$ it is the category of **Gray**-categories [22]. Both these categories are examples of enrichment over a closed symmetric monoidal category which comes from our homotopical center construction (see Examples 81 and 82). Combining the results of [10] with the approach of our paper, we hope to be able to construct an analogue of the Gray tensor product for all dimensions and therefore a good theory of semistrict n-categories. This is currently a work in progress with M. Weber and D.-C. Cisinski [11].

Acknowledgment. We would like to express our thanks to C. Berger, D.-C. Cisinski, S. Lack, R. Street, D. Tamarkin and M. Weber for many enlightening discussions.

2. Monoidal V-categories and duoidal V-categories

We fix from the beginning a complete and cocomplete closed symmetric monoidal category $(V, \otimes, I)$. Its underlying category is denoted $\mathcal{U}V$. For objects X, Y of an V-enriched category $\mathcal{A}$, we denote by $\mathcal{A}(X,Y) \in V$ the enriched hom and by $\mathcal{U}\mathcal{A}(X,Y) := \mathcal{U}V\big(I, \mathcal{A}(X,Y)\big)$ the set of homomorphism in the underlying category.

2.1. **Monoidal V-categories.** It is classical [19] that V-categories, V-functors and V-natural transformations form a 2-category $\mathcal{C}at(V)$. Moreover, this 2-category is a symmetric monoidal 2-category with respect to the tensor product $\times_V$ of V-categories:

$$Ob(K \times_V L) := Ob(K) \times Ob(L),$$
$$(K \times_V L)\big((X,Y),(Z,W)\big) := K(X,Z) \otimes_V L(Y,W).$$

The unit for this tensor product is the category 1 which has one object $*$ and $1(*,*) = I$.

When $V = \mathcal{S}et$ we will use the notation $\mathcal{C}at$ for $\mathcal{C}at(V)$. The underlying category functor provides then a symmetric lax-monoidal 2-functor

$$\mathcal{U} : \mathcal{C}at(V) \to \mathcal{C}at.$$

Recall that, for any monoidal 2-category or, more generally, for a monoidal bicategory there exists a concept of a pseudomonoid, i.e. of an object equipped with a coherently associative multiplication and a coherent unit which generalizes the notion of a monoidal category [29].

Definition 1. A *monoidal V-category* is a pseudomonoid in $\mathcal{C}at(V)$.

So, the definition is the usual definition of a monoidal category, but we require the tensor product to be a V-functor and the coherence constraint to be V-natural.

Definition 2. A lax-monoidal V-functor between monoidal V-categories $K = (K, \square_K, e_K)$ and $L = (L, \square_L, e_L)$ consists of

(i) a V-functor $F : K \to L$,
(ii) a V-natural transformation

$$\phi : F(X) \square_L F(Y) \to F(X \square_K Y)$$

and a morphism

$$\phi_e : e_L \to F(e_K)$$

which satisfy the usual coherence conditions.

A lax-monoidal functor is called *strong monoidal* if ϕ and ϕ_e are isomorphisms and it is called *strict monoidal* if they are identities.

Monoidal V-categories, lax-monoidal V-functors and their monoidal V-transformations form a 2-category $1\mathcal{C}at_{lax}(V)$. It is a monoidal 2-category with respect to the tensor product $\times_V$. Analogously, we have monoidal 2-subcategories

$$1\mathcal{C}at_{strict}(V) \subset 1\mathcal{C}at(V) \subset 1\mathcal{C}at_{lax}(V)$$

of strict monoidal and strong monoidal functors.

2.2. Duoidal V-categories.

Definition 3. A *duoidal V-category* is a pseudomonoid in $1\mathcal{C}at_{lax}(V)$. Explicitly, a duoidal V-category is a quintuple $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ such that

(i) $(\mathscr{D}, \square_0, e)$ and $(\mathscr{D}, \square_1, v)$ are monoidal V-categories, equipped with
(ii) a V-natural interchange transformation

$$(X \square_1 Y) \square_0 (Z \square_1 W) \to (X \square_0 Z) \square_1 (Y \square_0 W),$$

(iii) a map

$$e \to e \square_1 e,$$

(iv) a map

$$v \square_0 v \to v,$$

(v) and a map

$$e \to v.$$

The above data enjoy the coherence properties listed in [1, Definition 2.1], namely the *associativity* meaning that the diagrams

$$\begin{array}{ccc}
\big((A \square_1 B) \square_0 (C \square_1 D)\big) \square_0 (E \square_1 F) & \longrightarrow & (A \square_1 B) \square_0 \big((C \square_1 D) \square_0 (E \square_1 F)\big) \\
\downarrow & & \downarrow \\
\big((A \square_0 C) \square_1 (B \square_0 D)\big) \square_0 (E \square_1 F) & & (A \square_1 B) \square_0 \big((C \square_0 E) \square_1 (D \square_0 F)\big) \\
\downarrow & & \downarrow \\
\big((A \square_0 C) \square_0 E\big) \square_1 \big((B \square_0 D) \square_0 F\big) & \longrightarrow & \big(A \square_0 (C \square_0 E)\big) \square_1 \big(B \square_0 (D \square_0 F)\big)
\end{array}$$

$$\begin{array}{ccc}
\big((A \square_1 B) \square_1 C\big) \square_0 \big((D \square_1 E) \square_1 F\big) & \longrightarrow & \big(A \square_1 (B \square_1 C)\big) \square_0 \big(D \square_1 (E \square_1 F)\big) \\
\downarrow & & \downarrow \\
\big((A \square_1 B) \square_0 (D \square_1 E)\big) \square_1 (C \square_0 F) & & (A \square_0 D) \square_1 \big((B \square_1 C) \square_0 (E \square_1 F)\big) \\
\downarrow & & \downarrow \\
\big((A \square_0 D) \square_1 (B \square_0 E)\big) \square_1 (C \square_0 F) & \longrightarrow & (A \square_0 D) \square_1 \big((B \square_0 E) \square_1 (C \square_0 F)\big)
\end{array}$$

commute, and the *unitality* meaning the commutativity of

$$\begin{array}{ccc}
e \square_0 (A \square_1 B) & \longrightarrow & (e \square_1 e) \square_0 (A \square_1 B) \\
\uparrow & & \downarrow \\
A \square_1 B & \longrightarrow & (e \square_0 A) \square_1 (e \square_0 B)
\end{array}
\qquad
\begin{array}{ccc}
(A \square_1 B) \square_0 e & \longrightarrow & (A \square_1 B) \square_0 (e \square_1 e) \\
\uparrow & & \downarrow \\
A \square_1 B & \longrightarrow & (A \square_0 e) \square_1 (B \square_0 e)
\end{array}$$

$$\begin{array}{ccc}
v \square_1 (A \square_0 B) & \longleftarrow & (v \square_0 v) \square_1 (A \square_0 B) \\
\uparrow & & \uparrow \\
A \square_0 B & \longrightarrow & (v \square_1 A) \square_0 (v \square_1 B)
\end{array}
\qquad
\begin{array}{ccc}
(A \square_0 B) \square_1 v & \longleftarrow & (A \square_0 B) \square_1 (v \square_0 v) \\
\uparrow & & \uparrow \\
A \square_0 B & \longrightarrow & (A \square_1 v) \square_0 (B \square_1 v).
\end{array}$$

In the above diagrams, $A, \ldots, F$ are objects of $\mathscr{D}$ and the arrows are induced by the structure operations of $\mathscr{D}$ in an obvious way. Moreover, we require the units e, v to be *compatible* in the sense that v is a monoid in $(\mathscr{D}, \square_0, e)$ and e a comonoid in $(\mathscr{D}, \square_1, v)$.

Remark 4. Observe that (v) is redundant as the interchange map (ii) with $A = D = e$ and $B = C = v$ gives exactly (v).

Definition 5. A duoidal category $\mathscr{D}$ is called *strict* if both monoidal categories $(\mathscr{D}, \square_0, e)$ and $(\mathscr{D}, \square_1, v)$ are strict monoidal categories.

Example 6. Pseudomonoids in $1\mathcal{C}at(V)$ are the same as braided monoidal V-categories [24]. Any braided monoidal V-category can be considered as a duoidal V-category in which two tensor products and two units coincide.

Example 7. The iterated 2-monoidal categories of Balteanu-Fiedorowicz-Schwänzl-Vogt [3] are strict duoidal categories for which $e = v$.

Example 8. Forcey's 2-fold monoidal categories are duoidal categories for which (iii) and (iv) are isomorphisms.

Example 9. If $\mathscr{D}$ is a duoidal V-category, then its underlying category $\mathcal{U}(\mathscr{D})$ is a duoidal $\mathcal{S}et$-category which we simply call a duoidal category. The duoidal categories are exactly the 2-monoidal categories in the original sense of [1].

Definition 10. A *lax-duoidal V-functor* between duoidal V-categories $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ and $\mathscr{D}' = (\mathscr{D}', \square'_0, \square'_1, e', v')$ consists of

(i) a V-functor $F : \mathscr{D} \to \mathscr{D}'$,

(ii) V-natural transformation

$$\phi : F(A) \,\square'_0 F(B) \to F(A \,\square_0 B)$$

and a morphism

$$\phi_e : e' \to F(e)$$

which makes F a lax-monoidal functor from $(\mathscr{D}, \square_0, e)$ to $(\mathscr{D}', \square'_0, e')$,

(iii) a V-natural transformation

$$\gamma : F(A) \,\square'_1 F(B) \to F(A \,\square_1 B)$$

and a morphism

$$\gamma_v : v' \to F(v)$$

which makes F a lax-monoidal functor from $(\mathscr{D}, \square_1, v)$ to $(\mathscr{D}', \square'_1, v')$

and which enjoy coherence properties from [1, Definition 6.44]. Namely, we require the commutativity of the diagrams

$$\begin{array}{ccc}
\big((F(A) \,\square'_1 F(B)\big) \,\square'_0 \big((F(C) \,\square'_1 F(D)\big) & \longrightarrow & F(A \,\square_1 B) \,\square'_0 F(C \,\square_1 D) \\
\downarrow & & \downarrow \\
\big((F(A) \,\square'_0 F(C)\big) \,\square'_1 \big((F(B) \,\square'_0 F(D)\big) & & F\big((A \,\square_1 B) \,\square_0 (C \,\square_1 D)\big) \\
\downarrow & & \downarrow \\
(F(A \,\square_0 C) \,\square'_1 (F(B \,\square_0 D) & \longrightarrow & F\big((A \,\square_0 C) \,\square_1 (B \,\square_0 D)\big)
\end{array}$$

$$\begin{array}{ccccc}
e' & \to & F(e) \to F(e \,\square_1 e) \\
\downarrow & & \uparrow \\
e' \,\square'_1 e' & \longrightarrow & F(e) \,\square'_1 F(e)
\end{array}
\qquad
\begin{array}{ccc}
v' \to F(v) & \leftarrow & F(v \,\square_0 v) \\
\uparrow & & \uparrow \\
v' \,\square'_0 v' & \longrightarrow & F(v) \,\square'_0 F(v)
\end{array}
\qquad
\begin{array}{ccc}
F(e) & \longrightarrow & F(v) \\
\uparrow & & \uparrow \\
e' & \longrightarrow & v',
\end{array}$$

where A, B, C, D are objects of $\mathscr{D}$ and the meaning of the arrows is clear.

We call a lax-duoidal functor *strong* if $\phi, \phi_e, \gamma, \gamma_v$ are isomorphisms. A strong duoidal functor is *strict* if these isomorphisms are identities.

Definition 11 ([1], Definition 6.46)**.** A duoidal transformation $\phi : F \to G$ between two lax-duoidal functors is a natural transformation between F and G as functors, which is a monoidal transformation with respect to two lax-monoidal structures on F and G.

Duoidal categories, lax-duoidal (strong, strict) duoidal functors and their duoidal transformations form a 2-category $2\mathcal{C}at_{lax}(V)$ $(2\mathcal{C}at(V), 2\mathcal{C}at_{strict}(V))$.

2.3. **Duoids in duoidal V-categories.** The following definition coincides with the definition of a double monoid given in [1].

Definition 12. A *duoid* in a duoidal V-category $\mathscr{D}$ is a lax-duoidal V-functor

$$\mathtt{D} : 1 \to \mathscr{D}.$$

A *morphism* between duoids is a duoidal transformation between corresponding duoidal lax-functors.

It is easy to see that a duoid $\mathtt{D}$ is given by an object $\mathtt{D} \in \mathscr{D}$ together with

(i) a structure of a monoid

$$\mathtt{D} \,\square_0 \mathtt{D} \to \mathtt{D}\ ,\ \ e \to \mathtt{D}$$

with respect to the first monoidal structure, and

(ii) a structure of a monoid

$$\mathtt{D} \,\square_1 \mathtt{D} \to \mathtt{D}\ ,\ \ v \to \mathtt{D}$$

with respect to the second monoidal structure.

This data should satisfy the following conditions:

($\star$) The map $v \to \mathtt{D}$ is a monoid morphism with respect to the first structure and

($\star\star$) the diagram

$$\begin{array}{ccccc}
(\mathtt{D}\,\square_1\mathtt{D})\,\square_0(\mathtt{D}\,\square_1\mathtt{D}) & \longrightarrow & \mathtt{D}\,\square_0\mathtt{D} & & \\
 & & & \searrow & \\
\big\downarrow & & & & \mathtt{D} \\
 & & & \nearrow & \\
(\mathtt{D}\,\square_0\mathtt{D})\,\square_1(\mathtt{D}\,\square_0\mathtt{D}) & \longrightarrow & \mathtt{D}\,\square_1\mathtt{D} & &
\end{array}$$

commutes.

Example 13. If $\mathscr{D}$ is a braided monoidal category then a duoid in $\mathscr{D}$ is the same as a commutative monoid.

Example 14. The second unit $v \in \mathscr{D}$ is a duoid in $\mathscr{D}$ with the first monoid structure given by the canonical morphism $v \,\square_0 v \to v$ and with the second monoid structure given by the canonical isomorphism $v \,\square_1 v \to v$.

Example 15. Duoids in the duoidal category $\mathcal{S}p_2(\mathscr{C}, \mathcal{K})$ (see Section 6) are $\mathcal{K}$-enriched 2-categories whose 1-truncation is the category $\mathscr{C}$.

Any lax-duoidal V-functor $F : \mathscr{D} \to \mathscr{D}'$ maps duoids in $\mathscr{D}$ to duoids in $\mathscr{D}'$. If $\mathtt{D}$ is a duoid in a duoidal V-category $\mathscr{D}$ then $\mathtt{D}$ is also a duoid in the underlying duoidal category $\mathcal{U}(\mathscr{D})$. We will use the notation $u(\mathtt{D})$ for this duoid and will call it the *underlying duoid* of $\mathtt{D}$.

2.4. **Coherence for duoidal V-categories.** Duoidal V-categories are algebras of a 2-monad on the 2-category $\mathcal{C}at(V)$. The forgetful 2-functor $2\mathcal{C}at(V) \to \mathcal{C}at(V)$ reflects equivalences by [15], hence, the following definition makes sense:

Definition 16. A strong duoidal V-functor F is a *duoidal equivalence* if it is a V-equivalence of the underlying V-categories. Two duoidal V-categories are called *duoidal equivalent* if there is a duoidal equivalence between them.

Theorem 17. *Every duoidal V-category is duoidal equivalent to a strict duoidal V-category.*

Before we prove the theorem, we introduce the following auxiliary terminology. Let us call a *double monoidal V-category* a category $\mathscr{C}$ equipped with two monoidal structures $(\mathscr{C}, \square_0, e)$ and $(\mathscr{C}, \square_1, v)$ without any relations between two structures. We call a double monoidal V-category *strict* if both monoidal structures are strict. Likewise, we have a notion of double monoidal strong functor and equivalence between double monoidal V-categories.

Lemma 18. *Any double monoidal V-category is equivalent to a strict double monoidal V-category.*

Proof. Any strict monoidal V-category is an algebra of the nonsymmetric operad $M = \{M(n)\}_{n\geq 0}$ in $\mathcal{C}at(V)$ such that $M(n)$ is, for each n, the terminal category. A monoidal V-category is an algebra of another operad M_c in $\mathcal{C}at(V)$. There is an operadic map $\pi : M_c \to M$ recalled below which is an adjoint V-equivalence in each arity (operadic weak equivalence). One can prove that M_c is a cofibrant resolution of M in the category of nonsymmetric operads in $\mathcal{C}at(V)$ equipped with a model structure developed in [33] and [30]. Here we consider the model structure on V for which weak equivalences are isomorphisms. These data allow to prove the coherence result for monoidal V-categories (using bar-construction, for example).

More generally, one can prove by the same method that given a weak equivalence $\xi : A \to B$ of $\mathcal{C}at(V)$-operads, every A-algebra is equivalent to an algebra of the form $\xi^*(X)$, where ξ^* is the restriction functor induced by ξ.

Remark 19. One can prove that the adjunction between categories of algebras induced by ξ is in fact a Quillen equivalence.

Observe now that a double monoidal V-category is an algebra of the operad $M_c \coprod M_c$ and a strict double monoidal V-category is an algebra of $M \coprod M$. Therefore, the lemma will be proved if we establish that coproduct $\pi \coprod \pi$ is a weak equivalence of operads. This follows from the following explicit description of this coproduct.

A bicolored binary planar tree is a planar tree whose vertices have valencies three or one and have two colors white and black. A planar tree l without vertices (and, therefore, without coloring) is also considered as a binary bicolored tree. Let $BTree$ be the set of isomorphism classes of bicolored binary trees with n-leaves. The sequence $BTree := \{BTree(n)\}_{n\geq 0}$ is an operad in Set – the free operad on two 0-operations and two binary operations. A subtree S of a bicolored binary tree T is called *monocolored* if all its vertices have the same colors. Any monocolored subtree belongs to a unique maximal monocolored subtree.

An alternating bicolored planar tree is a planar tree whose vertices have valencies one or greater or equal than three and have two colors – white and black. It also must satisfy the following condition: there is no edge connecting two vertices of the same color. The tree l is also considered as an alternating bicolored tree. Leaves and roots of an alternating bicolored tree inherit the color by the following rule: a leaf (a root) has white (black) color if the unique vertex to which the leaf (the root) is attached has white (black) color.

Let $ATree(n)$ be the set of isomorphism classes of alternating bicolored trees with n-leaves. The sequence $ATree := \{ATree(n)\}_{n\geq 0}$ forms an operad. The operadic multiplication is given by grafting if we graft a tree to a leaf of another tree and the color of this leaf is different from the color of the root. In the case the colors coincide, we graft and contract the edge which has the endpoints of the same color. The unit of this operad is l.

There is an obvious operadic map $F : BTree \to ATree$. For a bicolored binary tree T, the tree $F(T)$ is obtained by contracting all maximal monocolored subtrees of T to corollas and preserving the colors.

We make $ATree$ a $\mathcal{C}at(V)$-operad by considering $ATree(n)$ as a discrete V-category. Likewise, we make $BTree$ a $\mathcal{C}at(V)$-operad by requiring that we have a unique isomorphism between two bicolored binary trees if and only if their imaged under F coincide. It is easy to see that F is indeed a weak equivalence of $\mathcal{C}at(V)$-operads. Indeed, one can easily check that $M_c \coprod M_c \simeq BTree$ and $M \coprod M \simeq ATree$ by considering generators and relations in these operads. Moreover, $F \simeq \pi \coprod \pi$, which completes the proof. □

Proof of Theorem 17. Let $\mathscr{D}$ be a duoidal category. It has an underlying double monoidal category D. By Lemma 18, one can find a strict double monoidal category D' and a double monoidal equivalence $F : D \to D'$. Using this equivalence one can transport the duoidal structure from $\mathscr{D}$ to D' without altering the tensor products and units in D'. In this way we obtain a strict duoidal category $\mathscr{D}'$ and F is lifted to a duoidal equivalence $F' : \mathscr{D} \to \mathscr{D}'$. □

Remark 20. Our proof of Lemma 17 was based on the fact that the coproduct of two weak equivalences of $\mathcal{C}at(V)$-operads is again a weak equivalence. This statement is a 'non-abelian' version of the Künneth formula for augmented dg-operads proved as Theorem 21 of [32].

3. $\mathscr{D}$-categories and monoidal $\mathscr{D}$-categories

If $\mathscr{D}$ is a duoidal category we will denote $\mathcal{C}at(\mathscr{D})$ the 2-category of $(\mathscr{D}, \Box_0, e)$-enriched categories. It was observed by Forcey [20] that $\mathcal{C}at(\mathscr{D})$ can be equipped with a monoidal structure. The tensor product $\times_1$ of two $\mathscr{D}$-categories $\mathcal{K}$ and $\mathcal{L}$ is given by the cartesian product on the objects level and

$$(\mathcal{K} \times_1 \mathcal{L})((X,Y),(Z,W)) = \mathcal{K}(X,Z)\,\Box_1 \mathcal{L}(Y,W), \text{ for } a,c \in S,\ b,d \in P.$$

The unit for this tensor product is the category $\mathbf{1}_v$ which has one object $*$ and $\mathbf{1}_v(*,*) = v$.

Definition 21. A monoidal $\mathscr{D}$-category $\mathcal{K} = (\mathcal{K}, \odot, \eta)$ is a pseudomonoid in the monoidal 2-category $(\mathcal{C}at(\mathscr{D}), \times_1, \mathbf{1}_v)$.

So we have a $\mathscr{D}$-functor $\odot : \mathcal{K} \times_1 \mathcal{K} \to \mathcal{K}$ fulfilling the expected associativity up to a $\mathscr{D}$-natural transformation, and a $\mathscr{D}$-functor $\eta : \mathbf{1}_v \to \mathcal{K}$. By abusing notations we will denote η the value of η on the unique object of $\mathbf{1}_v$.

A pseudomonoid structure therefore implies the existence of a monoid morphism

$$v \to \mathcal{K}(\eta,\eta) \tag{1}$$

and interchange morphisms

$$\mathcal{K}(X,Y)\,\Box_1 \mathcal{K}(Z,W) \to \mathcal{K}(X \odot Z, Y \odot W)$$

satisfying various coherence conditions.

Every $\mathscr{D}$-category $\mathcal{K}$ has an underlying V-category $\mathcal{U}\mathcal{K}$, with the same objects and morphisms given by

$$\mathcal{U}\mathcal{K}(X,Y) = \mathscr{D}(e, \mathcal{K}(X,Y)),\ \ X,Y \in \mathcal{K}.$$

This gives a 2-functor

$$\mathcal{U} : \mathcal{C}at(\mathscr{D}) \to \mathcal{C}at(V).$$

This is actually a lax-monoidal 2-functor. To see this we have to specify a transformation

$$\mathcal{U}\mathcal{K} \times_V \mathcal{U}\mathcal{L} \to \mathcal{U}(\mathcal{K} \times_1 \mathcal{L}).$$

On the object level this is an identity and on the morphisms level we have

$$(\mathcal{U}\mathcal{K} \times_V \mathcal{U}\mathcal{L})((X,Y),(Z,W)) = \mathcal{U}\mathcal{K}(X,Z) \otimes_V \mathcal{U}\mathcal{L}(Y,W) =$$

$$= \mathscr{D}(e, \mathcal{K}(X,Z)) \otimes_V \mathscr{D}(e, \mathcal{L}(Y,W)) \to \mathscr{D}(e \,\Box_1 e, \mathcal{K}(X,Z) \,\Box_1 \mathcal{L}(Y,W)) \to$$

$$\to \mathscr{D}(e, (\mathcal{K} \times_1 \mathcal{L})((X,Y),(Z,W)) = \mathcal{U}(\mathcal{K} \times_1 \mathcal{L})((X,Y),(Z,W)).$$

In this calculation we used the fact that $\Box_1$ is a V-functor and that e is a comonoid with respect to $\Box_1$. The unit constraint

$$1 \to \mathbf{1}_v$$

amounts to a morphism $I_V \to \mathscr{D}(e,v)$ which corresponds to the canonical morphism $e \to v$ in $\mathscr{D}$. We leave the verification of coherence conditions to the reader.

Proposition 22. *The* 2*-functor* $\mathcal{U}$ *maps monoidal* $\mathscr{D}$*-categories to monoidal* V*-categories.*

Proof. This is a direct consequence of lax-monoidality of $\mathcal{U}$. □

Definition 23. A *lax-monoidal functor* F from a monoidal $\mathscr{D}$-category $(\mathcal{K}, \odot, \eta)$ to a monoidal $\mathscr{D}$-category $(\mathcal{L}, \diamond, \iota)$ is a $\mathscr{D}$-functor $F : \mathcal{K} \to \mathcal{L}$ equipped with a $\mathscr{D}$-natural transformation:

$$F(X) \diamond F(Y) \to F(X \odot Y)$$

and a morphism

$$\iota \to F(\eta)$$

which make F a lax-monoidal functor between underlying monoidal V-categories and which satisfy the following additional coherence condition:

$$\begin{array}{ccc}
 & e \,\Box_0 \mathcal{K}(X \odot Z, Y \odot W) & \\
\nearrow & & \searrow \\
\mathcal{K}(X \odot Z, Y \odot W) & & \mathcal{L}(FX \diamond FZ, F(X \odot Z)) \,\Box_0 \mathcal{L}(F(X \odot Z), F(Y \odot W)) \\
\uparrow & & \downarrow \\
\mathcal{K}(X,Y) \,\Box_1 \mathcal{K}(Z,W) & & \mathcal{L}(FX \diamond FZ, F(Y \odot W)) \\
\downarrow & & \uparrow \\
\mathcal{L}(FX,FY) \,\Box_1 \mathcal{L}(FZ,FW) & & \mathcal{L}(FX \diamond FZ, FY \diamond FW) \,\Box_0 \mathcal{L}(FY \diamond FW, F(Y \odot W)) \\
\searrow & & \nearrow \\
 & (\mathcal{L}(FX,FY) \,\Box_1 \mathcal{L}(FZ,FW)) \,\Box_0 e &
\end{array}$$

As usual, we call a lax-monoidal $\mathscr{D}$-functor *strong* (*strict*) if its coherence constrains are isomorphisms (identities).

Definition 24. A *monoidal* $\mathscr{D}$*-transformation* between two lax-monoidal $\mathscr{D}$-functors is a $\mathscr{D}$-natural transformation which is a monoidal transformation between their underlying lax-monoidal V-functors.

Monoidal $\mathscr{D}$-categories, lax-monoidal (strong, strict) $\mathscr{D}$-functors and their monoidal $\mathscr{D}$-transformations form a 2-category $1\mathcal{C}at_{lax}(\mathscr{D})$ $(1\mathcal{C}at(\mathscr{D}), 1\mathcal{C}at_{strict}(\mathscr{D}))$. Every lax-monoidal $\mathscr{D}$-functor between monoidal $\mathscr{D}$-categories induces a lax-monoidal V-functor between the underlying monoidal V-categories. The same is true for $\mathscr{D}$-transformations.

Remark 25. The 2-category $1\mathcal{C}at(\mathscr{D})$ is not a monoidal 2-category. To make it monoidal, we need one more tensor product on $\mathscr{D}$ which would make it a trioidal category. If this is the case, the underlying V-category functor

$$\mathcal{U} : 1\mathcal{C}at(\mathscr{D}) \to 1\mathcal{C}at(V)$$

would be even a monoidal 2-functor.

Theorem 26. *Every monoidal $\mathscr{D}$-category is equivalent in $1\mathcal{C}at(\mathscr{D})$ to a strict monoidal $\mathscr{D}$-category.*

Proof. This follows from a general theorem of S. Lack about strictification of pseudomonoids in a Gray-monoid [29]. In our situation, the monoidal 2-category $(\mathcal{C}at(\mathscr{D}), \times_1, 1_v)$ is not a Gray-monoid but can be replaced by an equivalent Gray-monoid due to the tricategorical coherence theorem of Gordon-Power-Street [22]. □

Due to this coherence theorem we will assume that all objects of $1\mathcal{C}at_{lax}(\mathscr{D})$, $1\mathcal{C}at(\mathscr{D})$ and $1\mathcal{C}at_{strict}(\mathscr{D})$ are strict monoidal $\mathscr{D}$-categories.

Definition 27. A $\mathscr{D}$-enriched 2-category is a category enriched over $(\mathcal{C}at(\mathscr{D}), \times_1, 1_v)$.

Remark 28. As in the classical situation, we can identify a strict monoidal $\mathscr{D}$-category with a $\mathscr{D}$-enriched 2-category with one object. On the other hand, if $\mathcal{C}$ is a $\mathscr{D}$-enriched 2-category and x is an object of $\mathcal{C}$, then $\mathcal{C}(x, x) \in \mathcal{C}at(\mathscr{D})$ is a monoidal $\mathscr{D}$-category.

3.1. **Monoids in monoidal $\mathscr{D}$-categories.** Let $(\mathcal{K}, \odot, \eta)$ be a monoidal $\mathscr{D}$-category. The category $\mathbf{1}_v$ is canonically a monoidal $\mathscr{D}$-category (the tensor product is given by the structure isomorphism $v \square_1 v \to v$).

Definition 29. A *monoid in* $\mathcal{K}$ is a lax-monoidal $\mathscr{D}$-functor

$$\mathtt{M} : \mathbf{1}_v \to \mathcal{K}.$$

A morphism $f : \mathtt{M} \to \mathtt{N}$ of monoids is a monoidal transformation. More explicitly, a monoid in $\mathcal{K}$ is given by an object $\mathtt{M} \in \mathcal{K}$ together with:

(i) a morphism (neutral element) in $\mathcal{K}$

$$i : \eta \to \mathtt{M},$$

which is, by definition, a morphism $\overline{\nu} : e \to \mathcal{K}(\eta, \mathtt{M})$ in $\mathscr{D}$,

(ii) a morphism in $\mathcal{K}$ (multiplication)

$$m : \mathtt{M} \odot \mathtt{M} \to \mathtt{M}$$

that is, a morphism $\overline{\mu} : e \to \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M})$ in $\mathscr{D}$ and

(iii) a morphism in $\mathscr{D}$ (the *unit*)

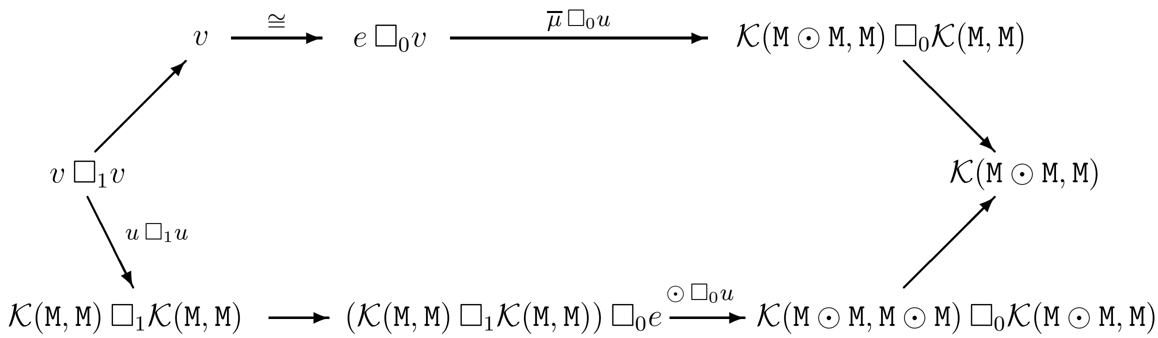
$$u : v \to \mathcal{K}(\mathtt{M}, \mathtt{M}).$$

This last unusual piece of data comes from the requirements that $\mathtt{M}$ is a $\mathscr{D}$-functor. These data should satisfy the following axioms:

($\star$) i and m make $\mathtt{M}$ a monoid in $\mathcal{UK}$,

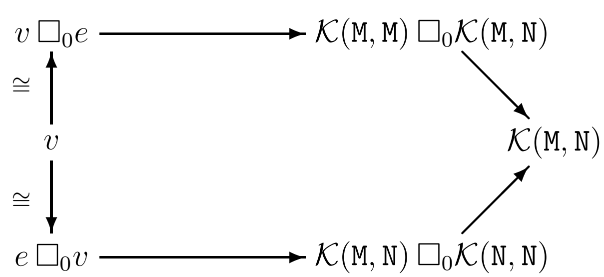
($\star\star$) u is a monoid morphism in $\mathscr{D}$, and

($\star\star\star$) the following diagram commutes:



A *monoid morphism* is a morphism $f : \mathtt{M} \to \mathtt{N}$ in $\mathcal{K}$ (i.e. a morphism $\bar{\phi} : e \to \mathcal{K}(\mathtt{M}, \mathtt{N})$ in $\mathscr{D}$) which satisfies the usual requirements for monoids morphism, and

($\star\star\star\star$) the following diagram commutes:



Monoids and their morphisms form the category $\mathcal{M}on(\mathcal{K})$.

3.2. **$\mathcal{K}$-enriched categories.** Classically, a monoid in a monoidal category $\mathcal{C}$ is the same as a one object $\mathcal{C}$-enriched category. We now introduce $\mathcal{K}$-enriched categories in a way that this property is preserved.

Definition 30. A *$\mathcal{K}$-enriched category* $\mathtt{M}$ consists of a set of objects $\mathtt{M}_0$ and, for each two objects $x, y \in \mathtt{M}_0$, an object $\mathtt{M}(x, y) \in \mathcal{K}$. The structure morphisms are:

(i) for each object $x \in \mathtt{M}_0$, a morphism in $\mathcal{K}$

$$i(x) : \eta \to \mathtt{M}(x, x),$$

(ii) for any $x, y, z \in \mathtt{M}_0$ a morphism

$$m(x, y, z) : \mathtt{M}(x, y) \odot \mathtt{M}(y, z) \to \mathtt{M}(x, z),$$

(iii) and, for any two objects $x, y \in \mathtt{M}_0$, a morphism in $\mathscr{D}$

$$u(x, y) : v \to \mathcal{K}(\mathtt{M}(x, y), \mathtt{M}(x, y)).$$

These data satisfy the obvious analogs of the axioms for monoids where, in the monoid coherence condition $(\star\star\star)$, we have to replace $\mathcal{K}(\mathtt{M}, \mathtt{M}) \,\square_1 \mathcal{K}(\mathtt{M}, \mathtt{M})$ by

$$\mathcal{K}(\mathtt{M}(x, y), \mathtt{M}(x, y)) \,\square_1 \mathcal{K}(\mathtt{M}(y, z), \mathtt{M}(y, z)).$$

The rest of the diagram is clear.

Analogously, one can define $\mathcal{K}$*-functors* and $\mathcal{K}$*-natural transformations.* So, a $\mathcal{K}$-functor $F : \mathtt{M} \to \mathtt{N}$ is given by a map of objects and an effect in $\mathscr{D}$ on morphisms expressed as a structure morphism

$$f_e(x, y) : e \to \mathcal{K}(\mathtt{M}(x, y), \mathtt{N}(F(x), F(y)),\ \ x, y \in \mathtt{M}_0, \tag{2}$$

satisfying the usual conditions and an obvious analogue of the extra coherence diagram $(\star\star\star\star)$ in which we have to replace $\mathtt{M}$ by $\mathtt{M}(x, y)$ and $\mathtt{N}$ by $\mathtt{N}(F(x), F(y))$.

Remark 31. We can replace the structure morphism (2) by the morphism

$$f_v(x, y) : v \to \mathcal{K}(\mathtt{M}(x, y), \mathtt{N}(F(x), F(y)),\ \ x, y \in \mathtt{M}_0, \tag{3}$$

defined as the composite

$$v \simeq e \,\square_0 v \xrightarrow{f_e \,\square_0 u} \mathcal{K}\big(\mathtt{M}(x, y), \mathtt{N}(F(x), F(y)\big) \,\square_0 \mathcal{K}\big(\mathtt{N}(F(x), F(y)), \mathtt{N}(F(x), F(y)\big) \longrightarrow$$
$$\longrightarrow \mathcal{K}\big(\mathtt{M}(x, y), \mathtt{N}(F(x), F(y)\big).$$

We can reconstruct f_e from f_v by precomposing with $e \to v$.

It is not difficult to check that $\mathcal{K}$-enriched categories, their $\mathcal{K}$-functors and $\mathcal{K}$-natural transformations form a 2-category which we will denote $\mathcal{C}at(\mathcal{K})$. By abusing the notation, we will also denote by $\mathcal{C}at(\mathcal{K})$ the 1-truncation of $\mathcal{C}at(\mathcal{K})$, i.e. the ordinary category of $\mathcal{K}$-categories and $\mathcal{K}$-functors when it does not lead to confusion.

With the definitions above we can identify a monoid in $\mathcal{K}$ with a $\mathcal{K}$-category with one object. This identification gives a functor

$$\Sigma : \mathcal{M}on(\mathcal{K}) \to \mathcal{C}at(\mathcal{K}).$$

4. Operads in duoidal categories

It is customary and convenient in the classical operad theory to assume that the base symmetric monoidal category is strict. This is possible due to MacLane coherence theorem. We follow this tradition and assume that our base duoidal category $\mathscr{D}$ is strict. Theorem 17 justifies this assumption.

The notion of a 2-fold operad in a 2-fold monoidal category was introduced by Forcey, Siehler and Sowers in [21]. Our notion of a duoidal category is weaker than Forcey's 2-fold monoidal category, so we need a slight modification of their definition.

Definition 32. A collection $A = \{A(n)\}_{n \geq 0}$ of objects of $\mathscr{D}$ is a *Forcey* 1*-operad* if, for each integers $n \geq 1$, $k_1, \ldots, k_n \geq 0$, one is given a morphism

$$\gamma : \big(A(k_1) \,\square_1 \cdots \square_1 A(k_n)\big) \,\square_0 A(n) \to A(k_1 + \cdots + k_n), \tag{4}$$

fulfilling the obvious version of the associativity for a nonsymmetric operad. One also requires a $\mathscr{D}$-map $j : e \to A(1)$ (the *unit*) such that the diagrams

$$\begin{array}{ccc} (\,\square_1^k e)\,\square_0 A(k) & \longleftarrow & e\,\square_0 A(k) \\ {\scriptstyle \square_1^k j\,\square_0 id}\downarrow & & \downarrow \\ (\,\square_1^k A(1))\,\square_0 A(k) & \xrightarrow{\ \gamma\ } & A(k) \end{array} \quad \text{and} \quad \begin{array}{ccc} A(k)\,\square_0 e & \xrightarrow{\ \cong\ } & A(k) \\ {\scriptstyle id\,\square_0 j}\downarrow & \nearrow{\scriptstyle \gamma} & \\ A(k)\,\square_0 A(1) & & \end{array}$$

commute for each $k \geq 0$. *Morphisms* of Forcey 1-operads are morphism of the underlying collections compatible with all structure operations.

We, therefore, have a category of Forcey 1-operads in $\mathscr{D}$. Every Forcey 1-operad determines a right action of A on $A(0)$ in the sense that there are morphisms $(\,\square_1^k A(0))\,\square_0 A(k) \to A(0), k \geq 1$, which satisfy the usual conditions for operad action.

Definition 33. A 1*-operad* in a duoidal category $\mathscr{D}$ is a Forcey 1-operad in $\mathscr{D}$ equipped with a left v-module structure $v\,\square_0 A(0) \to A(0)$ with respect to $\square_0$ on $A(0)$ such that it makes $A(0)$ a (v, A)-bimodule in the sense that the following diagram commutes:

$$\begin{array}{ccccc} (\,\square_1^k v)\,\square_0(\,\square_1^k A(0))\,\square_0 A(k) & \xrightarrow{\ \cong\ } & v\,\square_0(\,\square_1^k A(0))\,\square_0 A(k) & \longrightarrow & v\,\square_0 A(0) \\ \downarrow & & & & \downarrow \\ \big(\,\square_1^k(v\,\square_0 A(0))\big)\,\square_0 A(k) & \longrightarrow & (\,\square_1^k A(0))\,\square_0 A(k) & \longrightarrow & A(0). \end{array}$$

Remark 34. As it is clear from these definitions, 1-operads and Forcey 1-operads differ only in the treatment of composition (4) for $n = 0$. If we agree that $\square_1^0 = v$ and add the $n = 0$ case of γ to Definition 32, we will obtain exactly Definition 33.

Example 35. The associativity 1-operad $\underline{\mathcal{A}ss}$ in $\mathscr{D}$ is given by $\underline{\mathcal{A}ss}(n) = v$, with the unit and multiplication given by canonical morphisms $e \to v$ and $v \square_0 v \to v$.

Example 36. The Forcey 1-operad $\underline{\mathcal{A}s}$ is given by $\underline{\mathcal{A}s}(n) = e$. The unit is obvious. The multiplication is defined as the composition

$$(\underbrace{e \square_1 \dots \square_1 e}_{n}) \square_0 e \simeq (\underbrace{e \square_1 \dots \square_1 e}_{n})) \square_0((\underbrace{v \square_1 \dots \square_1 v}_{n-1} \square_1 e) \to$$
$$\to \underbrace{(e \square_0 v) \square_1 \dots \square_1 (e \square_0 v)}_{n-1} \square_1 (e \square_0 e) \simeq \underbrace{v \square_1 \dots \square_1 v}_{n-1} \square_1 e \simeq e.$$

For $n = 1$ we have just the canonical isomorphism

$$e \square_0 e \to e.$$

Example 37. If $\mathscr{D}$ is a braided monoidal category, then a 1-operad in $\mathscr{D}$ is a classical nonsymmetric operad. In this case there is no difference between Forcey operads and 1-operads.

4.1. **Endomorphism operads and algebras of operads.** Let $(\mathcal{K}, \odot, \eta)$ be a (strict) monoidal $\mathscr{D}$-category.

Definition 38. The *endomorphism* 1-*operad* of an object $X \in \mathcal{K}$ is determined by:

$$\mathcal{E}nd_X(n) = \mathcal{K}(\odot^n X, X)$$

with the obvious multiplication and unit data. The only unusual data is the action of v on $\mathcal{E}nd_X(0) = \mathcal{K}(\eta, X)$ given by the composition of the structure maps

$$v \square_0 \mathcal{E}nd_X(0) \to \mathcal{K}(\eta, \eta) \square_0 \mathcal{K}(\eta, X) \to \mathcal{K}(\eta, X) = \mathcal{E}nd_X(0).$$

Since the monoid $\mathcal{K}(\eta, \eta)$ acts on $\mathcal{K}(\eta, X)$, we have also an action of v via the morphism of monoids (1).

Definition 39. An *algebra* of a 1-operad A is an object X of a duoidal category $\mathcal{K}$ equipped with a 1-operad morphism $k : A \to \mathcal{E}nd_X$.

We define now the V-enriched category of algebras for a 1-operad A. To shorten the notations, we denote for two objects $X, Y \in \mathcal{K}$ and $n \geq 0$,

$$\mathcal{E}_X(n) = \mathcal{E}nd_X(n),\ \mathcal{E}_Y(n) = \mathcal{E}nd_Y(n),\ \mathcal{E}_{X,Y}(n) = \mathcal{K}(\odot^n X, Y).$$

Let now X, Y be two A-algebras and let $k_X^n : A(n) \to \mathcal{E}_X(n)$, $k_Y^n : A(n) \to \mathcal{E}_Y(n)$ be components of their structure morphisms. By definition, they are morphisms

$$k_X^n : I_V \to \mathscr{D}(A(n), \mathcal{E}_X(n)),\ k_Y^n : I_V \to \mathscr{D}(A(n), \mathcal{E}_Y(n))$$

in V. For any $n \geq 0$, we have obvious actions

$$\mathcal{E}_X(n)\,\Box_0\mathcal{E}_{X,Y}(1) \xrightarrow{a_0} \mathcal{E}_{X,Y}(n) \xleftarrow{a_1} \mathcal{E}_{X,Y}(1)\,\Box_0\mathcal{E}_Y(n).$$

Now we define, using a_0, for each n a morphism $d^0_n : \mathscr{D}(e, \mathcal{E}_{X,Y}(1)) \to \mathscr{D}(A(n), \mathcal{E}_{X,Y}(n))$ in V

$$\mathscr{D}(e,\mathcal{E}_{X,Y}(1)) \to I_V \otimes_V \mathscr{D}(e,\mathcal{E}_{X,Y}(1)) \to \mathscr{D}(A(n),\mathcal{E}_X(n)) \otimes_V \mathscr{D}(e,\mathcal{E}_{X,Y}(1)) \to$$
$$\to \mathscr{D}(A(n)\,\Box_0 e, \mathcal{E}_X(n))\,\Box_0\mathcal{E}_{X,Y}(1)) \to \mathscr{D}(A(n),\mathcal{E}_{X,Y}(n)).$$

Similarly we define $d^1_n : \mathscr{D}(e,\mathcal{E}_{X,Y}(1)) \to \mathscr{D}(A(n),\mathcal{E}_{X,Y}(n))$ using a_1. Finally, we define the V-enriched Hom from X to Y as the equalizer of the products of d^0_n and d^1_n,

$$\mathscr{D}(e,\mathcal{E}_{X,Y}(1)) \rightrightarrows \prod_{n\geq 0} \mathscr{D}(A(n),\mathcal{E}_{X,Y}(n)).$$

It is easy to see that the above construction defines a V-enriched category of algebras of A. As usual, the category of algebras of A is just the underlying category of this V-category. Analogously one can define V-category of algebras of any Forcey operad.

Proposition 40. *The category of algebras of the Forcey* 1*-operad* $\underline{\mathcal{A}s}$ *is isomorphic to the category of monoids in the underlying category* $\mathcal{UK}$*. The category of algebras of the* 1*-operad* $\underline{\mathcal{A}ss}$ *is isomorphic to the category of monoids in* $\mathcal{K}$*.*

Proof. The first statement of the proposition is classical. Let us prove the second statement. Let $\mathtt{M}$ be an algebra of $\underline{\mathcal{A}ss}$. It is obvious that the structure algebra map $\underline{\mathcal{A}ss} \to \mathcal{E}nd_{\mathtt{M}}$ is given by the following three maps in $\mathscr{D}$:

(i) the 'neutral element' $\nu : v \to \mathcal{K}(\eta, \mathtt{M})$,
(ii) the 'unit' $u : v \to \mathcal{K}(\mathtt{M},\mathtt{M})$, and
(iii) the 'multiplication' $\mu : v \to \mathcal{K}(\mathtt{M}\odot\mathtt{M},\mathtt{M})$,

such that the diagrams (5a)–(5e) of maps in $\mathscr{D}$ below are commutative.

(5a)
$$\begin{array}{ccc}
v\,\Box_0 v \cong (v\,\Box_1 v)\,\Box_0 v & \xrightarrow{\odot(u\,\Box_1\mu)\,\Box_0\mu} & \mathcal{K}(\mathtt{M}\odot(\mathtt{M}\odot\mathtt{M}),\mathtt{M}\odot\mathtt{M})\,\Box_0\,\mathcal{K}(\mathtt{M}\odot\mathtt{M},\mathtt{M}) \\
\uparrow & & \downarrow \circ \\
 & & \mathcal{K}(\mathtt{M}\odot(\mathtt{M}\odot\mathtt{M}),\mathtt{M}) \\
v\,\Box_0 e \cong v & & \updownarrow \cong \\
 & & \mathcal{K}((\mathtt{M}\odot\mathtt{M})\odot\mathtt{M},\mathtt{M}) \\
\downarrow & & \uparrow \circ \\
v\,\Box_0 v \cong (v\,\Box_1 v)\,\Box_0 v & \xrightarrow[\odot(\mu\,\Box_1 u)\,\Box_0\mu]{} & \mathcal{K}((\mathtt{M}\odot\mathtt{M})\odot\mathtt{M},\mathtt{M}\odot\mathtt{M})\,\Box_0\,\mathcal{K}(\mathtt{M}\odot\mathtt{M},\mathtt{M})
\end{array}$$

(5b)
$$\begin{array}{ccc}
v\,\Box_0 v \cong (v\,\Box_1 v)\,\Box_0 v & \xrightarrow{\odot(\nu\,\Box_1 u)\,\Box_0\mu} & \mathcal{K}(\mathtt{e}\odot\mathtt{M},\mathtt{M}\odot\mathtt{M})\,\Box_0\,\mathcal{K}(\mathtt{M}\odot\mathtt{M},\mathtt{M}) \\
\downarrow & & \downarrow \circ \\
v & \xrightarrow{u} & \mathcal{K}(\mathtt{M},\mathtt{M}) \cong \mathcal{K}(\mathtt{e}\odot\mathtt{M},\mathtt{M}) \cong \mathcal{K}(\mathtt{M}\odot\mathtt{e},\mathtt{M}) \\
\uparrow & & \uparrow \circ \\
v\,\Box_0 v \cong (v\,\Box_1 v)\,\Box_0 v & \xrightarrow[\odot(u\,\Box_1\nu)\,\Box_0\mu]{} & \mathcal{K}(\mathtt{M}\odot\mathtt{e},\mathtt{M}\odot\mathtt{M})\,\Box_0\,\mathcal{K}(\mathtt{M}\odot\mathtt{M},\mathtt{M})
\end{array}$$

$$
(5c)\qquad
\begin{array}{ccc}
v \,\square_0 v & \xrightarrow{\ \mu \,\square_0 u\ } & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M}, \mathtt{M}) \\
\downarrow & & \downarrow \circ \\
v & \xrightarrow{\ \mu\ } & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M}) \\
\uparrow & & \uparrow \circ \\
(v \,\square_1 v) \,\square_0 v & \xrightarrow[\ \odot(u \,\square_1 u) \,\square_0 \mu\ ]{} & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M} \odot \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M})
\end{array}
$$

$$
(5d)\qquad
\begin{array}{ccc}
v \,\square_0 v & \xrightarrow{\ \nu \,\square_0 u\ } & \mathcal{K}(\mathtt{e}, \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M}, \mathtt{M}) \\
\downarrow & & \downarrow \circ \\
v & \xrightarrow{\ u\ } & \mathcal{K}(\mathtt{e}, \mathtt{M})
\end{array}
$$

$$
(5e)\qquad
\begin{array}{ccc}
v \,\square_0 v & \xrightarrow{\ u \,\square_0 u\ } & \mathcal{K}(\mathtt{M}, \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M}, \mathtt{M}) \\
\downarrow & & \downarrow \circ \\
v & \xrightarrow{\ u\ } & \mathcal{K}(\mathtt{M}, \mathtt{M})
\end{array}
$$

Using the map $e \to v$ of Definition 3(v), one defines the morphisms $\overline{\nu} \in \mathcal{UK}(\mathtt{e}, \mathtt{M})$, $\overline{\mu} \in \mathcal{UK}(\mathtt{M} \odot \mathtt{M}, \mathtt{M})$ and $\overline{u} \in \mathcal{UK}(\mathtt{M}, \mathtt{M})$ of the underlying category as the compositions

$$
\overline{\nu} := e \to v \xrightarrow{\nu} \mathcal{K}(\mathtt{e}, \mathtt{M}),\ \overline{\mu} := e \to v \xrightarrow{\mu} \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M}) \ \text{ and } \ \overline{u} := e \to v \xrightarrow{u} \mathcal{K}(\mathtt{M}, \mathtt{M}).
$$

Diagram (5d) extends into

$$
\begin{array}{ccccc}
e \,\square_0 v & \longrightarrow & v \,\square_0 v & \xrightarrow{\ \nu \,\square_0 u\ } & \mathcal{K}(\mathtt{e}, \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M}, \mathtt{M}) \\
& \nwarrow \ \cong & \downarrow & & \downarrow \circ \\
& & v & \xrightarrow{\ \nu\ } & \mathcal{K}(\mathtt{e}, \mathtt{M})
\end{array}
$$

which shows that ν is determined by the composition $\overline{\nu} \,\square_0 u$ of the top two horizontal maps, i.e. by $\overline{\nu}$ and u. Similarly one proves, using (5c), that μ is determined by $\overline{\mu}$ and u. Finally, (5e) implies that $\overline{u}$ equals the unit map of the underlying category.

From (5a) and (5b) one concludes that $(\mathtt{M}, \overline{\mu}, \overline{\nu})$ is a unital monoid in the underlying category $\mathcal{UK}$. Diagram (5d) asserts that u is a monoid morphism. From (5c) one gets the last coherence condition:

$$
(6)\qquad
\begin{array}{ccc}
e \,\square_0 v & \xrightarrow{\ \overline{\mu} \,\square_0 u\ } & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M}, \mathtt{M}) \\
\updownarrow & & \downarrow \circ \\
\cong & & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M}) \\
\updownarrow & & \uparrow \circ \\
(v \,\square_1 v) \,\square_0 e & \xrightarrow[\ \odot(u \,\square_1 u) \,\square_0 \overline{\mu}\ ]{} & \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M} \odot \mathtt{M}) \,\square_0\, \mathcal{K}(\mathtt{M} \odot \mathtt{M}, \mathtt{M})
\end{array}
$$

This shows that $\underline{\mathcal{A}ss}$-algebras determine a monoid in $\mathcal{K}$. The opposite implication is now obvious as well as the statement about the isomorphism of categories. □

5. Center, δ-center and homotopy center of a monoid

5.1. **Multiplicative 1-operads in duoidal categories.** In the following definition, $\underline{\mathcal{A}ss}$ is the 1-operad in $\mathscr{D} = (\mathscr{D}, \square_0, \square_1, e, v)$ introduced in Example 35.

Definition 41. A 1-operad A in a duoidal category $\mathscr{D}$ is *multiplicative* if it is equipped with a 1-operad map $\underline{\mathcal{A}ss} \to A$.

By definition, a multiplicative structure on A is given by a system $v \to A(n)$, $n \geq 0$, of $\mathscr{D}$-morphisms satisfying appropriate compatibility conditions. Let Δ be the classical simplicial category whose objects are finite ordinals $[n] = \{0, \ldots, n\}$, $n \geq 1$, and morphisms in $\Delta(m, n)$ are order-preserving set maps $\{0, \ldots, m\} \to \{0, \ldots, n\}$.

Proposition 42. *Each multiplicative operad A determines a cosimplicial object $A : \Delta \to \mathscr{D}$ in $\mathscr{D}$ whose value at $n \in \Delta$ is $A(n)$.*

Proof. Assume that the multiplicative structure of A is given by a system $m_n : v \to A(n)$, $n \geq 0$, of $\mathscr{D}$-maps. We define the cosimplical structure on A by specifying the actions of the standard generating maps of Δ. The coboundary $d_0 : A(n) \to A(n+1)$ is the composition

$$A(n) \cong \big(v \,\square_1 A(n)\big) \,\square_0 e \to \big(v \,\square_1 A(n)\big) \,\square_0 v \xrightarrow{(m_1 \,\square_1 id) \,\square_0 m_2} \big(A(1) \,\square_1 A(n)\big) \,\square_0 A(2) \xrightarrow{\gamma} A(n+1),$$

where the second map uses the canonical morphism $c : e \to v$. Likewise, $d_{n+1} : A(n) \to A(n+1)$ is the composition

$$A(n) \cong \big(A(n) \,\square_1 v\big) \,\square_0 e \to \big(A(n) \,\square_1 v\big) \,\square_0 v \xrightarrow{(id \,\square_1 m_1) \,\square_0 m_2} \big(A(n) \,\square_1 A(1)\big) \,\square_0 A(2) \xrightarrow{\gamma} A(n+1).$$

For $1 \leq i \leq n$, the map $d_i : A(n) \to A(n+1)$ is the composition

$$\begin{aligned} A(n) \cong e \,\square_0 A(n) \xrightarrow{c \,\square_0 id} v \,\square_0 A(n) \cong (\,\square_1^n v) \,\square_0 A(n) \xrightarrow{f_i \,\square_0 id} \\ \xrightarrow{f_i \,\square_0 id} \big(\,\square_1^{i-1} A(1) \,\square_1 A(2) \,\square_1^{n-i} A(1)\big) \,\square_0 A(n) \xrightarrow{\gamma} A(n+1), \end{aligned}$$

where $f_i := \square_1^{i-1} m_1 \,\square_1 m_2 \,\square_1^{n-i} m_1$. The cosimplicial degeneracies $s_i : A(n+1) \to A(n)$, $0 \leq i \leq n$, are the compositions

$$\begin{aligned} A(n+1) \cong e \,\square_0 A(n+1) \xrightarrow{c \,\square_0 id} v \,\square_0 A(n+1) \cong (\,\square_1^{n+1} v) \,\square_0 A(n+1) \xrightarrow{g_i \,\square_0 id} \\ \xrightarrow{g_i \,\square_0 id} \big(\,\square_1^{i} A(1) \,\square_1 A(0) \,\square_1^{n-i} A(1)\big) \,\square_0 A(n+1) \xrightarrow{\gamma} A(n), \end{aligned}$$

where $g_i := \square_1^{i} m_1 \,\square_1 m_0 \,\square_1^{n-i} m_1$. ∎

5.2. **Hochschild object, center and homotopy center.** Assume that $\mathscr{D}$ is complete as a V-category. By [26, Theorem 3.73] this is equivalent to $\mathscr{D}$ having small conical limits and V-cotensors. The last condition means that, for any $a \in V$ and $y \in \mathscr{D}$, there exists an object $y^a \in \mathscr{D}$ such that

$$\mathscr{D}(x, y^a) \simeq V\big(a, \mathscr{D}(x,y)\big)$$

naturally for all $x \in \mathscr{D}$. We also fix a cosimplicial object $\delta : \Delta \to V$ in V. Then one defines the δ-*totalization* of a cosimplicial object $\phi : \Delta \to \mathscr{D}$ as the V-enriched end

$$\mathrm{Tot}_\delta(\phi) := \int_{n\in\Delta} \phi(n)^{\delta(n)} \in \mathscr{D}.$$

Let A be a multiplicative 1-operad in $\mathscr{D}$. By Proposition 42, it determines a cosimplicial object in $\mathscr{D}$ (denoted again by A).

Definition 43. The *Hochschild δ-object* of a multiplicative 1-operad A is defined as

$$CH_\delta(A) := \mathrm{Tot}_\delta(A).$$

The endomorphism operad $\mathcal{E}nd_{\mathtt{M}}$ of a monoid $\mathtt{M}$ in a $\mathscr{D}$-monoidal category $\mathcal{K}$ is, by Proposition 40, a multiplicative 1-operad in $\mathscr{D}$.

Definition 44. The δ-*center* of a monoid $\mathtt{M}$ is defined as

$$CH_\delta(\mathtt{M},\mathtt{M}) := CH_\delta(\mathcal{E}nd_{\mathtt{M}}).$$

If $\delta = I$ is the constant cosimplicial object that equals $I \in V$ for all n, then the δ-center $Z(\mathtt{M}) := CH_I(\mathtt{M},\mathtt{M})$ will be called the *center* of $\mathtt{M}$. Notice that, in general, the center of a monoid in $\mathcal{K}$ lives in the category $\mathscr{D}$, not in $\mathcal{K}$. It is not difficult to see that the center of a monoid $\mathtt{M}$ is given by the following equalizer in $\mathscr{D}$:

$$Z(\mathtt{M}) \to \mathcal{K}(\eta,\mathtt{M}) \rightrightarrows \mathcal{K}(\mathtt{M},\mathtt{M}). \tag{7}$$

Example 45. A trivial example of a monoid in $\mathcal{K}$ is the unit object η. It is obvious from (7) that $Z(\eta) = \mathcal{K}(\eta,\eta)$ and that $\mathcal{K}(\eta,\eta)$ is a duoid in $\mathscr{D}$.

Theorem 46. *Let A be a multiplicative operad and $\delta = I$. Then the Hochschild object $CH_I(A)$ has the canonical structure of a duoid in $\mathscr{D}$. In particular, the center of a monoid $\mathtt{M} \in \mathcal{K}$ has a canonical structure of a duoid in $\mathscr{D}$.*

Proof. We describe the structure morphisms for the duoid $CH_I(A)$. To construct a morphism

$$CH_I(A)\,\Box_0\, CH_I(A) \to CH_I(A)$$

it is enough to construct a morphism $CH_I(A)\,\Box_0\, CH_I(A) \to A(0)$ which equalizes the coboundary operators $d_0, d_1 : A(0) \to A(1)$. One can take the composite

$$CH_I(A)\,\Box_0\, CH_I(A) \xrightarrow{\pi\,\Box_0\pi} A(0)\,\Box_0 A(0) \xrightarrow{d_0\,\Box_0 d_0} A(1)\,\Box_0 A(1) \to A(1) \stackrel{s_0}{\to} A(0),$$

in which $\pi : CH_I(A) \to A(0)$ is the canonical map. Observe that, since π equalizes d_0 and d_1, instead of $d_0 \square_0 d_0$, one could have taken, in the above composition, $d_i \square_0 d_j$ with arbitrary $i, j \in \{0, 1\}$. Analogously, we construct a morphism

$$CH_I(A) \square_1 CH_I(A) \to CH_I(A)$$

using the composite

$$CH_I(A) \square_1 CH_I(A) \to A(0) \square_1 A(0) \simeq \big(A(0) \square_1 A(0)\big) \square_0 e \to \\ \to \big(A(0) \square_1 A(0)\big) \square_0 v \to \big(A(0) \square_1 A(0)\big) \square_0 A(2) \to A(0).$$

We define the unit

$$v \to CH_I(A)$$

for the second product using the composite

$$v \to A(1) \stackrel{s_0}{\to} A(0)$$

and the unit for the first product by composing

$$e \to v \to CH_I(A).$$

We leave a long, tedious, but straightforward verification of the correctness of our definitions, as well as the verification of the duoid axioms to the reader. $\square$

C. Barwick developed in [2] a notion of a model category enriched in a monoidal model category. We can easily adapt his definition to the situation of an enrichment over a duoidal category. So, let us assume that V is a monoidal model category, the category $\mathscr{D}$ is a model category which is a monoidal model V-category for each of the monoidal structures on $\mathscr{D}$, and $\mathcal{K}$ is a $\mathscr{D}$-monoidal model category. In this case one can speak about a *standard system of simplices* for V in the sense of [14, Definition A.6] (see also [7], Section 3.5). Let δ be such a standard system of simplices for V. We also assume that there is a model structure on the category of monoids in $\mathcal{K}$ and $Fb(\mathtt{M})$ is a fibrant replacement for a monoid $\mathtt{M}$.

Definition 47. The δ-center $CH_\delta\big(Fb(\mathtt{M}), Fb(\mathtt{M})\big)$ will be called the *homotopy center* of $\mathtt{M}$ and will be denoted $CH(\mathtt{M}, \mathtt{M})$.

Remark 48. The homotopy aspects of the theory of the center are out of the scope of this paper and will be considered in the sequel [9]. We will show there that, under some, not very restrictive, technical conditions the notion of homotopy center does not depend (up to homotopy) on the standard system of simplices we use (see Example 52 for illustration). This justifies our terminology.

Example 49. Let $V = \mathcal{S}et$ and let $\mathscr{D}$ be a closed braided monoidal category, i.e $\mathscr{D}$ is enriched over itself. Then the center of a monoid $\mathtt{M}$ in $\mathscr{D}$ is the equalizer

$$Z(\mathtt{M}) \to \mathtt{M} \simeq \mathscr{D}(e,\mathtt{M}) \rightrightarrows \mathscr{D}(\mathtt{M},\mathtt{M}).$$

where the two arrows are induced by the left and right multiplication in $\mathtt{M}$. Therefore, $Z(\mathtt{M})$ is the classical center of $\mathtt{M}$.

Example 50. Let $V = \mathscr{D} = \mathcal{K} = \mathcal{C}at$. A monoid $\mathtt{M}$ in $\mathcal{C}at$ is a strict monoidal category. Let δ be the cosimplicial object in $\mathcal{C}at$ whose n-th space is equal to the chaotic groupoid with $n+1$ objects. This is a standard system of simplices in $\mathcal{C}at$ if we equip $\mathcal{C}at$ with a Joyal-Tirney model structure for which weak equivalences are categorical equivalences. In this model structure all objects in Cat are fibrant, hence the δ-center of $\mathtt{M}$ is its homotopy center and is equal to the Joyal-Street center of $\mathtt{M}$ [24].

Example 51. If we use, in the previous example, δ which in dimension n equals to the free category on a linear graph with $n+1$ objects

$$\bullet_0 \to \bullet_1 \to \ldots \to \bullet_{n-1} \to \bullet_n$$

then the δ-center of $\mathtt{M}$ is its lax-center (or colax if we reverse the orientation in δ), see [18].

Example 52. Let k be a commutative ring and let $V = \mathscr{D} = \mathcal{K} = \mathcal{C}hain$ the category of chain complexes of k-modules. Let $\delta := C_*(\Delta^\bullet)$, the complex of normalized simplicial chains on the standard simplicial simplex $\Delta^\bullet$. This δ is a standard system of simplices. The δ-center of a monoid $\mathtt{M}$ (i.e. a unital differential graded algebra) is its normalized Hochschild complex. It is the homotopy center of $\mathtt{M}$.

Instead of δ given by normalized chains we can take $\tilde{\delta} = Lan_i(\delta_{un})$, the left Kan extension of $\delta_{un} : \Delta_{in} \to \mathcal{C}hain$ given by un-normalized chains. Here $i : \Delta_{in} \subset \Delta$ is the subcategory of injections. As follows from [8, Proposition A.6] and the discussion in the appendix to that paper, the $\tilde{\delta}$-center of $\mathtt{M}$ will be the unnormalized Hochschild complex of $\mathtt{M}$. It is classical that $CH_\delta(\mathtt{M})$ is weakly equivalent to $CH_{\tilde{\delta}}(\mathtt{M})$.

6. The duoidal category $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$

Let us fix a small (with respect to some universe $\mathbb{U}$) category $\mathscr{C}$ with the set of objects $\mathscr{C}_0$, set of arrows $\mathscr{C}_1$, source and target maps $s_\mathscr{C}, t_\mathscr{C} : \mathscr{C}_1 \to \mathscr{C}_0$, and the identity map $i_\mathscr{C} : \mathscr{C}_0 \to \mathscr{C}_1$. We also fix a duoidal V-category $\mathscr{D} = (\mathscr{D}, \Box_0, \Box_1, e, v)$ as in Definition 3. We assume, in addition, that $\mathscr{D}$ has small products and coproducts and both $\Box_0$ and $\Box_1$ commute with coproducts in each variable.

Remark 53. In our applications we often assume that $\mathscr{C}$ is a large category with respect to $\mathbb{U}$ such as the category of small categories. This is not, however, a big obstacle. Let $\mathbb{U}' \supset \mathbb{U}$ be a bigger universe with respect to which the set of objects of $\mathscr{C}$ is a small set i.e. $\mathscr{C}_0 \in \mathbb{U}'$. There is a standard procedure in enriched category theory described in [26, Section 3.11] known as the *universe enlargement* which allows to embed a symmetric monoidal category V in an essentially unique manner to a larger symmetric monoidal category V' in a way that this embedding preserves all limits and colimits which exist in V, but V' also admits large (with respect to $\mathbb{U}$) limits and colimits which are small with respect to $\mathbb{U}'$.

This embedding is based on the argument of Day [17] which uses the convolution tensor product (Day convolution) on the presheaf category $\mathcal{SET}^{V^{op}}$. Here $\mathcal{SET}$ is a version of the category of sets based on the universe $\mathbb{U}'$. It is not difficult to check that Day's argument works equally well for a duoidal category $\mathscr{D}$ so we can embed $\mathscr{D}$ to a larger duoidal category $\mathscr{D}'$ which admits all necessary limits and colimits. Due to this consideration, we can always assume that V and $\mathscr{D}$ are large enough to form limits and colimits we need.

Definition 54. A *globe* (in $\mathscr{C}$) is a diagram

$$\mathrm{glb}\,(A,B;f,g) := A \underset{g}{\overset{f}{\rightrightarrows}} B, \tag{8}$$

where $A, B \in \mathscr{C}_0$ are objects of $\mathscr{C}$ and $f, g : A \to B$ their morphisms.

For G as in (8) we set $s(\mathsf{G}) := f$, $t(\mathsf{G}) := g$, $S(\mathsf{G}) := A$ and $T(\mathsf{G}) := B$ (the *source, target, supersource and supertarget of* G, respectively). We will often need the 'trivial' globes

$$\mathsf{G}(A) \;:=\; \mathrm{glb}\,(A,A;id_A,id_A) = A \underset{id_A}{\overset{id_A}{\rightrightarrows}} A,\;\; A \in \mathscr{C}_0, \;\text{ and} \tag{9a}$$

$$\mathsf{G}(f) \;:=\; \mathrm{glb}\,(A,B;f,f) = A \underset{f}{\overset{f}{\rightrightarrows}} B,\;\; f : A \to B \in \mathscr{C}_1. \tag{9b}$$

Definition 55. Let $\mathscr{D}$ be a duoidal V-category as above. A *span $\mathscr{D}$-object over $\mathscr{C}$* (or simply a *span object*) is a system $Y = \{Y_\mathsf{G}\}$ of objects of $\mathscr{D}$ indexed by globes in $\mathscr{C}$. A *morphism* $F : Y' \to Y''$ of span objects is a system of morphisms $\{F_\mathsf{G} \in \mathscr{D}(Y'_\mathsf{G}, Y''_\mathsf{G})\}$ indexed by globes in $\mathscr{C}$.

We denote by $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ or simply by $\mathcal{S}p_2$ if $\mathscr{C}$ and $\mathscr{D}$ are understood, the V-category of span $\mathscr{D}$-objects over $\mathscr{C}$ and their morphisms.

Example 56. If $\mathscr{D} = V$ is the category $\mathcal{S}et$ of sets, a span $\mathcal{S}et$-object, or a *span set* for short, is the same as a diagram of sets

$$\begin{array}{c} Y \\ t \downarrow\;\downarrow s \\ \mathscr{C}_1 \\ t_{\mathscr{C}} \downarrow\;\downarrow s_{\mathscr{C}} \\ \mathscr{C}_0 \end{array} \tag{10}$$

in which $s_{\mathscr{C}} s = s_{\mathscr{C}} t$ and $t_{\mathscr{C}} t = t_{\mathscr{C}} s$. Indeed, for Y is as in the above diagram, we define the *fiber* over a globe G as

$$Y_{\mathsf{G}} := \{y \in Y;\ s(y) = s(\mathsf{G}),\ t(y) = t(\mathsf{G})\}. \tag{11}$$

The system $\{Y_{\mathsf{G}}\}$ of fibers is then a span set in the sense of Definition 55.

On the other hand, any collection Y_{G} of sets indexed by globes in $\mathscr{C}$ assembles into the disjoint union $Y := \bigcup_{\mathsf{G}} Y_{\mathsf{G}}$. The maps $s, t : Y \to \mathscr{C}_1$ defined by $s(y) := s(\mathsf{G})$, $t(y) := t(\mathsf{G})$ for $y \in Y_{\mathsf{G}}$, are then as in diagram (10).

We call the composition $S := s_{\mathscr{C}} s : Y \to \mathscr{C}_0$ (resp. $T := t_{\mathscr{C}} t : Y \to \mathscr{C}_0$) the *supersource* (resp. the *supertarget*) map.

Convention 57. We will visualize span $\mathscr{D}$-objects as diagrams (10) even when $\mathscr{D}$ is a general duoidal category so the disjoint union $Y := \bigcup_{\mathsf{G}} Y_{\mathsf{G}}$ does not have a formal sense. We can then think of Y as of a set fibered over the globes in $\mathscr{C}$, with the fibers objects of $\mathscr{D}$.

6.1. **The first monoidal structure.** Let $\mathcal{S}p_2 = \mathcal{S}p_2(\mathscr{C}, \mathscr{D})$ be as in Definition 55. For span objects $Y_i = \{Y_{i,\mathsf{G}}\} \in \mathcal{S}p_2$, $i = 1, 2$, define $Y_1 \times_0 Y_2 = \{(Y_1 \times_0 Y_2)_{\mathsf{G}}\} \in \mathcal{S}p_2$ by

$$(Y_1 \times_0 Y_2)_{\mathsf{G}} := \coprod_{\mathsf{G}_1, \mathsf{G}_2} Y_{1,\mathsf{G}_1} \Box_0 Y_{2,\mathsf{G}_2}, \tag{12}$$

where the coproduct is taken over all globes $\mathsf{G}_1, \mathsf{G}_2$ that decompose G in the sense that $T(\mathsf{G}_1) = S(\mathsf{G}_2)$ and

$$s(\mathsf{G}) = s(\mathsf{G}_2) s(\mathsf{G}_1),\ t(\mathsf{G}) = t(\mathsf{G}_2) t(\mathsf{G}_1) \ \text{ (the composition in } \mathscr{C}).$$

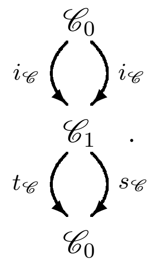
If we think of Y_1 and Y_2 in terms of diagrams (10), then $Y_1 \times_0 Y_2$ is the pullback

$$\begin{array}{ccccc} & & Y_1 \times_0 Y_2 & & \\ & \swarrow & & \searrow & \\ Y_1 & & & & Y_2 \\ & {\scriptstyle tt}\searrow & & \swarrow{\scriptstyle ss} & \\ & & \mathscr{C}_o & & \end{array} . \tag{13}$$

The above construction clearly extends into a functor $\times_0 : \mathcal{S}p_2 \times \mathcal{S}p_2 \to \mathcal{S}p_2$. Let $0 \in \mathscr{D}$ be the initial object and recall that $e \in \mathscr{D}$ is the unit for $\square_0$. Denote by $1_0 = \{1_{0\mathsf{G}}\} \in \mathcal{S}p_2$ the object defined by

$$1_{0\mathsf{G}} := \begin{cases} e, & \text{if } \mathsf{G} \text{ is the globe } \mathsf{G}(A) \text{ in (9a) for some } A \in \mathscr{C}_0, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that 1_0 is a two-sided unit for $\times_0$. Observe that, if $\mathscr{D} = V = \mathcal{S}et$, then 1_0 is given by the diagram



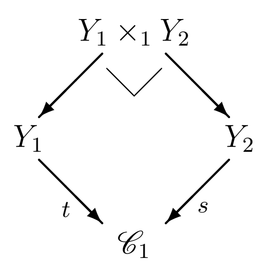
6.2. **The second monoidal structure.** For span V-objects $Y_i = \{Y_{i,\mathsf{G}}\} \in \mathcal{S}p_2$, $i = 1, 2$, define $Y_1 \times_1 Y_2 = \{(Y_1 \times_1 Y_2)_{\mathsf{G}}\} \in \mathcal{S}p_2$ by

$$(Y_1 \times_1 Y_2)_{\mathsf{G}} := \coprod_{\mathsf{G}_1,\mathsf{G}_2} Y_{1,\mathsf{G}_1} \square_1 Y_{2,\mathsf{G}_2}, \tag{14}$$

where $\square_1$ is the second monoidal structure of $\mathscr{D}$ and the coproduct is taken over all globes $\mathsf{G}_1, \mathsf{G}_2$ such that

$$s(\mathsf{G}) = s(\mathsf{G}_1),\ t(\mathsf{G}_1) = s(\mathsf{G}_2) \text{ and } t(\mathsf{G}) = t(\mathsf{G}_2). \tag{15}$$

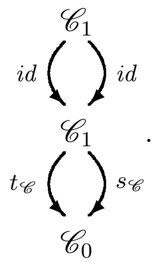
In terms of diagrams (10), $Y_1 \times_1 Y_2$ is the pullback



which has to be compared to the pullback (13) defining the $\times_0$-product. Let $1_1 = \{1_{1\mathsf{G}}\} \in \mathcal{S}p_2$ be the object with

$$1_{1\mathsf{G}} := \begin{cases} v, & \text{if } \mathsf{G} \text{ is the globe } \mathsf{G}(f) \text{ of (9b) for some } A \xrightarrow{f} B \in \mathscr{C}_1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In the diagrammatic language, 1_1 is the diagram



It is clear that 1_1 is a two-sided unit for $\times_1$. To construct the canonical map $1_1 \times_0 1_1 \to 1_1$, observe that $(1_1 \times_0 1_1)_{\mathsf{G}} \neq 0$ only if $\mathsf{G} = \mathsf{G}(f)$ for some morphism f in $\mathscr{C}$, in which case one has the composition

$$(1_1 \times_0 1_1)_{\mathsf{G}(f)} = \coprod_{f=f_2f_1} 1_{1\mathsf{G}(f_1)} \Box_0 1_{1\mathsf{G}(f_2)} = \coprod_{f=f_2f_1} v \Box_0 v \longrightarrow \coprod_{f=f_2f_1} v, \tag{16}$$

in which the last arrow is the map (iv) of Definition 3. We define the component $(1_1 \times_0 1_1)_{\mathsf{G}} \to 1_{1\mathsf{G}}$ of the structure map $1_1 \times_0 1_1 \to 1_1$ as the composition of the map (16) with the folding map $\coprod_{f=f_2f_1} v \to v = 1_{1\mathsf{G}}$ if $\mathsf{G} = \mathsf{G}(f)$ for some f, and as the unique map $0 \to 0$ in the remaining cases.

It is clear that, for the object 1_0 defined in Subsection 6.1, $(1_0 \times_1 1_0)_{\mathsf{G}} \neq 0$ only if $\mathsf{G} = \mathsf{G}(A)$ for some $A \in \mathscr{C}_0$, in which case

$$(1_0 \times_1 1_0)_{\mathsf{G}(A)} = e \Box_1 e.$$

We define the structure map $1_0 \to 1_0 \times_1 1_0$ to be the map induced, in the obvious way, by (iii) of Definition 3.

Let us describe the interchange law $(A \times_1 B) \times_0 (C \times_1 D) \to (A \times_0 C) \times_1 (B \times_0 D)$. From the definitions (12) and (14) of the products $\times_0$ and $\times_1$ we get that

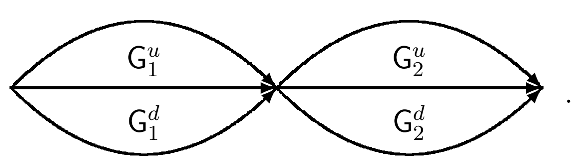
$$(A \times_1 B) \times_0 (C \times_1 D)_{\mathsf{G}} = \coprod_{(\mathsf{G}_1^u, \mathsf{G}_2^u, \mathsf{G}_1^d, \mathsf{G}_2^d) \in \mathsf{L}_{\mathsf{G}}} (A_{\mathsf{G}_1^u} \Box_1 B_{\mathsf{G}_1^d}) \Box_0 (C_{\mathsf{G}_2^u} \Box_1 D_{\mathsf{G}_2^d}),$$

where L_{G} is the set of all globes $\mathsf{G}_1^u, \mathsf{G}_2^u, \mathsf{G}_1^d, \mathsf{G}_2^d$ such that

$$T(\mathsf{G}_1^u) = S(\mathsf{G}_2^u),\ T(\mathsf{G}_1^d) = S(\mathsf{G}_2^d),\ s(\mathsf{G}) = s(\mathsf{G}_2^u)s(\mathsf{G}_1^u),\ t(\mathsf{G}) = t(\mathsf{G}_2^d)t(\mathsf{G}_1^d) \ \text{ and} \tag{17a}$$

$$t(\mathsf{G}_1^u) = s(\mathsf{G}_1^d),\ t(\mathsf{G}_2^u) = s(\mathsf{G}_2^d). \tag{17b}$$

The 'configuration' of the globes $(\mathsf{G}_1^u, \mathsf{G}_2^u, \mathsf{G}_1^d, \mathsf{G}_2^d) \in \mathsf{L}_{\mathsf{G}}$ is schematically depicted as



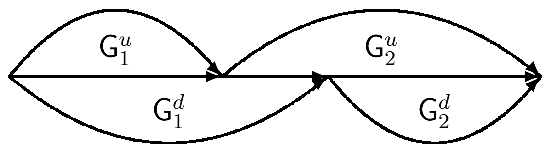
It is equally clear that

$$(A \times_0 C) \times_1 (B \times_0 D)_{\mathsf{G}} = \coprod_{(\mathsf{G}_1^u,\mathsf{G}_2^u,\mathsf{G}_1^d,\mathsf{G}_2^d)\in \mathsf{R}_{\mathsf{G}}} (A_{\mathsf{G}_1^u} \square_0 C_{\mathsf{G}_2^u}) \square_1 (B_{\mathsf{G}_1^d} \square_1 D_{\mathsf{G}_2^d}),$$

where R_{G} is the set of all globes $\mathsf{G}_1^u, \mathsf{G}_2^u, \mathsf{G}_1^d, \mathsf{G}_2^d$ satisfying (17a), but instead of (17b), a weaker condition

$$s(\mathsf{G}_2^u)s(\mathsf{G}_1^u) = t(\mathsf{G}_2^d)t(\mathsf{G}_1^d).$$

It is clear that $\mathsf{L}_{\mathsf{G}} \subset \mathsf{R}_{\mathsf{G}}$. A 'configuration' that belongs to R_{G} but not to L_{G} is portrayed below:



The G-component of the interchange law is defined as the map of coproducts induced by the inclusion $\mathsf{L}_{\mathsf{G}} \hookrightarrow \mathsf{R}_{\mathsf{G}}$ of the indexing sets, precomposed with the map

$$\coprod_{(\mathsf{G}_1^u,\mathsf{G}_2^u,\mathsf{G}_1^d,\mathsf{G}_2^d)\in \mathsf{L}_{\mathsf{G}}} (A_{\mathsf{G}_1^u} \square_1 B_{\mathsf{G}_1^d}) \square_0 (C_{\mathsf{G}_2^u} \square_1 D_{\mathsf{G}_2^d}) \longrightarrow \coprod_{(\mathsf{G}_1^u,\mathsf{G}_2^u,\mathsf{G}_1^d,\mathsf{G}_2^d)\in \mathsf{L}_{\mathsf{G}}} (A_{\mathsf{G}_1^u} \square_0 C_{\mathsf{G}_2^u}) \square_1 (B_{\mathsf{G}_1^d} \square_1 D_{\mathsf{G}_2^d})$$

induced by the interchange law in $\mathscr{D}$.

Theorem 58. *The object* $\mathcal{S}p_2(\mathscr{C},\mathscr{D}) = (\mathcal{S}p_2(\mathscr{C},\mathscr{D}), \times_0, \times_1, 1_0, 1_1)$ *constructed above is a duoidal V-category in the sense of Definition 3. Suppose moreover that $\mathscr{D}$ is V-complete. Then* $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ *is also V-complete.*

Proof. We leave the proof of the first part as an exercise. If V is complete, it is clear that $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ has all conical limits. Let us prove that $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ has V-cotensors if $\mathscr{D}$ has. For $v \in V$ and $Y = \{Y_{\mathsf{G}}\} \in \mathcal{S}p_2(\mathscr{C},\mathscr{D})$ put $Y^v := \{Y^v_{\mathsf{G}}\} \in \mathcal{S}p_2(\mathscr{C},\mathscr{D})$, where Y^v_{G} is the V-cotensor of $Y_{\mathsf{G}} \in \mathscr{D}$. Let us verify that this formula defines a cotensor in $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$. For $X = \{X_{\mathsf{G}}\} \in \mathcal{S}p_2(\mathscr{C},\mathscr{D})$ one has

$$\begin{aligned}\mathcal{S}p_2(X, Y^v) &\cong \textstyle\prod_{\mathsf{G}} \mathscr{D}(X_{\mathsf{G}}, Y^v_{\mathsf{G}}) \cong \prod_{\mathsf{G}} V\big(v, \mathscr{D}(X_{\mathsf{G}}, Y_{\mathsf{G}})\big) \cong V\big(v, \prod_{\mathsf{G}} \mathscr{D}(X_{\mathsf{G}}, Y_{\mathsf{G}})\big)\\ &\cong V\big(v, \mathcal{S}p_2(X,Y)\big)\end{aligned}$$

as required. □

Observe that, if $\mathscr{D} = V$, i.e. if $\square_0 = \square_1 = \otimes$ and $e = v = I$, then the structure map $1_0 \to 1_0 \times_1 1_0$ of $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ is an isomorphism. Also, the assumption of the second part of Theorem 58 is satisfied, therefore the category $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ has cotensors.

Example 59. If $\mathscr{C}$ is the one-object, one-morphism category, the duoidal category $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$ is isomorphic to the basic duoidal category $\mathscr{D}$.

Example 60. If $\mathscr{D} = \mathcal{S}et$, then $\mathcal{S}p_2(\mathscr{C}, \mathscr{D})$ is the category of derivation schemes introduced by R. Street in [35]. If, in addition, $\mathscr{C}$ is the free category on a graph G, then $\mathcal{S}p_2(\mathscr{C}, \mathscr{D})$ is the category of 2-computads in the sense of Street, whose 1-truncation is G.

7. Span categories and span operads

From now on we will assume that $\mathscr{D} = (\mathscr{D}, \Box_0, \Box_1, e, v)$ is a duoidal category which is V-complete, has coproducts, and both monoidal structures in $\mathscr{D}$ preserve coproducts in each variable.

Definition 61. A *span $\mathscr{D}$-category* is a category enriched over the V-monoidal category $\mathcal{S}p_2 = (\mathcal{S}p_2(\mathscr{C}, \mathscr{D}), \times_0, 1_0)$.

By expanding the above definition, one sees that a span category $\mathcal{A}$ consists of a class $\mathrm{Ob}(\mathcal{A})$ of objects and of span $\mathscr{D}$-sets $\mathcal{A}(\mathtt{E}, \mathtt{F})$ given for any $\mathtt{E}, \mathtt{F} \in \mathrm{Ob}(\mathcal{A})$, equipped with the composition

$$\circ : \mathcal{A}(\mathtt{E}, \mathtt{F})_{\mathsf{G}} \Box_0 \mathcal{A}(\mathtt{F}, \mathtt{G})_{\mathsf{F}} \to \mathcal{A}(\mathtt{E}, \mathtt{G})_{\mathsf{FG}} \quad \text{(a map in } \mathscr{D}) \tag{18a}$$

defined for all objects $\mathtt{E}, \mathtt{F}, \mathtt{G} \in \mathrm{Ob}(\mathcal{A})$ and globes G, F satisfying $S(\mathsf{F}) = T(\mathsf{G})$. In (18a), FG denotes the globe $\mathrm{glb}\,(S(\mathsf{G}), T(\mathsf{F}); s(\mathsf{F})s(\mathsf{G}), t(\mathsf{F})t(\mathsf{G}))$. Still more explicitly, the compositions are $\mathscr{D}$-maps

$$\circ : \mathcal{A}(\mathtt{E}, \mathtt{F})_{A \underset{g}{\overset{f}{\rightrightarrows}} B} \Box_0 \mathcal{A}(\mathtt{E}, \mathtt{F})_{B \underset{l}{\overset{h}{\rightrightarrows}} C} \to \mathcal{A}(\mathtt{E}, \mathtt{F})_{A \underset{lg}{\overset{hf}{\rightrightarrows}} B}, \tag{18b}$$

defined for arbitrary $A, B, C \in \mathscr{C}_0$ and $f, g, h, l \in \mathscr{C}_0$ for which the globes in the above display make sense.

The operation $\circ$ is assumed to fulfill the standard associativity whenever the iterated composition is defined. We also require, for each $\mathtt{E} \in \mathrm{Ob}(\mathcal{A})$, the *unit map* $i_{\mathtt{E}} \in \mathcal{A}(\mathtt{E}, \mathtt{E})$ having the standard unitality property with respect to the composition $\circ$.

Convention 62. As usual, by a *map* in an enriched category we understand a map in the underlying category (recalled below). So $i_{\mathtt{E}}$ is in fact a map in

$$\mathcal{S}p_2\big(1_0, \mathcal{A}(\mathtt{E}, \mathtt{E})\big) \cong \prod_{A \in \mathscr{C}_0} \mathscr{D}(e, \mathcal{A}(\mathtt{E}, \mathtt{E})_{\mathsf{G}(A)}), \quad \text{(cartesian product in } V)$$

where $\mathsf{G}(A)$ is the trivial globe (9a). Because $\mathscr{D}$ is V-enriched, we still have to descent one more step and interpret $i_{\mathtt{E}}$ as an element of the set

$$\mathcal{U}V\big(I, \prod_{A \in \mathscr{C}_0} \mathscr{D}(e, \mathcal{A}(\mathtt{E}, \mathtt{E})_{\mathsf{G}(A)})\big), \quad \text{(cartesian product of sets)}$$

where $\mathcal{U}V$ is the underlying category of V. This convention will be used throughout the rest of the paper.

If we interpret span objects as diagrams (10), a span category appears as a 'partial' category, in which the categorial composition $\phi \circ \psi$ of $\psi \in \mathcal{A}(\mathtt{E},\mathtt{F})$ and $\phi \in \mathcal{A}(\mathtt{F},\mathtt{G})$ is defined only if $T(\psi) = S(\phi)$. One then has

$$s(\phi \circ \psi) = s(\phi)s(\psi) \text{ and } t(\phi \circ \psi) = t(\phi)t(\psi),$$

which implies $T(\phi \circ \psi) = T(\phi)$ and $S(\phi \circ \psi) = S(\psi)$. The unit $i_{\mathtt{E}}$ is then represented by a map $i_{\mathtt{E}} : \mathscr{C}_0 \to \mathcal{A}(\mathtt{E},\mathtt{E})$ with $si_{\mathtt{E}} = ti_{\mathtt{E}} = i_{\mathscr{C}}$ such that

$$\phi \circ i_{\mathtt{E}}(S(\phi)) = i_{\mathtt{F}}(T(\phi)) \circ \phi = \phi,$$

for all $\phi \in \mathcal{A}(\mathtt{E},\mathtt{F})$ and $\mathtt{E},\mathtt{F} \in \mathrm{Ob}(\mathcal{A})$.

The *underlying category* of a span (= $\mathcal{S}p_2$-enriched) category $\mathcal{A}$ is defined in the usual manner as the V-enriched category $\mathcal{U}\mathcal{A}$ with the same set of objects, and morphism $\mathcal{U}\mathcal{A}(\mathtt{E},\mathtt{F}) := \mathcal{S}p_2(1_0, \mathcal{A}(\mathtt{E},\mathtt{F}))$. It follows from definition that

$$\mathcal{U}\mathcal{A}(\mathtt{E},\mathtt{F}) = \prod_{A \in \mathscr{C}_0} \mathscr{D}\big(e, \mathcal{A}(\mathtt{E},\mathtt{F})_{\mathsf{G}(A)}\big), \text{ (the product in } V)$$

where $\mathsf{G}(A)$ is the trivial A-globe (9a). In the diagrammatic interpretation (10) of span objects one has

$$\mathcal{U}\mathcal{A}(\mathtt{E},\mathtt{F}) := \{\lambda : \mathscr{C}_0 \to \mathcal{A}(\mathtt{E},\mathtt{F});\ s\lambda = t\lambda = i_{\mathscr{C}}\}.$$

So, the underlying category $\mathcal{U}\mathcal{A}$ of a span W-category $\mathcal{A}$ is a V-enriched category. It therefore has its own underlying category $\mathcal{U}^2\mathcal{A} := \mathcal{U}(\mathcal{U}\mathcal{A})$, which is this time an ordinary category (no enrichment). Objects of a span category $\mathcal{A}$ are *isomorphic* if and only if they are isomorphic as objects of $\mathcal{U}^2\mathcal{A}$.

Example 63. If $\mathscr{C}$ is the initial one-object, one-morphism category, then span $\mathscr{D}$-categories over $\mathscr{C}$ are ordinary $(\mathscr{D}, \square_0, e)$-enriched categories.

7.1. **Monoidal span-categories.** Monoidal categories over a duoidal category were introduced in Section 3. Here we address the particular case of the duoidal category $\mathcal{S}p_2(\mathscr{C},\mathscr{D})$. The 1-*product* of span $\mathscr{D}$-categories $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathscr{C}$ is the span V-category $\mathcal{A}_1 \times_1 \mathcal{A}_2$ over $\mathscr{C}$ whose class of objects is the cartesian product $\mathrm{Ob}(\mathcal{A}_1) \times \mathrm{Ob}(\mathcal{A}_2)$. The morphisms are

(19a) $$(\mathcal{A}_1 \times_1 \mathcal{A}_2)(\mathtt{E}_1 \times \mathtt{E}_2, \mathtt{F}_1 \times \mathtt{F}) := \mathcal{A}_1(\mathtt{E}_1,\mathtt{F}_1) \times_1 \mathcal{A}_2(\mathtt{E}_2,\mathtt{F}_2).$$

Explicitly, for a globe G and objects $\mathtt{E}_i, \mathtt{F}_i \in \mathcal{A}_i$, $i = 1, 2$, we have

(19b) $$(\mathcal{A}_1 \times_1 \mathcal{A}_2)(\mathtt{E}_1 \times \mathtt{E}_2, \mathtt{F}_1 \times \mathtt{F})_{\mathsf{G}} := \coprod_{\mathsf{G}_1,\mathsf{G}_2} \mathcal{A}_1(\mathtt{E}_1,\mathtt{F}_1)_{\mathsf{G}_1} \square_1 \mathcal{A}_2(\mathtt{E}_2,\mathtt{F}_2)_{\mathsf{G}_2},$$

with the coproduct over all globes $\mathsf{G}_1, \mathsf{G}_2$ as in (15).

Loosely speaking, the set of morphisms $(\mathcal{A}_1 \times_1 \mathcal{A}_2)(\mathtt{E}_1 \times \mathtt{E}_2, \mathtt{F}_1 \times \mathtt{F}_2)$ is generated by the products $\phi_1 \square_1 \phi_2 \in \mathcal{A}_1(\mathtt{E}_1,\mathtt{F}_1) \square_1 \mathcal{A}_2(\mathtt{E}_2,\mathtt{F}_2)$ satisfying $t(\phi_1) = s(\phi_2)$, see Figure 1. The

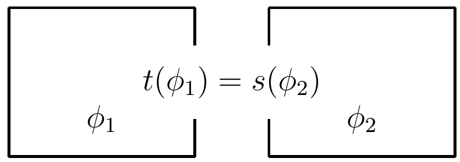


FIGURE 1. The source-target conditions for generators $\phi_1 \square_1 \phi_2 \in \mathcal{A}_1(\mathtt{E}_1, \mathtt{F}_1) \square_1 \mathcal{A}_2(\mathtt{E}_2, \mathtt{F}_2)$.

categorical composition is defined componentwise in the obvious manner. One clearly has, for span categories $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, an isomorphism

$$(\mathcal{A}_1 \times_1 \mathcal{A}_2) \times \mathcal{A}_3 \cong \mathcal{A}_1 \times_1 (\mathcal{A}_2 \times \mathcal{A}_3),$$

but, in general, $\mathcal{A}_1 \times_1 \mathcal{A}_2 \ncong (\mathcal{A}_2 \times_1 \mathcal{A}_1)$. The category $\mathbf{1}_v$ with one object 1 and the span set of morphisms $\mathbf{1}_v(1,1) := v$ is the unit for the multiplication $\times_1$.

Definition 64. A *span monoidal $\mathscr{D}$-category* is a $\mathcal{S}p_2(\mathscr{C}, \mathscr{D})$-monoidal category in the sense of Definition 21.

Observation 65. The functor $\eta : \mathbf{1}_v \to \mathcal{K}$ in Definition 21 is specified by an object $\mathtt{e} := \eta(1)$ together with a $\mathscr{C}_1$-family of $\mathscr{D}$-morphisms $v \to \mathcal{K}(\mathtt{e}, \mathtt{e})_{\mathsf{G}(f)}$, with $\mathsf{G}(f)$ as in (9b).

7.2. **Span operads.**

Definition 66. A *span operad* is a 1-operad, in the sense of Definition 33, in the duoidal category $\mathcal{S}p_2(\mathscr{C}, \mathscr{D})$.

Expanding the above definition, we see that a span operad is an $\mathcal{S}p_2$-collection $X = \{X(n)\}_{n \geq 0}$ such that, for $n \geq 1$, $k_1, \ldots, k_n \geq 0$, and globes G, G_i, $1 \leq i \leq n$, that satisfy

$$S(\mathsf{G}) = T(\mathsf{G}_1) = \cdots = T(\mathsf{G}_n)$$

and

$$t(\mathsf{G}_1) = s(\mathsf{G}_2),\ t(\mathsf{G}_2) = s(\mathsf{G}_3), \ldots, t(\mathsf{G}_{n-1}) = s(\mathsf{G}_n),$$

one has a $\mathscr{D}$-map

$$\gamma : \big(X(k_1)_{\mathsf{G}_1} \square_1 \cdots \square_1 X(k_n)_{\mathsf{G}_n}\big) \square_0 X(n)_{\mathsf{G}} \longrightarrow X(k_1 + \cdots + k_n)_{\mathsf{G}(\mathsf{G}_1, \ldots, \mathsf{G}_n)}, \tag{20}$$

where $\mathsf{G}(\mathsf{G}_1, \ldots, \mathsf{G}_n) := \mathrm{glb}\,(S(\mathsf{G}), T(\mathsf{G}_1); s(\mathsf{G})s(\mathsf{G}_1), t(\mathsf{G})t(\mathsf{G}_n))$, satisfying May's associativity (which, in this case, includes also the distributivity law in $\mathscr{D}$) whenever the iterated compositions are defined.

An operad unit is given by a map[2] $j \in \mathcal{S}p_2\big(e, X(1)\big)$, i.e. by maps $j_{\mathsf{G}(A)} \in \mathscr{D}\big(e, X(1)_{\mathsf{G}(A)}\big)$, $A \in \mathscr{C}_0$, such that

$$\gamma(x, j_{\mathsf{G}(B)}, \ldots, j_{\mathsf{G}(B)}) = x \ \text{ and } \ \gamma(j_{\mathsf{G}(A)}, x_1) = x_1,$$

for each $x \in X(n)$, $x_1 \in X(1)$ and $A, B \in \mathscr{C}_0$ such that $S(x) = B$ and $T(x_1) = A$.

[2]In the sense of Convention 62.

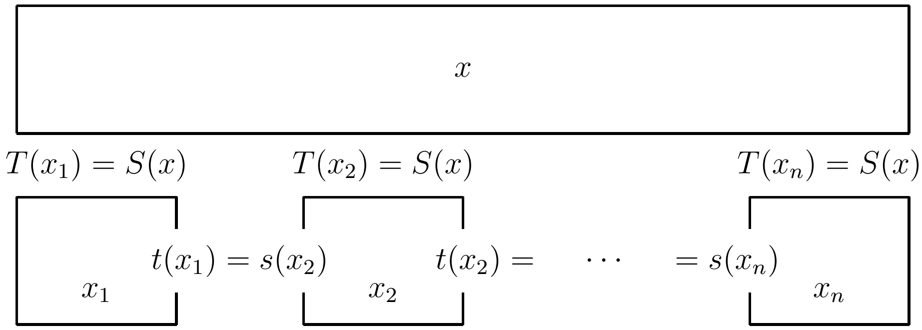


FIGURE 2. The source-target conditions of elements for which $\gamma(x, x_1, \ldots, x_n)$ is defined. The composition is assumed from the bottom to the top.

Informally, a span operad is a 'partial' operad with the composition $\gamma(x, x_n, \ldots, x_1)$ defined only if the *source and target conditions*

$$S(x) = T(x_1) = \cdots = T(x_n)$$

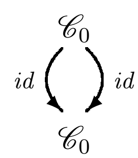
and

$$t(x_1) = s(x_2),\ t(x_2) = s(x_3), \ldots,\ t(x_{n-1}) = s(x_n),$$

are satisfied, see Figure 2.

Example 67. If $\mathscr{C}$ is the one object, one morphism category, then a span operad is a 1-operad in the duoidal category $\mathscr{D}$ in the sense of Definition 33.

Example 68. If $\mathscr{D} = V = \mathcal{S}et$, we immediately see from the above explicit description that a span operad is exactly a **fc**-operad of Leinster [31] with discrete graph of colors



whose 0-truncation is **Id**-operad $\mathscr{C}$.

Example 69. According to Example 35, one has a span operad $\underline{\mathcal{A}ss}$ with $\underline{\mathcal{A}ss}(n) := 1_1$ for $n \geq 0$. There is also a Forcey span-operad $\underline{\mathcal{A}s}$ with $\underline{\mathcal{A}s}(n) := 1_0$. If $\mathscr{C}$ is the initial one-object one-arrow category and $\mathscr{D} = V$, then $\underline{\mathcal{A}ss}$ and $\underline{\mathcal{A}s}$ coincide and are the non-Σ operads for unital associative V-algebras.

Example 70. Each object $\mathtt{E}$ of a span monoidal category $(\mathcal{A}, \odot, \mathtt{e})$ determines the span *endomorphism operad* $\mathcal{E}nd_{\mathtt{E}} = \{\mathcal{E}nd_{\mathtt{E}}(n)\}_{n \geq 0}$ with $\mathcal{E}nd_{\mathtt{E}}(n) := \mathcal{A}(\mathtt{E}^{\odot n}, \mathtt{E})$. We put, by definition, $\mathtt{E}^{\odot 0} := \mathtt{e}$ so $\mathcal{E}nd_{\mathtt{E}}(0) = \mathcal{A}(\mathtt{e}, E)$. The structure operations are given in an obvious way. The operadic unit $j : e \to \mathcal{E}nd_E(1) = \mathcal{A}(\mathtt{E}, \mathtt{E})$ is the unit map of the category $\mathcal{A}$.

8. The monoidal span category $\mathcal{J}(O,\mathcal{K})$

In this section we describe a construction providing examples of span categories. More precisely, we us fix a monoidal $\mathscr{D}$-category $\mathcal{K} = (\mathcal{K}, \odot, \eta)$ which admits coproducts and $\odot$ preserves them in each variable. Let us fix also a functor $O : \mathscr{C} \to \mathcal{S}et$. We will construct a monoidal span category $\mathcal{J}(O,\mathcal{K})$ which will play the rôle of the basis monoidal category for the construction of the Tamarkin complex.

Let $\mathcal{G}l(\mathscr{C})$ be a category of globes in $\mathscr{C}$, that is, the category whose set of objects is $\mathscr{C}_0$ and whose set of arrows is the set of globes in $\mathscr{C}$. The composition and identities are obvious.

Let $F : \mathcal{G}l(\mathscr{C}) \to \mathcal{C}at(\mathscr{D})$ be a functor. The enriched version of the Grothendieck construction $\int F$ is the category whose objects are pairs (A, X) where $A \in \mathscr{C}_0$ and X is an object of $F(A)$. The enriched hom $\int F\big((A,X),(B,Y)\big)$ is defined as

$$\coprod_{\mathsf{G} \in \mathcal{G}l(\mathscr{C})(A,B)} F(B)\big(F(\mathsf{G})(X), Y\big).$$

There is a projection $\pi_0 : (\int F)_0 \to \mathscr{C}_0$. One therefore has a span category $\Gamma(F)$ whose objects are sections of the map $\pi_0 : (\int F)_0 \to \mathscr{C}_0$. Given two such sections $\mathtt{E}, \mathtt{F}$, one puts

$$\Gamma(F)(\mathtt{E},\mathtt{F}) := \coprod_{A,B \in \mathscr{C}_0} \int F\big(\mathtt{E}(A), \mathtt{F}(B)\big)$$

with the obvious structure of a span $\mathscr{D}$-object. The composition and identities are obvious.

Now we will construct a canonical spanification functor $Sp(O,\mathcal{K}) : \mathcal{G}l(\mathscr{C}) \to \mathcal{C}at(\mathscr{D})$ out of a functor $O : \mathscr{C} \to \mathcal{S}et$ and a monoidal $\mathscr{D}$-category $\mathcal{K}$. The objects of the category $Sp(O,\mathcal{K})(A)$ are $\mathcal{K}$-spans of the form

$$\begin{array}{ccc} & X & \\ {}^{s}\swarrow & & \searrow^{t} \\ O(A) & & O(A) \end{array}, \quad A \in \mathscr{C}_0, \tag{22}$$

i.e. collections $X = \{X(a', a'')\}$ of objects of $\mathcal{K}$ indexed by elements $a', a'' \in O(A)$. The $\mathscr{D}$-enriched homs in $Sp(O,\mathcal{K})(A)$ are given by

$$Sp(O,\mathcal{K})(A)(X,Y) := \prod_{a',a'' \in O(A)} \mathcal{K}\big(X(a',a''), Y(a',a'')\big).$$

It is easy to see that the monoidal $\mathscr{D}$-structure $\odot$ of $\mathcal{K}$ induces a monoidal $\mathscr{D}$-structure $\circledast$ on $Sp(O,\mathcal{K})(A)$ by the formula

$$(X \circledast Y)(a', a'') := \coprod_{a \in O(A)} X(a', a) \odot Y(a, a''). \tag{23}$$

For a globe as in (8), the functor $Sp(O,\mathcal{K})(\mathsf{G})$ maps a span (22) to the span

$$O(B) \xleftarrow{O(f)} O(A) \xleftarrow{s} X \xrightarrow{t} O(A) \xrightarrow{O(g)} O(B).$$

So we have a span category $\Gamma(Sp(O,\mathcal{K}))$. This category is the main ingredient for the construction of the Tamarkin complex, so we describe it explicitly.

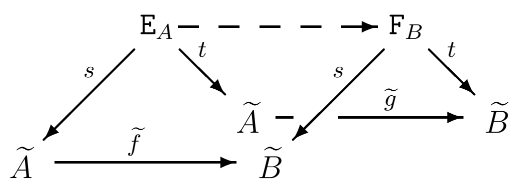
To simplify the notation, we will denote O by $\widetilde{(-)} : \mathscr{C} \to \mathcal{S}et$ and $\Gamma(Sp(O,\mathcal{K}))$ will be denoted $\mathfrak{J}(O,\mathcal{K})$. Objects of $\mathfrak{J}(O,\mathcal{K})$ are $\mathscr{C}_0$-families of $\mathcal{K}$-enriched spans

$$\begin{array}{ccc} & \mathtt{E}_A & \\ {}^{s}\swarrow & & \searrow^{t} \\ \widetilde{A} & & \widetilde{A} \end{array} \quad , \quad A \in \mathscr{C}_0, \tag{24}$$

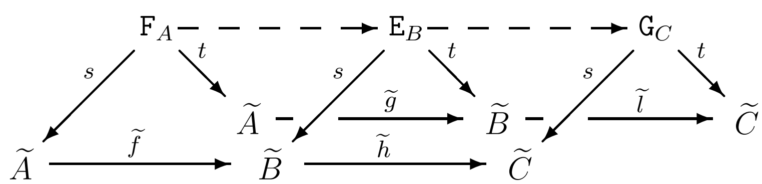
i.e. families $\mathtt{E} = \{\mathtt{E}_A(a',a'')\}$, $a',a'' \in \widetilde{A}$, $A \in \mathscr{C}_0$, $\mathtt{E}(a',a'') \in \mathcal{K}$. The span $\mathscr{D}$-sets of morphisms $\mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{F}) = \{\mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{F})_{\mathtt{G}}\}$ have fibers the products

$$\mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{F})_{A \overset{f}{\underset{g}{\rightrightarrows}} B} := \prod_{a',a'' \in \widetilde{A}} \mathcal{K}\Big(\mathtt{E}_A(a',a''), \mathtt{F}_B(\widetilde{f}(a'), \widetilde{g}(a''))\Big) \tag{25}$$

of $\mathscr{D}$-enriched homs in $\mathcal{K}$. Less formally, (25) is the set of dashed arrows in the commutative diagram



The structure maps (18b) are then compositions of the dashed arrows in



In terms of the fibers, the structure maps are compositions of the maps in the following display:

$$\prod_{a',a''\in\widetilde{A}} \mathcal{K}\left(\mathtt{E}_A(a',a''), \mathtt{F}_B(\widetilde{f}(a'),\widetilde{g}(a''))\right) \Box_0 \prod_{b',b''\in\widetilde{B}} \mathcal{K}\Big(\mathtt{F}_B(b',b''), \mathtt{G}_C(\widetilde{h}(b'),\widetilde{l}(b''))\Big)$$
$$\downarrow$$
$$\prod_{a',a''\in\widetilde{A}} \mathcal{K}\left(\mathtt{E}_A(a',a''), \mathtt{F}_B(\widetilde{f}(a'),\widetilde{g}(a''))\right) \Box_0 \mathcal{K}\left(\mathtt{F}_B(\widetilde{f}(a'),\widetilde{g}(a'')), \mathtt{G}_C(\widetilde{hf}(a')), \widetilde{lg}(a''))\right)$$
$$\downarrow$$
$$\prod_{a',a''\in\widetilde{A}} \mathcal{K}\left(\mathtt{E}_A(a',a''), \mathtt{G}_C(\widetilde{hf}(a'), \widetilde{lg}(a''))\right).$$

The upper map above is the canonical one and the lower map is the categorial composition in $\mathcal{K}$. The unit map $i_{\mathtt{E}}$ in $\mathcal{S}p_2(e, \mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{E})$ is the the product in

$$\prod_{A\in\mathscr{C}_0} \mathscr{D}\big(e, \mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{E})_{\mathtt{G}(A)}\big) = \prod_{A\in\mathscr{C}_0} \prod_{a',a''\in\widetilde{A}} \mathscr{D}\big(e, \mathcal{K}(\mathtt{E}_A(a',a''), \mathtt{E}_A(a',a''))\big)$$

of the enriched units $i_{\mathtt{E}(a',a'')} \in \mathscr{D}\big(e, \mathcal{K}(\mathtt{E}_A(a',a''), \mathtt{E}_A(a',a''))\big)$ of the category $\mathcal{K}$.

Example 71. The following particular case will be relevant to our interpretation of the Tamarkin construction addressed in Section 10. Let $\mathscr{C}$ be the category of small dg-categories, $\mathcal{K} = \mathscr{D} = V = \mathcal{C}hain$ and $O : \mathscr{C} \to \mathcal{S}et$ be the object functor. The objects of the corresponding category $\mathfrak{J}(O,\mathcal{K})$ will then be collections of chain complexes $\mathtt{E} = \{\mathtt{E}_A(a',a'')\}$, indexed by objects $a', a'' \in A$ of dg-categories $A \in \mathscr{C}_0$.

We will often drop the indices $A, B, C, ... \in \mathscr{C}_0$ and write simply $\{\mathtt{E}(a',a'')\}$ instead of $\{\mathtt{E}_A(a',a'')\}$, &c.

Let us prove that the category $\mathfrak{J}(O,\mathcal{K})$ constructed above has a natural monoidal structure. The functor $\circledast : \mathfrak{J}(O,\mathcal{K}) \times_1 \mathfrak{J}(O,\mathcal{K}) \to \mathfrak{J}(O,\mathcal{K})$ assigns to objects $\mathtt{E}_1 = \{\mathtt{E}_1(a',a'')\}$ and $\mathtt{E}_2 = \{\mathtt{E}_2(a',a'')\}$ of $\mathfrak{J}(O,\mathcal{K})$ the object $\mathtt{E}_1 \circledast \mathtt{E}_2 = \{(\mathtt{E}_1 \circledast \mathtt{E}_2)(a',a'')\} \in \mathfrak{J}(O,\mathcal{K})$ where in the right hand side we use the 'local' product defined by (23). Informally, $\mathtt{E}_1 \circledast \mathtt{E}_2$ is the $\mathscr{C}_0$-family of the pull-backs

$$\begin{array}{ccccccc} & & & \mathtt{E}_1 \circledast \mathtt{E}_2 & & & \\ & & \swarrow & & \searrow & & \\ & \mathtt{E}_1 & & & & \mathtt{E}_2 & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & & {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ \widetilde{A} & & & \widetilde{A} & & & \widetilde{A} \end{array} .$$

Before we explain how the functor $\circledast$ acts on morphisms, we need to expand some definitions. For objects $\mathtt{E}_1, \mathtt{E}_2, \mathtt{F}_1, \mathtt{F}_2 \in \mathfrak{J}(O,\mathcal{K})$ and a globe $\mathsf{G} = \mathrm{glb}\,(A,B;f,g) \in \mathcal{G}\mathrm{l}(\mathscr{C})$, one sees that the fiber $(\mathfrak{J}(O,\mathcal{K}) \times_1 \mathfrak{J}(O,\mathcal{K}))(\mathtt{E}_1 \times \mathtt{E}_2, \mathtt{F}_1 \times \mathtt{F}_2)_{\mathsf{G}}$ of the mapping space in $\mathfrak{J}(O,\mathcal{K}) \times_1 \mathfrak{J}(O,\mathcal{K})$ equals

$$\coprod_{l \in \mathscr{C}(A,B)} \left(\prod_{a'_1, a''_1 \in \widetilde{A}} \mathcal{K}\big(\mathtt{E}_1(a'_1,a''_1), \mathtt{F}_1(\widetilde{f}(a'_1), \widetilde{l}(a''_1))\big) \,\square_1 \prod_{a'_2, a''_2 \in \widetilde{A}} \mathcal{K}\big(\mathtt{E}_2(a'_2,a''_2), \mathtt{F}_2(\widetilde{l}(a'_2), \widetilde{g}(a''_2))\big) \right),$$

since all globes G_1, G_2 such that $t(\mathsf{G}_1) = s(\mathsf{G}_2)$ as in (19b) have the form

$$\mathsf{G}_1 = \mathrm{glb}\,(A,B;f,l)\,,\ \mathsf{G}_2 = \mathrm{glb}\,(A,B;l,g)\,,$$

for some $l : A \to B \in \mathscr{C}_1$. On the other hand, the fiber $\mathfrak{J}(O,\mathcal{K})(\mathtt{E}_1 \circledast \mathtt{E}_2, \mathtt{F}_1 \circledast \mathtt{F}_2)_{\mathsf{G}}$ of the hom space in $\mathfrak{J}(O,\mathcal{K})$ equals

$$\prod_{a',a'' \in \widetilde{A}} \mathcal{K}\Big(\coprod_{a \in \widetilde{A}} \mathtt{E}_1(a',a) \odot \mathtt{E}_2(a,a''), \coprod_{a \in \widetilde{A}} \mathtt{F}_1(\widetilde{f}(a'),a) \odot \mathtt{F}_2(a, \widetilde{g}(a''))\Big).$$

To define the functor $\circledast$ on morphisms, one needs to specify, for objects $\mathtt{E}_1, \mathtt{E}_2, \mathtt{F}_1, \mathtt{F}_2 \in \mathfrak{J}(O,\mathcal{K})$ and a globe G as above, a $\mathscr{D}$-morphism

$$\circledast : (\mathfrak{J}(O,\mathcal{K}) \times_1 \mathfrak{J}(O,\mathcal{K}))(\mathtt{F}_1 \times \mathtt{F}_2, \mathtt{E}_1 \times \mathtt{E}_2)_{\mathsf{G}} \to \mathfrak{J}(O,\mathcal{K})(\mathtt{F}_1 \circledast \mathtt{F}_2, \mathtt{E}_1 \circledast \mathtt{E}_2)_{\mathsf{G}}.$$

One defines this $\mathscr{D}$-morphism as the composition of the canonical maps

$$\coprod_{l\in\mathscr{C}(A,B)}\left(\prod_{a_1',a_1''\in\widetilde{A}}\mathcal{K}\big(\mathtt{E}_1(a_1',a_1''),\mathtt{F}_1(\widetilde{f}(a_1'),\widetilde{l}(a_1''))\big)\ \Box_1 \prod_{a_2',a_2''\in\widetilde{A}}\mathcal{K}\big(\mathtt{E}_2(a_2',a_2''),\mathtt{F}_2(\widetilde{l}(a_2'),\widetilde{g}(a_2''))\big)\right)$$

$$\downarrow$$

$$\coprod_{l\in\mathscr{C}(A,B)}\ \prod_{a_1',a_1'',a_2',a_2''\in\widetilde{A}}\Big(\mathcal{K}\big(\mathtt{E}_1(a_1',a_1''),\mathtt{F}_1(\widetilde{f}(a_1'),\widetilde{l}(a_1''))\big)\,\Box_1\,\mathcal{K}\big(\mathtt{E}_2(a_2',a_2''),\mathtt{F}_2(\widetilde{l}(a_2'),\widetilde{g}(a_2''))\big)\Big)$$

$$\downarrow$$

$$\coprod_{l\in\mathscr{C}(A,B)}\ \prod_{a_1',a_1'',a_2',a_2''\in\widetilde{A}}\mathcal{K}\big(\mathtt{E}_1(a_1',a_1'')\odot\mathtt{E}_2(a_2',a_2''),\ \mathtt{F}_1(\widetilde{f}(a_1'),\widetilde{l}(a_1''))\odot\mathtt{F}_2(\widetilde{l}(a_2'),\widetilde{g}(a_2''))\big)$$

$$\downarrow$$

$$\coprod_{l\in\mathscr{C}(A,B)}\ \prod_{a_1',a,a_2''\in\widetilde{A}}\mathcal{K}\big(\mathtt{E}_1(a_1',a)\odot\mathtt{E}_2(a,a_2''),\ \mathtt{F}_1(\widetilde{f}(a_1'),\widetilde{l}(a))\odot\mathtt{F}_2(\widetilde{l}(a),\widetilde{g}(a_2''))\big)$$

$$\downarrow$$

$$\prod_{a_1',a,a_2''\in\widetilde{A}}\mathcal{K}\big(\mathtt{E}_1(a_1',a)\odot\mathtt{E}_2(a,a_2''),\ \coprod_{a'\in\widetilde{A}}\mathtt{F}_1(\widetilde{f}(a_1'),a')\odot\mathtt{F}_2(a',\widetilde{g}(a_2''))\big)$$

$$\downarrow$$

$$\prod_{a',a''\in\widetilde{A}}\mathcal{K}\big(\coprod_{a\in\widetilde{A}}\mathtt{E}_1(a',a)\odot\mathtt{E}_2(a,a''),\ \coprod_{a\in\widetilde{A}}\mathtt{F}_1(\widetilde{f}(a'),a)\odot\mathtt{F}_2(a,\widetilde{g}(a''))\big).$$

Observe the necessity of the source-target condition $t(\mathsf{G}_1)=l=s(\mathsf{G}_2)$ for the existence of the above composed map.

The first piece of data specifying the unit functor $\eta:\mathbf{1}\to\mathfrak{J}(O,\mathcal{K})$ as in Observation 65 is the object $\mathtt{e}=\{\mathtt{e}(a',a'')\}\in\mathfrak{J}(O,\mathcal{K})$ defined by

$$\mathtt{e}(a',a''):=\begin{cases}\eta, & \text{if } a'=a'', \text{ and}\\ 0, & \text{otherwise.}\end{cases}\tag{26}$$

In the diagrammatic language, $\mathtt{e}$ is the span

$$\begin{array}{ccc} & \widetilde{A} & \\ {\scriptstyle id}\swarrow & & \searrow{\scriptstyle id} \\ \widetilde{A} & & \widetilde{A}\end{array}.$$

To define $v\to\mathfrak{J}(O,\mathcal{K})(\mathtt{e},\mathtt{e})_{\mathsf{G}(f)}$, notice that, for $f\in\mathscr{C}_1$,

$$\mathfrak{J}(O,\mathcal{K})(\mathtt{e},\mathtt{e})_{\mathsf{G}(f)}=\prod_{a',a''\in\widetilde{A}}\mathcal{K}(\mathtt{e}(a',a''),\mathtt{e}(\widetilde{f}(a'),\widetilde{f}(a''))\cong\prod_{a\in\widetilde{A}}\mathcal{K}(\eta,\eta).$$

With this identification, the $\mathscr{D}$-morphism $v\to\mathfrak{J}(O,\mathcal{K})(\mathtt{e},\mathtt{e})_{\mathsf{G}(f)}$ is the product of the $\mathscr{D}$-morphisms $v\to\mathcal{K}(\eta,\eta)$ of (1).

The underlying category $\mathcal{U}\mathfrak{J}(O,\mathcal{K})$ has the same objects as $\mathfrak{J}(O,\mathcal{K})$, i.e. families of $\mathcal{K}$-spans $\mathtt{E}=\{\mathtt{E}_A\}$, $A\in\mathscr{C}_0$, as in (24). We leave as an exercise to prove that $\mathcal{U}\mathfrak{J}(O,\mathcal{K})(\mathtt{E},\mathtt{F})$ consists

$\mathscr{C}_0$-families $\{\varphi_A : \mathtt{E}_A \to \mathtt{F}_A\}$ of morphisms of $\mathcal{K}$-enriched spans, with the componentwise composition. By Proposition 22, $\mathcal{U}\mathcal{J}(O,\mathcal{K})$, is a monoidal $\mathscr{D}$-category.

9. Factorization of functors, and monoids in $\mathcal{J}(O,\mathcal{K})$

In this section we analyze a correspondence between monoids in the span monoidal category $\mathcal{J}(O,\mathcal{K})$ introduced and further studied in Section 8, and factorizations of the defining functor $O = \widetilde{\ } : \mathscr{C} \to \mathcal{S}et$. Let $\mathcal{G}rph(\mathcal{K})_0$ be the set of $\mathcal{K}$-enriched graphs, i.e. objects

$$\begin{array}{ccc} & A & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ S & & S \end{array}$$

in which S is a set and A a collection of objects of $\mathcal{K}$ indexed by $S \times S$. Suppose that the object map $\widetilde{(-)}_0 : \mathscr{C}_0 \to \mathcal{S}et_0$ of the functor $\widetilde{(-)} : \mathscr{C} \to \mathcal{S}et$ factorizes as

$$\begin{array}{ccc} & & \mathcal{G}rph(\mathcal{K})_0 \\ & \overset{L_0}{\nearrow} & \downarrow{\scriptstyle vrt_0} \\ \mathscr{C}_0 & \xrightarrow{\widetilde{(-)}_0} & \mathcal{S}et_0 \end{array} \tag{27a}$$

where vrt_0 assigns to each $\mathcal{K}$-graph its set of vertices. This factorization determines a *distinguished object* of $\mathcal{J}(O,\mathcal{K})$, namely the $\mathscr{C}_0$-family $\mathtt{M} = \{\mathtt{M}_A\}$, with $\mathtt{M}_A := L_0(A)$, for $A \in \mathscr{C}_0$.

Suppose that there is a map $F_0 : \mathscr{C}_0 \to \mathcal{C}at(\mathcal{K})_0$ assigning to each object $A \in \mathscr{C}_0$ a small $\mathcal{K}$-category F_0A such that $\widetilde{(-)}_0 : \mathscr{C}_0 \to \mathcal{S}et_0$ further factorizes as

$$\begin{array}{ccc} \mathcal{C}at(\mathcal{K})_0 & \xrightarrow{gr_0} & \mathcal{G}rph(\mathcal{K})_0 \\ {\scriptstyle F_0}\uparrow & & \downarrow{\scriptstyle vrt_0} \\ \mathscr{C}_0 & \xrightarrow{\widetilde{(-)}_0} & \mathcal{S}et_0 \end{array} \tag{27b}$$

where gr_0 is the underlying graph map.

Proposition 72. *Suppose that the map $\widetilde{(-)}_0 : \mathscr{C}_0 \to \mathcal{S}et_0$ factorizes as in (27b). Then the distinguished object $\mathtt{M} \in \mathcal{J}(O,\mathcal{K})$ constructed above is a monoid in the underling category $\mathcal{U}\mathcal{J}(O,\mathcal{K})$.*

Proof. A monoid structure on $\mathtt{M}$ is given by $\mathcal{S}p_2$-maps $\overline{\mu} : e \to \mathcal{J}(O,\mathcal{K})(\mathtt{M} \circledast \mathtt{M}, \mathtt{M})$ and $\overline{\nu} : e \to \mathcal{J}(O,\mathcal{K})(\mathtt{e},\mathtt{M})$. It is an exercise on definitions that these maps are given by specifying, for each $A \in \mathscr{C}_0$, elements

$$\overline{\mu}_{\mathsf{G}(A)} \in \prod_{a',a,a'' \in \widetilde{A}} \mathcal{K}\big(F_0A(a',a) \odot F_0A(a,a''), F_0A(a',a'')\big) \ \text{ and } \ \overline{\nu}_{\mathsf{G}(A)} \in \prod_{a \in \widetilde{A}} \mathcal{K}\big(\eta, F_0A(a,a)\big),$$

where $\mathsf{G}(A)$ is as in (9a). Since F_0A is a $\mathcal{K}$-category with the set of objects $\widetilde{A}$, one can take as $\overline{\mu}_{\mathsf{G}(A)}$ the element determined by the $\mathcal{K}$-category composition of F_0A and as $\overline{\nu}_{\mathsf{G}(A)}$ the element determined by the $\mathcal{K}$-category identities of F_0A. One easily verifies that this choice gives a monoid in $\mathcal{U}\mathcal{J}(O,\mathcal{K})$. □

Assume that the factorization (27b) is induced by a factorization

$$\begin{array}{ccc} \mathcal{C}at(\mathcal{K}) & \xrightarrow{gr} & \mathcal{G}rph(\mathcal{K}) \\ {\scriptstyle F}\uparrow & & \downarrow{\scriptstyle vrt} \\ \mathscr{C} & \xrightarrow{\;\sim\;} & \mathcal{S}et \end{array} \tag{27c}$$

of the *functor* $\widetilde{\ } : \mathscr{C} \to \mathcal{S}et$ via *functors*. One then has

Proposition 73. *Suppose that the functor* $\widetilde{\ } : \mathscr{C} \to \mathcal{S}et$ *factorizes via functors as in (27c). Then the object* $\mathtt{M} \in \mathcal{J}(O,\mathcal{K})$ *is a monoid, in the sense of Definition 29, in the span-monoidal category* $\mathcal{J}(O,\mathcal{K})$*.*

Proof. Since factorization (27c) implies factorization (27b) $\mathtt{M}$ has, by Proposition 72, an induced structure of a monoid in $\mathcal{U}\mathcal{J}(O,\mathcal{K})$. By Proposition 40, it remains to specify an $\mathcal{S}p_2$-map $u : v \to \mathcal{J}(O,\mathcal{K})(\mathtt{M},\mathtt{M})$. Such a map is determined by a choice, for each $A \stackrel{f}{\to} B \in \mathscr{C}_1$, of an element

$$u_{\mathsf{G}(f)} \in \prod_{a',a'' \in \widetilde{A}} \mathscr{D}\big(v, \mathcal{K}\big(FA(a',a''), FB(f(a'),f(a''))\big)\big),$$

where $\mathsf{G}(f)$ is as in (9b). We take as $u_{\mathsf{G}(f)}$ the product of the $\mathscr{D}$-maps $F(f)_v$ of (3) determining the functor $F(f) : FA \to FB$. □

The correspondences described above can be organized into the scheme:

functor $\widetilde{\ } : \mathscr{C} \to \mathcal{S}et$	$\longmapsto$	category $\mathcal{J}(O,\mathcal{K})$
factorization (27a)	$\longmapsto$	object of $\mathcal{J}(O,\mathcal{K})$
factorization (27b)	$\longmapsto$	monoid in the underlying category $\mathcal{U}\mathcal{J}(O,\mathcal{K})$
factorization (27c)	$\longmapsto$	monoid in $\mathcal{J}(O,\mathcal{K})$

The table above can be 'categorified' as follows. Let us denote by $\mathcal{F}ct_1(O)$ the category whose objects are factorizations L_0 as in (27a) and whose morphisms $L_0' \to L_0''$ are $\mathscr{C}_0$-families $\{\alpha_A : L_0'(A) \to L_0''(A)\}$ of graph morphisms. Likewise, let $\mathcal{F}ct_2(O)$ be the category whose objects are factorizations F_0 as in (27b) and morphisms $F_0' \to F_0''$ are $\mathscr{C}_0$-families $\{\beta_A : F_0'(A) \to F_0''(A)\}$ of $\mathcal{K}$-functors. Finally, let $\mathcal{F}ct_3(O)$ be the category of functors F as in (27c), and their natural transformations.

Proposition 74. *One has the following natural isomorphisms of categories:*

$$\mathcal{F}ct_1(O) \cong \text{ the underlying category } \mathcal{U}\mathfrak{I}(O,\mathcal{K}) \text{ of } \mathfrak{I}(O,\mathcal{K}),$$
$$\mathcal{F}ct_2(O) \cong \text{the category of monoids in } \mathcal{U}\mathfrak{I}(O,\mathcal{K}),$$
$$\mathcal{F}ct_3(O) \cong \text{the category of monoids in } \mathfrak{I}(O,\mathcal{K}).$$

The correspondence $O \to \mathfrak{I}(O,\mathcal{K})$ behaves functorially as well:

Proposition 75. *The correspondence $O \mapsto \mathfrak{I}(O,\mathcal{K})$ extends to a contravariant functor $\mathfrak{I}(\mathcal{K})$ from the category of functors $[\mathscr{C}, \mathcal{S}et]$ and their natural transformations to the category of monoidal span $\mathscr{D}$-categories and their span $\mathscr{D}$-functors.*

Proof. To prove the proposition, we need to construct in a functorial manner, for an arbitrary natural transformation $\Phi : O_1 \to O_2$ of functors $O_1, O_2 : \mathscr{C} \to \mathcal{S}et$, a span-functor $\Phi^* : \mathfrak{I}(O_2) \to \mathfrak{I}(O_1)$.

Let $\mathtt{E} = \{\mathtt{E}_A(a', a'')\}$, $A \in \mathscr{C}_0$, $a', a'' \in O_2(A)$, be an object of $\mathfrak{I}(O_2,\mathcal{K})$. We then define $\Phi^*\mathtt{E} \in \mathfrak{I}(O_1,\mathcal{K})$ to be the object $\Phi^*\mathtt{E} = \{\Phi^*\mathtt{E}_A(b', b'')\}$, $A \in \mathscr{C}_0$, $b', b'' \in O_1(A)$, with

$$\Phi^*\mathtt{E}_A(b', b'') := \mathtt{E}_A(\Phi_A(b'), \Phi_A(b'')),$$

where $\Phi_A : O_1(A) \to O_2(A)$ is the set map induced by the transformation Φ. To finish the definition of Φ^* we need to specify, for each globe in $\mathscr{C}$ and each $\mathtt{E}, \mathtt{F} \in \mathfrak{I}(O_2,\mathcal{K})$, a $\mathscr{D}$-map

$$\mathfrak{I}(O_2,\mathcal{K})(\mathtt{E},\mathtt{F})_{A \overset{f}{\underset{g}{\rightrightarrows}} B} \longrightarrow \mathfrak{I}(O_1,\mathcal{K})(\Phi^*\mathtt{E},\Phi^*\mathtt{F})_{A \overset{f}{\underset{g}{\rightrightarrows}} B}.$$

Expanding definitions, we see that we need to construct a $\mathscr{D}$-map from the product

$$\prod_{a',a'' \in O_2(A)} \mathscr{D}\left(\mathtt{E}_A(a', a''), \mathtt{F}_B(O_2(f)(a'), O_2(g)(a''))\right) \tag{28}$$

to the product

$$\prod_{b',b'' \in O_1(A)} \mathscr{D}\left(\mathtt{E}_A(\Phi_A(b'), \Phi_A(b''), \mathtt{F}_B(\Phi_B O_1(f)(b'), \Phi_B O_1(g)(b''))\right).$$

Since, of course, $\Phi_B O_1(f)(b') = O_2(f)(\Phi_A(b'))$ and $\Phi_B O_1(g)(b'') = O_2(g)(\Phi_A(b''))$, the product in the last display equals

$$\prod_{b',b'' \in O_1(A)} \mathscr{D}\left(\mathtt{E}_A(\Phi_A(b'), \Phi_A(b''), \mathtt{F}_B(O_2(f)(\Phi_A(b')), O_2(g)(\Phi_A(b'')))\right), \tag{29}$$

so we need to construct a map from the product (28) to the product (29). We take the map induced by the map

$$\Phi_A \times \Phi_A : O_1(A) \times O_1(A) \to O_2(A) \times O_2(A)$$

of the indexing sets. It is not difficult to verify that the above constructions indeed assemble into a span $\mathscr{D}$-functor $\Phi^* : \mathfrak{I}(O_2,\mathcal{K}) \to \mathfrak{I}(O_1,\mathcal{K})$. □

10. Tamarkin complex of a functor $F : \mathscr{C} \to \mathcal{C}at(\mathcal{K})$

Let $F : \mathscr{C} \to \mathcal{C}at(\mathcal{K})$ be a functor. Then we have factorization (27c) and therefore a distinguished monoid $\mathtt{M}(F) \in \mathfrak{J}(O, \mathcal{K})$. Let $\delta : \Delta \to V$ be a fixed cosimplicial object in V.

Definition 76. The *Tamarkin complex* of a functor $F : \mathscr{C} \to \mathcal{C}at(\mathcal{K})$ relative to δ is the δ-center $CH_\delta(F, F) := CH_\delta(\mathtt{M}(F), \mathtt{M}(F))$.

Example 77. Let $\mathscr{C} = 1$. A functor $\mathcal{C} : 1 \to \mathcal{C}at(\mathcal{K})$ picks up a $\mathcal{K}$-category $\mathcal{C}$. Let $\delta = I$ so the δ-center of a monoid is its center. Then $CH_\delta(\mathcal{C}, \mathcal{C})$ can be identified with the duoid of $\mathcal{K}$-natural transformations of the identity functor $Id : \mathcal{C} \to \mathcal{C}$. If $\mathcal{K} = \mathscr{D} = V$, we get the classical center of $\mathcal{C}$.

Example 78. Take, in the previous example, $\mathcal{K} = \mathscr{D} = V = \mathcal{C}hain$ and δ as in Example 52. Then $CH_\delta(\mathcal{C}, \mathcal{C})$ is the classical Hochschild complex of the dg-category $\mathcal{C}$ [25].

The following result shows that the Tamarkin complex is a powerful tool for constructing enrichments.

Theorem 79. *Let $\mathscr{C} = \mathcal{C}at(\mathcal{K})$ and $F = Id : \mathcal{C}at(\mathcal{K}) \to \mathcal{C}at(\mathcal{K})$. Let again $\delta = I$. Then $CH_\delta(Id, Id)$ is a duoid in $\mathcal{S}p_2(\mathscr{C}, \mathscr{D})$ (i.e. a $\mathscr{D}$-enriched 2-category) with the property that its underlying duoid $u\big(CH_\delta(Id, Id)\big)$ is equal to $\mathcal{C}at(\mathcal{K})$, the 2-category of $\mathcal{K}$-categories, $\mathcal{K}$-functors and $\mathcal{K}$-natural transformations.*

Proof. Direct verification. □

From now on we will use the notation $\mathcal{C}at(\mathcal{K})$ for the $\mathscr{D}$-enriched 2-category of $\mathcal{K}$-categories, $\mathcal{K}$-functors and $\mathcal{K}$-natural transformations. Theorem 79 provides a classical interpretation of the center of a monoid as the object of natural transformations of the identity functor. Indeed, if $\mathtt{M}$ is a monoid in $\mathcal{K}$, then $\Sigma(\mathtt{M})$ is a one object $\mathcal{K}$-category. Remark 28 shows that

$$\mathcal{C}at(\mathcal{K})(\Sigma(\mathtt{M}), \Sigma(\mathtt{M}))$$

is a monoidal $\mathscr{D}$-category with the unit object given by the identity functor Id.

Corollary 80. *The following four duoids in $\mathscr{D}$ are naturally isomorphic:*

1. *the center $Z(\mathtt{M})$ of a monoid $\mathtt{M}$ in $\mathcal{K}$,*
2. *the center $Z(Id)$ in $\mathcal{C}at(\mathcal{K})\big(\Sigma(\mathtt{M}), \Sigma(\mathtt{M})\big)$,*
3. *the duoid $\mathcal{C}at(\mathcal{K})\big(\Sigma(\mathtt{M}), \Sigma(\mathtt{M})\big)(Id, Id)$ in $\mathscr{D}$ and*
4. *the duoid $CH_I\big(\Sigma(\mathtt{M}), \Sigma(\mathtt{M})\big)$ (see Example 77).*

Proof. By Example 45, $Z(Id) \in \mathscr{D}$ equals $\mathcal{C}at(\mathcal{K})(\Sigma(\mathtt{M}), \Sigma(\mathtt{M}))(Id, Id)$. On the other hand, we establish by direct calculation that this object coincides with the equalizer (7) and, therefore, is the center $Z(\mathtt{M})$. □

Example 81. If $\mathcal{K} = \mathscr{D} = V = \mathcal{S}et$, then $CH_I(Id, Id)$ is the 2-category of categories $\mathcal{C}at$ with its cartesian closed structure. On the other hand, if we define $\delta : \Delta \to \mathcal{S}et$ in dimension n as the set $\{0, \ldots, n\}$ with the obvious coface and codegeneracy operators, then $CH_\delta(Id, Id)$ is the sesquicategory of categories, functors and their unnatural transformations (so it is $\mathcal{C}at$ with its second closed symmetric monoidal structure [27]).

Example 82. Let $\mathcal{K} = \mathscr{D} = V = \mathcal{C}at$ with its cartesian closed monoidal structure and δ be as in Example 50. Let $F = Id : \mathcal{C}at(\mathcal{C}at) \to \mathcal{C}at(\mathcal{C}at)$. Then $CH_\delta(Id, Id)$ is the **Gray**-category **Gray** of 2-categories, 2-functors and pseudonatural transformations [22].

Example 83. Replacing δ from (50) by δ from (51), we obtain a nonsymmetric version of **Gray** which consists of 2-categories, 2-functors and lax-natural (or colax-natural if we change the orientation in δ) transformations [23].

We end up this section by showing that when $\mathcal{K} = \mathscr{D} = V$ is the category $\mathcal{C}hain$ of chain complexes, $\mathscr{C}$ is the category $\mathcal{C}at(\mathcal{C}hain)$ of small dg-categories, $F = Id : \mathcal{C}at(\mathcal{C}hain) \to \mathcal{C}at(\mathcal{C}hain)$ and δ is again as in Example 52, the resulting Tamarkin complex indeed coincides with the original Tamarkin's construction $\mathbf{Rhom}(-,-)$ from [36, Definition 3.0.2]. Let us recall its definition.

For small dg-categories A, B and dg-functors $f, g : A \to B$, one defines, for each $n \geq 0$,

$$\mathbf{hom}^n(f, g) := \prod_{a_0,\ldots,a_n \in A} \mathcal{C}hain\big(A(a_0, a_1) \otimes \cdots \otimes A(a_{n-1}, a_n), B(f(a_0), g(a_n))\big),$$

with the product taken over all $(n+1)$-tuples $(a_0, \ldots, a_n)$ of objects of A. As shown in [36], the objects $\mathbf{hom}^n(f, g)$ assemble into the cosimplicial chain complex $\mathbf{hom}^*(f, g)$.

Let $O = \widetilde{(-)} : \mathcal{C}at(\mathcal{C}hain) \to \mathcal{S}et$ be the object functor. As explained in Example 71, the corresponding category $\mathcal{J}(O, \mathcal{C}hain)$ consists of collections of chain complexes $\mathtt{E} = \{\mathtt{E}_A(a', a'')\}$, indexed by objects $a', a'' \in A$ of small dg-categories A.

Theorem 73 therefore gives a distinguished monoid $\mathtt{M}$ in the span-monoidal category $\mathcal{J}(O, \mathcal{C}hain)$. The monoid $\mathtt{M} = \{\mathtt{M}_A(a', a'')\}$ has a simple explicit description. For $A \in \mathscr{C}$, one has

$$\mathtt{M}_A(a', a'') := A(a', a''),$$

the $\mathcal{C}hain$-enriched hom-functor in A. To specify a monoid structure of $\mathtt{M}$, one needs to choose, for any dg-category $A \in \mathscr{C}$ and any dg-functor $f : A \to B \in \mathscr{C}$, the following three pieces of data:

$$\overline{\mu}_{\mathsf{G}(A)} \in \prod_{a',a,a'' \in A} \mathcal{C}hain\big(A(a', a) \otimes A(a, a''), A(a', a'')\big)$$

$$\overline{\nu}_{\mathsf{G}(A)} \in \prod_{a \in A} \mathcal{C}hain\big(k, A(a, a)\big), \text{ and}$$

$$u_{\mathsf{G}(f)} \in \prod_{a',a'' \in A} \mathcal{C}hain\big(A(a', a''), B(f(a'), f(a''))\big).$$

The element $\overline{\mu}_{\mathsf{G}(A)}$ is given by the enriched composition in A, $\overline{\nu}_{\mathsf{G}(A)}$ by the enriched unit and $u_{\mathsf{G}(f)}$ is part of the definition of the $\mathcal{C}hain$-enriched functor f.

One can consider, as in Definition 38, the endomorphism 1-operad $\mathcal{E}nd_{\mathtt{M}}$ of $\mathtt{M}$. The monoid structure of $\mathtt{M}$ is, by Proposition 40, equivalent to a 1-operad map $\underline{\mathcal{A}ss} \to \mathcal{E}nd_{\mathtt{M}}$, i.e. $\mathcal{E}nd_{\mathtt{M}}$ is a multiplicative operad in the sense of Definition 41. By Proposition 42, $\mathcal{E}nd_{\mathtt{M}}$ therefore carries a natural structure of a cosimplicial object in $\mathcal{C}hain$. The following statement of this section is now obvious:

Theorem 84. *Let $f, g : A \to B$ be dg-functors between dg-categories. Then the cosimplicial hom-functor* $\mathbf{hom}^\bullet(f,g)$ *defined in* [36] *and recalled above, is the fiber over the globe*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad (30)$$

of the cosimplicial span-object associated to the multiplicative endomorphism 1*-operad* $\mathcal{E}nd_{\mathtt{M}}$ *of the distinguished monoid* $\mathtt{M}$ *in the span-monoidal category* $\mathcal{J}(O, \mathcal{C}hain)$.

Tamarkin defined, in [36], the right derived hom-functor $\mathbf{Rhom}(f,g)$ as the totalization $|\mathbf{hom}^\bullet(f,g)|$ of the cosimplicial hom-functor $\mathbf{hom}^\bullet(f,g)$. In the terminology of Subsection 5.2, $|\mathbf{hom}^\bullet(f,g)|$ is therefore the G-fiber, where G is the globe in (30), of $CH_\delta(Id, Id)$.

11. The Deligne conjecture in monoidal $\mathscr{D}$-categories

For n a positive integer, n-operads are higher analogs of (nonsymmetric) operads. Their pieces have arities given by trees with n-levels. While ordinary operads live in monoidal categories, n-operads live in augmented monoidal n-globular categories. We begin this section by introducing, for $n = 0, 1, 2$, a simplified version of n-operads tailored for the needs of the present paper. A general approach can be found in [4]. The relation between our restricted case and the general one is addressed in Remark 92.

11.1. **2-operads and their algebras in duoidal V-categories.** Let us recall the definition of the category Ω_k of k-trees, for $k \leq 2$. The category of 0-*trees* Ω_0 is the terminal category 1. Its unique object is denoted U_0.

The category of 1-*trees* Ω_1 is the category of finite ordinals $(n) := \{1, \ldots, n\}$, $n \geq 0$, and their order-preserving maps. As usual, we interpret $\{1, \ldots, n\}$ for $n = 0$ as the empty set. The terminal object of Ω_1 is denoted $U_1 := (1)$. When the meaning is clear from the context, we will simplify the notation and denote the object $(n) \in \Omega_1$ simply by n.

Notice that Ω_1 is isomorphic to the 'algebraic' version Δ_{alg} of the basic simplicial category Δ, i.e. to Δ augmented by the empty set, and can be characterized as the free strict monoidal category generated by a monoid. The category Ω_1 can also be interpreted as the subcategory of open maps of Joyal's (skeletal) category of intervals $\mathcal{I}$, whereas Δ is isomorphic to $\mathcal{I}^{op}$, see [12, Section 2].

The definition of the category Ω_2 of 2-trees is more involved. A *2-tree* T is a morphism $t : n \to m$ in Ω_1. *Leaves of height* 2 of the tree T are, by definition, elements from $\{1, \ldots, n\}$. *Leaves of height* 1 of T are those elements $i \in \{1, \ldots, m\}$ for which $t^{-1}(i) = \emptyset$. The set of all leaves of the tree T has a natural linear order defined by counting leaves when we are going around the tree in the clockwise direction.

There are exactly two 2-trees with one leaf. The tree $U_2 := (1 \to 1)$ has one leaf of height 2 while the tree $zU_1 := (0 \to 1)$ has one leaf of height 1. The tree $z^2U_0 := (0 \to 0)$ has no leaves. A *map of 2-trees*

$$\sigma : T = (n \to m) \to S = (p \to q) \tag{31}$$

is a commutative diagram of maps in $\mathcal{S}et$:

$$\begin{array}{ccc} \{1, \ldots, n\} & \xrightarrow{t} & \{1, \ldots, m\} \\ \Big\downarrow{\scriptstyle \sigma_2} & & \Big\downarrow{\scriptstyle \sigma_1} \\ \{1, \ldots, p\} & \xrightarrow{s} & \{1, \ldots, q\} \end{array}$$

such that

(i) σ_1 is order preserving and
(ii) for any $i \in \{1, \ldots, m\}$, the restriction of σ_2 to $t^{-1}(i)$ is order preserving.

We denote by Ω_2 the category of 2-*trees*. Its terminal object is the tree $U_2 = (1 \to 1)$. The category Ω_2 is a monoidal category with the structure $+$ given by the fiberwise ordinal sum (gluing the roots of 2-trees in geometric terms). The unit of this monoidal product is z^2U_0.

Remark 85. There are obvious truncation functors

$$\Omega_2 \stackrel{\partial}{\to} \Omega_1 \stackrel{\partial}{\to} \Omega_0. \tag{32}$$

If we consider them as the source-target functors $s = t = \partial$, then (32) becomes a strict monoidal 2-globular category i.e. the 2-categorical object in $\mathcal{C}at$. It can be characterized as being the free strict monoidal globular 2-category generated by an internal 2-category [13].

Similarly, one can characterize the strict monoidal 1-globular category

$$\Omega_1 \stackrel{\partial}{\to} \Omega_0$$

as the free strict monoidal globular 1-category generated by an internal 1-category. Notice that this universal property is an 'extension' of the universal property of Δ_{alg} since any strict monoidal category can be considered as a one object monoidal globular 1-category [4].

Any morphism of 2-trees $\sigma : T \to S$ as in (31) has its *fibers*. Given a leaf $i \in \{1, \ldots, p\}$ of height 2, the restriction of t determines an order preserving map

$$t_i : \sigma_2^{-1}(i) \to \sigma_1^{-1}(s(i))$$

which we consider as a 2-tree and call the *fiber* over the leaf i. In the case of a leaf $j \in \{1, \ldots, q\}$ of height 1, $\sigma_1^{-1}(j)$ is an ordered subset of $\{1, \ldots, m\}$ which determines a 1-tree in Ω_1, the *fiber* over the leaf j. With a slight abuse of notation, we will denote this fiber by $\sigma_1^{-1}(j)$ and use the same convention throughout the rest of this section.

Since the set of leaves of S has a natural linear order, the set of fibers of σ also inherits this order. So, for a $\sigma : T \to S$ we will denote $T_1, \ldots, T_k$ the set of its fibers in this order. Let us fix $n \in \{0, 1, 2\}$.

In item $(\star)$ of the next definition we consider the composition $T \stackrel{\sigma}{\to} S \stackrel{\omega}{\to} R$ of maps of n-trees, $n \leq 2$. We will use the following notation. Let $S_1, \ldots, S_k$ be the fibers of ω and $T_i := \sigma^{-1}(S_i)$, $1 \leq i \leq k$. Denote also by $\sigma_i : T_i \to S_i$ the restriction $\sigma|_{T_i}$ and $T_{i,1}, \ldots, T_{i,m_i}$ the fibers of σ_i. Clearly $T_1, \ldots, T_k$ are precisely the fibers of the composition $\omega\sigma$ and

$$\{T_{1,1}, \ldots, T_{1,m_1}, \ldots, T_{k,1}, \ldots, T_{k,m_k}\}$$

the set of fibers of σ.

In the following definition where $V = (V, \otimes, I)$ is the basic monoidal category, we introduce our restricted version of n-operads. The terminology will be justified in Remark 92.

Definition 86. Let $0 \leq n \leq 2$. An *n-operad in* $V^{(n)}$ is a collection $A(T),\ T \in \Omega_i\ ,\ i \leq n$, of objects of V equipped with the following structure:

(i) morphisms $\xi_i : I \to A(U_i)\ ,\ i \leq n$ (the units);
(ii) for every morphism $\sigma : T \to S$ in $\Omega_i,\ i \leq n$, with fibers $T_1, \ldots, T_k$ a morphism

$$m_\sigma : A(T_1) \otimes \cdots \otimes A(T_k) \otimes A(S) \to A(T) \ \ \text{(the multiplication)}.$$

The structure operations are required to satisfy the following conditions.

$(\star)$ For any composite $T \stackrel{\sigma}{\to} S \stackrel{\omega}{\to} R$, the associativity diagram

$$\begin{array}{ccc}
\bigotimes_{1 \leq i \leq k} \big(A(T_{i,\bullet}) \otimes A(S_i)\big) \otimes A(R) & \xrightarrow{\bigotimes_{i=1}^k m_{\sigma_i} \otimes id} & A(T_\bullet) \otimes A(R) \\
 & & \downarrow m_{\omega\sigma} \\
\simeq \updownarrow & & A(T) \\
 & & \uparrow m_\sigma \\
\bigotimes_{1 \leq i \leq k} A(T_{i,\bullet}) \otimes A(S_\bullet) \otimes A(R) & \xrightarrow{id \otimes m_\omega} & \bigotimes_{1 \leq i \leq k} A(T_{i,\bullet}) \otimes A(S)
\end{array}$$

in which

$$\begin{aligned} A(S_\bullet) &:= A(S_1) \otimes \cdots \otimes A(S_k), \\ A(T_{i,\bullet}) &:= A(T_{i,1}) \otimes \cdots \otimes A(T_{i,m_i}),\ 1 \leq i \leq k, \quad \text{and} \\ A(T_\bullet) &:= A(T_1) \otimes \cdots \otimes A(T_k); \end{aligned}$$

commutes.

($\star\star$) For the identity $\sigma = id : T \to T$, the diagram

$$\begin{array}{ccc} A(U_{i_0}) \otimes \cdots \otimes A(U_{i_n}) \otimes A(T) & \longleftarrow & I \otimes \cdots \otimes I \otimes A(T) \\ \downarrow{\scriptstyle m_{id}} & \swarrow{\scriptstyle id} & \\ A(T) & & \end{array}$$

commutes.

($\star\star\star$) For $0 \leq i \leq n$ and the unique morphism $T \to U_i$ in Ω_i, the diagram

$$\begin{array}{ccc} A(T) \otimes A(U_i) & \longleftarrow & A(T) \otimes I \\ \downarrow & \swarrow{\scriptstyle id} & \\ A(T) & & \end{array}$$

commutes.

Example 87. A 0-operad in $V^{(0)}$ consists of an object $A(U_0)$. The structure maps equip it with a monoid structure in V.

Example 88. A 1-operad A in $V^{(1)}$ is given by a nonsymmetric operad A' in V (which is the same as a 1-operad in V if we interpret V as a duoidal category) with $A'(k) := A((k))$, $k \geq 0$, and a monoid $A(U_0)$. The map of 1-trees $id : (0) \to (0)$ induces an operadic multiplication

$$A(U_0) \otimes A((0)) \to A((0))$$

which equips $A'(0)$ with a $A(U_0)$-module structure. This structure is compatible with the rest of the operadic structure of A' as in Definition 33 with v replaced by $A(U_0)$.

Definition 89. The 0-operad $\underline{\mathcal{A}ss}_0$ defined as the monoid $I \in V$. The classical associativity 1-operad $\underline{\mathcal{A}ss}_1$ is such that $\underline{\mathcal{A}ss}_1(T) := I$ for each n-tree T, $n \leq 1$. Similarly, we define the 2-operad $\underline{\mathcal{A}ss}_2$ with $\underline{\mathcal{A}ss}_2(T) := I$ for any n-tree T, $n \leq 2$, with all structure maps being the canonical isomorphisms.

Definition 90. Let $k < n$. The restriction of an n-operad A on $\Omega_i, i \leq k$, is a k-operad $tr_k(A)$ in $V^{(k)}$ called the *k-truncation* of A.

Definition 91. An n-operad in $V^{(n)}$ is called 0-*terminal* if $tr_0(A) = \underline{\mathcal{A}ss}_0$. An n-operad in $V^{(n)}$ is called 1-*terminal* if $tr_1(A) = \underline{\mathcal{A}ss}_1$.

Nonsymmetric operads in V are therefore exactly 0-terminal 1-operads in $V^{(1)}$.

Remark 92. According to [4], general n-operads live in augmented monoidal n-globular categories. The above notion of an n-operad in $V^{(n)}$ is the specialization of this general notion to the augmented monoidal n-globular category $V^{(n)}$ defined, for $n \leq 2$, as follows.

The category $V^{(0)}$ is just the category V with its monoidal structure. The category $V^{(1)}$ is the following monoidal 1-globular category: in dimension 0 we have V, in dimension 1 we have $V \times V$. The source and target functors coincide and equal to the projection on the second variable. The functor $z : V \to V \times V$ is defined by $z(x) = (I, x)$. The monoidal structure is induced the by monoidal structure of V in an obvious manner.

To construct $V^{(2)}$, we add to $V^{(1)}$ the product $V \times V \times V$ in dimension 2, with the projection to the second and third coordinates as its 1-source and 1-target functors. We leave to the interested reader to describe the rest of the augmented monoidal structure of $V^{(2)}$.

Remark 93. There is another construction of an augmented monoidal n-globular category associated with a symmetric monoidal category V. This augmented monoidal n-globular category $\Sigma^n V$ has terminal category $\mathbf{1}$ in dimensions strictly less than n and V in dimension n. An n-operad in Σ^n was called $(n-1)$-*terminal operad in* V [6]. The relation between our terminology here and the terminology of [6] is following.

There is a globular functor $\Sigma^n(V) \to V^{(n)}$ which in dimension $k < n$ sends a unique object of $\mathbf{1}$ to the $(k+1)$-tuple $(I, \ldots, I)$ and in dimension n it sends an object $X \in V$ to the tuple $(X, I, \ldots, I)$. It is not hard to check that this is an augmented monoidal globular inclusion. An n-operad in $V^{(n)}$ is $(n-1)$-terminal in the present terminology if it takes values in the subcategory $\Sigma^n(V)$. Therefore, our terminology is compatible with the terminology of [6].

A 2-tree $T = (n \stackrel{t}{\to} m)$ is called *pruned* if t is an epimorphism. Equivalently, a 2-tree is pruned if all its leaves are in height 2. Any 2-tree T contains the maximal pruned subtree $\iota : T^{(p)} \to T1$. It is obvious that the fibers of ι are U_2 or $z^2 U_0$. For any 1-terminal 2-operad A one therefore has the morphism

$$A(T) \to I \otimes \cdots \otimes I \otimes A(T) \to A(U_2) \otimes \cdots \otimes A(U_2) \otimes A(T) \to A(T^{(p)}). \tag{33}$$

Definition 94. A 1-terminal 2-operad is *pruned* if (33) is an isomorphism for any $T \in \Omega_2$.

If $\mathscr{C}$ is a V-category, then with every object $X \in \mathscr{C}$ we can associate a 0-operad $\mathcal{E}nd_{X0}$, which is just the endomorphism monoid $\mathscr{C}(X, X)$.

Let now $\mathscr{E} = (\mathscr{E}, \Box, e)$ be a monoidal V-category. For any object $X \in \mathscr{E}$ we will define its endomorphism 1-operad $\mathcal{E}nd_{X1}$ in $V^{(1)}$. To do this, we introduce the tensor power of X as follows.

(0) With a unique 0-tree U_0 we associate its tensor power as

$$X^{U_0} := e.$$

(1) With a 1-tree $n \in \Omega_1$ we associate the tensor power

$$X^n := \underbrace{X \,\square \cdots \square X}_{n}$$

with the convention

$$X^0 = X^{zU_0} := e.$$

The definition of X^n above looks tautological, but notice that n abbreviates $(n) \in \Omega_1$.

Definition 95. The endomorphism 1-operad of $X \in \mathscr{E}$ is given by

$$\mathcal{E}nd_{X1}(T) := \mathscr{E}(X^T, X^{U_i}),$$

where $T \in \Omega_i,\ i = 0, 1$.

Let now $\mathscr{D}$ be a duoidal V-category. For any object $X \in \mathscr{D}$ we will define its endomorphism 2-operad $\mathcal{E}nd_{X2}$ in $V^{(2)}$. To do this, we define first the tensor power of an object X.

(0) With a unique 0 tree U_0 we associate the tensor power

$$X^{U_0} := e;$$

(1) With a 1-tree $n \in \Omega_1$ we associate the tensor power

$$X^n := \underbrace{v \,\square_0 \cdots \square_0 v}_{n}$$

with the convention

$$X^0 = X^{zU_0} := e.$$

(2) With a 2-tree T we associate the tensor power X^T as follows. Let $T = (n \xrightarrow{t} m) \neq z^2U_0$ and let $n_i := t^{-1}(i)$ for $1 \leq i \leq m$. Then we put

$$X^T := (X^{\square_1^{n_1}}) \,\square_0 \cdots \square_0 (X^{\square_1^{n_m}}).$$

We use here the convention that $X^{\square_1^0} := v$. We complete the definition by putting

$$X^{z^2U_0} := e.$$

We believe that the 'ideological' portrait of X^T in Figure 3 clarifies our definition.

Definition 96. The endomorphism 2-operad of $X \in \mathscr{D}$ is given by

$$\mathcal{E}nd_{X2}(T) = \mathscr{D}(X^T, X^{U_i}),$$

where $T \in \Omega_i,\ i = 0, 1, 2$.

Lemma 97. *The collection $\mathcal{E}nd_{X2}(T), T \in \Omega_i, i = 0, 1, 2$, has a natural structure of a 2-operad in $V^{(2)}$.*

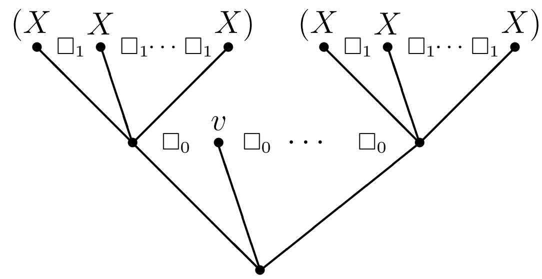


FIGURE 3. An 'ideological' picture of X^T. Leaves of height 2 (resp. 1) are decorated by X (resp. v). The decorations of vertices of height 2 (resp. 1) are then multiplied by $\square_1$ (resp. $\square_0$), with the $\square_1$-multiplication performed first.

Proof. We construct first the units $\xi_i : I \to \mathcal{E}nd_{X2}(U_i)$, $i = 0, 1, 2$. We have $\mathcal{E}nd_{X2}(U_0) = \mathscr{D}(X^{U_0}, X^{U_0}) = \mathscr{D}(e, e)$, and we define $\xi_0 := id_e : I \to \mathscr{D}(e, e)$. Analogously we define $\xi_1 := id_v : I \to \mathscr{D}(v, v)$ and $\xi_2 := id_X : I \to \mathscr{D}(X, X)$.

The 0-truncation of $\mathcal{E}nd_X$ is clearly the endomorphism monoid of $e \in \mathscr{D}$. The 1-truncation is the endomorphism operad of the monoid $v \in \mathscr{D}$ with the obvious multiplication.

To define the multiplication with respect to morphisms of 2-trees, we begin with the special case when the codomain of $\sigma : T \to S$ has the form $S = (k \to 1)$. We will say that such a tree is a *suspension* of the 1-tree k. If $k = 0$, then $T = (0 \to m)$ and the unique fiber of σ is equal to the 1-tree m. We define the operadic multiplication as the composite in $\mathscr{D}$:

$$\mathscr{D}(\underbrace{v \,\square_0 \cdots \square_0 v}_{m}, v) \otimes \mathscr{D}(v, X) \to \mathscr{D}(\underbrace{v \,\square_0 \cdots \square_0 v}_{m}, X).$$

Suppose $k > 0$ and $T = (n \to m)$. Then the fiber over a leaf $i \in \{1, \dots, k\}$ has the form $T_i = (t_i : n_i \to m)$. For $j \in \{1, \dots, m\}$ let $n_{ij} := t_i^{-1}(j)$. There is then a canonical morphism

$$X^\sigma : X^T \to X^{T_1} \,\square_1 \cdots \square_1 X^{T_k}. \tag{34}$$

To see it, we observe that

$$X^T \cong \big((X^{\square_1^{n_{1,1}}}) \,\square_1 \cdots \square_1 (X^{\square_1^{n_{k,1}}})\big) \,\square_0 \dots \square_0 \big((X^{\square_1^{n_{1,m}}}) \,\square_1 \cdots \square_1 (X^{\square_1^{n_{k,m}}})\big),$$

and

$$X^{T_i} \cong (X^{\square_1^{n_{i,1}}}) \,\square_0 \cdots \square_0 (X^{\square_1^{n_{i,m}}}),\ 1 \le i \le k.$$

We define X^σ as the interchange morphism

$$\begin{aligned} X^T &\cong \big((X^{\square_1^{n_{1,1}}}) \,\square_1 \cdots \square_1 (X^{\square_1^{n_{k,1}}})\big) \,\square_0 \cdots \square_0 \big((X^{\square_1^{n_{1,m}}}) \,\square_1 \cdots \square_1 (X^{\square_1^{n_{k,m}}})\big) \longrightarrow \\ &\longrightarrow \big((X^{\square_1^{n_{1,1}}}) \,\square_0 \cdots \square_0 (X^{\square_1^{n_{1,m}}})\big) \,\square_1 \cdots \square_1 \big((X^{\square_1^{n_{k,1}}}) \,\square_0 \cdots \square_0 (X^{\square_1^{n_{k,m}}})\big) \cong \\ &\cong X^{T_1} \,\square_1 \cdots \square_1 X^{T_k}. \end{aligned}$$

The operadic multiplication $m_\sigma : \mathcal{E}nd_{X2}(T_1) \otimes \cdots \otimes \mathcal{E}nd_{X2}(T_k) \otimes \mathcal{E}nd_{X2}(S) \to \mathcal{E}nd_{X2}(T)$ is now defined as the composition

$$\begin{aligned}&\mathscr{D}(X^{T_1}, X) \otimes \cdots \otimes \mathscr{D}(X^{T_k}, X) \otimes \mathscr{D}(X^S, X) \to\\ &\to \mathscr{D}(X^{T_1} \square_1 \cdots \square_1 X^{T_k}, X^{\square_1^k}) \otimes \mathscr{D}(X^{\square_1^k}, X) \to\\ &\to \mathscr{D}(X^T, X^S) \otimes \mathscr{D}(X^S, X) \to \mathscr{D}(X^T, X).\end{aligned}$$

The first map in the above composition exists because $\square_1$ is a V-functor, the second map is induced by X^σ.

Let now S be a general 2-tree. If $S = z^2U_0$ then $\sigma = id_{z^2U_0}$ and m_σ is simply the composite

$$\mathscr{D}(e, e) \otimes \mathscr{D}(e, X) \to \mathscr{D}(e, X).$$

Let $S \neq z^2U_0$. Then S is canonically the ordinal sum of trees, $S = P_1 + \cdots + P_l$, where P_i is, for $1 \leq i \leq l$, a suspension of a 1-tree k_i. Moreover, there obviously exist 2-trees $Q_1, \ldots, Q_l$ such that $T = Q_1 + \cdots + Q_l$ and $\sigma : T \to S$ is the sum $\sigma = \sigma_1 + \cdots + \sigma_l$, for some $\sigma_i : Q_i \to P_i$, $1 \leq i \leq l$. We denote $T_{i,j}$ the fiber of σ over a leaf $j \in P_i$.

Observe that $X^T = X^{Q_1} \square_0 \cdots \square_0 X^{Q_l}$ and $X^S = X^{P_1} \square_0 \cdots \square_0 X^{P_l}$. We now define

$$X^\sigma : X^T \to (X^{T_{1,1}} \square_1 \cdots \square_1 X^{T_{1,k_1}}) \square_0 \cdots \square_0 (X^{T_{l,1}} \square_1 \cdots \square_1 X^{T_{l,k_l}})$$

as the product $X^\sigma = X^{\sigma_1} \square_0 \cdots \square_0 X^{\sigma_l}$. Finally, we define the operadic multiplication m_σ as

$$\begin{aligned}&\mathscr{D}(X^{T_{1,1}}, X) \otimes \cdots \otimes \mathscr{D}(X^{T_{l,k_l}}, X) \otimes \mathscr{D}(X^S, X) \to\\ &\to \mathscr{D}(X^{T_{1,1}} \square_1 \cdots \square_1 X^{T_{1,k_1}}, X^{P_1}) \otimes \cdots \otimes \mathscr{D}(X^{T_{l,1}} \square_1 \cdots \square_1 X^{T_{l,k_l}}, X^{P_l}) \otimes \mathscr{D}(X^S, X) \to\\ &\to \mathscr{D}\big((X^{T_{1,1}} \square_1 \cdots \square_1 X^{T_{1,k_1}}) \square_0 \cdots \square_0 (X^{T_{l,1}} \square_1 \cdots \square_1 X^{T_{l,k_l}}), X^S\big) \otimes \mathscr{D}(X^S, X) \to\\ &\to \mathscr{D}(X^T, X^S) \otimes \mathscr{D}(X^S, X) \to \mathscr{D}(X^T, X).\end{aligned}$$

We used again that $\square_0$ and $\square_1$ are V-functors. We leave the tedious but obvious verification of the associativity of thus defined operadic multiplication to the reader. $\square$

Observe that the 1-truncation of the 2-operad $\mathcal{E}nd_{X2}$ is the endomorphism 1-operad of the monoid $v \in (\mathscr{D}, \square_0, e)$. So we have a canonical operadic map

$$k_v : \underline{\mathcal{A}ss}_1 \to tr_1(\mathcal{E}nd_{X2}).$$

Definition 98. An *algebra* of a pruned 2-operad A in $V^{(2)}$ is an object $X \in \mathscr{D}$ equipped with a map of 2-operads

$$k : A \to \mathcal{E}nd_{X2}$$

such that $tr_1(k) = k_v$.

As in Subsection 4.1, one can show that A-algebras form a V-category. Notice also that a more precise name for algebras in Definition 98 would be 1-*terminal* A-algebras, but we opted for a simpler terminology.

Example 99. We leave as an exercise for the reader to show that algebras of $\underline{\mathcal{A}ss}_2$ are exactly duoids in $\mathscr{D}$.

A proof of the following theorem will be given in [9]:

Theorem 100. *Let δ be a fixed cosimplicial object in V. Then there is a pruned 2-operad $Coend_{\mathcal{T}am_2}(\delta)$ with a canonical action on $CH_\delta(A)$ for any multiplicative 1-operad A in $\mathscr{D}$. In particular, such an action exists on the δ-center $CH_\delta(\mathtt{M},\mathtt{M})$ of a monoid $\mathtt{M}$ in a monoidal $\mathscr{D}$-category.*

If $\delta = I$, then $Coend_{\mathcal{T}am_2}(\delta) = \underline{\mathcal{A}ss}_2$ and the action of $Coend_{\mathcal{T}am_2}(\delta)$ recovers the canonical duoid structure on $CH_I(A)$ constructed in Theorem 46.

Remark 101. The notation $Coend_{\mathcal{T}am_2}(\delta)$ comes from [7]. In that paper the authors developed techniques of condensation of symmetric colored operads. The operad $Coend_{\mathcal{T}am_2}(\delta)$ is also a condensation, but we condense a colored 2-operad $\mathcal{T}am_2$ instead of a colored symmetric operad. The colored operad $\mathcal{T}am_2$ was actually constructed by Tamarkin in [36]. A different and short description of $\mathcal{T}am_2$ can be found in [7, page 25].

11.2. **Deligne's conjecture for δ-center of a monoid.** Algebras of contractible 1-operads in $\mathcal{C}hain$ are known as A_∞-algebras. In fact, we usually replace an action of a contractible 1-operad by a minimal cofibrant resolution of $\underline{\mathcal{A}ss}_1$ to get a canonical notion of an A_∞-algebra. We use the same philosophy and think about algebras of a contractible 2-operad in $V^{(2)}$ as duoids in $\mathscr{D}$ up to all higher homotopies (see Example 99).

Definition 102. Let V be a monoidal model category and $n \leq 2$. An $(n-1)$-terminal n-operad A equipped with an operad map $A \to \underline{\mathcal{A}ss}_n$ is called *contractible* if, for each n-tree T, the map $A(T) \to \underline{\mathcal{A}ss}_n(T) = I$ is a weak equivalence.

Theorem 103. *Let V be a monoidal model category and δ be a standard system of simplices for V such that the lattice path operad is strongly δ-reductive in the sense of* [7, Definition 3.7]. *Then the operad $Coend_{\mathcal{T}am_2}(\delta)$ is contractible. In particular, the δ-center $CH_\delta(\mathtt{M},\mathtt{M})$ of a monoid $\mathtt{M}$ in a monoidal $\mathscr{D}$-category is an algebra of a contractible 2-operad.*

Corollary 104 (duoidal Deligne's conjecture)**.** *There is a canonical action of a contractible 2-operad on the homotopical center of a monoid $\mathtt{M}$ which lifts the duoid structure on the center of $\mathtt{M}$.*

Theorem 104 has been proved by Tamarkin in [36] for the particular case of the Tamarkin complex of the functor

$$Id : Cat(\mathcal{C}hain) \to Cat(\mathcal{C}hain)$$

thus answering the question *'what do DG-categories form?'* in the title of that paper. The proofs of Theorems 103 and 104 will be addressed in [9].

11.3. **Relation to the classical Deligne's conjecture.** Let $\mathscr{D}$ be a cocomplete symmetric monoidal category. Then $e = v$ in $\mathscr{D}$ and, for any $X \in \mathscr{D}$ and a 2-tree T, one clearly has $X^T \cong X^{T^{(p)}}$. This implies that $\mathcal{E}nd_{X2}$ satisfies condition (33). The operad $\mathcal{E}nd_{X2}$ is, however, not 1-terminal, so it is not pruned in the sense of Definition 94. But one can still construct a modified pruned endomorphism 2-operad End_{X2} together with an operadic map $End_{X2} \to \mathcal{E}nd_{X2}$ which is an isomorphism for all trees of height 2. The operad End_{X2} is determined by these conditions uniquely. Moreover, any morphism from a pruned 2-operad A to $\mathcal{E}nd_{X2}$ can be factorized through End_{X2}.

The operad End_{X2} is the endomorphism 2-operad of X in the monoidal 2-globular category $\Sigma^2 V$. Therefore, the results of [5, 6] are applicable and an action $A \to End_{X2}$ of A on X is equivalent to a (classical) action of the symmetric operad $sym_2(A)$ on X.

If V is a monoidal model category satisfying the requirements of Theorem 8.6 or 8.7 of [5] and A is a contractible cofibrant 2-operad, then the symmetrization $sym_2(A)$ has the homotopy type of the little 2-disks operad [5]. So we have

Theorem 105. *If $\mathscr{D}$ is a symmetric monoidal V-category and the assumptions of Theorem 103 are satisfied, then $CH_\delta(A)$ of a multiplicative operad A in $\mathscr{D}$ admits a structure of an algebra of a E_2-operad. In particular, such an action exists on the homotopy center $CH(\mathtt{M},\mathtt{M})$ of a monoid $\mathtt{M}$ in a monoidal model $\mathscr{D}$-category.*

This form of Deligne's conjecture generalizes the classical one. As a corollary we have

Corollary 106. *The Hochschild complex of the dg-category $\mathcal{C}$ (see Example 78) is an algebra of an E_2-operad. In particular, if $\mathcal{C} = \Sigma A$ for a unital associative algebra A, we get the classical Deligne conjecture for the Hochschild complex of A.*

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