

# FOUNDATIONS AS SUPERSTRUCTURE<sup>1</sup>

*(Reflections of a practicing mathematician)*

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*ABSTRACT.* This talk presents foundations of mathematics as a historically variable set of principles appealing to various modes of human intuition and devoid of any prescriptive/prohibitive power. At each turn of history, foundations crystallize the accepted norms of interpersonal and intergenerational transfer and justification of mathematical knowledge.

## Introduction

**Foundations vs Metamathematics.** In this talk, I will interpret the idea of Foundations in the wide sense. For me, Foundations at each turn of history embody currently recognized, but historically variable, principles of organization of mathematical knowledge and of the interpersonal/transgenerational transferral of this knowledge. When these principles are studied using the tools of mathematics itself, we get a new chapter of mathematics, *metamathematics*.

Modern philosophy of mathematics is often preoccupied with *informal interpretations of theorems, proved in metamathematics of the XX-th century*, of which the most influential was probably Gödel’s incompleteness theorem that aroused considerable existential anxiety.

In metamathematics, Gödel’s theorem is a discovery that a certain class of finitely generated structures (statements in a formal language) contains substructures that are not finitely generated (those statements that are true in a standard interpretation).

It is no big deal for an algebraist, but certainly interesting thanks to a new context.

Existential anxiety can be alleviated if one strips “Foundations” from their rigid *prescriptive/prohibitive*, or *normative* functions and considers various foundational

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matters simply from the viewpoint of their mathematical content and on the background of whatever historical period.

Then, say, the structures/categories controversy is seen in a much more realistic light: contemporary studies fuse (Bourbaki type) structures and categories freely, naturally and unavoidably.

For example, in the definition of *abelian categories* one starts with *structurizing sets of morphisms*: they become abelian groups. In the definition of 2-categories, the sets of morphisms are even *categorified*: they become objects of categories, whose morphisms become then the *morphisms of the second level* of initial category. Since in this way one often obtains vast mental images of complex combinatorial structure, one applies to them principles of *homotopy topology* (structural study of topological structures up to homotopy equivalence) in order to squeeze it down to size etc.

I want to add two more remarks to this personal credo.

First, the recognition of quite restrictive and historically changing normative function of Foundations makes this word somewhat too expressive for its content. In a figure of speech such as “Crisis of Foundations” it suggests a looming crash of the whole building (cf. similar concerns expressed by R. Hersh, [He]).

But, second, the first “Crisis of Foundations” occurred in a very interesting historical moment, when the images of formal mathematical reasoning and algorithmic computation became so precise and detailed that they could be, and were, described as *new mathematical structures*: formal languages and their interpretations, partial recursive functions. They could easily fit Bourbaki’s universe, even if Bourbaki himself was too slow and awkward to really appreciate the new development.

At this juncture, contemporary “*foundations*” morphed into a *superstructure*, high level floor of mathematics building itself. This is the reason why I keep using the suggestive word “metamathematics” for it.

This event generated a stream of philosophical thought striving to recover the lost normative function. One of the reasons of my private mutiny against it (see e.g. [Ma1]) was my incapability to find any of the philosophical arguments more convincing that even the simplest mathematical reasonings, whatever “forbidden” notions they might involve.

In particular, whatever doubts one might have about the scale of Cantorial cardinal and ordinal infinities, the basic idea of set embodied in Cantor’s famous

“definition”, as a collection of definite, distinct objects of our thought, is as alive as ever. Thinking about a topological space, a category, a homotopy type, a language or a model, we start with imagining such a collection, or several ones, and continue by adding new types “of distinct objects of our thought”, derivable from the previous ones or embodying fresh insights.

To summarize: good metamathematics is good mathematics rather than shackles on good mathematics.

**Plan of the article.** Whatever one’s attitude to mathematical Platonism might be, it is indisputable that human minds constitute an important part of habitat of mathematics. In the first section, I will postulate three basic types of mathematical intuition and argue that one can recognize them at each scale of study: personal, interpersonal and historical ones.

The second section is concerned with historical development of the dichotomy *continuous/discrete* and evolving interrelations between its terms.

Finally, in the third section I briefly recall the discrete structures of linear languages studied in *classical metamathematics*, and then sketch the growing array of language-like non-discrete structures that gradually become the subject-matter of *contemporary metamathematics*.

## 1. Modes of mathematical intuition

**1.1. Three modes.** I will adopt here the viewpoint according to which *at the individual level* mathematical intuition, both primary and trained one, has three basic sources, that I will describe as *spatial*, *linguistic*, and *operational* ones.

The neurobiological correlates of the *spatial/linguistic* dichotomy were elaborated in the classical studies of *lateral asymmetry of brain*. When its mathematical content is objectivized, one often speaks about the opposition *continuous/discrete*.

The *linguistic/operational* dichotomy is observed in many experiments studying proto-mathematical abilities of animals. Animals, when they solve and communicate solutions of elementary problems related to counting, use not words but *actions*: cf. some expressive descriptions by Stanislas Dehaene in [De], Chapter 1: “Talented and gifted animals”. Operational mode, when it is externalized and codified, becomes a powerful tool for social expansion of mathematics. Learning by rote of “multiplication table” became almost a symbol of democratic education.

The sweeping subdivision of mathematics into Geometry and Algebra, to which at the beginning of modern era was added Analysis (or Calculus) can be considered as a correlate on the scale of whole (Western) civilization of the trichotomy that we postulated above on the scale of an individual (cf. [At]).

It is less widely recognized that *even at the civilization scale*, at various historical periods, each of the spatial, linguistic and operational modes of mathematical intuition can dominate and govern the way that basic mathematical abstractions are perceived and treated.

I will consider as an example “natural” numbers. Most of us nowadays immediately associate them with their *names*: decimal notation 1,2,3, ... , 1984, ... , perhaps completed by less systemic signs such as  $10^6$  or *XIX*.

This was decidedly not always so as the following examples stretching over centuries and millennia show.

### 1.2. Euclid and his “Elements”: spatial and operational *vs* linguistic.

For Euclid, a number was a “magnitude”, a potential result of measurement. Measurement of a geometric figure  $A$  by a “unit”, another geometric figure  $U$ , was conceived as a “physical activity in mental space”: moving a segment of line inside another segment, step by step; paving a square by smaller squares etc. Inequality  $A < B$  roughly speaking, meant that a figure  $A$  could be moved to fit inside  $B$  (eventually, after cutting  $A$  into several pieces and rearranging them in the interior of  $B$ ).

In this sense, Euclidean geometry might be conceived as “physics of solid bodies in the dimensions one, two and three” (or more precisely, after Einstein, physics *in gravitational vacuum* of respective dimension). This pervasive identification of Euclidean space with our physical space probably influenced the history of Euclid’s “fifth postulate”. This history includes repeating attempts *to prove it*, that is, to deduce properties of space “at infinity” from observable ones at a finite distance, and then only reluctantly accepting the Bolyai and Lobachevsky non-Euclidean spaces as “non-physical” ones.

As opposed to addition and subtraction, *the multiplication of numbers* naturally required passage into *a higher dimension*: multiplying two lengths, we get a surface. This was a great obstacle, but, I think, also opened for trained imagination the door to higher dimensions. At least, when Euclid has to speak about the product of an arbitrary large finite set of primes (as in his proof involving  $p_1 \dots p_n + 1$ ), he is

careful to explain his general reasoning by the case of *three* factors, but without doubt, he had some mental images overcoming this restriction.

In fact, the strength of spatial and operational imagination required and achieved by modern mathematics can be glimpsed on a series of examples, starting, say with Morse theory and reaching Perelman's proof of Poincaré conjecture. Moreover, physicists could produce such wonders as Feynman's path integral and Witten's topological invariants, which mathematicians include in their more rigidly organized world only with considerable efforts.

At first sight, it might seem strange that the notion of a *prime number*, theorem about (potential) *infinity of primes*, and theorem about *unique decomposition* could have been stated and proved by Euclid in his geometric world, when no systematic notation for integers was accepted as yet, and no computational rules *dealing with such a notation* rather than numbers themselves were available.

But trying to rationalize this historical fact, one comes to a somewhat paradoxical realization that an efficient notation, such as Hindu–Arabic numerals, actually *does not help*, and even *hinders* the understanding of properties related to divisibility, primality etc. that is, all properties that refer to numbers themselves rather than their *names*.

In fact, the whole number theory could come into being only unencumbered by any efficient *notation* for numbers.

**1.3. “Algorist and Abacist”:** *linguistic vs operational*. The dissemination of a positional number system in Europe after the appearance of Leonardo Fibonacci's *Liber Abaci* (1202) was, in essence, the beginning of the expansion of a universal, truly global language. Its final victory took quite some time.

The book by Gregorio Reisch, *Margarita Philosophica*, was published in Strasbourg in 1504. One engraving in this book shows a female figure symbolizing Arithmetics. She contemplates two men, sitting at two different tables, an *abacist* and an *algorist*.

The abacist is bent over his *abacus*. This primitive calculating device survived until the days of my youth: every cashier in any shop in Russia, having accepted a payment, would start calculating change clicking movable balls of her abacus.

The algorist is computing something, writing Hindu–Arabic numerals on his desk. The words “algorist” and modern “algorithm” are derived from the name of the great Al Khwarezmi (born in Khorezm c. 780).

In the context of this subsection, abacus illustrates the operational mode whereas computations with numerals do the same for linguistic one (although in other contexts the operational side of such computations might dominate).

This engraving in the reception of contemporary readers was more politicised. It symbolized coming of a new epoch of democratic learning.

Catholic Church supported the Roman tradition, usage of Roman numerals. They were fairly useless for practical commercial bookkeeping, calender computations such as dates of Easter and other moveable feasts etc. Here abacus was of great help.

The competing tribe of algorists were able to compute things by writing strange signs on paper or sand, and their art was associated with dangerous, magical, secret Muslim knowledge. Al Khwarezmi teaching became their (and our) legacy.

Arithmetics blesses both practitioners.

**1.4. John Napier and Alan Turing: operational.** The nascent programming languages for centuries existed only as informal subdialects of a natural language. They had a very limited (but crucially important) sphere of applicability, and were addressed to human calculators, not electronic or mechanical ones. Even Alan Turing in the 20th century, when speaking of his universal formalization of computability, later called Turing machine, used the word “computer” to refer to *a person* who mechanically follows a finite list of instructions lying before him/her.

The ninety–page table of natural logarithms that John Napier published in his book *Mirifici Logarithmorum Canonis Descriptio* in 1614 was a paradoxical example of this type of activity that became a cultural and historical monument on a global scale. Napier, who computed the logarithms manually, digit by digit, combined in one person the role of creator of new mathematics and that of computer–clerk who followed his own instructions. His assistant Henry Briggs later performed this function.

Napier’s tables were tables of (approximate values of) *natural logarithms*, with the base  $e = 2,718281828\dots$ . However, it seems that he neither referred to  $e$  explicitly, nor even recognized its existence. Roughly speaking, after having chosen the precision which he wanted to calculate logarithms, say with error  $< 10^{-7}$ , he dealt with integer powers of the number  $1 + 10^{-8}$ , whose  $10^8$  power was close to  $e$ .

This is one more example of the seemingly paradoxical fact, that an efficient and unified notation for objects of mathematical world can *hinder a theoretical understanding* of this world.

All the more amazing was the philosophical insight of Leibniz, who in his famous exhortation *Calculamus!* postulated that not only numerical manipulations, but any rigorous, logical sequence of thoughts that derives conclusions from initial axioms can be reduced to computation. It was the highest achievement of the great logicians of the 20th century (Hilbert, Church, Gödel, Tarski, Turing, Markov, Kolmogorov,...) to draw a precise map of the boundaries of the Leibnizian ideal world, in which

*reasoning is equivalent to computation;*

*truth can be formalized, but cannot always be verified formally;*

*the “whole truth” even about the smallest infinite mathematical universe, natural numbers, exceeds potential of any finitely generated language to generate true theorems.*

The central concept of this program, *formal languages*, inherited the basic features of both natural languages (written form fixed by an alphabet) and the positional number systems of arithmetic. In particular, any classical formal language is one-dimensional (linear) and consists of discrete symbols that explicitly express the basic notions of logic.

Euclid found the remedy for the deficiencies of this linearity by strictly restricting role of natural language to the expression of *logic* of his proofs. The *content* of his mathematical imagination was transmitted by pictures.

## 2. Continuous or discrete?

### From Euclid to Cantor to homotopy theory

**2.1. From continuous to discrete in “Elements”.** As we have seen, integers (and a restricted amount of other real numbers) for Euclid were results of (mental) measurement: *discrete came from continuous*. This was one-way road: continuous could not be produced from discrete. The idea that a line “consists” of points, so familiar to us today, does not seem to belong to Euclid’s mental world and, in fact, to mental worlds of many subsequent generations of mathematicians until Georg Cantor. For Euclid, a point can be (a part of) the boundary of a (segment) of line, but such a segment cannot be scattered to a heap of points.

Geometric images are the source and embodiment not only of numbers, but of logical reasoning as well: in “Elements” at least a comparable part of its logic is encoded in figures rather than in words.

This was made very clear in the London publication of 1847, entitled

The first six books of  
*THE ELEMENTS OF EUCLID*  
 in which coloured diagrams and symbols  
 are used instead of letters for the  
 greater ease of learners

whose author was Oliver Byrne, “*Surveyor of her Majesty’s settlements in the Falklands Islands*”.

Byrnes literally writes algebraic formulas whose main components are triangles, colored sectors of circle, segments of line etc. connected by more or less conventional algebraic signs.

**2.2. From discrete to continuous: Cantor, Dedekind, Hausdorff, Bourbaki ...** This way is so familiar to my contemporaries that I do not have to spend much time to its description. The description of a mathematical structure, such as a group, or a topological space, according to Bourbaki starts with one or several unstructured sets, to which one adds elements of a these sets or derived sets satisfying restrictions formulated in terms of set theory.

Thus the twentieth century idea of “continuous” is based upon two parallel notions: that of *topological space*  $X$  (a set with the system of “open” subsets) and that of a “continuous map”  $f : X \rightarrow Y$  between topological spaces. Further elaboration involving sheaves, topoi etc does not part with this basic intuition.

However, the set–theoretic point of depart helped enrich the geometric intuition by images that were totally out of reach earlier. The discovery of difference between *continuous* and *measurable* (from Lebesgue integral to Brownian motion to Feynman integral) was a radical departure from Euclidean universe.

In a finite–dimensional context, one could now think about Cantor sets, Hausdorff dimension and fractals, curves filling a square, Banach–Tarski theorem. In infinite–dimensional contexts wide new horizons opened, starting with topologies of Hilbert and Banach linear spaces and widening in an immense universe of topology and measure theory of non–linear function spaces.

**2.3. From continuous to discrete: homotopy theory.** One of the most important development of topology was the discovery of main definitions and results of homotopy theory. Roughly speaking, a homotopy between two topological spaces



$X, Y$  is a continuous deformation producing  $Y$  from  $X$ , and similarly a homotopy between two continuous maps  $f, g : X \rightarrow Y$  is a continuous deformation producing  $g$  from  $f$ . A *homotopy type* is the class of spaces that are homotopically equivalent pairwise. To see how drastically the homotopy can change a space, one can note that a ball, or a cube, of any dimension is *contractible*, that is, can be homotopically deformed to a point, so that dimension ceases to be invariant.

The basic discrete invariant of the homotopy type of  $X$  is the set of its connected components  $\pi_0(X)$ . To see, how this invariant gives rise to one of the basic structures of mathematics, ring of integers  $\mathbf{Z}$ , consider a real plane  $P$  with a fixed orientation, a point  $x_0$  on it, different from  $(0, 0)$ , and the set of homotopy classes of loops (closed paths) in  $P$ , starting and ending at  $x_0$  and avoiding  $(0, 0)$ . This latter set can be canonically identified with  $\mathbf{Z}$ : just count the number of times the loop goes around  $(0, 0)$ . Each loop going in the direction of orientation counts as  $+1$ , where as the “counter-clockwise” loops count as  $-1$ .

On a very primitive level, this identification shows how the ideas of homotopy naturally introduce negative numbers. In the historically earlier periods when integers were measuring geometric figures (or counting real/mental objects) even idea of zero was very difficult and slowly gained ground in the symbolic framework of positional notation. Introduction of negative numbers required appellation to an extra-mathematical reality, such as *debt* in economics.

More generally, Voevodsky in his research project [Vo] introduces the following hierarchy of homotopy types graded by their  $h$ -levels. Zero level homotopy type consists of one point representing contractible spaces. If types of level  $n$  are already defined, types of level  $n + 1$  consist of spaces such that the space of paths between any two points belongs to type of level  $n$ .

He further interprets types of level 1, represented by one point and empty sets, as *truth values*, and types of level 2 as sets. All sets in this universe are thus of the form  $\pi_0(X)$ .

Higher levels are connected with theory of categories, poly-categories etc, and we will return to them in the next section. At this point, we mention only that Voevodsky hierarchy does not *replace* sets but rather systematically embeds set-theoretical and categorical constructions and intuitions into a vaster universe where continuous and discrete are treated on an equal footing.

### 3. Language–like mathematical structures and metamathematics

**3.1. Metamathematics: mathematical studies of formalized languages of mathematics.** Philosophy of mathematics in the XX–th century had to deal with lessons of *metamathematics*, especially of Gödel’s incompleteness theorem.

As I have already said, I will consider metamathematics as *a special chapter of mathematics itself*, whose subject is the study of *formal languages and their interpretations*. On the foreground here were the first order formal languages, a formalization of Euclid’s and Aristotle’s legacy. Roughly speaking, to Euclid we owe the mathematics of spatial imagination (and/or kinematics of solid bodies), whereas Aristotle founded *the mathematics of logical deduction*, expressed in “Elements” by natural language *and* creative usage of drawings.

An important parallel development of formal languages involved languages formalizing *programs for and processes of computation*, of which chronologically first in the XX–th century was *Church’s lambda calculus*.

An important feature of lambda calculus is the absence of formal distinctions between the language of programs and the language of input/output data (unlike Turing’s machines, where a machine “is” the program, whereas input/output are represented by binary words). When, due to von Neumann’s insights, this feature became implemented in hardware, lambda calculus was rediscovered and became in the 1960’s the basis of development of programming languages.

These languages are *linear*, in the following sense: the set of all syntactically correct expressions in a formal language  $L$  could be described as a Bourbaki structure consisting of a certain *words*, – finite sequences of letters in a given *alphabet*, and finite sequences of such words, *expressions*. Words and expressions must be *syntactically correct* (precise description of this is a part of definition of each concrete language). Letters of alphabet are subdivided into *types*: variables, connectives and quantifiers, symbols for operations, relations ... Syntactically correct expressions can be terms, formulas, ...

Such Bourbaki structures can be sufficiently rich to produce formal versions of real mathematical texts, existing and potential ones, and make them an object of study.

I will explain how the advent of category theory (and, to a lesser degree, theory of computability) required enriched languages, that after formalization become at first *non–linear*, and then *multidimensional*. Such languages require for their study

*homotopy theory* and suggest a respective enrichment of the universe in which interpretations/models are supposed to live, from Sets to Homotopy Types, as in the Voevodsky’s project (cf. above).

**3.2. One-dimensional languages of diagrams and graphs.** With the development of homological algebra and category theory in the second half of the XX-th century, the language of commutative diagrams began to penetrate ever wider realms of mathematics. It took some time for mathematicians to get used to “diagram-chasing.” A simply looking algebraic identity  $kg = hf$ , when it expresses a property of four morphisms in a category, means that we are contemplating a simple *commutative diagram*, in which, besides morphisms  $f, g, k, h$ , also the objects  $A, B, C, D$  invisible in the formula  $kg = hf$  play key roles:

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ f \downarrow & & \downarrow k \\ C & \xrightarrow{h} & D \end{array}$$

Although this square is not a “linear expression”, one may argue that it, and its various generalizations of growing size (even the whole relevant category), are still “one-dimensional”. This means that they can be encoded in a graph, whose vertices are labeled by (names of) objects of our category, whereas edges are labeled by pairs consisting of an orientation and a morphism between the relevant objects.

Similarly, a program written in a linear programming language can be encoded in a graph whose vertices are labeled by (names of) elementary operations that can be performed over the relevant data. To understand labeling of (oriented) edges, one must imagine that they encode channels, forwarding output data calculated by the operation at the start (input) of the edge to the its endpoint where they become input of the next operation (or the final output, if the relevant vertex is labeled respectively). Labels of edges might then include *types* of the relevant data.

**3.3. From graphs to higher dimensions.** Generally, a square of morphisms as above need not be commutative (i. e. it is possible that  $kg \neq hf$ ). In order to distinguish these two cases graphically, we may decide to associate with a *commutative square* the *two-dimensional picture*, by glueing the interior part of the square to the relevant graph.

A well known generalization of this class of spaces are *cell complexes*, or, in more

combinatorial and therefore more language-like version, *simplicial complexes*. Of course, we must allow labels of cells as additional structures.

In this way, we can get, for example, a geometric encoding of the category  $\mathcal{C}$  by a simplicial complex, in which labeled  $n$ -complexes are sequences of morphisms

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n$$

whereas the face map  $\partial^i$  omits one of the objects  $X_i$  and, if  $1 \leq i \leq n-1$ , replaces the pair of arrows around  $X_i$  by one arrow labeled by the composition of the relevant morphisms. The resulting simplicial space encodes the whole category in a simplicial complex that is called *the nerve* of the category. Clearly, not only objects and morphisms, but also all compositions of morphisms and relations between them can be read off it.

Thus the language of commutative diagrams becomes a chapter of algebraic topology, and when the study of functors is required, the chapter of homotopical topology.

### 3.4. Quillen’s homotopical algebra and univalent foundations project.

In his influential book [Qu] Quillen developed the idea that the natural language for homotopy theory should appeal *not* to the initial intuition of continuous deformation itself, but rather to a codified list of properties of category of topological spaces stressing those that are relevant for studying homotopy.

Quillen defined a *closed model category* as a category endowed with three special classes of morphisms: *fibrations*, *cofibrations*, and *weak equivalences*. The list of axioms to which these three classes of morphisms must satisfy is not long but structurally quite sophisticated. They can be easily defined in the category of topological spaces using homotopy intuition but remarkably admit translation into many other situations. An interesting new preprint [GaHa] even suggests the definition of these classes in appropriate categories of discrete sets, contributing new insights to old Cantorian problems of the scale of infinities.

Closed model categories become in particular a language of preference for many contexts in which objects of study are quotients of “large” objects by “large” equivalence relations, such as homotopy.

It is thus only natural that the most recent Foundation/Superstructure, Voevodsky’s Univalent Foundations Project (cf. [Vo] and [Aw]) is based on direct axiomatization of the world of homotopy types.

As a final touch of modernism, the metalanguage of this project is a version of typed lambda calculus, because its goal is to develop a tool for the computer assisted verification of programs and proofs. Thus computers become more and more involved in the interpersonal habitat of “theoretical” mathematics.

It remains to hope that humans will not be finally excluded from this habitat, as some aggressive proponents of databases replacing science suggest (cf. [An]).

### Post Scriptum: Truth and Proof in Mathematics

As I have written in [Ma2], the notion of “truth” in most philosophical contexts is a reification of a certain relationship between humans and *texts/utterances/statements*, the relationship that is called “belief”, “conviction” or “faith”.

Professor Blackburn in [Bl] in his keynote speech to the Balzan Symposium on “Truth” (where [Ma2] was delivered) extensively discussed other relationships of humans to texts, such as *scepticism, conservatism, relativism, deflationism*. However, in the long range all of them are secondary in the practice of a researcher in mathematics.

I will only sketch here what must be said about texts, sources of conviction, and methods of conviction peculiar to mathematics.

*Texts.* Alfred North Whitehead said that all of Western philosophy was but a footnote to Plato.

The underlying metaphor of such a statement is: “Philosophy is a text”, the sum total of all philosophic utterances.

Mathematics decidedly is *not* a text, at least not in the same sense as philosophy. There are no authoritative books or articles to which subsequent generations turn again and again for wisdom. Already in the XX-th century, researchers did not read Euclid, Newton, Leibniz or Hilbert in order to study geometry, calculus or mathematical logic. The life span of any contemporary mathematical paper or book can be years, in the best (and exceptional) case decades. Mathematical wisdom, if not forgotten, lives as an invariant of all its (re)presentations in a permanently self-renewing discourse.

*Sources and methods of conviction.* Mathematical truth is not revealed, and its acceptance is not imposed by any authority.

Ideally, the truth of a mathematical statement is ensured by *a proof*, and the ideal picture of a proof is a sequence of elementary arguments whose rules of formation are explicitly laid down before the proof even begins, and ideally are common for all proofs that have been devised and can be devised in future. The admissible starting points of proofs, “axioms”, and terms in which they are formulated, should also be discussed and made explicit.

This ideal picture is so rigid that it became the subject of mathematical study in metamathematics.

But in the creative mathematics, the role of proof is in no way restricted to its function of carrier of conviction. Otherwise, there would be no need for Carl Friedrich Gauss to consider eight (!) different proofs of the quadratic reciprocity law (cf. [QuRL] for an extended bibliography; I am grateful to Prof. Yuri Tschinkel for this reference).

One metaphor of proof is a route, which might be a desert track boring and unimpressive until one finally reaches the oasis of one’s destination, or a foot path in green hills, exciting and energizing, opening great vistas of unexplored lands and seductive offshoots, leading far away even after the initial destination point has been reached.

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