Exactness of the reduction on étale modules

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Abstract

We prove the exactness of the reduction map from étale (φ,Γ)-modules over completed localized group rings of compact open subgroups of unipotent p-adic algebraic groups to usual étale (φ,Γ)-modules over Fontaine's ring. This reduction map is a component of a functor from smooth p-power torsion representations of p-adic reductive groups (or more generally of Borel subgroups of these) to (φ,Γ)-modules. Therefore this gives evidence for this functor—which is intended as some kind of p-adic Langlands correspondence for reductive groups—to be exact. We also show that the corresponding higher Tor-functors vanish. Moreover, we give the example of the Steinberg representation as an illustration and show that it is acyclic for this functor to (φ,Γ)-modules whenever our reductive group is $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ for some $d\geq 1$.

1 Introduction

1.1 Colmez' work

In recent years it has become increasingly clear that some kind of p-adic version of the local Langlands conjectures should exist. However, a precise formulation is still missing. It is all the more remarkable that Colmez has recently managed to establish such a correspondence between 2-dimensional p-adic Galois representations of $\mathbb{Q}_p$ and continuous irreducible unitary p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In fact, Colmez [3, 4] constructed a functor from smooth torsion P-representations to étale (φ,Γ)-modules where P is the standard parabolic subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. Whenever we are given a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation V, we may find a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant lattice L inside it. Hence we can take the restriction to P of the reduction L/p^mL mod p^m for some positive integer m and pass to (φ,Γ)-modules using Colmez' functor. The (φ,Γ)-module corresponding to the initial representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ will be the projective limit of these (φ,Γ)-modules when m tends to infinity. The miracle is that whenever we started with an irreducible supercuspidal GL_2-representation in characteristic p the resulting (φ,Γ)-module will be 2-dimensional and hence correspond to a 2-dimensional modulo p Galois representation of the field $\mathbb{Q}_p$. The image of 1-dimensional and pricipal series representations is, however, 0 and 1 dimensional, respectively (see Thm. 10.7 in [14] and Thm. 10.4 in [9]).

1.2 The Schneider-Vigneras functors

Even more recently, Schneider and Vigneras [11] managed to generalize Colmez' functor to general p-adic reductive groups. Their context is the following. Let G be the group of

$\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$ whose centre is also assumed to be connected. To review their construction we fix a Borel subgroup $P = TN$ with split torus T and unipotent radical N. We also fix an appropriate compact open subgroup N_0 which gives rise to the 'dominant' submonoid $T_+ := \{t \in T \mid tN_0t^{-1} \subseteq N_0\}$ in T. On the one side we consider the abelian category $\mathcal{M}_{o-tor}(P)$ of all smooth o-torsion representations of the group P where o is the ring of integers in a fixed finite extension $K/\mathbb{Q}_p$. On the other side a monoid ring $\Lambda(P_+)$ is introduced for the monoid $P_+ := N_0T_+$ and we denote the category of all (left unital) $\Lambda(P_+)$-modules by $\mathcal{M}(\Lambda(P_+))$. Such a module M is called étale if every $t \in T_+$ acts, informally speaking, with slope zero on M. The universal δ-functor $V \mapsto D^i(V)$ for $i \geq 0$ from $\mathcal{M}_{o-tor}(P)$ to the category $\mathcal{M}_{et}(\Lambda(P_+))$ of étale $\Lambda(P_+)$-modules is constructed the following way. D^i are the derived functors of a contravariant functor $D\colon \mathcal{M}_{o-tor}(P) \to \mathcal{M}(\Lambda(P_+))$ which is not exact in the middle, but takes surjective, resp. injective maps to injective, reps. surjective maps. (Hence $D \subsetneq D^0$ in general.) Note that the $\Lambda(P_+)$-module $D(V)$ is not étale in general, but $D^i(V)$ always lies in $\mathcal{M}_{et}(\Lambda(P_+))$ for any $i \geq 0$. The modules $D^i(V)$ are not expected to have good properties in general. This is why it is natural to pass to some topological localization $\Lambda_\ell(P_\star)$ of the group ring $\Lambda(P_\star)$ of a submonoid $P_\star$ of P_+ generated by P_0, φ, and Γ. The corresponding abelian category $\mathcal{M}_{et}(\Lambda_\ell(P_\star))$ of étale $\Lambda_\ell(P_\star)$-modules is a generalization of Fontaine's (φ, Γ)-modules. Indeed, whenever $G = \mathrm{GL}_2(\mathbb{Q}_p)$ (in this case we denote by $S_\star$ the standard monoid inside $\mathrm{GL}_2(\mathbb{Q}_p)$ and note that $N_0 \cong \mathbb{Z}_p$) then the objects that are finitely generated over the smaller localized ring $\Lambda_\ell(N_0) \cong \Lambda_F(\mathbb{Z}_p)$ are exactly Fontaine's (φ, Γ)-modules. This construction leads to the universal δ-functor $D^i_\ell(V)$. The fundamental open question in [11] is for which class of P-representations V are the modules $D^i_\ell(V)$ finitely generated over $\Lambda_\ell(N_0)$. Moreover, with the help of a Whittaker type functional ℓ one may pass to the category $\mathcal{M}_{et}(\Lambda_F(S_\star))$ for the standard monoid $S_\star$ in $\mathrm{GL}_2(\mathbb{Q}_p)$. This way one obtains a δ-functor $D^i_{\Lambda_F(S_\star)}$ from $\mathcal{M}_{o-tor}(P)$ to the category of not necessarily finitely generated (φ, Γ)-modules à la Fontaine. For the group $G = \mathrm{GL}_2(\mathbb{Q}_p)$ Colmez' original functor coincides with $D^0_{\Lambda_F(S_\star)}$ and the higher $D^i_{\Lambda_F(S_\star)}$ vanish.

1.3 Outline of the paper

The aim of this short note is to investigate the exactness properties of the functors constructed by Schneider and Vigneras [11]. Whenever $G \neq \mathrm{GL}_2(\mathbb{Q}_p)$ then the reduction map

$$\ell\colon \Lambda_\ell(N_0) \twoheadrightarrow \Lambda_F(\mathbb{Z}_p)$$

has a nontrivial kernel and hence is not flat since $\Lambda_\ell(N_0)$ has no zero divisors and therefore any flat module is torsion-free. However, the extra étale φ-structure allows us to show that the reduction functor from étale φ-modules over $\Lambda_\ell(N_0)$ to étale φ-modules over $\Lambda_F(\mathbb{Z}_p)$ induced by ℓ is still exact if we restrict ourselves to *pseudocompact* $\Lambda_\ell(N_0)$-modules which includes those finitely generated. The proof relies on the existence of a descending separated filtration of (two-sided) ideals J_n of $\Lambda_\ell(N_0)$ such that $\varphi(J_n) \subseteq J_{n+1}$. In section 3.4 we use this to show that in fact the higher Tor-functors $\mathrm{Tor}^i_{\Lambda_\ell(N_0)}(\Lambda_F(\mathbb{Z}_p), M)$ vanish for $i \geq 1$ whenever M is a pseudocompact étale φ-module over $\Lambda_\ell(N_0)$.

In section 4 we investigate the example of the Steinberg representation V_{St}. We show that in this case we have $D^0(V_{St}) = D(V_{St})$ and, in particular $D^0(V_{St})$ is finitely generated over $\Lambda_\ell(N_0)$. Moreover, we prove that all the higher $D^i(V_{St})$ vanish for $i \geq 1$. This is the first known example of a smooth o-torsion P-representation with finitely generated D^0_ℓ and with known D^i_ℓ for all $i \geq 0$ apart from those for $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence V_{St} is acyclic for the functor D_ℓ and also for the functor $D_{\Lambda_F(S_\star)}$ by the first part of the paper. It also follows that the functor in the other direction from $\mathcal{M}_{et}(\Lambda(P_+))$ to $\mathcal{M}_{o-tor}(P)$ sends $D^0(V_{St})$ back to V_{St}. We expect that the method of computing $D^i(V_{St})$ for $i \geq 0$ generalizes to a wider class of smooth o-torsion P-representations. For technical reasons we restrict ourselves in this section to the general linear group $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ with $d \geq 1$. The case $\mathrm{GL}_2(\mathbb{Q}_p)$ is also formally included—however, the functor D is known [3, 4] to be exact in this case.

2 Preliminaries and notations

2.1 Basic notations

We are going to use the notations of [11], but for the convenience of the reader we recall them here, as well. Let G be the group of $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. Assume further that the centre of this reductive group is also connected. We fix a Borel subgroup $P = TN$ in G with maximal split torus T and unipotent radical N. Let Φ^+ denote, as usual, the set of roots of T positive with respect to P and let $\Delta \subseteq \Phi^+$ be the subset of simple roots. For any $\alpha \in \Phi^+$ we have the root subgroup $N_\alpha \subseteq N$. We recall that $N = \prod_{\alpha \in \Phi^+} N_\alpha$ for any total ordering of Φ^+. Let $T_0 \subseteq T$ be the maximal compact subgroup. We fix a compact open subgroup $N_0 \subseteq N$ which is totally decomposed, ie. $N_0 = \prod_\alpha N_0 \cap N_\alpha$ for any total ordering of Φ^+. Then $P_0 := T_0 N_0$ is a group. We introduce the submonoid $T_+ \subseteq T$ of all $t \in T$ such that $tN_0t^{-1} \subseteq N_0$, or equivalently, such that $\alpha(t)$ is integral for any $\alpha \in \Delta$. Then $P_+ := N_0 T_+ = P_0 T_+ P_0$ is obviously a submonoid of P.

We also fix a finite extension $K/\mathbb{Q}_p$ with ring of integers o, prime element π, and residue class field k. For any profinite group H let $\Lambda(H) := o[[H]]$, resp. $\Omega(H) := k[[H]] = \Lambda(H)/\pi\Lambda(H)$ be the Iwasawa algebra of H with coefficients in o, resp. k.

2.2 The functors D and D^i

By a representation we will always mean a linear action of the group (or monoid) in question in a torsion o-module V. It is called smooth if the stabilizer of each element in V is open in the group. We put $V^* := \mathrm{Hom}_o(V, K/o)$ the Pontryagin dual of V which is a compact linear-topological o-module. Following [11] we define

$$D(V) := \varinjlim_M M^*$$

where M runs through all the generating P_+-subrepresentations of V. Whenever V is compactly induced it is equipped with an action of the ring $\Lambda(P_+)$ which is by definition the image of the natural map

$$\Lambda(P_0) \otimes_{o[P_0]} o[P_+] \to \varprojlim_Q o[Q \setminus P_+]$$

where Q runs through all open normal subgroups $Q \subseteq P_0$ which satisfy $bQb^{-1} \subseteq Q$ for any $b \in P_+$ (cf. Proposition 3.4 in [11]). The $\Lambda(P_+)$-modules $D^i(V)$ for a general smooth P-representation V and $i \geq 0$ are obtained as the cohomology groups $D^i(V) := h^i(D(\mathcal{I}_\bullet(V)))$

for some resolution

$$\mathcal{I}_\bullet(V)\colon \cdots \to \operatorname{ind}_{P_0}^P(V_n) \to \cdots \to \operatorname{ind}_{P_0}^P(V_0) \to V \to 0$$

of V by compactly induced representations. This is independent of the choice of the resolution by Corollary 4.4 in [11]. Note that $D(V) \subsetneq D^0(V)$ in general.

2.3 The ring $\Lambda_\ell(N_0)$

As in [11] we fix once and for all isomorphisms of algebraic groups

$$\iota_\alpha\colon N_\alpha \xrightarrow{\cong} \mathbb{Q}_p$$

for $\alpha \in \Delta$, such that

$$\iota_\alpha(tnt^{-1}) = \alpha(t)\iota_\alpha(n)$$

for any $n \in N_\alpha$ and $t \in T$. Since $\prod_{\alpha\in\Delta} N_\alpha$ is naturally a quotient of $N/[N,N]$ we now introduce the group homomorphism

$$\ell := \sum_{\alpha\in\Delta} \iota_\alpha\colon N \to \mathbb{Q}_p.$$

Moreover, for the sake of convenience we normalize the ι_α such that

$$\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p$$

for any α in Δ. In particular, we then have $\ell(N_0) = \mathbb{Z}_p$. We put $N_1 := \operatorname{Ker}(\ell_{|N_0})$. The group homomorphism ℓ also induces a map

$$\Lambda(N_0) \twoheadrightarrow \Lambda(\mathbb{Z}_p)$$

which we still denote by ℓ. By [2] the multiplicatively closed subset $S := \Lambda(N_0) \setminus (\pi, \operatorname{Ker}(\ell))$ is a left and right Ore set in $\Lambda(N_0)$ and we may define the localization $\Lambda(N_0)_S$ of $\Lambda(N_0)$ at S. We define the ring $\Lambda_\ell(N_0) := \Lambda_{N_1}(N_0)$ as the completion of $\Lambda(N_0)_S$ with respect to the ideal $(\pi, \operatorname{Ker}(\ell))\Lambda(N_0)_S$. This is a strict-local ring with maximal ideal $(\pi, \operatorname{Ker}(\ell))\Lambda_\ell(N_0)$. Moreover, it is pseudocompact (c.f. Thm 4.7 in [10]).

2.4 Generalized (φ, Γ)-modules

Now since we assume that the centre of G is connected, the quotient $X^*(T)/\bigoplus_{\alpha\in\Delta}\mathbb{Z}\alpha$ is free. Hence we find a cocharacter ξ in $X_*(T)$ such that $\alpha \circ \xi = \mathrm{id}_{\mathbb{G}_m}$ for any α in Δ. It is injective and uniquely determined up to a central cocharacter. We fix once and for all such a ξ. It satisfies

$$\xi(\mathbb{Z}_p \setminus \{0\}) \subseteq T_+$$

and

$$\ell(\xi(a)n\xi(a^{-1})) = a\ell(n)$$

for any a in $\mathbb{Q}_p^\times$ and n in N. Put $\Gamma := \xi(1 + p^{\epsilon(p)}\mathbb{Z}_p)$ and $\varphi := \xi(p)$.

The group Γ and the semigroup generated by φ naturally act on the ring $\Lambda_\ell(N_0)$. Hence we may define (φ,Γ)-modules (resp. φ-modules) over $\Lambda_\ell(N_0)$ as $\Lambda_\ell(N_0)$-modules together with a commuting and compatible action of φ and Γ (resp. just a compatible action of φ). The notion of $(\Lambda_\ell(N_0),\Gamma,\varphi)$-module refers to (φ,Γ)-modules that are *finitely generated* over $\Lambda_\ell(N_0)$. We call a φ-module M étale if the map

$$\begin{array}{rcl}\Lambda_\ell(N_0)\otimes_\varphi M & \to & M\\ \nu\otimes m & \mapsto & \nu\varphi_M(m)\end{array}$$

is bijective.

The map ℓ induces a φ- and Γ-equivariant ring homomorphism

$$\Lambda_\ell(N_0)\twoheadrightarrow\Lambda_F(\mathbb{Z}_p)$$

onto Fontaine's ring $\Lambda_F(\mathbb{Z}_p)$ which is the p-adic completion of the Laurent series ring $o[[T]][T^{-1}]$. Hence it gives rise to a functor from (étale) (φ,Γ)-modules over $\Lambda_\ell(N_0)$ to not necessarily finitely generated (étale) (φ,Γ)-modules over $\Lambda_F(\mathbb{Z}_p)$. We may restrict this functor to pseudocompact (or less generally to finitely generated) étale modules. The main result of this short note is that this restriction is exact.

3 Exactness of reduction on pseudocompact modules

3.1 A p-valuation on N_0

In this section we are going to define a p-valuation ω on the group N_0 which we will restrict to N_1 in order to produce ideals J_n in $\Lambda_\ell(N_0)$. Since N_0 is totally decomposed we can fix topological generators n_α of $N_0\cap N_\alpha$ for any α in Δ such that $\ell(n_\alpha)=1$. Further, we fix topological generators n_β of $N_0\cap N_\beta$ for each $\beta\in\Phi^+\setminus\Delta$. Now we define a p-valuation ω on N_0 as follows. Any β in Φ^+ can be written as a positive integer combination $\beta=\sum_{\alpha\in\Delta}m_{\alpha\beta}\alpha$ of simple roots α. We denote by $m_\beta:=\sum_\alpha m_{\alpha\beta}$ the degree of $\beta\circ\xi$ which is a positive integer and is equal to 1 if and only if β lies in Δ. Further, we fix a total ordering $<$ of Φ^+ such that whenever $m_\alpha<m_\beta$ for roots α,β in Φ^+ then also $\alpha<\beta$. As N_0 is totally decomposed we may write any element g in N_0 uniquely as a product

$$g=\prod_{\alpha\in\Phi^+}n_\alpha^{g_\alpha}$$

where g_α are in $\mathbb{Z}_p$ and the product is taken in the ordering $<$ of Φ^+ defined above. We put

$$\omega(g):=\min_{\alpha\in\Phi^+}m_\alpha(v_p(g_\alpha)+1)$$

for any $1\neq g$ in N_0, and $\omega(1):=\infty$. Here v_p denotes the additive p-adic valuation on $\mathbb{Z}_p$.

Lemma 1. *The function ω is a p-valuation on N_0. In other words, we have*

(i) $\omega(gh^{-1})\geq\min(\omega(g),\omega(h))$.

(ii) $\omega(g^{-1}h^{-1}gh)\geq\omega(g)+\omega(h)$.

(*iii*) $\omega(g^p) = \omega(g) + 1$.

Proof. For the proof of (*i*) we are going to use triple induction. At first by induction on the number of non-zero coordinates among $(h_\alpha)_{\alpha\in\Phi}$ we are reduced to the case when h is of the form $n_\alpha^{h_\alpha}$ for some α in Φ^+. Let now g be of the form $g = \prod_{k=1}^r n_{\beta_k}^{g_{\beta_k}}$ with $\beta_1 < \beta_2 < \cdots < \beta_r$ in Φ^+. If $\beta_r \leq \alpha$ then (*i*) is trivially satisfied. On the other hand, if $\beta_r > \alpha$ then we write

$$\prod_{k=1}^{r} n_{\beta_k}^{g_{\beta_k}} \cdot n_\alpha^{-h_\alpha} = \prod_{k=1}^{r-1} n_{\beta_k}^{g_{\beta_k}} n_\alpha^{-h_\alpha} \cdot n_{\beta_r}^{g_{\beta_r}} [n_{\beta_r}^{g_{\beta_r}}, n_\alpha^{-h_\alpha}].$$

Now we use (descending) induction on α in the chosen ordering of Φ^+ and suppose that the statement (*i*) is true for any g and any h' of the form $h' = n_{\alpha'}^{h'_{\alpha'}}$ with $\alpha' > \alpha$. For this we remark that the set Φ^+ is finite and totally ordered. Note that for any $\alpha < \beta$ in Φ^+ we have

$$[n_\beta^{g_\beta}, n_\alpha^{-h_\alpha}] = \prod_{i\beta+j\alpha\in\Phi^+; i,j>0} n_{i\beta+j\alpha}^{c_{\beta,\alpha,i,j} g_\beta^i (-h_\alpha)^j} \tag{1}$$

by the commutator formula in Proposition 8.2.3 in [12]. Here the constants $c_{\beta,\alpha,i,j}$ a priori only lie in $\mathbb{Q}_p$ but since our parametrization and the fact that N_0 is a subgroup of N they are forced to be in $\mathbb{Z}_p$. Moreover, we have $m_{i\beta+j\alpha} \geq m_\beta + m_\alpha > m_\alpha$, so we may apply the inductional hypothesis for $n_{\beta_r}^{g_{\beta_r}}$ and for all the terms on the right hand side of (1) (with the choice $\beta := \beta_r$) in order to obtain

$$\omega(gh^{-1}) \geq \min\left(\omega\left(\prod_{k=1}^{r-1} n_{\beta_k}^{g_{\beta_k}} n_\alpha^{-h_\alpha}\right), \omega(n_{\beta_r}^{g_{\beta_r}}), \omega\left([n_{\beta_r}^{g_{\beta_r}}, n_\alpha^{-h_\alpha}]\right)\right). \tag{2}$$

On the other hand, we compute

$$\begin{aligned} \omega(n_{i\beta+j\alpha}^{c_{\beta,\alpha,i,j} g_\beta^i (-h_\alpha)^j}) &= m_{i\beta+j\alpha}(v_p(c_{\beta,\alpha,i,j} g_\beta^i (-h_\alpha)^j) + 1) \geq \\ \geq (m_\beta + m_\alpha)(v_p(g_\beta) + v_p(h_\alpha) + 1) &\geq \omega(n_\beta^{g_\beta}) + \omega(n_\alpha^{h_\alpha}) \geq \min(\omega(n_\beta^{g_\beta}), \omega(n_\alpha^{h_\alpha})). \end{aligned} \tag{3}$$

Hence combining (2) and (3) (with the choice $\beta := \beta_r$) we are done by a third induction on r.

Since we know (*i*) it suffices to check (*ii*) in the case $g = n_\beta^{g_\beta}$ and $h = n_\alpha^{h_\alpha}$. For these we are done by (1) and (3). The assertion (*iii*) is clear from the definition using (*i*) and (*ii*). □

3.2 The ideals J_n

In view of Lemma 1 we define for each positive integer n the normal subgroup $N_{0,n}$ in N_0 as the set of elements g in N_0 with $\omega(g) \geq n$. Further we put $N_{1,n} := N_1 \cap N_{0,n}$. In particular, $N_{1,1} = N_1$ and $N_{0,1} = N_0$. We define $J_n(\Lambda(N_1))$ to be the kernel of the natural surjection $\Lambda(N_1) \twoheadrightarrow \Lambda(N_1/N_{1,n})$. Moreover, we denote by J_n the ideal generated by $J_n(\Lambda(N_1))$ in $\Lambda_\ell(N_0)$. We further have the following

Lemma 2. *$N_{1,n}$ is a normal subgroup in P_0 for any $n \geq 1$. In particular, J_n is the kernel of the natural surjection from $\Lambda_\ell(N_0) = \Lambda_{N_1}(N_0)$ onto $\Lambda_{N_1/N_{1,n}}(N_0/N_{1,n})$. Further, we have $\varphi N_{1,n}\varphi^{-1} \subseteq N_{1,n+1}$. Therefore there is an induced φ-action on each $\Lambda_\ell(N_0)/J_n$ such that the module J_n/J_{n+1} is killed by φ for any $n \geq 1$.*

Proof. Since N_1 is normal in P_0 it suffices to verify that $N_{0,n}$ is normal in P_0. For this note that $N_{0,n}$ is normal in N_0 as ω is a p-valuation and if t is in T_0 then we have $tn_\alpha t^{-1} = n_\alpha^{t_\alpha}$ with t_α in $\mathbb{Z}_p^\times$. Hence the first part of the statement. For the second part we note that $\varphi n_\alpha \varphi^{-1} = n_\alpha^{p^{m_\alpha}}$. $\square$

Note that the Jacobson radical $Jac(\Lambda_\ell(N_0))$ is equal to the ideal (π, J_1) by definition of J_1. Moreover, any element g in $N_{1,n\max_{\beta\in\Phi^+} m_\beta}$ is a product of p^nth powers of elements in N_1, hence $J_{n\max_{\beta\in\Phi^+} m_\beta} \subseteq Jac(\Lambda_\ell(N_0))^{n+1}$. Indeed, if g is any element in N_1 then $g^{p^n} - 1 = \prod_{j=0}^n \Phi_{p^j}(g)$ (with $\Phi_{p^j}(x)$ being the p^jth cyclotomic polynomial) and $\Phi_{p^j}(g)$ clearly lies in $Jac(\Lambda_\ell(N_0))$ as both p and $g-1$ lie in $Jac(\Lambda_\ell(N_0))$. In particular, $\bigcap_n J_n = 0$.

Recall that $\Lambda_\ell(N_0)$ is a pseudocompact ring (c.f. [10] Thm. 4.7).

Lemma 3. *If M is any pseudocompact module over $\Lambda_\ell(N_0)$ then J_nM and M/J_nM are also pseudocompact in the subspace, resp. quotient topologies.*

Proof. It suffices to show that J_nM is closed in M. By Lemma 1.6 in [1] and by the fact that the pseudocompact modules form an abelian category ([6] IV.3. Thm. 3) we are reduced to the case when $M = \prod_{i\in I}\Lambda_\ell(N_0)$ with the product topology. However, in this case we have $J_n\prod_{i\in I}\Lambda_\ell(N_0) = \prod_{i\in I} J_n$ as J_n is finitely generated ($\Lambda_\ell(N_0)$ is noetherian), and this is closed in the product topology as J_n is closed in $\Lambda_\ell(N_0)$ using once again that it is finitely generated and hence pseudocompact in the subspace topology of $\Lambda_\ell(N_0)$. $\square$

Lemma 4. *If M is any pseudocompact module over the ring $\Lambda_\ell(N_0)$ then the natural map induces an isomorphism*

$$M \cong \varprojlim_n M/J_nM.$$

Proof. By Lemma 3 the submodules J_nM are closed, and since $J_{n\max_{\beta\in\Phi^+} m_\beta} \subseteq Jac(\Lambda_\ell(N_0))^n$ we have $\bigcap_n J_nM = 0$. The statement follows from IV.3. Proposition 10 in [6]. $\square$

3.3 Main result

Proposition 5. *Let M and N be pseudocompact étale φ-modules over $\Lambda_\ell(N_0)$. Then injective continuous maps (in the pseudocompact topology) $M \hookrightarrow N$ reduce to injective maps $M/J_1M \hookrightarrow N/J_1N$ between the φ-modules over $\Lambda_F(\mathbb{Z}_p)$.*

Proof. Let K_n be the kernel of the induced map from M/J_nM to N/J_nN. We assume indirectly that $K_1 \neq 0$. We show that the natural map from K_n to K_1 is surjective for any n. For this we are going to use the following commutative diagram with some X_n and Y_n.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_n & \longrightarrow & K_n & \longrightarrow & K_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1M/J_nM & \longrightarrow & M/J_nM & \longrightarrow & M/J_1M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1N/J_nN & \longrightarrow & N/J_nN & \longrightarrow & N/J_1N & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & Y_n & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{4}
$$

We remark immediately that by Lemma 3 all the modules in the diagram (4) are pseudocompact modules over $\Lambda_\ell(N_0)$, and all the maps are continuous in the pseudocompact topologies. Indeed, the pseudocompact modules form an abelian category ([6] IV.3. Thm. 3).

By the snake lemma we obtain the exact sequence

$$0 \to X_n \to K_n \to K_1 \overset{\delta_n}{\to} Y_n.$$

We claim that there does not exist any nonzero φ-equivariant $\Lambda_\ell(N_0)$-homomorphism from K_1 to Y_n. This would show that K_n surjects onto K_1 for any n. Note that $\Lambda_\ell(N_0)$ acts on K_1 via its quotient $\Lambda_F(\mathbb{Z}_p)$ hence we may view K_1 as a φ-module over both rings. As φ is flat over $\Lambda_F(\mathbb{Z}_p)$, étale φ-modules form an abelian category over $\Lambda_F(\mathbb{Z}_p)$. In particular, K_1 is étale as a φ-module over $\Lambda_F(\mathbb{Z}_p)$ since it is the kernel of a homomorphism between the étale modules M/J_1M and N/J_1N. We remark that K_1 is not étale as a φ-module over $\Lambda_\ell(N_0)$ since K_1 is annihilated by J_1 and $\Lambda_\ell(N_0) \otimes_{\varphi,\Lambda_\ell(N_0)} K_1$ is not. The latter is only annihilated by $\varphi(J_1) \subseteq J_2 \subsetneq J_1$. So the map

$$\Lambda_\ell(N_0) \otimes_{\varphi,\Lambda_\ell(N_0)} K_1 \to K_1$$

is surjective (since K_1 is étale over $\Lambda_F(\mathbb{Z}_p)$), but not injective. Therefore if there is a surjective φ-equivariant $\Lambda_\ell(N_0)$-homomorphism from K_1 to some φ-module A over $\Lambda_\ell(N_0)$ then we also have that $\varphi(A)$ generates A as a $\Lambda_\ell(N_0)$-module. On the other hand, J_1N/J_nN admits the filtration $Fil^k(J_1N/J_nN) := J_kN/J_nN$ for $1 \leq k \leq n$. This induces a filtration $Fil^k(Y_n)$ on Y_n via the above surjection in (4). Let us assume now that δ_n is nonzero. Then there is an integer $k < n$ such that $\delta_n(K_1) \subseteq Fil^k(Y_n)$ but $\delta_n(K_1) \not\subseteq Fil^{k+1}(Y_n)$. Hence we get a nonzero φ-equivariant $\Lambda_\ell(N_0)$-homomorphism from K_1 to $Fil^k(Y_n)/Fil^{k+1}(Y_n)$ which we denote by δ'_n. However, we claim that φ acts as zero on the latter which will contradict to the fact that $\varphi(\delta'_n(K_1))$ generates $\delta'_n(K_1)$. Indeed, we have a surjective composite map

$$(J_k/J_{k+1}) \otimes_{\Lambda_\ell(N_0)} N \twoheadrightarrow J_kN/J_{k+1}N \twoheadrightarrow Fil^k(Y_n)/Fil^{k+1}(Y_n),$$

hence $\varphi(Fil^k(Y_n)/Fil^{k+1}(Y_n)) = 0$ as we have $\varphi(J_k) \subseteq J_{k+1}$ by Lemma 2.

Now we have a map from the projective system $(K_n)_n$ to the projective system $(K_1)_n$ which is surjective on each layer, hence its projective limit is also surjective by the exactness of $\varprojlim$ on pseudocompact modules ([6] IV.3. Thm. 3). The statement follows from the completeness of M (Lemma 4). □

Whenever M and N are finitely generated over $\Lambda_\ell(N_0)$ then they admit a unique pseudocompact topology (since $\Lambda_\ell(N_0)$ is pseudocompact and noetherian [10] Thm. 4.7 and Lemma 4.2(ii)) and any homomorphism between them is continuous. So we obtain the main result of this paper as corollary of Proposition 5.

Proposition 6. *The functor from the category of étale $(\Lambda_\ell(N_0), \Gamma, \varphi)$-modules to the category of étale $(\Lambda_F(S_0), \Gamma, \varphi)$-modules induced by the natural surjection*

$$\ell\colon \Lambda_\ell(N_0) \twoheadrightarrow \Lambda_F(\mathbb{Z}_p)$$

is exact.

3.4 Vanishing of higher Tor-functors

Let M be a pseudocompact étale φ-module over $\Lambda_\ell(N_0)$. Then $M/(\pi, J_1)M$ is also a pseudocompact étale φ-module over the field $\Lambda_\ell(N_0)/(\pi, J_1) \cong k((t))$. Hence there is an index set I such that we have an isomorphism of pseudocompact modules

$$M/(\pi, J_1)M \cong \prod_{i\in I} \Lambda_\ell(N_0)/(\pi, J_1)$$

by Lefschetz's Structure Theorem for linearly compact vector spaces ([8], p. 83 Thm. (32.1), see also [5]). Note that over fields the notion of linearly compact vectorspaces coincides with the notion of pseudocompact modules. Indeed, pseudocompact modules over fields are by definition the projective limits of finite dimensional vectorspaces with the projective limit topology of the discrete topology on each finite dimensional vectrospace. However, the category of linearly compact vectorspaces is closed under products and factors by closed subspaces (properties *c*) and *d*) on page 1 of [5]), hence also under projective limits. Moreover, we have $(\pi, J_1) = Jac(\Lambda_\ell(N_0))$, therefore we obtain a projective cover of M

$$f\colon \prod_{i\in I} \Lambda_\ell(N_0) \twoheadrightarrow M \tag{5}$$

which is an isomorphism modulo (π, J_1). Note that in fact f is a right minimal morphism. Indeed, whenever $g\colon \prod_{i\in I} \Lambda_\ell(N_0) \to \prod_{i\in I} \Lambda_\ell(N_0)$ is a $\Lambda_\ell(N_0)$-homomorphism such that $f \circ g = f$ then g is the identity modulo $Jac(\Lambda_\ell(N_0)) = (\pi, J_1)$. Hence g is invertible by the completeness of $\prod_{i\in I} \Lambda_\ell(N_0)$ with respect to the $Jac(\Lambda_\ell(N_0))$-adic filtration.

In this section we need to assume that φ acts continuously on the pseudocompact module M. Note that this is automatic if M is finitely generated over $\Lambda_\ell(N_0)$.

Lemma 7. *Let $F = \prod_{i\in I} \Lambda_\ell(N_0)$ be a φ-module over $\Lambda_\ell(N_0)$ with continuous φ-action. Then F is étale if and only if so is $F/Jac(\Lambda_\ell(N_0))F$ over $k((t))$.*

Proof. If F is étale then by definition so is $F/Jac(\Lambda_\ell(N_0))F$. Now assume that $F/Jac(\Lambda_\ell(N_0))F$ is étale. In other words the map

$$1 \otimes \varphi \colon \Lambda_\ell(N_0) \otimes_{\varphi,\Lambda_\ell(N_0)} F \to F \tag{6}$$

is isomorphism modulo $Jac(\Lambda_\ell(N_0))$. Therefore (6) is for instance surjective as its cokernel is pseudocompact and killed by $Jac(\Lambda_\ell(N_0))$. On the other hand, since F is topologically free, we have a continuous section of the map (6). Since (6) is an isomorphism modulo $Jac(\Lambda_\ell(N_0))$, so is this section. However, by the same argument as above this section also has to be surjective and therefore is an inverse to the map (6). $\square$

Proposition 8. *Let M be an étale pseudocompact φ-module over $\Lambda_\ell(N_0)$ with continuous φ-action. Then the action of φ on M can be lifted continuously to $F := \prod_{i\in I} \Lambda_\ell(N_0)$ via the surjection f in* (5)*. Any such lift makes F an étale φ-module.*

Proof. Let us define another continuous $\Lambda_\ell(N_0)$-homomorphism

$$\begin{aligned} g\colon \prod_{i\in I} \Lambda_\ell(N_0) &\to M \\ e_i &\mapsto \varphi(f(e_i)). \end{aligned}$$

We need to check that $\lim_{i\in I} \varphi(f(e_i)) = 0$ in the pseudocompact topology of M so that g really defines a continuous homomorphism. This is, however, clear by the continuity of φ and f. By the projectivity of F (Lemma 1.6 in [1]) we obtain a lift φ_{lin}

$$\begin{array}{ccc} & & F \\ & \overset{\varphi_{\mathrm{lin}}}{\nearrow} & \downarrow f \\ F & \xrightarrow{\ g\ } & M \end{array}$$

which we define as the linearization of φ on F. Hence we define

$$\varphi(e_i) := \varphi_{\mathrm{lin}}(e_i)$$

and extend it σ_φ-linearly and continuously to the whole F. By construction this is a lift of $\varphi_{|M}$. The étaleness follows from Lemma 7 noting that by construction of (5) we have $F/Jac(\Lambda_\ell(N_0))F = M/Jac(\Lambda_\ell(N_0))M$ and the latter is étale as so is M. $\square$

Corollary 9. *For any pseudocompact étale φ-module M over $\Lambda_\ell(N_0)$ with continuous φ and any $i \geq 1$ we have*

$$\mathrm{Tor}^i_{\Lambda_\ell(N_0)}(\Lambda_\ell(N_0)/J_1, M) = 0.$$

Proof. By Proposition 8 there is a projective resolution $(F_i)_{i\in\mathbb{N}}$ of M in the category of pseudocompact $\Lambda_\ell(N_0)$-modules, such that the F_i are étale φ-modules and the resolution is φ-equivariant. By Proposition 5, the functor $\Lambda_\ell(N_0)/J_1 \otimes_{\Lambda_\ell(N_0)} \cdot$ is exact on this resolution. The result follows noting that the modules $\prod_{i\in I} \Lambda_\ell(N_0)$ are flat over the noetherian ring $\Lambda_\ell(N_0)$ as in this case an arbitrary direct product of flat modules is flat again. $\square$

Corollary 10. *Let M be a pseudocompact étale module over $\Lambda_\ell(N_0)$ with continuous φ such that $\pi M = 0$. Then there exists an index set I such that $M \cong \prod_{i\in I} \Lambda_\ell(N_0)/\pi$. In particular, M is a projective object in the category of pseudocompact modules over $\Lambda_\ell(N_0)/\pi$.*

Proof. By Proposition 8 we obtain a minimal projective cover F of M with F admitting an étale lift of the φ-action on M. Since $\pi M = 0$ this factors through $F/\pi F$ which is also étale in the induced φ-action. Now we denote by K the kernel of the map from $F/\pi F$ onto M. Then K is also étale as these form an abelian category. Hence by Proposition 5 we obtain an exact sequence

$$0 \to K/J_1K \to F/(\pi, J_1)F \to M/J_1M \to 0.$$

However, the map $F/(\pi, J_1)F \to M/J_1M$ is an isomorphism by the construction of F (5) showing that $K/J_1K = 0$ whence $K = 0$ as K is pseudocompact. □

4 An example

In this section we are going to investigate the so called Steinberg representation. For the sake of simplicity (of the Bruhat-Tits building) we let G be $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ in this section for some $d \geq 1$ and P be its standard Borel subgroup of lower triangular matrices. Recall that the group $P = NT$ acts on N by $(nt)(n') = ntn't^{-1}$. This induces an action of P on the vector space $V_{St} := C_c^\infty(N)$ of k-valued locally constant functions with compact support on N. It is straightforward to see (cf. Example on p. 8 in [11] and [13] Lemme 4) that the subspace $M := C^\infty(N_0)$ of locally constant functions on N_0 is generating and P_+-invariant. Moreover, it is shown in [11] Lemma 2.6 that we have $D(V_{St}) = \Lambda(N_0)/\pi\Lambda(N_0)$. We have the following refinement of this.

Proposition 11. *Let V_{St} be the smooth modulo p Steinberg representation of the group P. Then we have $D^0(V_{St}) = D(V_{St}) = \Lambda(N_0)/\pi\Lambda(N_0)$, and $D^i(V_{St}) = 0$ for any $i \geq 1$.*

For the proof of Proposition 11 we are going to construct an explicit resolution

$$\mathcal{I}_\bullet \colon 0 \to \mathrm{ind}_{P_0Z}^P(V_d) \to \cdots \to \mathrm{ind}_{P_0Z}^P(V_1) \to \mathrm{ind}_{P_0Z}^P(V_0) \to V_{St} \to 0$$

of V_{St} using the Bruhat-Tits building of G. Here Z denotes the centre of G that will act trivially on each V_i $(0 \leq i \leq d)$. Since $Z \cong \mathbb{Q}_p^\times$, Lemma 11.8 in [11] generalizes to this case with the same proof, so we have $D^0(\mathrm{ind}_{P_0Z}^P(V_i)) = D(\mathrm{ind}_{P_0Z}^P(V_i))$ and $D^i(\mathrm{ind}_{P_0Z}^P(V_i)) = 0$ for all $0 \leq i \leq d$. In particular, we may compute $D^i(V_{St}) = h^i(D(\mathcal{I}_\bullet))$.

Recall that the Bruhat-Tits building $\mathcal{BT}$ of G is the simplicial complex whose vertices are the similarity classes $[L]$ of $\mathbb{Z}_p$-lattices in the vector space $\mathbb{Q}_p^{d+1}$ and whose q-simplices are given by families $\{[L_0], \ldots, [L_q]\}$ of similarity classes such that

$$pL_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_q \subsetneq L_0.$$

Let $\mathcal{BT}_q$ denote the set of all q-simplices of $\mathcal{BT}$. We also fix an orientation of $\mathcal{BT}$ with the corresponding incidence numbers $[\eta : \eta']$. We choose a basis $e_0, \ldots, e_d$ of $\mathbb{Q}_p^{d+1}$ in which P is the Borel subgroup of lower triangular matrices and denote the origin of $\mathcal{BT}$ by $x_0 := [\sum_{i=0}^d \mathbb{Z}_p e_i]$. Further, for all $1 \leq i \leq d$ let φ_i be the dominant diagonal matrix $\mathrm{diag}(1, \ldots, 1, p, \ldots, p)$ with i entries equal to 1 and $d + 1 - i$ entries equal to p and put $x_i := \varphi_i x_0$. Then T_+/T_0Z is clearly generated by the elements $\{\varphi_i T_0 Z\}_{i=1}^d$ as a monoid. Moreover, for each subset $J = \{j_1 < \cdots < j_q\} \subseteq \{1, \ldots, d\}$ we define the (oriented) q-simplex

$$\eta_J := \{x_0, x_{j_1}, \ldots, x_{j_q}\}.$$

Now we define the coefficient system

$$V_{nt\eta_J} := C_c^\infty\left(nt\left(\bigcap_{j\in J}\varphi_j N_0\varphi_j^{-1}\right)t^{-1}\right)$$

for any n in N, t in T, and $J\subseteq\{1,\dots,d\}$; and $V_x := 0$ if $\eta\neq b\eta_J$ for any b in P and $J\subseteq\{1,\dots,d\}$. The restriction maps are the natural inclusion maps. Indeed, for any two simplices $\eta_1\subseteq\eta_2$ such that $V_{\eta_2}\neq 0$ we have a $b=nt$ in P such that $\eta_i=b\eta_{J_i}$ for $J_1\subseteq J_2\subseteq\{1,\dots,d\}$ and $i=1,2$ therefore $V_{\eta_2}=ntV_{\eta_{J_2}}$ is naturally contained in $V_{\eta_1}=ntV_{\eta_{J_1}}$ by extending the functions f in $V_{nt\eta_J}$ to the whole N by putting $f_{|N\setminus nt\left(\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}\right)t^{-1}}=0$. Later on we will often view elements of $V_{nt\eta_J}$ as functions on N with support in $\mathrm{supp}(V_{nt\eta_J}) = nt\left(\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}\right)t^{-1}$.

Note that V_η is either zero or equal to $\bigcap_{x\in\eta\cap\mathcal{BT}^0}V_x$. It might, however, happen that this intersection is nonzero but $V_\eta=0$ as η is not in the P-orbit of η_J for any $J\subseteq\{1,\dots,d\}$. We also see immediately that P acts naturally on the coefficient system (V_η) and this action is compatible with the boundary maps. Moreover, we claim

Lemma 12. *We have*

$$\bigoplus_{\eta\in\mathcal{BT}_q}V_\eta\cong\mathrm{ind}_{P_0Z}^P(V_q)\tag{7}$$

with

$$V_q := \sum_{\substack{b_0\in P_0,\ |J|=q\\ J\subseteq\{1,\dots,d\}}}V_{b_0\eta_J} = \bigoplus_{|J|=q,J\subseteq\{1,\dots,d\}}\ \bigoplus_{n_0\in N_0/\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}}V_{n_0\eta_J}.$$

Proof. By construction V_q is a P_0-subrepresentation of $\bigoplus_{\eta\in\mathcal{BT}_q}V_\eta$ so we clearly have a P-equivariant map from the right hand side of (7) to the left hand side. Since V_q contains V_{η_J} for any q-element subset J of $\{1,\dots,d\}$ this map is surjective.

For the injectivity let b be in P with $b\eta_{J_1}=\eta_{J_2}$ for two (not necessarily distinct) subsets J_1 and J_2 of $\{1,\dots,d\}$. Assume that b does not lie in P_0Z. Then we have $bx_0=\varphi_ix_0$ and $b\varphi_jx_0=x_0$ for some $1\leq i,j\leq d$. Hence $b=\varphi_ib_0$ for some b_0 in $\mathrm{Stab}_P(x_0)=P_0Z$ with $\varphi_ib_0\varphi_j$ lying also in P_0Z. This is a contradiction as $\varphi_ib_0\varphi_i^{-1}$ is in P_0Z, but $\varphi_i\varphi_j$ is not. It follows that η_{J_1} and η_{J_2} are in different P-orbits of $\mathcal{BT}$ if $J_1\neq J_2$ (since $\dim_{\mathbb{F}_p}L_{\varphi_ix_0}/pL_0=p^i$ for all $1\leq i\leq d$) and $\mathrm{Stab}_P(\eta_J)\subseteq P_0Z$. The statement follows. $\square$

Lemma 13. *The coefficient system $(V_\eta)_\eta$ defines an acyclic resolution of the representation V_{St}, ie. $H_0((V_\eta)_\eta)=V_{St}$ and $H_i((V_\eta)_\eta)=0$ for all $i\geq 1$.*

Proof. By Lemma 12 we note immediately that the natural map

$$\bigoplus_{\eta\in\mathcal{BT}_0}V_\eta\cong\mathrm{ind}_{P_0Z}^P(V_0)=\mathrm{ind}_{P_0Z}^P(M)\to V_{St}\tag{8}$$

is surjective since M generates V_{St}. On the other hand, if an element f in $\mathrm{ind}_{P_0Z}^P(M)$ lies in the kernel of the above map (8) then for some t in T_+ the support of tf lies in P_+. Hence for proving that f lies in the image of $\bigoplus_{\eta\in\mathcal{BT}_1}V_\eta$ we may assume that f has support in P_+.

However, we claim that for any b in P_+ and any element v in V_{bx_0} there is an element v_0 in V_{x_0} such that $v - v_0$ lies in the image of $\bigoplus_{\eta\in\mathcal{BT}_1} V_\eta$. Indeed, if $b = n_0 t$ for some n_0 in N_0 and t in T_+ (since $P_+ = N_0 T_+$) then v has support in $n_0 t N_0 t^{-1} \subseteq n_0 t\varphi_j^{-1} N_0 \varphi_j t^{-1}$ for any j with $t\varphi_j^{-1} \in T_+$. Hence v lies in $V_{\{n_0 t\varphi^{-1}x_0, n_0 t x_0\}}$ and the claim follows by induction on $K = \sum_{i=1}^d k_i$ with $tT_0Z = \prod_{i=1}^d \varphi_i^{k_i} T_0 Z$. This shows that $H_0((V_\eta)_\eta) = V_{St}$.

For the acyclicity of the resolution $(V_\eta)_\eta$ we are going to use Grosse-Klönne's local criterion [7]. To recall his result we need to introduce some terminology. Let $\hat\eta$ be a pointed $(q-1)$-simplex with underlying $(q-1)$-simplex η. Let $N_{\hat\eta}$ be the set of vertices z of $\mathcal{BT}$ such that $(\hat\eta, z)$ is a pointed q-simplex. Each element z in $N_{\hat\eta}$ corresponds to a lattice L_z with $L_{q-1} \subsetneq L_z \subsetneq L_0$ where $(L_0, \dots, L_{q-1})$ represents η. We call a subset M_0 of $N_{\hat\eta}$ stable with respect to $\hat\eta$ if for any two z, z' in M_0 the lattice $L_z \cap L_{z'}$ represents an element in M_0, as well. (By Lemma 2.2 in [7] this is equivalent to the original definition of stability in the case of the Bruhat-Tits building.) By Theorem 1.7 in [7] we need to verify that for any $1 \leq q \leq d$, any pointed $(q-1)$-simplex $\hat\eta$, and any subset M_0 of $N_{\hat\eta}$ that is stable with respect to $\hat\eta$ the sequence

$$\bigoplus_{\substack{z, z' \in M_0 \\ \{z,z'\} \in \mathcal{BT}_1}} V_{\{z,z'\}\cup\eta} \to \bigoplus_{z\in M_0} V_{\{z\}\cup\eta} \to V_\eta \tag{9}$$

is exact. Since our coefficient system is P-equivariant, we may assume without loss of generality that $\eta = \eta_J$ for some subset $J \subseteq \{1, \dots, d\}$ with $|J| = q-1$. Let $M_0 \subseteq N_{\hat\eta_J}$ be stable with respect to $\hat\eta_J$ (here $\hat\eta_J$ corresponds to any fixed vertex of η_J). Since the stabilizer of $\eta = \eta_J$ is contained in P_0Z, for any simplex $\nu \supset \eta$ we have $\nu = n_\nu \eta_{J'}$ for some $J' \supset J$ and n_ν in N_0 stabilizing η. In particular, $\mathrm{supp}(V_\nu) = n_\nu \left(\bigcap_{j\in J'} \varphi_j N_0 \varphi_j^{-1}\right)$. Hence for any n_0 in N_0 the coset $n_0 \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}$ is either contained in $\mathrm{supp}(V_\nu)$ or disjoint from $\mathrm{supp}(V_\nu)$. This means that we have

$$V_\nu = C_c^\infty\left(n_\nu \bigcap_{j\in J'} \varphi_j N_0 \varphi_j^{-1}\right) = \bigoplus_{n_0 \in n_\nu \bigcap_{i\in J'} \varphi_i N_0 \varphi_i^{-1} / \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}} C_c^\infty\left(n_0 \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}\right)$$

and it suffices to check the exactness of the restriction of (9) to each coset $n_0 \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}$. For any fixed n_0 we multiply the restriction of (9) to the coset $n_0 \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}$ by n_0^{-1} and obtain the sequence

$$\bigoplus_{z\neq z' \in n_0^{-1} M_0 \cap \{x_0,\dots,x_d\}} C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}) \to \bigoplus_{z \in n_0^{-1} M_0 \cap \{x_0,\dots,x_d\}} C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}) \to C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}) \tag{10}$$

since the condition on n_0 lying in $n_\nu \bigcap_{i\in J'} \varphi_i N_0 \varphi_i^{-1} / \bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1}$ is equivalent to that $n_0^{-1}\nu$ is a subsimplex of $\{x_0, \dots, x_d\}$. However, (10) is clearly exact and the lemma follows. □

Proof of Proposition 11. At first we note that Lemma 11.8 in [11] generalizes to our case with the same proof, ie. $D(\mathrm{ind}_{P_0Z}^P(V)) = D^0(\mathrm{ind}_{P_0Z}^P(V))$ and $D^i(\mathrm{ind}_{P_0Z}^P(V)) = 0$ for $i \geq 1$ for any smooth P-representation V with central character since $Z \cong \mathbb{Q}_p^\times$ here, as well. So by Lemmas

12 and 13 (and noting that Z acts trivially on each V_q) we may compute

$$D^i(V_{St}) = h^i(D(\bigoplus_{\eta\in\mathcal{BT}_\bullet} V_\eta)).$$

By Lemma 2.5 in [11] it suffices to show that for any $0 \leq q \leq d-1$ and any generating P_+-subrepresentation M_{q+1} of $\mathrm{ind}_{P_0Z}^P(V_{q+1})$ there exists a generating P_+-subrepresentation M_q of $\mathrm{ind}_{P_0Z}^P(V_q)$ such that $M_q \cap \partial_{q+1}(\mathrm{ind}_{P_0Z}^P(V_{q+1})) \subseteq \partial_{q+1}(M_{q+1})$. By (the analogue of) Lemma 3.2 in [11] (see the proof of Lemma 11.8 in [11]) we may assume that M_{q+1} is of the form $M_{q+1} = M_{q+1,\sigma}$ for some order reversing map σ from T_+/T_0Z to $\mathrm{Sub}(V_{q+1})$ satisfying

$$\bigcup_{t\in T_+/T_0Z} \sigma(t) = V_{q+1}.$$

Here $\mathrm{Sub}(V_{q+1})$ denotes the partially ordered set of P_0-subrepresentations of V_{q+1} and

$$M_{q+1,\sigma} = \bigoplus_{t\in T_+/T_0Z} \mathrm{ind}_{P_0Z}^{N_0tP_0Z}\sigma(t)$$

where $\mathrm{ind}_{P_0Z}^X(V)$ denotes the set of functions with support in X from P to V as a subset of $\mathrm{ind}_{P_0Z}^P(V)$ for any P_0Z-representation V and P_0Z-invariant subset X of P.

Moreover, since we have for any n_0 in N_0

$$V_{n_0\eta_J} = C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}\right) = \bigcup_{n=0}^{\infty} C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}/\bigcap_{j'=1}^{d}\varphi_{j'}N_0^{p^n}\varphi_{j'}^{-1}\right)$$

with finite sets

$$C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}/\bigcap_{j'=1}^{d}\varphi_{j'}N_0^{p^n}\varphi_{j'}^{-1}\right),$$

we may further assume (making M_{q+1} possibly even smaller) that σ is induced by an unbounded order reversing map $\sigma_0\colon T_+/T_0Z \to \mathbb{N}\cup\{-1\}$ with

$$\sigma(t) = \sum_{\substack{n_0\in N_0,\ |J|=q+1\\ J\subseteq\{1,\dots,d\}}} V_{n_0\eta_J}(\sigma_0(t))$$

where

$$V_{n_0\eta_J}(\sigma_0(t)) := C_c^\infty\left(n_0\bigcap_{j\in J}\varphi_jN_0\varphi_j^{-1}/\bigcap_{j'=1}^{d}\varphi_{j'}N_0^{p^{\sigma_0(t)}}\varphi_{j'}^{-1}\right) \tag{11}$$

for $\sigma_0(t)\geq 0$ and $V_{n_0\eta_J}(-1) := 0$. Now we put

$$M_q := M_{q,\sigma_0} := \bigoplus_{t\in T_+/T_0Z} \mathrm{ind}_{P_0Z}^{N_0tP_0Z} \sum_{\substack{n_0\in N_0,\ |J|=q\\ J\subseteq\{1,\dots,d\}}} V_{n_0\eta_J}(\sigma_0(t))$$

with $V_{n_0\eta_J}(\sigma_0(t))$ defined as in (11). We claim that

$$M_q \cap \partial_{q+1}(\mathrm{ind}_{P_0Z}^P(V_{q+1})) = \partial_{q+1}(M_{q+1}). \tag{12}$$

We now distinguish two cases whether $q=0$ or bigger. In the case $q>0$ the proof of (12) is completely analogous to that of Lemma 13. We see by construction that $\partial_{q+1}(M_{q+1}) \subseteq M_q$. Hence we have the following coefficient system on $\mathcal{BT}$ concentrated in degrees $q+1$, q, and $q-1$. In degrees $q+1$ and q we put M_{q+1} and M_q, respectively as subspaces of $\bigoplus_{\eta\in\mathcal{BT}_{q+1}} V_\eta = \mathrm{ind}_{P_0Z}^P(V_{q+1})$ and $\bigoplus_{\eta\in\mathcal{BT}_q} V_\eta = \mathrm{ind}_{P_0Z}^P(V_q)$, respectively. Indeed, we have by construction

$$\begin{aligned} M_{q+1} &= \bigoplus_{\eta\in\mathcal{BT}_{q+1}} M_{q+1}\cap V_\eta; \\ M_q &= \bigoplus_{\eta\in\mathcal{BT}_q} M_q \cap V_\eta. \end{aligned}$$

In degree $q-1$ we put the whole $\mathrm{ind}_{P_0Z}^P(V_{q-1})$. We use Grosse-Klönne's criterion in order to show that the sequence

$$M_{q+1} \to M_q \to \mathrm{ind}_{P_0Z}^P(V_{q-1})$$

is exact which implies (12) as the kernel of the map from M_q to $\mathrm{ind}_{P_0Z}^P(V_{q-1})$ is exactly the left hand side of (12) by Lemma 13. The proof proceeds the same way as in Lemma 13, but here all the functions are constant modulo the subgroup $\bigcap_{j=1}^d \varphi_j N_0^{p^{\sigma_0(t)}}\varphi_j^{-1}$ where t only depends on η (except for the case $\sigma_0(t)=-1$ whence all the functions are zero and the exactness is trivial). The sequence (10) remains exact if we replace $C_c^\infty(\bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1})$ by

$$C_c^\infty\left(\bigcap_{j=1}^d \varphi_j N_0 \varphi_j^{-1} / \bigcap_{j=1}^d \varphi_j N_0^{p^{\sigma_0(t)}}\varphi_j^{-1}\right)$$

hence the statement.

For $q=0$ we have to be a bit more careful, since the inductional argument in the proof of Lemma 13 does not work here as it is not true that any v in $M_0\cap V_{n_0tx_0}$ is equivalent to some v_0 in $M_0 \cap V_{x_0}$ modulo $\partial_1(M_1)$. (Note that $M_0\cap V_{x_0} = V_{x_0}(\sigma_0(1))$ but $M_0\cap V_{n_0tx_0} = V_{n_0tx_0}(\sigma_0(t))$ and $\sigma_0(t)$ could be much bigger than $\sigma_0(1)$.) However, we claim that for any v_t in $M_0\cap V_{n_0tx_0}$ with n_0 in N_0 and any $t'\leq t$ in T_+ there exists an element $v_{t'}$ in

$$M_0 \cap \left(\bigoplus_{n_1\in N_0/t'N_0t'^{-1}} V_{n_1t'x_0}\right)$$

such that $v_t - v_{t'}$ lies in $\partial_1(M_1)$. The statement is derived from this the following way. Any element m in M_0 is supported on finitely many vertices $\{b_it_ix_0\}_{i=1}^l$ of $\mathcal{BT}$ with t_i in T_+ and b_i in N_0. Moreover, there is a common t' in T_+ with $t'\leq t_i$ for any $1\leq i\leq l$. Now if m lies in $M_0\cap\partial_1(\mathrm{ind}_{P_0Z}^P(V_1))$ then by our claim there exists an m' in

$$M_0 \cap \left(\bigoplus_{n_1\in N_0/t'N_0t'^{-1}} V_{n_1t'x_0}\right) \tag{13}$$

such that $m - m'$ lies in $\partial_1(M_1)$. However, the map from (13) to V_{St} is injective since the supports of functions in $V_{n_1 t' x_0}$ and in $V_{n_1' t' x_0}$ are disjoint for $n_1 n_1'^{-1}$ not in $t' N_0 t'^{-1}$. It follows that $m' = 0$ hence m is in $\partial_1(M_1)$.

For the proof of the claim let v_t be in $M_0 \cap V_{n_0 t x_0}$ for some n_0 in N_0 and t in T_+. Then by definition of $V_{n_0 t x_0}$ the function v_t is supported on

$$n_0 t N_0 t^{-1} = \dot{\bigcup_{n_1 \in t N_0 t^{-1} / t' N_0 t'^{-1}}} n_0 n_1 t' N_0 t'^{-1} \tag{14}$$

since $t' \leq t$ implies $t' N_0 t'^{-1} \subseteq t N_0 t^{-1}$. Moreover, v_t is constant on the cosets of

$$t \left(\bigcap_{j=1}^{d} \varphi_j N_0^{p^{\sigma_0(t)}} \varphi_j^{-1} \right) t^{-1}$$

by the definition of M_0. We may assume by induction that $t' = t\varphi_i$ for some $1 \leq i \leq d$. Hence for any n_1 in $t N_0 t^{-1} / t\varphi_i N_0 \varphi_i^{-1} t^{-1}$ the pair $\{x_0, t^{-1} n_1 t \varphi_i x_0\}$ represents an edge of $\mathcal{BT}$. Therefore we have

$$M_1 \cap V_{\{n_0 t x_0, n_0 n_1 t \varphi_i x_0\}} = C_c^\infty(n_0 n_1 (t\varphi_i N_0 \varphi_i^{-1} t^{-1} / t \bigcap_{j=1}^{d} \varphi_j N_0^{p^{\sigma_0(t)}} \varphi_j^{-1} t^{-1})) \tag{15}$$

and the map

$$\pi_{n_0 t x_0} \circ \partial_1 \colon M_1 \cap \left(\bigoplus_{n_1 \in t N_0 t^{-1} / t' N_0 t'^{-1}} V_{\{n_0 t x_0, n_0 n_1 t \varphi_i x_0\}} \right) \to M_0 \cap V_{n_0 t x_0}$$

is surjective comparing (14) and (15). (Here $\pi_{n_0 t x_0}$ denotes the projection of M_0 onto $M_0 \cap V_{n_0 t x_0}$.) The claim follows noting that

$$\partial_1(M_1 \cap V_{\{n_0 t x_0, n_0 n_1 t \varphi_i x_0\}}) \subseteq M_0 \cap (V_{n_0 t x_0} \oplus V_{n_0 n_1 t \varphi_i x_0}).$$

□

The following is an immediate corollary of Remark 6.4 in [11] using Proposition 11.

Corollary 14. *The natural transformation a_V defined in section 6 of [11] gives an isomorphism*

$$a_{V_{St}} \colon V_{St}^* \to \psi^{-\infty}(D^0(V_{St})).$$

Remark. Proposition 11 (and also Lemmas 12 and 13) remain true in the following more general setting with basically the same proof. Let G still be $\mathrm{GL}_{d+1}(\mathbb{Q}_p)$ and V be a smooth o-torsion P-representation with a unique minimal generating P_+-subrepresentation M. Assume further that $nM \cap M = 0$ for any n in $N \setminus N_0$. Then we have $D^0(V) = D(V)$ and $D^i(V) = 0$ for all $i \geq 1$.

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