ANALYTIC DIRAC APPROXIMATION FOR REAL LINEAR ALGEBRAIC GROUPS

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Abstract. For a real linear algebraic group G let $\mathcal{A}(G)$ be the algebra of analytic vectors for the left regular representation of G on the space of superexponentially decreasing functions. We present an explicit Dirac sequence in $\mathcal{A}(G)$. Since $\mathcal{A}(G)$ acts on E for every Fréchet-representation (π, E) of moderate growth, this yields an elementary proof of a result of Nelson that the space of analytic vectors is dense in E.

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1. introduction

In this paper we provide an explicit Dirac sequence of superexponentially decrasing analytic functions on a linear algebraic group. This yields an elementary proof of a theorem of Nelson [2] that the space of analytic vectors is dense. In order to keep the exposition self contained we recall basic constructions from [5, 6]

Let (π, E) be a representation of a Lie group G on a Fréchet-space E. For a vector $v \in E$ we denote by γ_v the corresponding orbit map

$$\gamma_v : \ G \to E, \quad g \mapsto \pi(g)v.$$

A vector $v \in E$ is called *analytic* if the orbit map γ_v is a real analytic E-valued map. We denote the space of all analytic vectors by E^ω.

Let $\mathbf{g}$ be a left invariant Riemannian measure on G. To $\mathbf{g}$ we associate a Riemannian distance d on G: $d(g)$ is defined as the infimum lenght of all arcs joining g and 1.

Let $\mathcal{R}(G)$ be the space of superexponentially decrasing smooth functions on G with respect to the distance d, i.e

$$\mathcal{R}(G) = \left\{ f \in C(G) \mid \forall n \in \mathbb{N} : p_n(f) := \sup_{g \in G} |f(g)| e^{nd(g)} < \infty \right\}.$$

The space $\mathcal{R}(G)$ is a Fréchet algebra under convolution and is independent of the choice of the left invariant metric.

Let us assume that (π, E) is a F-representation, i.e the representation is of moderate growth. In particular every Banach representation is a F-representation.

For every continuous seminorm q on E exists a continuous seminorm q' and constants $C, c > 0$ such that

$$q(\pi(g)v) \leq C e^{cd(g)} q'(v) \quad (\forall g \in G, \forall v \in E).$$

Furthermore there exists a constant $r' > 0$ such that $\forall r > r'$

$$\int_G e^{-rd(g)}\ dg < \infty.$$

Hence there is a corresponding algebra representation Π of $\mathcal{R}(G)$ which is given by

$$\Pi(f)v = \int_G f(g)\pi(g)v\ dg \quad (f \in (G), v \in E).$$

We denote the space of analytic vectors $\mathcal{R}(G)^\omega$ for the left regular representation L by $\mathcal{A}(G)$.
A function $f \in \mathcal{R}(G)$ is in $\mathcal{A}(G)$ if and only if it satisfies the following two conditions.

(1) There exists a neighborhood $U \subset G_\mathbb{C}$ of 1 and a $F \in \mathcal{O}(U^{-1}G)$ with $F|_G = f$.
(2) For every compact subset $Q \subset U$ we have $\sup_{k\in Q} p_n\left(L_k(F)\right) < \infty$ for all $n \in \mathbb{N}$.

Throughout this text we refer to these conditions as condition (1) and condition (2).
We define a positive function on $GL_n(\mathbb{R})$ by

$$|g| = \sqrt{\mathrm{tr}\,(g^t g)} \quad (g \in GL_n(\mathbb{R})).$$

Let K be the maximal compact subgroup $O(n)$ of $GL_n(\mathbb{R})$ and $K_\mathbb{C}$ its complexification. Then $|\cdot|$ is $K_\mathbb{C}$-bi-invariant and sub-multiplicative. Note that for a matrix $g = (a_{ij})_{1\le i,j\le n}$ we have $|g| = \sqrt{\sum_{1\le i,j\le n} a_{ij}^2}$. Hence $|\cdot|^2$ is holomorphic on $GL_n(\mathbb{C})$.
Let G be a real linear algebraic group, then G is a closed subgroup of some $GL_n(\mathbb{R})$.
We define a norm in the sense of [1] on G by

$$\|g\| = \max\{|g|, |g^{-1}|\}, \qquad (g \in G)\,.$$

For $t > 0$ we consider the function

$$\varphi_t : G \to \mathbb{R}, \quad g \mapsto C_t e^{-t^2\left(|g-1|^{4n} + |g^{-1}-1|^{4n}\right)},$$

with constants $C_t > 0$ such that $\|\varphi_t\|_{\mathrm{L}^1(G)} = 1$.
Recall that a sequence $(f_k)_{k>0}$ is called a Dirac sequence if it satisfies the following three conditions:

(a) $f \geq 0$, $\forall k \in \mathbb{N}$
(b) $\int_G f_k(g)\ dg = 1$, $\forall k \in \mathbb{N}$
(c) For every $\varepsilon > 0$ and every neighborhood U of 1 in G exists a $M \geq 1$ such that $\int_{G\setminus U} f_m(g)\ dg < \varepsilon$, $\forall m \geq M$.

We prove the following theorem

Theorem 1.1. (a) *$\varphi_t \in \mathcal{A}(G)$ for all $t > 0$.*
(b) *The sequence $(\varphi_t)_{t>0}$ forms a Dirac sequence.*

As a corollary we obtain a result of Nelson [2] for real linear algebraic groups.

Corollary 1.2. *Let (π, E) be a F-representation of a real linear algebraic group G on a Fréchet space E, then the space of analytic vectors E^ω is dense in E.*

Remark 1.3. *In fact* [5] *every analytic vector is a finite sum of vectors of the form $\Pi(f)v$ with $f \in \mathcal{A}(G)$ and $v \in E$.*

Remark 1.4. *If G is a real reductive group then Theorem 1.1 holds even for*

$$\varphi_t' : G \to \mathbb{R}, \quad g \mapsto C_t e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)},$$

with constants $C_t > 0$ such that $\|\varphi_t'\|_{\mathrm{L}^1(G)} = 1$.

2. Proofs

The function φ_t posses an holomorphic continuation to $G_\mathbb{C}$ which we also denote by φ_t, but φ_t does not satisfy condition (2) on the whole of $G_\mathbb{C}$. We now describe for $GL_n(\mathbb{R})$ a $K_\mathbb{C} \times GL_n(\mathbb{R})$-invariant domain in $GL_n(\mathbb{C})$ where φ_t satisfies condition (2). It turns out that this domain is a subdomain of the *crown domain* Ξ [3, 4]. Therefore let $\Omega = \{\mathrm{diag}(d_1, \ldots, d_n) \ : \ d_k \in \mathbb{R}, |d_k| < \frac{\pi}{4}, \forall\ k = 1 \ldots, n\} \subset \mathbb{R}^{n^2}$.

Remark 2.1. *Note that this Ω is not the same as in* [4]*. Let us denote by Ω_{ss} the Omega used in* [4] *for $SL_n(\mathbb{R})$. Then Ω is related to Ω_{ss} in the following way: Ω has the property that $\Omega_{ss} + \mathbb{R}e = \Omega + \mathbb{R}e$ with $e = \mathrm{diag}(1, \ldots, 1)$. In other words, up to central shift the Omegas coincide.*

We define Ξ_n by $\Xi_n = GL_n(\mathbb{R}) \exp\left(i\frac{1}{n+1}\Omega\right) K_\mathbb{C}$.

Remark 2.2. *Let us remark that if G be a real reductive group, i.e a closed transposition stable subgroup of $GL_n(\mathbb{R})$, then $d(g)$ and $\log\|g\|$ are comparable in the sense that there are constants $c_1, c_2 > 0$ and $C_1, C_2 \in \mathbb{R}$ such that*

$$c_1 d(g) + C_1 \le \log\|g\| \le c_1 d(g) + C_2.$$

Hence we can give an alternative characterization of the space $\mathcal{R}(G)$ in terms of $\|\cdot\|$:

$$\mathcal{R}(G) = \left\{ f \in C(G) \mid \forall n \in \mathbb{N} : p_n(f) := \sup_{g\in G} \|g\|^n |f(g)| < \infty \right\}.$$

In the proof of the next proposition we need the following notations: For a matrix $g = (a_{ij})_{1\le i,j\le n}$ we denote by g_i the i-th column vector $(a_{1i}, \ldots, a_{ni})^T$ and for a vector $w \in \mathbb{C}^n$ we denote by $\|w\|_2$ the euclidean norm.

Proposition 2.3. *The function φ_t satisfies condition (2) on Ξ_n.*

Proof. Let $Q \subset \Xi_n$ be compact. We show that there exists a constant $C > 0$ such that

$$|\varphi_t(gq)| \le e^{-C\|g\|^{4n}}, \qquad (\forall g \in G, \forall q \in Q). \tag{2.1}$$

There exists a $\Omega' \subset \Omega$ which satisfies the following properties.

(a) $Q \subset GL_n(\mathbb{R}) \exp(i\frac{1}{n+1}\Omega')K_\mathbb{C}$.

(b) There exists a constant $C_1' > 0$ such that for all $d = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \in \exp(i\frac{1}{n+1}\Omega')$ we have $\cos(2(\theta_{\alpha_1} + \ldots + \theta_{\alpha_{n+1}})) \geq C_1'$ for all $\alpha_j \in \{1, \dots, 2n\}$.

This implies that for $k = 1, \dots, 2n$ there exists a constant $C_1 > 0$ such that

$$\operatorname{Re}\left(|gq|^{2k}\right) \geq C_1 |g|^{2k}, \quad (g \in GL_n(\mathbb{R}), q \in Q)\,. \tag{2.2}$$

Therefore let $d = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \in \exp(i\Omega')$ and $g' \in GL_n(\mathbb{R})$ then

$$|g'd|^{2k} = \left(e^{2\theta_1 i}\|g_1'\|^2 + \cdots + e^{2\theta_n i}\|g_n'\|^2\right)^k \tag{2.3}$$

Hence $\operatorname{Re}\left(|g'q|^{2k}\right) \geq C_1'|g'|^{2k}$ according to (b).
Let $q = hdk$ with $h \in GL_n(\mathbb{R})$ and $k \in K_{\mathbb{C}}$.
Then $\operatorname{Re}\left(|gq|^2\right) = \operatorname{Re}\left(|ghdk|^2\right) = \operatorname{Re}\left(|ghd|^2\right) \geq C_1'|gh|^2$. Since Q is compact there exists a constant $C_1 > 0$ such that $C_1'|gh|^2 > C_1|g|^2$ for all $q \in Q$. Thus we obtain (2.2).
Likewise we can show that for $k = 1, \dots, 2n$ there exists a constant $C_2 > 0$ such that

$$\operatorname{Re}\left(\left|(gq)^{-1}\right|^{2k}\right) \geq C_2|g^{-1}|^{2k}, \quad (g \in GL_n(\mathbb{R}), q \in Q)\,. \tag{2.4}$$

Note that for $k = 1, \dots, 4n$ there exists a constant $C_3 > 0$ such that

$$\operatorname{Re}\left(\operatorname{tr}(gq)^k\right) \leq |\operatorname{tr}(gq)|^k \leq C_3|g|^k, \ (g \in GL_n(\mathbb{R}), q \in Q)\,. \tag{2.5}$$

Since $|g-1|^{4n} = (|g|^2 - 2\operatorname{tr}(g) + n)^{2n}$ we obtain the upper bound (2.1) for some $C > 0$ by expanding the $2n$-th power and combining the estimates.
Since $GL_n(\mathbb{R})$ is real reductive Remark 2.2 implies that φ_t satisfies condition (2) on Ξ. $\square$

Hence $\varphi_t \in \mathcal{A}(GL_n(\mathbb{R}))$. Now we show that for every real linear algebraic group $G \subset GL_n(\mathbb{R})$ the functions φ_t are elements of $\mathcal{A}(G)$.

Proposition 2.4. *Let $G \subset GL_n(\mathbb{R})$ be a real linear algebraic group then $\varphi_t \in \mathcal{A}(G)$.*

Proof. The set $G_{\mathbb{C}} \cap \Xi_n$ is an open neighborhood of $1 \in G$ to which φ_t extends holomorphically.
We give an upper bound for $d(g)$ which implies that φ_t satisfies (2) on this neighborhood.
Every algebraic group G can be decomposed as a semidirect product $G = \operatorname{Rad}_u G \rtimes L$ of a connected unipotent group $\operatorname{Rad}_u G$ and a reductive group L. We write $g = ur$ with u unipotent and r reductive, then $d(g) = d(ur) = d(u) + d(r)$.
Remark 2.2 implies that there exists a constant $C > 0$ such that $d(r) \leq C\log(\|r\|) + C$. Note that the unipotent radical $\operatorname{Rad}_u G$ is connected and u has a real logarithm. The path $\gamma(t) = \exp(t\log(u))$ connects 1 and $\log(u)$ and has length $|\log(u)|$, thus $d(u) \leq |\log(u)|$. Since $\log(u) = \sum_{k=0}^n \frac{(-1)^k(u-1)^k}{k}$ and $|u-1|^k \leq 1 + |u-1|^n \leq 1 + |u|^n$ for $k = 0, \dots, n$ we obtain $|\log(u)| \leq 1 + n + n\|u\|^n$. Let $J = D + N$ be the Jordan normal form of g with D a diagonal and N a nilpotent matrix and let $P \in GL_n(\mathbb{C})$ be the change of basis matrix.

Since the Jordan-Chevalley decomposition is unique, $u = P(1 + D^{-1}N)P^{-1}$ and $r = PDP^{-1}$. Therefore $\|u\| \leq \|P\|^2(\|1\| + \|D^{-1}N\|) \leq \|P\|^2(\|1\| + \|D\|) \leq \|P\|^2(\|1\| + \|g\|)$. The last inequality follows from the fact that the sum of the absolute values of the squares of the eigenvalues is less or equal than the sum of the squares of the singular values. Likewise we obtain $\|r\| \leq \|P\|^2\|g\|$. Since the column vectors of the matrices P and P^{-1} are chains of generalized eigenvectors of g we obtain $\|P\|^2 \leq n2^n\|g\|^2$.
Combining these estimates we obtain that there exists a constant $R > 0$ such that

$$e^{nd(g)} \leq Re^{R\|g\|^{3n}}\|g\|^R.$$

Hence φ_t satisfies condition (2) on $G_{\mathbb{C}} \cap \Xi_n$. $\square$

Proposition 2.5. *The family $(\varphi_t)_{t\geq 1}$ forms for $t \to \infty$ a Dirac sequence.*

Proof. Let V be a neighborhood of 0 in $\mathfrak{g}$ such that the exponential map is a diffeomorphism of V with some neighborhood U of 1 in G. Then

$$\int_G e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)}\, dg \geq \int_U e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)}\, dg.$$

The differential of exp at X is given by

$$dL_{\exp(X)} \circ \tfrac{1-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}.$$

Therefore

$$\int_U e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)}\, dg = \int_V e^{-t^2\left(|e^X-1|^2+|e^{-X}-1|^2\right)} \left|\det\left(\tfrac{1-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}\right)\right|\, dX$$

There exists a constant $C > 0$ with

$$\left|\det\left(\tfrac{1-e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)}\right)\right| \geq C, \quad \forall X \in V.$$

Hence

$$\int_G e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)}\, dg \geq C\int_V e^{-t^2\left(|e^X-1|^2+|e^{-X}-1|^2\right)}\, dX$$

There exists a constant $C' > 0$ such that

$$|e^X - 1|^2 + |e^{-X} - 1|^2 \leq C'|X|^2, \quad \forall X \in V.$$

Thus

$$\begin{aligned}\int_V e^{-t^2\left(|e^X-1|^2+|e^{-X}-1|^2\right)}\, dX &\geq \int_V e^{-t^2C'|X|^2}\, dX \\ &= \int_V e^{-C'|tX|^2}\, dX \\ &= \frac{1}{t^{\dim\mathfrak{h}}}\int_V e^{-C'|X|^2}\, dX\end{aligned}$$

Therefore

$$\int_H e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)}\, dg \geq \int_V e^{-C'|X|^2}\, dX \geq C_1 t^{-\dim\mathfrak{h}}$$

with $C_1 = C\int_V e^{-C'|X|^2}\, dX < \infty$.
Let U be a neighborhood of 1 in G, there exists a constant $R > 0$ such that

$$|g - 1|^2 + |g^{-1} - 1|^2 \geq R, \quad \forall g \in G\backslash U.$$

Hence
$$e^{-\frac{1}{2}t^2\left(|g-1|^2+|g^{-1}-1|^2\right)} \leq e^{-\frac{1}{2}t^2 R}, \quad \forall g \in G\backslash U.$$
Therefore
$$\begin{aligned}
&\int_{G\backslash U} e^{-t^2\left(|g-1|^2+|g^{-1}-1|^2\right)} dg \\
&= \int_{G\backslash U} e^{-\frac{1}{2}t^2\left(|g-1|^2+|g^{-1}-1|^2\right)} e^{-\frac{1}{2}t^2\left(|g-1|^2+|g^{-1}-1|^2\right)} dg \\
&\leq e^{-\frac{1}{2}t^2 R} \int_{G\backslash U} e^{-\frac{1}{2}t^2\left(|g-1|^2+|g^{-1}-1|^2\right)} dg \\
&\leq e^{-\frac{1}{2}t^2 R} \int_{G\backslash U} e^{-\frac{1}{2}\left(|g-1|^2+|g^{-1}-1|^2\right)} dg \\
&= C_2 e^{-\frac{1}{2}t^2 R}
\end{aligned}$$
with $C_2 = \int_{G\backslash U} e^{-\frac{1}{2}\left(|g-1|^2+|g^{-1}-1|^2\right)} dg < \infty$. Hence
$$\int_{G\backslash U} \varphi_t(g) dh \leq e^{-\frac{1}{2}t^2 R} t^{-\dim \mathfrak{g}} \frac{C_2}{C_1}.$$
The expression on the right hand side tends to 0 as t tends to infinity. $\square$

Lemma 2.6.
$$\Pi(\mathcal{A}(G))E \subset E^{\omega}$$

Proof. Let $f \in \mathcal{A}(G), v \in E$. Then the orbit map $\gamma_{\Pi(f)v}$ is given by
$$\begin{aligned}
\gamma_{\Pi(f)v}(g) &= \pi(g) \int_H f(x)\pi(x)v \; d\mu(x) \\
&= \int_H f(x)\pi(gx)v \; d\mu(x) \\
&= \int_H f(g^{-1}x)\pi(x)v \; d\mu(x) \\
&= \pi\big(L_g(f)\big)v.
\end{aligned}$$
Hence the orbit map is equal to to the composition
$$G \to \mathcal{R}(G) \to E$$
Here the first arrow denotes the map $g \mapsto L_g(f)$ and the second the map $\varphi \mapsto \Pi(\varphi)v$. The first map in this composition is analytic and the last is linear. Hence the whole map is an analytic map from G to E. $\square$

Theorem 2.7. *For every real linear algebraic group G exists an analytic Dirac sequence, i.e a Dirac sequence which members are elements of $\mathcal{A}(G)$.*

Proof. The sequence of functions $(\varphi_t)_{t\geq 1}$ on G provides a Dirac sequence, as we have seen in Proposition 2.5. $\square$

Corollary 2.8. *Let (π, E) be a F-representation of a real linear algebraic group G on a Fréchet space E. Then the space E^{ω} of analytic vectors is dense in E.*

Proof. Let $v \in E$ and let $(\varphi_t)_{t\geq 1}$ be an analytic Dirac sequence. Then $\pi(\varphi_t)v$ is, according to Lemma 2.6, a sequence of analytic vectors which tends to v in E. $\square$

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