Localization algebras and deformations of Koszul algebras

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Abstract. We show that the center of a flat graded deformation of a standard Koszul algebra A behaves in many ways like the torus-equivariant cohomology ring of an algebraic variety with finite fixed-point set. In particular, the center of A acts by characters on the deformed standard modules, providing a "localization map." We construct a universal graded deformation of A, and show that the spectrum of its center is supported on a certain arrangement of hyperplanes which is orthogonal to the arrangement coming from the algebra Koszul dual to A. This is an algebraic version of a duality discovered by Goresky and MacPherson between the equivariant cohomology rings of partial flag varieties and Springer fibers; we recover and generalize their result by showing that the center of the universal deformation for the ring governing a block of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_n$ is isomorphic to the equivariant cohomology of a Spaltenstein variety. We also identify the center of the deformed version of the "category $\mathcal{O}$" of a hyperplane arrangement (defined by the authors in a previous paper) with the equivariant cohomology of a hypertoric variety.

1 Introduction

In 1976, Bernstein, Gelfand, and Gelfand introduced the category $\mathcal{O}$ of representations of a semisimple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ [BGG76]. Over the course of the next decade, several new techniques appeared in the algebraic and geometric study of this category. Two of the most important were

- the use of a deformed category $\widehat{\mathcal{O}}$, which consists of families of representations over a formal neighborhood of 0 in the weight space $\mathfrak{h}^*$ of $\mathfrak{g}$
- connections to the geometry of the flag variety G/B, especially through the localization theorem of Beilinson and Bernstein [BB81].

The first of these was used by Soergel [Soe90] to show that an integral block of category $\mathcal{O}$ is equivalent to the module category of a certain finite dimensional algebra A. Furthermore, Soergel showed that the center of A is isomorphic to the cohomology ring of $H^*(G/P)$, where $P \subset G$ is a parabolic subalgebra that depends on the block. This fact reflects a second connection to geometry:

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each block of category $\mathcal{O}$ not only has a geometric interpretation via the localization theorem, it is also Koszul dual to the category of Schubert smooth perverse sheaves on G/P, which is proved independently of the localization theorem in [BGS96]. In later work [Soe92], Soergel showed that $\widehat{\mathcal{O}}$ can be described using the T-equivariant geometry of G/P, where T is a maximal torus of G. In particular, he computed a deformation $\widehat{A}$ of A whose module category is isomorphic to the corresponding block of $\widehat{\mathcal{O}}$, and he showed that the center of $\widehat{A}$ is isomorphic to a completion of the equivariant cohomology ring $H^*_T(G/P)$.

Our aim in this paper is to study how the first of these techniques, that of studying the representation theory of a finite dimensional Koszul algebra by deforming it, can be applied in a general algebraic context, without the benefit of the geometric or Lie theoretic interpretations of category $\mathcal{O}$. In Section 4 we use the work of Braverman and Gaitsgory [BG96] to show that any Koszul algebra A has a universal flat graded deformation $\tilde{A}$, so that any other graded flat deformation is obtained from $\tilde{A}$ by a unique base change. In Sections 9 and 10, we use techniques of Soergel and Fiebig [Soe90, Soe92, Fie03, Fie06, Fie08] to show that Soergel's deformed algebra is in fact the completion of the universal deformation.

For what follows, we will need to assume not just that our algebras are Koszul, but also **standard Koszul** (Definition 3.4), which roughly means that its module category is highest weight in the sense of [CPS94]. The general study of standard Koszul algebras was initiated in [ÁDL03], and the main examples are the algebras A introduced above, the generalizations obtained by replacing $\mathcal{O}$ with its parabolic version, and the algebras which were defined in [BLPWa] using hyperplane arrangements.

We focus our attention primarily on the center of the universal deformation $\tilde{A}$ of a standard Koszul algebra A. Recall that Soergel identified the center of his deformed algebra with the (completed) equivariant cohomology ring of a partial flag variety. We generalize this result by showing that the center of $\tilde{A}$ always "behaves like" a torus-equivariant cohomology ring. More precisely, we introduce a structure called a **localization algebra** (Definition 2.1), which is an abstraction of the data given by a torus-equivariant cohomology ring and the localization map to the fixed point set, and we prove the following theorem (Corollary 5.7).

Theorem 1.1. *The center $\mathcal{Z}(A) := Z(\tilde{A})$ of the universal deformation canonically admits the structure of a localization algebra.*

For many of the examples of standard Koszul algebras mentioned above, the localization algebras that we obtain are in fact isomorphic to equivariant cohomology rings. We have already addressed the case where A is the algebra whose module category is a block of $\mathcal{O}$. If $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathcal{O}$ is replaced by its parabolic version, then Stroppel and Brundan [Str, Bru08a] show that the center of A is isomorphic to the cohomology ring of a Spaltenstein variety.[4] In this case, we show that the center of $\tilde{A}$ is isomorphic to the torus-equivariant cohomology ring of the same variety (Theorem 9.9),

[4] A Spaltenstein variety is a certain subvariety of G/P, where once again the choice of P depends on the choice of central character. If the central character is generic, then P will be a Borel subalgebra, and the Spaltenstein variety will be a Springer fiber; this is the case proven in [Str]. See Section 9 for more details.

generalizing Soergel's result in the non-parabolic case and Brundan's result for the un-deformed algebra. Finally, in the case where A is one of the algebras introduced in [BLPWa], we show that the center of $\tilde{A}$ is isomorphic to the cohomology ring of a hypertoric variety (Theorem 8.5), generalizing the un-deformed, non-equivariant result of [BLPWa, 4.16].

Once we have established that the center of the universal deformation is a localization algebra (Theorem 1.1), we study the relationship between the localization algebras associated to a dual pair of standard Koszul algebras. This problem is motivated by an observation made by Goresky and MacPherson in a paper that has, *a priori*, nothing to do with Koszul algebras [GM].[5] If X is a variety equipped with the action of a torus T with isolated fixed points, then the localization map in equivariant cohomology may be used to define a finite collection of vector subspaces of $H_2^T(X)$, each of which is isomorphic to the Lie algebra $\mathfrak{t}$ of T. Goresky and MacPherson observed that the arrangement associated to a partial flag variety for $\mathfrak{gl}_n$ is dual to the one associated to a Springer fiber, in the sense that the equivariant second homology groups are dual as vector spaces, and the subspaces appearing on one side are the perpendicular spaces to the subspaces appearing on the other side.

Our approach to the problem is to interpret and generalize the examples of [GM] in a purely algebraic context. First, we define what it means for two localization algebras to be **dual**, so that the result of [GM] may be formulated as the duality of a certain pair of localization algebras. We next introduce one more technical hypothesis: we call an algebra A **flexible** if it is standard Koszul and the natural map from the center of $\tilde{A}$ to the center of A is surjective in degree 2. Our main result (Theorem 7.1 and Corollary 7.5) is the following.

Theorem 1.2. *If A is flexible, then so is the dual algebra $A^!$, and the localization algebras $\mathcal{Z}(A)$ and $\mathcal{Z}(A^!)$ are canonically dual.*

In light of Theorem 9.9 and the fact that a regular block of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_n$ is Koszul dual to a singular block of ordinary category $\mathcal{O}$ [BGS96], the examples found by Goresky and MacPherson follow from Theorem 1.2. In fact, a theorem of Backelin's [Bac99] says that an integral block of parabolic category $\mathcal{O}$ is dual to another such block, and so the Goresky-MacPherson phenomenon generalizes to all $\mathfrak{gl}_n$ Spaltenstein varieties.

Example 1.3. Consider the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "x_1"] & \bullet \arrow[l, bend left, "y_1"] \arrow[r, bend left, "x_2"] & \bullet \arrow[l, bend left, "y_2"] \end{tikzcd}$$

and let A be the path algebra modulo the relations x_1y_1 and $y_1x_1 - x_2y_2$. This is a noncommutative graded algebra, and it is standard Koszul. is equivalent to a singular integral block of category $\mathcal{O}$ for $\mathfrak{sl}_3$; Soergel [Soe90] tells us that its center is isomorphic to the cohomology ring of $\mathbb{P}^2$. Indeed, the center $Z(A)$ is generated by the degree 2 class $x_2y_2 + y_2x_2$, whose cube is zero. The center

[5] In particular, it is unrelated to the appearance of Koszul duality in [GKM98], which is very different in flavor from anything in this paper.

$\mathcal{Z}(A) := Z(\tilde{A})$ of the universal deformation of A is isomorphic to the T^2-equivariant cohomology ring of $\mathbb{P}^2$.

Now consider the Koszul dual ring $A^!$ of A, which is a quotient of the path algebra of the dual quiver

$$\begin{tikzcd} \bullet \arrow[r, bend right, "y_1^!"'] & \bullet \arrow[l, bend right, "x_1^!"'] \arrow[r, bend right, "y_2^!"'] & \bullet \arrow[l, bend right, "x_2^!"'] \end{tikzcd}$$

by the relations $x_1^! y_1^! + y_2^! x_2^!$, $x_2^! y_2^!$, $x_2^! x_1^!$, and $y_1^! y_2^!$. The module category of $A^!$ is equivalent to a regular block of parabolic category $\mathcal{O}$ for the parabolic subalgebra of $\mathfrak{sl}_3$ preserving a line in $\mathbb{C}^3$ [Str03, 5.2.1]. The results of Stroppel or Brundan [Str, Bru08a] tell us that the center of A should be isomorphic to the cohomology ring of a certain Springer fiber, namely the one consisting of two projective lines that touch in a single point. Indeed, the center of $A^!$ is generated by the degree 2 classes $y_1^! x_1^!$ and $y_2^! x_2^!$, with all products trivial. The center $\mathcal{Z}(A^!) := Z(\tilde{A}^!)$ of the universal deformation of $A^!$ is isomorphic to the T^1-equivariant cohomology ring of the same variety.

The equivariant cohomology rings associated to these two algebras constitute the simplest non-trivial example of a dual pair from [GM]. More details from the perspective of [GM] are given in Examples 2.4, 2.5, and 2.7.

Remark 1.4. The following is meant only to provide some additional geometric motivation for our results. In both of the families of standard Koszul algebras considered in Sections 8-10, the localization algebras that arise are isomorphic to equivariant cohomology rings of certain algebraic symplectic manifolds or orbifolds.[6] We expect that the algebra itself will be isomorphic to the Ext-algebra of a certain module over a quantization of the structure sheaf of the manifold. The map from the cohomology ring to the center of our algebra will then be induced by the action of the constant sheaf on this module. For the case studied in Section 8, this program is being carried out in [BLPWb]. In the case of an algebra whose module category is equivalent to an integral block of ordinary category $\mathcal{O}$, it is well-known: the manifold in question is $T^*(G/P)$, and the module is the microlocalization of the direct sum of all of the Schubert-smooth simple D-modules on G/P. The authors plan to treat the case of parabolic category $\mathcal{O}$ in a forthcoming paper.

When two algebraic symplectic manifolds give rise to dual standard Koszul rings in this way, we refer to them as a **symplectic dual** pair. So the main result of this paper could be interpreted as saying that symplectic dual pairs have equivariant cohomology rings that are dual as localization algebras. Beside the pairs of hypertoric varieties and pairs of resolved Slodowy slices that we consider in this paper, other conjectural examples include Hilbert schemes on ALE spaces, which we expect to be dual to certain moduli spaces of instantons on $\mathbb{C}^2$, and quiver varieties of simply laced Dynkin type, which we expect to be dual to resolutions of slices to certain subvarieties of the affine Grassmannian. We expect further examples to arise from physics as Higgs branches of the moduli space of vacua for mirror dual 3-dimensional $\mathcal{N} = 4$ superconformal field theories, or as

[6] Hypertoric varieties are symplectic orbifolds, and Spaltenstein varieties are torus-equivariant deformation retracts of resolved Slodowy slices, which are symplectic manifolds.

the Higgs and Coulomb branches of a single such theory. That hypertoric varieties occur in mirror dual theories was observed by Kapustin and Strassler in [KS99].

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2 Localization algebras

In this section we introduce localization algebras and define a notion of duality between them. Before giving the general definition of a localization algebra, we consider a motivating setup from equivariant topology.

Let X be a complex algebraic variety and T be an algebraic torus acting on X with the fixed point set X^T finite and non-empty. The equivariant cohomology ring[7] $H^*_T(X)$ is a graded algebra over the polynomial ring $\operatorname{Sym}\mathfrak{t}^*$ (the T-equivariant cohomology of a point), where the elements of $\mathfrak{t}^*$ lie in degree 2. From these data, we obtain natural graded algebra homomorphisms

$$\operatorname{Sym}\mathfrak{t}^* \hookrightarrow \operatorname{Sym} H^2_T(X) \to H^*_T(X) \to H^*_T(X^T) \cong H^*(X^T) \otimes \operatorname{Sym}\mathfrak{t}^*, \tag{1}$$

where the second map is given by multiplying classes of degree 2 together, and the third, often called the **localization map**, is given by restriction to X^T. If the ordinary cohomology of X vanishes in odd degrees (for example if X is a smooth projective variety), then $H^*_T(X)$ is a free $\operatorname{Sym}\mathfrak{t}^*$-module, the localization map is injective, and the cokernel of the localization map is a torsion $\operatorname{Sym}\mathfrak{t}^*$-module. With this example in mind, we formulate the following general definitions.

Definition 2.1. A **localization algebra** is a quadruple $\mathcal{Z} = (U, Z, \mathcal{I}, h)$, where U is a finite-dimensional complex vector space, Z is a finitely generated graded $\operatorname{Sym} U$-algebra, $\mathcal{I}$ is a finite set, and

$$h : Z \to \bigoplus_{\alpha\in\mathcal{I}} \operatorname{Sym} U$$

is a homomorphism of $\operatorname{Sym} U$-algebras. If the kernel and cokernel of h are torsion $\operatorname{Sym} U$-modules, then we call $\mathcal{Z}$ **strong**. If Z is free of rank $|\mathcal{I}|$ as a $\operatorname{Sym} U$-module, we call $\mathcal{Z}$ **free**. When there is no chance for confusion, we may refer to Z itself as a localization algebra.

Example 2.2. By the preceding discussion, the equivariant cohomology ring $H^*_T(X)$ carries a natural structure of a localization algebra, with $U = \mathfrak{t}^*$ and $\mathcal{I} = X^T$. It is both strong and free if and only if $H^{\mathrm{odd}}(X) = 0$.

[7] All cohomology groups in this paper will be taken with complex coefficients.

If $H^{\mathrm{odd}}(X) = 0$, it is often easier to think of the morphisms of (1) in terms of the dual morphisms of schemes:

$$\mathfrak{t} \twoheadleftarrow H_2^T(X) \leftarrow \operatorname{Spec} H_T^*(X) \twoheadleftarrow X^T \times \mathfrak{t}.$$

The composite map $\operatorname{Spec} H_T^*(X) \to \mathfrak{t}$ is a flat family of schemes, with zero fiber equal to the fat point $\operatorname{Spec} H^*(X)$ and with general fiber isomorphic to $\operatorname{Spec} H^*(X^T) \cong X^T$. For each $\alpha \in X^T$, let H_α be the image of $\{\alpha\} \times \mathfrak{t}$ in $H_2^T(X)$, a linear subspace that projects isomorphically onto $\mathfrak{t}$. The union of all of these subspaces is equal to the spectrum of the subring of $H_T^*(X)$ that is generated by the degree two part $H_T^2(X)$; equivalently, it is the image of the map from $\operatorname{Spec} H_T^*(X)$ to $H_2^T(X)$ [GM, 3.2]. This leads us naturally to the following definition, which can be found in [GM, §8.1].

Definition 2.3. A **fibered arrangement** is a surjective map of finite-dimensional complex vector spaces $E \twoheadrightarrow F$ along with a finite set $\mathcal{I}$ and a collection $\{H_\alpha \mid \alpha \in \mathcal{I}\}$ of linear subspaces of E that project isomorphically onto F. For example, a localization algebra $(U, Z, \mathcal{I}, h)$ gives rises to a fibered arrangement by taking $E = Z_2^*$, $F = U^*$, and H_α equal to the image of the dual of the degree 2 part of the α component of the localization map h.

Example 2.4. Let $X = \mathbb{P}^2$, and let $T \subset \mathrm{PGL}_2$ be the diagonal subgroup. Then T acts on X with three isolated fixed points. The ring $H_T^*(X)$ is isomorphic to $\mathbb{C}[b_1, b_2, b_3]/\langle b_1 b_2 b_3\rangle$, where b_i is a degree 2 generator represented by the coordinate projective line $L_i \subset X$. The subring $\operatorname{Sym} \mathfrak{t}^*$ is generated by the classes $b_1 - b_2$ and $b_2 - b_3$. The vector space $H_2^T(X)$ is 3-dimensional, with coordinates b_1, b_2, and b_3. The kernel of the map to $\mathfrak{t}$ is generated by the $(1, 1, 1)$ vector, and the three subspaces H_α are the coordinate hyperplanes.

Example 2.5. Let X be a pair of projective lines touching at a single point, and let T be a one-dimensional torus. We consider the action of T on X such that T acts effectively on each component, and the double point is an attracting fixed point for one component and a repelling fixed point for the other. The ring $H_T^*(X)$ is isomorphic to $\mathbb{C}[c_1, c_2, c_3]/\langle c_1c_2, c_1c_3, c_2c_3\rangle$, where c_i is a degree 2 generator whose restriction to the fixed point p_j is δ_{ij} times a fixed generator of $H_T^2(pt)$. The vector space $H_2^T(X)$ is 3-dimensional, with coordinates c_1, c_2, and c_3. The kernel of the map to $\mathfrak{t}$ is defined by the equation $c_1 + c_2 + c_3 = 0$, and the three subspaces H_α are the coordinate lines.

Examples 2.4 and 2.5 motivate the notion of dual fibered arrangements and dual localization algebras.

Definition 2.6. Consider a fibered arrangement with notation as in Definition 2.3. Its **dual** is given by $E^* \twoheadrightarrow E^*/F^*$, along with the linear subspaces $H_\alpha^\perp \subset E^*$, indexed by the same finite set $\mathcal{I}$. A **duality** between two localization algebras $\mathcal{Z}$ and $\mathcal{Z}^\vee$ is an isomorphism between the fibered arrangement associated to $\mathcal{Z}^\vee$ and the dual of the fibered arrangement associated to $\mathcal{Z}$. Thus it consists of an identification of $\mathcal{I}^\vee$ with $\mathcal{I}$ and a perfect pairing between Z_2^* and $(Z_2^\vee)^*$ such that each $H_\alpha \subset Z_2^*$ is the perpendicular space to $H_\alpha^\vee \subset (Z_2^\vee)^*$, and the kernels of the projections to U^* and $(U^\vee)^*$ are also perpendicular to each other.

Example 2.7. The localization algebras in Examples 2.4 and 2.5 are dual via the perfect pairing of vector spaces with respect to which b_1, b_2, b_3 and c_1, c_2, c_3 are dual coordinate systems, and the bijection of fixed point sets that takes $L_i \cap L_j$ to p_k for i, j, k distinct.

3 Koszul, quasi-hereditary, and standard Koszul algebras

In this section we review the well-known definitions of quadratic, Koszul, and quasi-hereditary algebras, along with the slightly less well-known notion of a standard Koszul algebra. Let $\mathcal{I}$ be a finite set of order n, and let $R := \mathbb{C}\{e_\alpha \mid \alpha \in \mathcal{I}\}$ be a ring spanned by pairwise orthogonal idempotents. Let M be a finitely generated R-bimodule, and let $W \subset M$ be a sub-bimodule.

Let

$$A := T_R(M) \big/ \langle W \rangle$$

be the associated quadratic algebra. For all $\alpha \in \mathcal{I}$, let

$$L_\alpha := A \Big/ A_+ \oplus \mathbb{C}\{e_\beta \mid \beta \neq \alpha\}$$

be the simple right A-module indexed by α, and let $P_\alpha := e_\alpha A$ be its projective cover.

Definition 3.1. A complex $\cdots \to M_{i+1} \to M_i \to M_{i-1} \to \cdots$ of graded right A-modules is called **linear** if M_i is generated in degree i. The algebra A is called **Koszul** if each simple module L_α admits a linear projective resolution.

Suppose that we are given a partial order $\leq$ on $\mathcal{I}$, and consider the idempotents

$$\varepsilon_\alpha := \sum_{\gamma \not\leq \alpha} e_\gamma \text{ and } \varepsilon'_\alpha := \varepsilon_a + e_\alpha.$$

The **right-standard module** V_α is defined to be the largest quotient of P_α that is supported at or below α, that is

$$V_\alpha := e_\alpha A \Big/ e_\alpha A \, \varepsilon_\alpha A.$$

Left-projective modules and left-standard modules are defined similarly.

Definition 3.2. Consider the natural surjections

$$P_\alpha \xrightarrow{\Pi_\alpha} V_\alpha \xrightarrow{\pi_\alpha} L_\alpha.$$

The algebra A is called **quasi-hereditary** if the following two conditions hold for all $\alpha \in \mathcal{I}$:

- $\ker \pi_\alpha$ admits a filtration with each subquotient isomorphic to L_β for some $\beta < \alpha$
- $\ker \Pi_\alpha$ admits a filtration with each subquotient isomorphic to V_γ for some $\gamma > \alpha$.

Remark 3.3. This is equivalent to asking that the regular right A-module admits a filtration with standard subquotients, and that the endomorphism algebra of each right-standard module V_α is a division algebra [ÁDL03, §1]. It is also equivalent to requiring that the standard modules form an exceptional sequence with respect to the partial order on $\mathcal{I}$ [Bez03, Proposition 2].

Definition 3.4. The algebra A is called **standard Koszul** if it is finite-dimensional and quasi-hereditary,[8] each right-standard module V_α admits a linear projective resolution, and the analogous condition holds for left-standard modules, as well.

Theorem 3.5. [ÁDL03, Theorem 1] *If A is standard Koszul, then it is Koszul.*

The **quadratic dual** $A^!$ of A is defined as the quotient

$$A^! := T_R M^*/\langle W^\perp \rangle,$$

where $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$, with the natural R-bimodule structure for which $e_\alpha M^* e_\beta = (e_\beta M e_\alpha)^*$. It is a well-known fact that $A^!$ is Koszul if and only if A is. The analogous fact holds for standard Koszulity, as well.

Theorem 3.6. [ÁDL03, Theorem 3] *If A is standard Koszul, then $A^!$ is standard Koszul with respect to the opposite partial order on $\mathcal{I}$.*

Remark 3.7. In fact, it is shown in [ÁDL03, Theorem 3] that any finite-dimensional, quasi-hereditary, Koszul algebra is standard Koszul if and only if its dual is quasi-hereditary. Thus standard Koszul algebras form the largest class of simultaneously Koszul and quasi-hereditary finite-dimensional algebras that is closed under the operation of quadratic duality.

We conclude with two technical lemmas that we will need in Section 6. The first says that if we express a standard Koszul algebra A as a quadratic quotient of the path algebra of a quiver with vertex set $\mathcal{I}$, that quiver has no loops, and it only has arrows between nodes that are comparable in our partial order. The second says that any path of length 2 that starts and ends at α may be uniquely expressed as a sum of paths that avoid all nodes that lie below α.

Lemma 3.8. *If A is standard Koszul and $e_\alpha M e_\beta \neq 0$, then either $\alpha < \beta$ or $\beta < \alpha$. In particular, $e_\alpha M e_\alpha = 0$ for all $\alpha \in \mathcal{I}$.*

Proof. For any A-module N, the **cosocle** of N is defined to be the largest semisimple quotient of N. Consider the right A-module $N = (P_\alpha)_{\geq 1}/(P_\alpha)_{\geq 2}$, which is isomorphic as an R-module to $e_\alpha M$. Since N has a grading that is concentrated in a single degree, it is semisimple, and therefore a quotient of the cosocle of $(P_\alpha)_{\geq 1}$.

The standard filtration of P_α induces a filtration of $(P_\alpha)_{\geq 1}$ with each subquotient isomorphic to either $\ker(\pi_\alpha)$ or V_γ for some $\gamma > \alpha$. This in turn induces a filtration of the cosocle of $(P_\alpha)_{\geq 1}$, with subquotients isomorphic to the cosocle of $\ker(\pi_\alpha)$ or of V_γ for some $\gamma > \alpha$. We know that $\ker(\pi_\alpha)$ only has composition factors of the form L_β for $\beta < \alpha$, and that the cosocle of V_γ is isomorphic to L_γ. Thus the simple modules that appear in the cosocle of $(P_\alpha)_{\geq 1}$ are all of the form L_β for $\beta < \alpha$ or $\beta > \alpha$. Since $N = e_\alpha M$ is a quotient of the cosocle of $(P_\alpha)_{\geq 1}$, the same is true for N. □

[8] [ÁDL03] is internally inconsistent about whether or not a standard Koszul algebra should be required to be either finite-dimensional or quasi-hereditary. For the purposes of this paper, we require both.

Lemma 3.9. *If A is standard Koszul, then the projection $e_\alpha M\varepsilon_\alpha \otimes_R \varepsilon_\alpha M e_\alpha \to e_\alpha A_2 e_\alpha$ is an isomorphism for every $\alpha \in \mathcal{I}$.*

Proof. Since A is standard Koszul, it is **lean** [ÁDL03, 1.4], which means that

$$\varepsilon'_\alpha(\operatorname{rad} A)\varepsilon'_\alpha(\operatorname{rad} A)\varepsilon'_\alpha = \varepsilon'_\alpha(\operatorname{rad} A)^2\varepsilon'_\alpha$$

for all $\alpha \in \mathcal{I}$. Multiplying on both the left and the right by e_α and looking in degree 2, we have a surjection

$$e_\alpha M\varepsilon_\alpha \otimes_R \varepsilon_\alpha M e_\alpha = e_\alpha M\varepsilon'_\alpha \otimes_R \varepsilon'_\alpha M e_\alpha \twoheadrightarrow e_\alpha A_1 \varepsilon'_\alpha A_1 e_\alpha = e_\alpha A_2 e_\alpha,$$

where the equality on the left follows from Lemma 3.8. For all $\beta \in \mathcal{I}$, let $V_\beta^\vee$ be the right costandard module associated to β (that is, the right module dual to the left standard module). Then

$$\begin{aligned}
\dim e_\alpha A_2 e_\alpha &= \sum_{\beta>\alpha} [L_\alpha : V_\beta]_1 [V_\beta : P_\alpha]_1 && \text{(by quasi-hereditarity)}\\
&= \sum_{\beta>\alpha} [L_\alpha : V_\beta]_1 [L_\alpha : V_\beta^\vee]_1 && \text{(by reciprocity [CPS94, 1.2.4])}\\
&= \sum_{\beta>\alpha} (\dim e_\alpha M e_\beta)(\dim e_\beta M e_\alpha)\\
&= \dim e_\alpha M\varepsilon_\alpha \otimes_R \varepsilon_\alpha M e_\alpha,
\end{aligned}$$

where $[L_\alpha : -]_1$ and $[V_\beta : -]_1$ denote graded filtration multiplicities. Thus the projection must be an isomorphism. □

4 Flat deformations of Koszul algebras

In this section we study graded deformations of Koszul algebras. To begin we consider a quadratic algebra $A = T_R(M)/\langle W\rangle$ which is not necessarily Koszul. Let U be a finite-dimensional $\mathbb{C}$-vector space and let $S = \operatorname{Sym} U$, graded so that elements of U have degree two. Now suppose given a graded deformation of A over U^*, that is, a graded R-algebra $\tilde{A}$ together with graded homomorphisms

$$S \xrightarrow{j} \tilde{A} \xrightarrow{\pi} A$$

so that j maps into the center of $\tilde{A}$ and π induces an isomorphism $A \cong \tilde{A}/\langle j(U)\rangle$ of graded algebras.

Since S is generated in degree two, the map π is an isomorphism in degrees zero and one, so we have $\tilde{A}_0 \cong R$ and $\tilde{A}_1 \cong M$. In degree two we have the right exact sequence

$$R \otimes_{\mathbb{C}} U \to \tilde{A}_2 \to A_2 \to 0.$$

We will make the additional assumption that $\tilde{A}$ is flat over S; in particular, this implies that the sequence above is in fact short exact.

To simplify notation, we will write $\otimes$ in place of $\otimes_{\mathbb{C}}$ throughout this section. Since W is contained in the kernel of the projection $T_R(M) \to A$, we get a map of R-bimodules $\Phi\colon W \to R \otimes U$ by letting $\Phi(w)$ be the image of w under the multiplication map $T^2_R(M) = T^2_R(\tilde{A}_1) \to \tilde{A}$.

Note that if N is any R-bimodule and V is a $\mathbb{C}$-vector space, then any map $\Psi\colon N \to R \otimes V$ of R-bimodules must annihilate any "off-diagonal" summand $e_\alpha N e_\beta \subset N$ with $\alpha \neq \beta$. In particular, this means that such bimodule maps are in bijection with linear maps $\Psi^\circ\colon N \to V$ that kill the off-diagonal summands, via the formula

$$\Psi(e_\alpha x e_\beta) = e_\alpha \otimes \Psi^\circ(e_\alpha x e_\beta).$$

For any map $\Psi\colon W \to R \otimes U$ of R-bimodules, we define

$$\tilde{A}_\Psi := T_R(M) \otimes S / \langle w \otimes 1 - 1 \otimes \Psi(w) \mid w \in W \rangle. \tag{2}$$

We then get a surjective map $\tilde{A}_\Phi \to \tilde{A}$ of graded algebras which becomes an isomorphism upon tensoring over S with $\mathbb{C}$. Since we are assuming that $\tilde{A}$ is flat over S, this map must be an isomorphism even before tensoring. Thus every flat graded deformation of A with deformation parameters in degree two arises from a bimodule map $\Psi\colon W \to R \otimes U$ as in Equation (2). It is not the case, however, that every bimodule map gives a flat graded deformation. More precisely, we have the following criterion for flatness.

Theorem 4.1. *Let $\Psi : W \to R \otimes U$ be a bimodule map, and let $\tilde{A}_\Psi$ be the graded deformation of A given in Equation* (2)*. This deformation is flat if and only if Ψ° factors through the quotient map $W \cong (A^!_2)^* \to Z(A^!)^*_2$.*

Remark 4.2. Since $Z(A^!)_2$ has no off-diagonal summands, Theorem 4.1 implies that graded flat deformations of A over U^* with deformation parameters in degree two are in bijection with linear maps $\psi\colon Z(A^!)^*_2 \to U$. If we take U to be $Z(A^!)^*_2$ and ψ to be the identity map, we call the resulting ring $\tilde{A}$ the **universal deformation** of A. It is universal in the sense that if $\psi'\colon Z(A^!)^*_2 \to U'$ is another linear map, then the corresponding deformation is isomorphic to $\tilde{A} \otimes_S \operatorname{Sym} U'$, where the map $S \to \operatorname{Sym} U'$ is induced by ψ'.

In Section 10, we will also need the analogous statement for deformations over power series rings. If $\widehat{A}$ is a flat deformation of A over the spectrum of the power series ring $\prod_{i=0}^\infty \operatorname{Sym}^i U'$ and $\widehat{A}$ admits a **formal grading** $\widehat{A} = \prod_{i=0}^\infty \widehat{A}_i$ compatible with its algebra structure, then $\widehat{A}$ may be obtained from the universal deformation of A via a base change to U^* followed by a completion at the unique graded maximal ideal of $\operatorname{Sym} U$.

Proof of Theorem 4.1. For any linear map $\chi\colon U \to \mathbb{C}$ let $\mathbb{C}_\chi$ denote the associated one-dimensional $\operatorname{Sym} U$-algebra, and consider the specialization

$$A_\chi := \tilde{A}_\Psi \otimes_{\operatorname{Sym} U} \mathbb{C}_\chi$$

of $\tilde{A}_\Psi$ at the point $\chi \in U^*$. Explicitly, the ring A_χ is the quotient of the tensor algebra $T_R M$ by the two-sided ideal $\langle w - \chi_R \circ \Psi(w) \mid w \in W \rangle$, where $\chi_R\colon R \otimes U \to R$ is given by $\chi_R(e_\alpha \otimes u) = e_\alpha \chi(u)$

for all $\alpha \in \mathcal{I}$ and $u \in U$. The grading on $\tilde{A}_\Psi$ induces a filtration on A_χ, and the ring $\tilde{A}_\Psi$ is flat if and only if the natural surjection $A \to \operatorname{gr} A_\chi$ is an isomorphism for all χ.

Our theorem now follows directly from a result of Braverman and Gaitsgory [BG96]. They study a more general situation, taking the quotient Q of the tensor algebra $T_R(M)$ by the two-sided ideal $\langle w - a(w) - b(w) \mid w \in W\rangle$, where $a\colon W \to M$ and $b\colon W \to R$ are maps of R-bimodules. Their main result[9] [BG96, 4.1] gives necessary and sufficient conditions on a and b to have $\operatorname{gr} Q \cong A$. In our case, we have $a = 0$ and $b = \chi_R \circ \Psi$. In this situation their conditions reduce to the statement that the map

$$b \otimes \mathrm{id} - \mathrm{id} \otimes b\colon (W \otimes_R M) \cap (M \otimes_R W) \to M$$

vanishes.

To relate this condition to the dual ring, note that

$$A^!_3 = M^* \otimes_R M^* \otimes_R M^* / (W^\perp \otimes_R M + M \otimes_R W^\perp)$$

is naturally dual to $(W \otimes_R M) \cap (M \otimes_R W)$. Thus we may identify $Z(A^!)_2$ with the set of $\mathbb{C}$-linear maps $b^\circ\colon W \to \mathbb{C}$ that kill the off-diagonal terms (this implies that b° commutes with the idempotents) and for which

$$b^\circ \otimes \gamma - \gamma \otimes b^\circ\colon (W \otimes_R M) \cap (M \otimes_R W) \to \mathbb{C}$$

vanishes for any $\gamma \in M^*$. Here we consider $W \otimes_R M$ as a subspace of $W \otimes M$ in the obvious way, so $(b^\circ \otimes \gamma)(w \otimes m) = \sum_\alpha b^\circ(e_\alpha w e_\alpha)\gamma(e_\alpha m)$, and similarly for $\gamma \otimes b^\circ$. Then if $b\colon W \to R \otimes \mathbb{C} = R$ is the bimodule map corresponding to a $\mathbb{C}$-linear map b°, we have $b^\circ \otimes \gamma - \gamma \otimes b^\circ = \gamma \circ (b \otimes \mathrm{id} - \mathrm{id} \otimes b)$, so b° represents an element of $Z(A^!)_2$ if and only if $b \otimes \mathrm{id} - \mathrm{id} \otimes b = 0$.

In other words, we have shown that $\tilde{A}_\Psi$ is flat if and only if $(\chi_R \circ \Psi)^\circ = \chi \circ \Psi^\circ$ is central for all $\chi \in U^*$, which is equivalent to saying that Ψ factors through $Z(A^!)_2^*$. $\square$

Remark 4.3. If we drop the Koszulity hypothesis, then the "if" direction of Theorem 4.1 becomes false; it fails, for example, when $|\mathcal{I}| = 1$ and $A = \mathbb{C}\langle x, y\rangle / \langle x^2, y^2 - xy\rangle$.[10]

Remark 4.4. Theorem 4.1 can be understood more abstractly using the fact that Koszul duality induces an equivalence of derived categories of graded modules

$$D^b(A - \mathsf{gmod}) \cong D^b(A^! - \mathsf{gmod}).$$

Since Hochschild cohomology is equal to the Ext-algebra of the identity functor on the derived category [Toë07, 1.6], this also induces an isomorphism of Hochschild cohomology groups. The behavior of this equivalence on grading shift functors is such that the group $HH^r(A^!)_s$ is identified with $HH^{r+s}(A)_{-s}$ [BGS96, 1.2.6]. In particular, if A is Koszul, $Z(A^!)_2 = HH^0(A^!)_2$ is naturally

[9] Braverman and Gaitsgory only treat the case when R is a field, but their results easily generalize to our semisimple ring R. Their condition (I) forces a and b to be bimodule maps.

[10] We thank Andrew Connor for this example.

isomorphic to $HH^2(A)_{-2}$. In fact, the proof of the main result of [BG96] proceeds by showing that an element of $Z(A^!)_2$ lifts to an R-bimodule map $M \otimes_{\mathbb{C}} M \to R$ satisfying a cocycle condition which allows it to represent a class in $HH^2(A)_{-2}$.

5 Deformed standard modules and malleable algebras

Throughout this section we assume that A is a standard Koszul algebra, $S = \operatorname{Sym} U$ is a polynomial ring, and $\tilde{A}$ is a flat graded deformation of A over U^*. Consider the right $\tilde{A}$-modules

$$\tilde{P}_\alpha := e_\alpha \tilde{A} \quad \text{and} \quad \tilde{V}_\alpha := \tilde{P}_\alpha \Big/ e_\alpha \tilde{A}\, \varepsilon_\alpha \tilde{A}.$$

Since $\tilde{P}_\alpha$ is a summand of $\tilde{A}$, it is a flat deformation of P_α. The purpose of this section is to show that the center of $\tilde{A}$ acts on each $\tilde{V}_\alpha$ via a central character

$$h_\alpha : Z(\tilde{A}) \longrightarrow S.$$

It will follow that the data $\mathcal{Z}(\tilde{A}) := (U,\, Z(\tilde{A}),\, \mathcal{I},\, \oplus h_\alpha)$ form a localization algebra.

For any $\alpha \in \mathcal{I}$, consider the algebra $C_\alpha := \varepsilon'_\alpha A\, \varepsilon'_\alpha \subset A$ and its deformation $\tilde{C}_\alpha := \varepsilon'_\alpha \tilde{A}\, \varepsilon'_\alpha \subset \tilde{A}$. By [ÁDL03, 3.9], C_α is standard Koszul. The deformed algebra $\tilde{C}_\alpha$ is a direct summand of $\tilde{A}$ as an S-module, so it is flat over U^*. For any $\alpha \leq \beta$, let

$$V^\alpha_\beta := e_\beta C_\alpha \Big/ e_\beta C_\alpha \varepsilon_\beta C_\alpha$$

be the standard cover of L_β in the category of right C_α-modules, and consider its deformation

$$\tilde{V}^\alpha_\beta := e_\beta \tilde{C}_\alpha \Big/ e_\beta \tilde{C}_\alpha \varepsilon_\beta \tilde{C}_\alpha.$$

Lemma 5.1. *We have an isomorphism $V^\alpha_\beta \otimes_{C_\alpha} \varepsilon'_\alpha A \cong V_\beta$ of right A-modules, and an isomorphism $\tilde{V}^\alpha_\beta \otimes_{\tilde{C}_\alpha} \varepsilon'_\alpha \tilde{A} \cong \tilde{V}_\beta$ of right $\tilde{A}$-modules.*

Proof. Using the equalities $e_\beta \varepsilon'_\alpha = e_\beta$ and $\varepsilon_\beta \varepsilon'_\alpha = \varepsilon_\beta = \varepsilon'_\alpha \varepsilon_\beta$, we have

$$\begin{aligned} V^\alpha_\beta \otimes_{C_\alpha} \varepsilon'_\alpha A &= \left(e_\beta C_\alpha \Big/ e_\beta C_\alpha \varepsilon_\beta C_\alpha \right) \otimes_{C_\alpha} \varepsilon'_\alpha A \\ &= \left(e_\beta A\, \varepsilon'_\alpha \Big/ e_\beta A \varepsilon_\beta A\, \varepsilon'_\alpha \right) \otimes_{C_\alpha} \varepsilon'_\alpha A \\ &\cong e_\beta A \Big/ e_\beta A\, \varepsilon_\beta A = V_\beta. \end{aligned}$$

The proof of the second statement is identical. □

Remark 5.2. The most important case of Lemma 5.1, and also the easiest one to think about, is the case in which $\alpha = \beta$, so that $V^\alpha_\beta = V^\alpha_\alpha$ is isomorphic to the simple module for C_α supported at the node α.

Proposition 5.3. *For all $\alpha \in \mathcal{I}$, $\tilde{V}_\alpha$ is a flat deformation of V_α.*

Proof. We first consider the case where α is a minimal element of our poset. In this case $V_\alpha \cong L_\alpha$ is one-dimensional. Then by semicontinuity, the $\tilde{A}_\chi$-module $(\tilde{V}_\alpha)_\chi = e_\alpha \tilde{A}_\chi \big/ e_\alpha \tilde{A}_\chi \, \varepsilon_\alpha \tilde{A}_\chi$ has dimension 0 or 1 for every $\chi \in U^*$. We must show that that dimension is equal to 1 for every χ, or equivalently that $e_\alpha \notin e_\alpha \tilde{A}_\chi \, \varepsilon_\alpha \tilde{A}_\chi e_\alpha$. Since α is minimal, Lemma 3.9 tells us that there are no nontrivial relations among loops of length 2 based at α. In particular there are no relations to deform, and the conclusion follows.

In the general case, α is a minimal element of the poset of simples for the subalgebra C_α, so $\tilde{V}^\alpha_\alpha$ is a flat deformation of V^α_α. This then implies the result for $\tilde{V}_\alpha \cong \tilde{V}^\alpha_\alpha \otimes_{\tilde{C}_\alpha} \varepsilon'_\alpha \tilde{A}$. □

Lemma 5.4. *The regular right $\tilde{A}$-module $\tilde{A}$ admits a filtration for which:*

(i) each subquotient is isomorphic to a direct sum of deformed standard modules, and

(ii) no deformed standard module $\tilde{V}_\alpha$ appears in two consecutive subquotients.

Proof. Choose a maximal index $\alpha \in \mathcal{I}$. Let $B = A/Ae_\alpha A$, and let $W_B \hookrightarrow B_1 \otimes_R B_1$ be the space of relations for B. The algebra B is standard Koszul by [ÁDL03, 3.9]. Let $\tilde{B} = \tilde{A}/\tilde{A}e_\alpha \tilde{A}$. We first use the results of Section 4 to show that $\tilde{B}$ is a flat deformation of B over U^*.

Let $\bar{e} = 1 - e_\alpha$. The surjection $A \twoheadrightarrow B$ induces an inclusion $B^! \hookrightarrow A^!$ with image contained in $\bar{e}A^!\bar{e} \subset A^!$. By [ÁDL03, 2.5], the image is equal to $\bar{e}A^!\bar{e}$. The map $q\colon A^!_2 \to B^!_2$ given by $q(x) = \bar{e}x\bar{e}$ is a left inverse for the inclusion $B^!_2 \hookrightarrow A^!_2$. It follows that if the deformation $\tilde{A}$ is described by an A_0-bimodule map $\Psi\colon W_A \to A_0 \otimes_{\mathbb{C}} U$ as in Section 4, then $\tilde{B}(\mathcal{V})$ is described by the composition

$$W_B \xrightarrow{q^*} W_A \xrightarrow{\Psi} A_0 \otimes_{\mathbb{C}} U \xrightarrow{\cdot \bar{e}} B_0 \otimes_{\mathbb{C}} U.$$

It is easy to check that q sends $Z(A^!)_2$ into $Z(B^!)_2$, so Theorem 4.1 implies that $\tilde{B}(\mathcal{V})$ is flat over U^*.

A deformed standard module over $\tilde{B}$ becomes a deformed standard $\tilde{A}$-module under the quotient homomorphism $\tilde{A} \to \tilde{B}$. Thus, inducting on the size of our poset, we may assume that the right $\tilde{A}$-module $\tilde{B}$ has a filtration by deformed standard $\tilde{A}$-modules $\tilde{V}_\beta$ with $\beta \neq \alpha$, so that no deformed standard module appears in consecutive subquotients.

Consider the exact sequence

$$0 \to \tilde{A}e_\alpha \tilde{A} \to \tilde{A} \to \tilde{B} \to 0. \tag{3}$$

We have

$$\tilde{A}e_\alpha \tilde{A} = \tilde{A}e_\alpha \otimes_{e_\alpha \tilde{A} e_\alpha} e_\alpha \tilde{A} = \tilde{A}e_\alpha \otimes_{e_\alpha \tilde{A} e_\alpha} \tilde{P}_\alpha = \tilde{A}e_\alpha \otimes_{e_\alpha \tilde{A} e_\alpha} \tilde{V}_\alpha,$$

where the last equality follows from the maximality of α. Maximality also implies that $e_\alpha \tilde{A} e_\alpha \cong S$ (by Lemma 3.9), thus the $\tilde{A}$-module $\tilde{A}e_\alpha \tilde{A}$ is isomorphic to a direct sum of $\dim Ae_\alpha$ copies of $\tilde{V}_\alpha$. The result then follows from the exact sequence (3). □

Corollary 5.5. *Suppose that $\tilde{V}$ is a graded $\tilde{A}$-module which is a flat deformation of V_α; that is, it is free as an S-module and $\tilde{V} \otimes_S \mathbb{C}_0$ is isomorphic to V_α as an A-module. Then $\tilde{V} \cong \tilde{V}_\alpha$.*

Proof. Consider the surjection $\tilde{V} \twoheadrightarrow \tilde{V} \otimes_S \mathbb{C}_0 \cong V_\alpha$. Since $\tilde{P}_\alpha$ is projective, we can lift the map $\tilde{P}_\alpha \to \tilde{V}_\alpha \to V_\alpha$ to a map $\phi\colon \tilde{P}_\alpha \to \tilde{V}$. The fiber of ϕ over 0 is a surjection, so Nakayama's lemma tells us that ϕ itself must be surjective. For any $\beta \in \mathcal{I}$, the natural map

$$\operatorname{Hom}_{\tilde{A}}(\tilde{P}_\beta, \tilde{V}) \otimes_S \mathbb{C}_0 \to \operatorname{Hom}_A(P_\beta, \tilde{V} \otimes_S \mathbb{C}_0)$$

is injective, thus $\operatorname{Hom}_{\tilde{A}}(\tilde{P}_\beta, \tilde{V}) = 0$ for any $\beta \not\leq \alpha$. It follows that $\ker \phi$ contains $e_\alpha \tilde{A}\, \varepsilon_\alpha \tilde{A}$, hence $\tilde{V}$ is a quotient of $\tilde{V}_\alpha$. Since they are both free S-modules and their fibers over 0 are isomorphic, we have $\tilde{V} \cong \tilde{V}_\alpha$. $\square$

The next result says that our deformed standard modules have well-defined central characters.

Proposition 5.6. *For each $\alpha \in \mathcal{I}$, there is an S-algebra homomorphism $h_\alpha : Z(\tilde{A}) \to S$ such that $v \cdot \zeta = h_\alpha(\zeta) v$ for all $v \in \tilde{V}_\alpha$ and $\zeta \in Z(\tilde{A})$.*

Proof. If α is minimal, then $V_\alpha \cong L_\alpha$ and $\tilde{V}_\alpha \cong S$ by Proposition 5.3, and the claim follows from the fact that $\tilde{A}$ acts on $\tilde{V}_\alpha$ by S-module endomorphisms. The general case follows from the isomorphism $\tilde{V}_\alpha \cong \tilde{V}^\alpha_\alpha \otimes_{\tilde{C}_\alpha} \varepsilon'_\alpha \tilde{A}$ of Lemma 5.1. $\square$

As a consequence, we can construct a localization algebra from a flat graded deformation $\tilde{A}$ of a standard Koszul algebra A, thus completing the proof of Theorem 1.1.

Corollary 5.7. *The data $\mathcal{Z}(\tilde{A}) := (U, Z(\tilde{A}), \mathcal{I}, \oplus h_\alpha)$ form a localization algebra.*

Let K be the fraction field of S, and for any S-module N, let $N^\infty = N \otimes_S K$. The filtration of $\tilde{A}$ from Lemma 5.4 induces a filtration of $\tilde{A}^\infty$ with subquotients isomorphic to direct sums of modules of the form $\tilde{V}^\infty_\alpha$.

Theorem 5.8. *Suppose that $A \cong A^{op}$ as R-algebras. The following are equivalent:*

1. *The deformed standard filtration of $\tilde{A}^\infty$ splits.*
2. *The action map $\tilde{A} \to \bigoplus_\alpha \operatorname{End}_K(\tilde{V}^\infty_\alpha)$ is an isomorphism.*
3. *The map $\bigoplus_\alpha h^\infty_\alpha : Z(\tilde{A})^\infty \to \bigoplus_\alpha K$ is an isomorphism.*
4. *The maps $\{h_\alpha \mid \alpha \in \mathcal{I}\}$ from $Z(\tilde{A})$ to S are all distinct.*

Proof. (1) $\Rightarrow$ (2): Since the deformed standard filtration of $\tilde{A}^\infty$ splits, an element of $\tilde{A}^\infty$ which kills every $\tilde{V}^\infty_\alpha$ must also act trivially on $\tilde{A}^\infty$, and must therefore be zero. Thus the action map is injective. Comparing dimensions, we have

$$\dim_K \tilde{A}^\infty = \dim_{\mathbb{C}} A = \sum_{\alpha,\beta} [L_\alpha : P_\beta] = \sum_{\alpha,\beta,\gamma} [L_\alpha : V_\gamma][V_\gamma : P_\beta] = \sum_{\alpha,\beta,\gamma} [L_\alpha : V_\gamma][L_\beta : V_\gamma] = \sum_\gamma (\dim V_\gamma)^2,$$

where the penultimate equality follows from BGG reciprocity [CPS88, 3.11] and the isomorphism $A \cong A^{\mathrm{op}}$. Thus our map must be an isomorphism.

(2) $\Rightarrow$ (3): This follows from the fact that $Z(\tilde{A})^\infty \cong Z(\tilde{A}^\infty)$.

(3) $\Rightarrow$ (4): Immediate from injectivity.

(4) $\Rightarrow$ (1): If all of the deformed standard modules have different central characters, then the filtration of $\tilde{A}$ can be split by taking the isotypic decomposition for the action of the center. Here we use the fact that there is no overlap between central characters of consecutive subquotients, which follows from Lemma 5.4. □

Definition 5.9. If $\tilde{A}$ satisfies the conditions of Theorem 5.8, we will call $\tilde{A}$ **malleable**.

Proposition 5.10. *Suppose that $A \cong A^{op}$. Then $\mathcal{Z}(\tilde{A})$ is a strong localization algebra iff $\tilde{A}$ is malleable.*

Proof. By Definition 2.1, $\mathcal{Z}(\tilde{A})$ is strong if and only if the kernel and cokernel of $\oplus_\alpha h_\alpha$ are torsion. This is equivalent to asking that $\oplus_\alpha h_\alpha^\infty$ be an isomorphism, which is condition (3) above. □

Recall from Definition 2.1 that a localization algebra is called free if it is a flat deformation.

Proposition 5.11. *If $\tilde{A}$ is malleable and $\dim_{\mathbb{C}} Z(A) = |\mathcal{I}|$, then $\mathcal{Z}(\tilde{A})$ is a free localization algebra.*

Proof. By Theorem 5.8, $\dim_K Z(\tilde{A})^\infty = |\mathcal{I}|$. On the other hand, flatness of $\tilde{A}$ implies that the natural map $Z(\tilde{A}) \otimes_S \mathbb{C} \to Z(A)$ is injective, thus $\dim_{\mathbb{C}} Z(\tilde{A}) \otimes_S \mathbb{C} \leq \dim_{\mathbb{C}} Z(A) = |\mathcal{I}|$. By semicontinuity, this inequality must be an equality, hence $Z(\tilde{A})$ is flat over S. □

6 Flexible algebras

In this section we define and study flexible algebras in preparation for the next section, which contains proofs of our central results, Theorem 7.1 and Corollary 7.5. Let A be a standard Koszul algebra, let $S = \operatorname{Sym} U$ be a polynomial ring generated in degree 2, and let $\tilde{A}$ be a flat graded deformation of A over U^*.

Definition 6.1. We say that $\tilde{A}$ is **flexible** if the natural projection $Z(\tilde{A})_2 \to Z(A)_2$ is surjective.

Example 6.2. Let $\mathcal{I} = \{1, 2\}$ be the nodes of a quiver with $r > 0$ arrows $x_1, \ldots, x_r$ from 1 to 2 and $s > 0$ arrows $y_1, \ldots, y_s$ from 2 to 1. Let A_{rs} be the path algebra modulo the quadratic relations $y_j x_i = 0$ for all i and j. Thus a right A_{rs}-module is a representation of the quiver for which every loop based at the node 2 acts trivially. We have standard modules $V_1 = L_1$ and $V_2 = P_2$ with $\ker \Pi_1 \cong P_2^{\oplus r}$ and $\ker \pi_2 \cong L_1^{\oplus s}$, so A is quasi-hereditary with respect to the order $1 < 2$. It is clear that both standard modules have linear projective resolutions. Furthermore, since the opposite algebra of A_{rs} is isomorphic to A_{sr}, the same is true for left-standard modules. Hence A_{rs} is standard Koszul. It is also easy to see that A_{rs} is isomorphic to its own quadratic dual.

The center $Z(A_{rs})$ is spanned by the unit and the $r \times s$ elements $x_i y_j$. The universal deformation $\tilde{A}_{rs}$ has central generators u_{ij} in degree 2 and relations $y_j x_i = u_{ij} e_2$. It is easy to check that the generators $x_i y_j$ of $Z(A_{rs})_2$ lift to central elements of $\tilde{A}_{rs}$ if and only if $r = s = 1$, thus only $\tilde{A}_{11}$ is flexible. We note that $\tilde{A}_{11}$ is also malleable in the sense of Definition 5.9.

If $\tilde{A}$ is flexible, each homomorphism $h_\alpha : Z(\tilde{A})_2 \to S_2 = U$ of Proposition 5.6 splits the exact sequence

$$0 \to U \to Z(\tilde{A})_2 \to Z(A)_2 \to 0. \tag{4}$$

The difference between any two splittings vanishes on U, and thus induces a map $j_{\alpha\beta} : Z(A)_2 \to U$ given by

$$j_{\alpha\beta}(z) := h_\beta(\zeta) - h_\alpha(\zeta) \quad \text{for any lift } \zeta \text{ of } z.$$

Define maps

$$\mu : Z(A)_2 \to \bigoplus_{\alpha\in\mathcal{I}} e_\alpha M \varepsilon_\alpha \otimes_R \varepsilon_\alpha M e_\alpha \qquad \text{and} \qquad \nu : Z(A)_2 \to \tilde{A}_2 \tag{5}$$

by setting $\mu(z)$ equal to the unique expression for z as a sum of loops that go first up and then down (which exists by Lemmas 3.8 and 3.9), and $\nu(z)$ to the image of $\mu(z)$ in $\tilde{A}_2$.

Proposition 6.3. *Suppose that $\tilde{A}$ is flexible. For all $z \in Z(A)_2$ and $\tilde{a} \in e_\alpha \tilde{A} e_\beta$, we have*

$$[\nu(z), \tilde{a}] = j_{\alpha\beta}(z)\, \tilde{a}.$$

Proof. Let z be given, and let ζ be a lift of z to $Z(\tilde{A})_2$. Since the kernel of the projection from $\tilde{A}_2$ to A_2 is equal to $R \otimes_{\mathbb{C}} U$, there exist elements $u_\alpha \in U$ such that $\zeta = \nu(z) + \sum_\alpha u_\alpha e_\alpha$. Since the deformed standard module $\tilde{V}_\alpha$ is supported on and below α, and $\nu(z)$ is expressed in terms of paths that avoid such nodes, we have $h_\alpha(\zeta) = u_\alpha$ for all $\alpha \in \mathcal{I}$, and therefore $j_{\alpha\beta}(z) = u_\beta - u_\alpha$ for all $\alpha, \beta \in \mathcal{I}$. Since ζ is central, we have

$$[\nu(z),\, \tilde{a}] = [\nu(z) - \zeta,\, \tilde{a}] = (u_\beta - u_\alpha)\, \tilde{a} = j_{\alpha\beta}(z)\, \tilde{a}$$

for all $\tilde{a} \in e_\alpha \tilde{A} e_\beta$. □

Proposition 6.4 may be regarded as a converse to Proposition 6.3.

Proposition 6.4. *Suppose that there exists a collection of linear maps*

$$\left\{ j'_{\alpha\beta} : Z(A)_2 \to U \;\middle|\; \alpha, \beta \in \mathcal{I} \right\}$$

satisfying the following two conditions:

- *$j'_{\alpha\beta} + j'_{\beta\gamma} = j'_{\alpha\gamma}$ for all $\alpha, \beta, \gamma \in \mathcal{I}$ (in particular $j'_{\alpha\beta} = -j'_{\beta\alpha}$ for all $\alpha, \beta \in \mathcal{I}$)*
- *for all $z \in Z(A)_2$ and $\tilde{a} \in e_\alpha \tilde{A} e_\beta$, we have $[\nu(z), \tilde{a}] = j'_{\alpha\beta}(z)\, \tilde{a}$.*

Then $\tilde{A}$ is flexible, and $j'_{\alpha\beta} = j_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{I}$.

Proof. Let $z \in Z(A)_2$ be given; we must show that we can lift it to $Z(\tilde{A})_2$. Choose an element $\delta \in \mathcal{I}$ arbitrarily, and let

$$\zeta = \nu(z) + \sum_{\gamma \in \mathcal{I}} j'_{\delta\gamma}(z) e_\gamma \in \tilde{A}_2.$$

Then for any $\tilde{a} \in e_\alpha \tilde{A} e_\beta$, we have

$$[\zeta, \tilde{a}] = [\nu(z),\, \tilde{a}] + j'_{\delta\alpha}(z)\,\tilde{a} - j'_{\delta\beta}(z)\,\tilde{a} = \Big(j'_{\alpha\beta}(z) + j'_{\delta\alpha}(z) - j'_{\delta\beta}(z)\Big)\tilde{a} = 0,$$

where the vanishing of the expression inside the parentheses follows from the first condition on the homomorphisms $j'_{\alpha\beta}$. Thus ζ is central.

By the same argument that we used in the proof of Proposition 6.3, we have $h_\gamma(\zeta) = j'_{\delta\gamma}(z)$ for all γ, thus $j_{\alpha\beta}(z) = h_\beta(\zeta) - h_\alpha(\zeta) = j'_{\delta\beta}(z) - j'_{\delta\alpha}(z) = j'_{\alpha\beta}(z)$. □

We conclude with a lemma that we will use in the last section to show that a certain flexible deformation of a standard Koszul algebra is universal, or at least has the universal deformation as a quotient. Let $\psi\colon Z(A^!)_2^* \to U$ be given, and let $\tilde{A}$ be the graded flat deformation provided by Theorem 4.1. Suppose that this deformation is flexible, so that we can define the maps $j_{\alpha\beta}\colon Z(A)_2 \to U$.

Lemma 6.5. *For any $\alpha, \beta \in \mathcal{I}$ such that $e_\alpha T_R(M) e_\beta \neq 0$, $\operatorname{Im} j_{\alpha\beta}$ is contained in $\operatorname{Im}\psi$.*

Proof. First suppose that $e_\alpha M e_\beta \neq 0$, and let $\tilde{a}$ be any nonzero element of this space. Then for any $z \in Z(A)_2$, Proposition 6.3 implies that $j_{\alpha\beta}(z)\tilde{a}$ is a nonzero element of the subring of $\tilde{A}$ generated by the degree 0 and 1 parts, thus $j_{\alpha\beta}(z) \in \operatorname{Im}(\psi)$. The general case follows from the identity $j_{\alpha\beta} + j_{\beta\gamma} = j_{\alpha\gamma}$. □

7 Koszul duality and GM duality

In this section we explore the relation between Koszul duality and the localization algebras of flexible deformations. Let A be a standard Koszul algebra, and let $\tilde{A}$ be its universal deformation over $U^* = Z(A^!)_2$. On the dual side, let $\tilde{A}^!$ be the universal deformation of the dual ring $A^!$ over $(U^!)^* = Z(A)_2$. Let $S = \operatorname{Sym} U$ and $S^! = \operatorname{Sym} U^!$.

We will call a standard Koszul algebra flexible if its universal deformation is flexible. If A is flexible, then we have the maps $j_{\alpha\beta}\colon Z(A)_2 \to U = Z(A^!)_2^*$ constructed in the previous section. If $A^!$ is flexible, then the same construction gives maps $j^!_{\alpha\beta}\colon Z(A^!)_2 \to U^! = Z(A)_2^*$.

Theorem 7.1. *If A is flexible, then so is $A^!$. Furthermore, for all $\alpha, \beta \in \mathcal{I}$, we have an identity $j^!_{\alpha\beta} = j^*_{\beta\alpha}$ of maps from $Z(A^!)_2 = U^*$ to $U^! = Z(A)_2^*$.*

Proof. By Proposition 6.4, it is enough to show that

$$[\nu^!(z^!), \tilde{a}^!] = j^*_{\beta\alpha}(z^!)\,\tilde{a}^!$$

for all $z^! \in Z(A^!)_2$ and $\tilde{a}^! \in e_\alpha \tilde{A} e_\beta$. Retracing the computation in the proof of Proposition 6.4, this is equivalent to showing that, for a fixed $\delta \in \mathcal{I}$,

$$\zeta^! := \nu^!(z^!) + \sum_{\gamma \in \mathcal{I}} j^*_{\gamma\delta}(z^!) e_\gamma \in \tilde{A}^!_2$$

is central. It clearly commutes with the idempotents, so it is enough to show that it commutes with elements of M^*.

We need to work with explicit representatives in the deformed tensor algebra

$$\widetilde{T_R(M^*)} := T_R(M^*) \otimes_{\mathbb{C}} \mathrm{Sym}(U^!).$$

Since elements of $U^!$ have degree 2, we have

$$\widetilde{T_R(M^*)}_2 = T_R(M^*)_2 \oplus (R \otimes U^!) \text{ and } \widetilde{T_R(M^*)}_3 = T_R(M^*)_3 \oplus (M \otimes U^!).$$

Define a lift $\eta^! \in \widetilde{T_R(M^*)}$ of $\zeta^!$ by

$$\eta^! := \mu^!(z^!) + \sum_{\gamma \in \mathcal{I}} j^*_{\gamma\delta}(z^!) e_\gamma \ \in \ T_R(M^*)_2 \oplus (R \otimes U^!),$$

where $\mu^!$ is defined as in Equation (5). Fix a pair of indices $\alpha, \beta \in \mathcal{I}$, and let $x^!$ be any element of $e_\beta M^* e_\alpha$. We need to show that the commutator $[\zeta^!, x^!] \in T_R(M^*)_3 \oplus (M^* \otimes U^!)$ reduces to zero in $\tilde{A}^!_3$.

Let $Q_1 = M \otimes W$ and $Q_2 = W \otimes M \subset T_R(M)_3$. By the definition of the quadratic dual, the kernel of the map from $T_R(M^*)_3$ to $A^!_3$ is equal to

$$M^* \otimes W^\perp + W^\perp \otimes M^* = Q_2^\perp + Q_1^\perp = (Q_1 \cap Q_2)^\perp.$$

We now need a similar expression for the relations in $\tilde{A}^!_3$. Let

$$O_1 = \mathbb{C}\{\, x\mu(z) + x \otimes z \mid x \in M,\ z \in Z(A)_2\} \subset T_R(M)_3 \oplus (M \otimes Z(A)_2)$$

and

$$O_2 = \mathbb{C}\{\, \mu(z)x + x \otimes z \mid x \in M,\ z \in Z(A)_2\} \subset T_R(M)_3 \oplus (M \otimes Z(A)_2).$$

Then the kernel of the quotient map $\widetilde{T_R(M^*)}_3 = T_R(M^*)_3 \oplus (M^* \otimes U^!) \to \tilde{A}^!_3$ is equal to

$$(Q_2 + O_2)^\perp + (Q_1 + O_1)^\perp = \Big((Q_2 + O_2) \cap (Q_1 + O_1)\Big)^\perp.$$

Consider any pair of elements $x \in M$ and $z \in Z(A)_2$. Since $\mu(z)$ reduces to a central element of A, there exist elements $x_1, \ldots, x_k, y_1, \ldots, y_\ell \in M$ and $r_1, \ldots, r_k, s_1, \ldots, s_\ell \in W \subset M \otimes M$ such that

$$[\mu(z), x] = \sum_{i=1}^{k} x_i r_i + \sum_{j=1}^{\ell} s_j y_j. \tag{6}$$

(To avoid unwanted cancellations, we choose our elements in a way that minimizes $k+\ell$, and we make a choice once and for all for each pair $(x, z) \in M \otimes Z(A)_2$.) Now consider the element

$$\kappa(x, z) := \mu(z)x - \sum s_j y_j + x \otimes z = x\mu(z) + \sum x_i r_i + x \otimes z \in (Q_2 + O_2) \cap (Q_1 + O_1).$$

The vector space $(Q_1+O_1)\cap(Q_2+O_2)$ is spanned by the elements $\kappa(x, z)$ and the subspace $Q_1 \cap Q_2$, hence any class in $T_R(M^*)_3 \oplus (M^* \otimes U^!)$ that reduces to zero in $A^!$ and is orthogonal to all of the elements $\kappa(x, z)$ must also reduce to zero in $\tilde{A}^!$. The commutator $[\eta^!, x^!]$ clearly reduces to zero in $A^!$, since $\eta^!$ is a lift of $z^!$, which was chosen to be central. Thus it remains only to show that it is orthogonal to each $\kappa(x, z)$.

Lemma 7.2. *If $x \in e_\alpha M e_\beta$ and $\alpha < \beta$, then*

$$\langle z^!, j_{\alpha\beta}(z)\rangle\, x = \sum_{i=1}^{k} \langle \mu^!(z^!), r_i\rangle\, x_i \in M \quad \textit{and} \quad s_j \in e_\alpha M \varepsilon_\alpha M \varepsilon_\beta \textit{ for all } 1 \le j \le \ell.$$

If $x \in e_\alpha M e_\beta$ and $\beta < \alpha$, then

$$\langle z^!, j_{\alpha\beta}(z)\rangle\, x = \sum_{j=1}^{\ell} \langle \mu^!(z^!), s_j\rangle\, y_j \in M \quad \textit{and} \quad r_i \in \varepsilon_\alpha M \varepsilon_\beta M e_\beta \textit{ for all } 1 \le i \le k.$$

Proof. We will prove only the case where $\alpha < \beta$ (the proof of the opposite case is identical). From the definition of $\mu(z)$ in Equation (5), we have

$$[\mu(z), x] \in e_\alpha M \varepsilon_\alpha M e_\alpha M e_\beta + e_\alpha M e_\beta M \varepsilon_\beta M e_\beta \subset e_\alpha M \varepsilon_\alpha M (e_\alpha + \varepsilon_\beta) M e_\beta.$$

It follows that $s_j \in e_\alpha M \varepsilon_\alpha M (e_\alpha + \varepsilon_\beta)$ for all j. For any j, the element $s_j e_\alpha$ lies in the subspace $\iota(W) \cap e_\alpha M \varepsilon_\alpha M e_\alpha$, which is trivial by Lemma 3.9. Hence we have $s_j \in e_\alpha M \varepsilon_\alpha M \varepsilon_\beta$ as claimed.

Consider the specialization $\tilde{A}_{z^!}$ of $\tilde{A}$. Since A is quadratic, the natural map from M to $\tilde{A}_{z^!}$ is an inclusion, thus we may regard M as a subspace of $\tilde{A}_{z^!}$. The image of $[\mu(z), x]$ in $\tilde{A}_{z^!}$ lies in M, and Equation (6) tells us that it is equal to the element

$$\sum_{i=1}^{k} \langle z^!,\, r_i\rangle\, x_i + \sum_{j=1}^{\ell} \langle z^!,\, s_j\rangle\, y_j = \sum_{i=1}^{k} \langle z^!,\, r_i\rangle\, x_i,$$

where the second equality follows from the fact that $e_\gamma s_j e_\gamma = 0$ for all j and all $\gamma \in \mathcal{I}$. On the other hand, it is also equal to $\langle z^!, j_{\alpha\beta}(z)\rangle\, x$ by Proposition 6.3, thus we obtain the desired identity. $\square$

We now use Lemma 7.2 to show that $\Big\langle [\eta^!, x^!],\, \kappa(x, z)\Big\rangle = 0$ for all $x \in e_\alpha M e_\beta$, $x^! \in e_\beta M^* e_\alpha$, and $z \in Z(A)_2$, and thus complete the proof of Theorem 7.1. By Lemma 3.8, we may assume that either $\alpha < \beta$ or $\beta < \alpha$. We have

$$\begin{aligned}\Big\langle [\eta^!, x^!],\, \kappa(x,z)\Big\rangle &= \langle \mu^!(z^!)x^!, \kappa(x,z)\rangle - \langle x^!\mu^!(z^!), \kappa(x,z)\rangle + \langle x^! \otimes j^*_{\beta\delta}(z^!) - x^! \otimes j^*_{\alpha\delta}(z^!),\, x\otimes z\rangle \\ &= \langle \mu^!(z^!)x^!, x\mu(z) + \sum x_i r_i\rangle - \langle x^!\mu^!(z^!), \mu(z)x - \sum s_i y_i\rangle + \langle x^!, x\rangle \cdot \langle j^*_{\beta\alpha}(z^!),\, z\rangle \\ &= \langle \mu^!(z^!)x^!, \sum x_i r_i\rangle + \langle x^!\mu^!(z^!), \sum s_j y_j\rangle + \langle x^!, x\rangle \cdot \langle z^!,\, j_{\beta\alpha}(z)\rangle \\ &= \sum \langle x^!, x_i\rangle \cdot \langle \mu^!(z^!), r_i\rangle + \sum \langle x^!, y_i\rangle \cdot \langle \mu^!(z^!), s_i\rangle - \langle x^!, x\rangle \cdot \langle z^!,\, j_{\alpha\beta}(z)\rangle.\end{aligned}$$

First assume that $\alpha < \beta$. By Lemma 7.2, each s_i pairs to zero with any loops, thus the second term of the last line vanishes. The lemma also tells us that the first and third terms cancel, so the entire expression is equal to zero. Similarly, if $\beta < \alpha$, the first term vanishes and the second and third terms cancel. $\square$

Suppose that A and $A^!$ are flexible. For all $\zeta \in Z(\tilde{A})_2$ and $\zeta^! \in Z(\tilde{A}^!)_2$, let $\pi(\zeta)$ and $\pi^!(\zeta^!)$ denote their images in $Z(A)_2$ and $Z(A^!)_2$, respectively. For all $\alpha \in \mathcal{I}$, the splitting h_α of the exact sequence (4) and the analogous splitting $h^!_\alpha$ on the dual side induce a perfect pairing

$$\langle\, ,\rangle_\alpha : Z(\tilde{A})_2 \times Z(\tilde{A}^!)_2 \to \mathbb{C}$$

given by the formula

$$\langle \zeta, \zeta^! \rangle_\alpha := \langle h_\alpha(\zeta), \pi(\zeta^!)\rangle + \langle \pi(\zeta), h^!_\alpha(\zeta^!)\rangle,$$

where we once again exploit the fact that $h_\alpha(z) \in U \cong Z(A^!)_2^*$ and $h^!_\alpha(\zeta^!) \in U^! \cong Z(A)_2^*$.

Proposition 7.3. *All of these pairings coincide.*

Proof. By definition of $j_{\alpha\beta}$, we have

$$\begin{aligned}\langle \zeta, \zeta^! \rangle_\alpha - \langle \zeta, \zeta^! \rangle_\beta &= \big\langle j_{\beta\alpha} \circ \pi(\zeta),\ \pi(\zeta^!)\big\rangle + \big\langle \pi(\zeta),\ j^!_{\beta\alpha} \circ \pi^!(\zeta^!)\big\rangle \\ &= \big\langle \pi(\zeta),\ j^*_{\beta\alpha} \circ \pi(\zeta^!)\big\rangle - \big\langle \pi(\zeta),\ j^!_{\alpha\beta} \circ \pi^!(\zeta^!)\big\rangle,\end{aligned}$$

which vanishes by Theorem 7.1. $\square$

Example 7.4. We illustrate Proposition 7.3 for the algebra $A = A_{11}$ from Example 6.2. Though the dual algebra $A^!$ is isomorphic to A, we will use separate notation in order to keep track of the two sides. The algebra A is generated by $x \in e_1Ae_2$ and $y \in e_2Ae_1$, which satisfy the relation $yx = 0$. Its deformation $\tilde{A}$ is generated by x, y, and a central variable u, which satisfy the relation $yx = ue_2$. On the dual side, $A^!$ is generated by $x^! \in e_2A^!e_1$ and $y^! \in e_1A^!e_2$, subject to the relation $y^!x^! = 0$. Its deformation $\tilde{A}^!$ is generated by $x^!, y^!$, and a central variable $u^!$, which satisfy the relation $y^!x^! = u^!e_1$. The generator $u \in U$ pairs to 1 with the generator $x^!y^! \in Z(A^!)_2$, while $u^! \in U^!$ pairs to 1 with $xy \in Z(A)_2$.

The vector space $Z(\tilde{A})_2$ is spanned by the elements $xy + ue_2$ and $ue_1 - xy$, and we have

$$(h_1 \oplus h_2)(xy + ue_2) = (0, u) \quad \text{and} \quad (h_1 \oplus h_2)(ue_1 - xy) = (u, 0).$$

On the dual side, $Z(\tilde{A}^!)_2$ is spanned by $x^! y^! + u^! e_1$ and $u^! e_2 - x^! y^!$, and we have

$$(h_1^! \oplus h_2^!)(x^! y^! + u^! e_1) = (u^!, 0) \quad \text{and} \quad (h_1^! \oplus h_2^!)(u^! e_2 - x^! y^!) = (0, u^!).$$

Let $\zeta = a(xy + ue_2) + b(ue_1 - xy) \in Z(\tilde{A})_2$ and $\zeta^! = a^!(x^! y^! + u^! e_1) + b^!(u^! e_2 - x^! y^!) \in Z(\tilde{A}^!)_2$. Then

$$\langle \zeta, \zeta^! \rangle_1 = \langle bu, (a^! - b^!) x^! y^! \rangle + \langle a^! u^!, (a - b) xy \rangle = b(a^! - b^!) + a^!(a - b) = aa^! - bb^!,$$

and

$$\langle \zeta, \zeta^! \rangle_2 = \langle au, (a^! - b^!) x^! y^! \rangle + \langle b^! u^!, (a - b) xy \rangle = a(a^! - b^!) + b^!(a - b) = aa^! - bb^!.$$

Thus the two pairings are the same.

We may now use Proposition 7.3 to prove Theorem 1.2, which we restate here.

Corollary 7.5. *If A is flexible, then $\mathcal{Z}(A)$ and $\mathcal{Z}(A^!)$ are canonically GM dual.*

Proof. By Definition 2.3, we associate to the localization algebra $\mathcal{Z}(A)$ the fibered arrangement consisting of the subspaces

$$H_\alpha := h_\alpha^*(U^*) \subset Z(\tilde{A})_2^*,$$

each of which projects isomorphically onto U^*. Definition 2.6 tells us that a duality between $\mathcal{Z}(A)$ and $\mathcal{Z}(A^!)$ is a perfect pairing between $Z(\tilde{A})_2^*$ and $Z(\tilde{A}^!)_2^*$ such that the kernels of the two projections are perpendicular to each other, as are H_α and $H_\alpha^!$ for each $\alpha \in \mathcal{I}$.

For each $\alpha \in \mathcal{I}$, we have constructed a perfect pairing

$$\langle\, , \rangle_\alpha : Z(\tilde{A})_2 \times Z(\tilde{A}^!)_2 \to \mathbb{C},$$

which induces a dual pairing

$$\langle\, , \rangle_\alpha^* : Z(\tilde{A})_2^* \times Z(\tilde{A}^!)_2^* \to \mathbb{C}.$$

It is clear from the definition of the pairing that the kernels of the two projections are perpendicular spaces of each other, and that H_α is the perpendicular space to $H_\alpha^!$. By Proposition 7.3, the pairings $\langle\, , \rangle_\alpha$ all coincide, therefore we have one canonical pairing satisfying all of the required properties. □

8 Example: Polarized arrangements and hypertoric varieties

In this section and the next, we consider two families of examples of Koszul dual pairs of flexible algebras, along with the associated dual pairs of localization algebras. As we will see, most of our examples have cohomological interpretations in addition to algebraic ones. For our first example, we use a ring that we introduced in an earlier paper [BLPWa], constructed from the following linear algebra data.

Definition 8.1. A **polarized arrangement** $\mathcal{V}$ is a triple (V, η, ξ), where V is a linear subspace of a coordinate vector space $\mathbb{R}^n$, $\eta \in \mathbb{R}^n/V$, and $\xi \in V^*$.

It is convenient to think of these data as describing an affine space $V_\eta \subseteq \mathbb{R}^n$ given by translating V away from the origin by η, together with an affine linear functional on V_η given by ξ and a finite hyperplane arrangement $\mathcal{H}$ in V_η, whose hyperplanes are the (possibly empty) restrictions of the coordinate hyperplanes in $\mathbb{R}^n$. We will assume that η and ξ are chosen generically enough so that $\mathcal{H}$ is simple (any set of m hyperplanes intersects in codimension m or not at all) and ξ is non-constant on any positive-dimensional intersection of the hyperplanes. For now (until Remark 8.10) we will also assume that $\mathcal{V}$ is **rational**, meaning that V, η, and ξ are all defined over $\mathbb{Q}$.

In [BLPWa, §4], we explained how to associate to this data a standard Koszul algebra $B(\mathcal{V})$. We sketch this construction here; many more details are given in [BLPWa]. For all $\alpha \in \{\pm 1\}^n$, let

$$\Delta_\alpha = \big\{ v \in V_\eta \subset \mathbb{R}^n \;\big|\; \alpha(i) \cdot v_i \geq 0 \text{ for all } i = 1, \dots n \big\}.$$

Geometrically, Δ_α is the chamber of $\mathcal{H}$ consisting of vectors that lie on a fixed side of each hyperplane. Let the indexing set $\mathcal{I}$ be the set of sign vectors α such that Δ_α is nonempty and the affine linear functional ξ is bounded above on Δ_α. To each $\alpha \in \mathcal{I}$, we may associate a toric variety X_α, with an effective action of the algebraic torus T whose Lie algebra is equal to $V_{\mathbb{C}}^*$ and whose character lattice is $\mathbb{Z}^n \cap V \subset V_{\mathbb{C}}$. The action of the maximal compact subtorus is hamiltonian, and Δ_α is the moment polyhedron for this action.

For all $\alpha, \beta \in \mathcal{I}$, let $d_{\alpha\beta}$ be codimension of $\Delta_\alpha \cap \Delta_\beta$ in V_η, and let $X_{\alpha\beta}$ be the toric variety with moment polyhedron $\Delta_\alpha \cap \Delta_\beta$. As a graded vector space, $B(\mathcal{V})$ is defined as the sum

$$\bigoplus_{\alpha, \beta \in \mathcal{I}} H^*(X_{\alpha\beta})[-d_{\alpha\beta}]. \tag{7}$$

The product that we define is a convolution product: to multiply an element of $H^*(X_{\alpha\beta})$ with an element of $H^*(X_{\beta\gamma})$, we pull both classes back to the toric variety with moment polyhedron $\Delta_\alpha \cap \Delta_\beta \cap \Delta_\gamma$, multiply them there, and then push forward to $X_{\alpha\gamma}$. It is an easy combinatorial exercise to check that this product respects the grading. Showing that it is associative is more subtle, and in fact is only true if we push forward not with respect to the complex orientations, but with respect to a collection of combinatorially defined orientations on the various toric varieties [BLPWa, 4.10].

Remark 8.2. The motivation for this definition comes from the geometry of the **hypertoric variety** $\mathfrak{M}(\mathcal{V})$ associated to $\mathcal{V}$, which is a complex symplectic algebraic variety of dimension $2 \dim V$ (or a hyperkähler manifold of real dimension $4 \dim V$). It comes equipped with an effective hamiltonian action of T (or a tri-hamiltonian action of the maximal compact subtorus). The variety itself depends only on $\mathcal{H}$, and the covector ξ determines an action of $\mathbb{C}^*$ on $\mathfrak{M}(\mathcal{V})$. For each $\alpha \in \mathcal{I}$, the toric variety X_α sits inside of $\mathfrak{M}(\mathcal{V})$ as a Lagrangian subvariety. The union of all of these subvarieties is equal to the set of points $p \in \mathfrak{M}(\mathcal{V})$ such that $\lim_{\lambda\to\infty} \lambda \cdot p$ exists. We conjecture that the algebra $B(\mathcal{V})$ is isomorphic to the Ext-algebra in the Fukaya category of $\mathfrak{M}(\mathcal{V})$ of the sum of the objects associated to the Lagrangian subvarieties X_α. For more information on hypertoric varieties, see the survey article [Pro08].

Given a polarized arrangement $\mathcal{V}$, we define its **Gale dual** $\mathcal{V}^\vee = (V^\perp, -\xi, -\eta)$, where $V^\perp$ sits inside of the dual coordinate vector space $(\mathbb{R}^n)^*$, $-\xi \in V^* \cong (\mathbb{R}^n)^*/V^\perp$, and $-\eta \in \mathbb{R}^n/V \cong (V^\perp)^*$.

Theorem 8.3. [BLPWa, 3.11, 4.14, 4.16, 5.23, & 5.24] *The algebra $B(\mathcal{V})$ is standard Koszul, and its center is isomorphic as a graded ring to the cohomology ring of $\mathfrak{M}(\mathcal{V})$. The algebras $B(\mathcal{V})$ and $B(\mathcal{V}^\vee)$ are Koszul dual to each other.*

Remark 8.4. Note that for $B(\mathcal{V})$ and $B(\mathcal{V}^\vee)$ to be Koszul dual, their degree 0 parts must be isomorphic. The degree 0 part of $B(\mathcal{V})$ is spanned by the unit elements $1_{\alpha\alpha} \in H^*(X_{\alpha\alpha})$ for all $\alpha \in \mathcal{I}$. Let $\mathcal{I}^\vee$ be the corresponding set for $\mathcal{V}^\vee$, that is, the set of sign vectors that give chambers of $\mathcal{V}^\vee$ on which $-\eta$ is bounded above. We prove in [BLPWa, 2.4] that $\mathcal{I}^\vee = \mathcal{I}$, and therefore that there is a canonical isomorphism between $B(\mathcal{V})_0$ and $B(\mathcal{V}^\vee)_0$.

In [BLPWa] we also define a deformation $\tilde{B}(\mathcal{V})$ of $B(\mathcal{V})$. (In that paper we denoted the deformation by $B'(\mathcal{V})$; we use the notation $\tilde{B}(\mathcal{V})$ here to agree with the notation in the rest of this paper.) The ring $\tilde{B}(\mathcal{V})$ is defined by replacing all of cohomology rings in (7) with T-equivariant cohomology rings:

$$\tilde{B}(\mathcal{V}) := \bigoplus_{\alpha,\beta\,\in\mathcal{I}} H^*_T(X_{\alpha\beta})[-d_{\alpha\beta}], \tag{8}$$

with a convolution product defined as for $B(\mathcal{V})$. By [BLPWa, 4.5 & 4.10] it is a flat deformation of $B(\mathcal{V})$ over $V^*_{\mathbb{C}}$, where $\tilde{B}(\mathcal{V}) \to B(\mathcal{V})$ is the map forgetting the equivariant structure, and the map

$$S := \mathrm{Sym}(V_{\mathbb{C}}) \cong H^*_T(pt) \to \tilde{B}(\mathcal{V})$$

which sends an element of S to the sum of its images in $H^*_T(X_{\alpha\alpha})$ over all $\alpha \in \mathcal{I}$.

Proposition 8.5. *The deformation $\tilde{B}(\mathcal{V})$ is flexible and malleable. Its center (with localization algebra structure defined in Corollary 5.7) is isomorphic as a localization algebra to the T-equivariant cohomology ring of $\mathfrak{M}(\mathcal{V})$ (with localization algebra structure defined in Example 2.2).*

Proof. The isomorphism of S-algebras between the center of $\tilde{B}(\mathcal{V})$ and $H^*_T(\mathfrak{M}(\mathcal{V}))$ is given in [BLPWa, 4.16], where we show that both rings are quotients of the polynomial ring $\mathbb{C}[u_1, \ldots, u_n]$ by the same ideal. This result also shows that $Z(\tilde{B}(\mathcal{V})) \to Z(B(\mathcal{V}))$ is surjective, so $\tilde{B}(\mathcal{V})$ is a flexible deformation. We also exhibit in [BLPWa, §2.6] a natural bijection between $\mathcal{I}$ and the fixed point set $\mathfrak{M}(\mathcal{V})^T$; it sends α to the fixed point $x_\alpha \in X_\alpha^T \subset \mathfrak{M}(\mathcal{V})^T$ corresponding to the vertex of Δ_α on which ξ attains its maximum.

The standard modules over $B(\mathcal{V})$ are described geometrically by [BLPWa, 5.22]; we have

$$V_\alpha \cong \bigoplus_{\beta\in\mathcal{I}} H^*(\{x_\alpha\}\cap X_\beta)[-d_{\alpha\beta}],$$

with a natural right action of $B(\mathcal{V})$ by convolution. Corollary 5.5 now implies that

$$\tilde{V}_\alpha := \bigoplus_{\beta\in\mathcal{I}} H_T^*(\{x_\alpha\}\cap X_\beta)[-d_{\alpha\beta}],$$

with the action of $\tilde{B}(\mathcal{V})$ by convolution, is the deformed standard object defined in Section 5. It follows immediately that the map $h_\alpha\colon Z(\tilde{B}(\mathcal{V}))_2 \to V_{\mathbb{C}}$ of Proposition 5.6 coincides with the localization map $H_T^2(\mathfrak{M}(\mathcal{V})) \to H_T^2(x_\alpha)$.

To see that $\tilde{B}(\mathcal{V})$ is malleable, first note that the anti-involution induced by the isomorphism of $X_{\alpha\beta}$ with $X_{\beta\alpha}$ induces an isomorphism $B(\mathcal{V}) \cong B(\mathcal{V})^{\mathrm{op}}$. Malleability now follows from the fact that the localization map $H_T^*(\mathfrak{M}(\mathcal{V})) \to H_T^*\big(\mathfrak{M}(\mathcal{V})^T\big)$ is an isomorphism over the generic point of $\mathfrak{t}$. $\square$

Using this, we have a simple description of the degree two part of the maps h_α from the localization algebra structure, and hence of the associated fibered arrangement. For simplicity we will assume that V is not contained in any coordinate hyperplane, so there are no empty hyperplanes in our arrangement.

For each $\alpha\in\mathcal{I}$, let $p_\alpha\in\Delta_\alpha$ be the point at which ξ attains its maximum and let $b_\alpha\subset\{1,\ldots,n\}$ be the set of indices i for which the i^{th} hyperplane of $\mathcal{H}$ contains p_α. The collection $\{b_\alpha \mid \alpha\in\mathcal{I}\}$ consists precisely of all subsets of $\{1,\ldots,n\}$ for which then the composition of the inclusion $\iota\colon V\hookrightarrow \mathbb{R}^n$ with the coordinate projection $\pi_\alpha\colon\mathbb{R}^n\to\mathbb{R}^{b_\alpha}$ is an isomorphism. Such subsets are known as the **bases** of $\mathcal{V}$.

Proposition 8.6. *There is an isomorphism of $H_T^2(\mathfrak{M}(\mathcal{V}))$ with $\mathbb{C}^n$ such that the inclusion*

$$V_{\mathbb{C}} \cong H_T^2(pt) \hookrightarrow H_T^2(\mathfrak{M}(\mathcal{V})) \cong \mathbb{C}^n$$

is the complexification of ι. Under this identification, the restriction of the localization algebra map h_α to degree 2 is the complexification of $(\pi_\alpha\circ\iota)^{-1}\circ\pi_\alpha$. The fibered arrangement associated to $\tilde{B}(\mathcal{V})$ is the union of the coordinate subspaces $(\mathbb{C}^{b_\alpha})^*$ of the dual space $(\mathbb{C}^n)^*\cong H_2^T(\mathfrak{M}(\mathcal{V}))$.

Proof. The first statement follows from the standard description of the equivariant cohomology of a hypertoric variety [Pro08, 3.2.2]. The remaining statements follow easily from the fact that $\mathbb{C}^{b_\alpha^c}$ must be in the kernel of h_α. $\square$

Theorem 8.7. *Suppose that the subspace $V\subset\mathbb{R}^n$ contains no coordinate line. Then $\tilde{B}(\mathcal{V})$ is isomorphic to the universal deformation of $B(\mathcal{V})$.*

Proof. Since V contains no coordinate line, $V^\perp$ is not contained in any coordinate plane, and so Theorem 8.3 implies that

$$Z(B(\mathcal{V})^!)_2 \cong Z(B(\mathcal{V}^\vee))_2 \cong H^2(\mathfrak{M}(\mathcal{V}^\vee)) \cong V_{\mathbb{C}}^*,$$

where the last isomorphism comes from the formula for the cohomology of a hypertoric variety [Kon00, HS02, Pro08], which gives $H^2(\mathfrak{M}(\mathcal{V}^\vee)) \cong \mathbb{C}^n/V_{\mathbb{C}}^\perp = V_{\mathbb{C}}^*$.

Thus to prove that the map $\psi\colon V_{\mathbb{C}} \to Z(B(\mathcal{V})^!)_2^*$ associated to the deformation $\tilde{B}(\mathcal{V})$ by Theorem 4.1 is an isomorphism, it is enough to show that it is surjective. We can do this using Lemma 6.5. Using Proposition 8.6 and a little linear algebra, it is not hard to show that for any $\alpha, \beta \in \mathcal{I}$, the value of $j_{\alpha\beta}$ on the parameter $\eta \in \mathbb{C}^n/V_{\mathbb{C}} \cong Z(B(\mathcal{V}))_2$ is given by

$$j_{\alpha\beta}(\eta) = p_\beta - p_\alpha \in V \subset V_{\mathbb{C}} \cong Z(B(\mathcal{V})^!)_2^*.$$

(Note that p_α and p_β both lie in the affine space V_η, so their difference lies in the vector space V.) The surjectivity of ψ now follows from Lemma 6.5, using the fact that the points p_α form an affine spanning set for V_η (this is where we use the assumption that V contains no coordinate line). □

Corollary 8.8. *The localization algebras $\mathcal{Z}(\tilde{B}(\mathcal{V}))$ and $\mathcal{Z}(\tilde{B}(\mathcal{V}^\vee))$ are dual.*

Corollary 8.8 follows immediately from Corollary 7.5, but the concrete description of these fibered arrangements in Proposition 8.6 makes it easy to see this directly: if $b_\alpha^\vee \subset \{1,\ldots,n\}$ is the basis for $\mathcal{V}^\vee$ indexed by α, then we have $b_\alpha^\vee = b_\alpha^c$ [BLPWa, 2.9], and so $H_\alpha^\vee = \mathbb{C}^{b_\alpha^\vee}$ is perpendicular to $(\mathbb{C}^{b_\alpha})^* = H_\alpha$.

Example 8.9. If V is $(n-1)$-dimensional and $\mathcal{H}$ consists of a collection of n hyperplanes in general position, then the hypertoric variety $\mathfrak{M}(\mathcal{V})$ is isomorphic to the cotangent bundle of $\mathbb{P}^{n-1}$. Dually, if V is one-dimensional and $\mathcal{H}$ consists of n points on a line, then $\mathfrak{M}(\mathcal{V})$ is isomorphic to the minimal resolution of the symplectic surface singularity $\mathbb{C}^2/\mathbb{Z}_n$, which retracts onto a chain of $n-1$ projective lines. Thus the duality of $\mathcal{Z}(\tilde{B}(\mathcal{V}))$ and $\mathcal{Z}(\tilde{B}(\mathcal{V}^\vee))$ generalizes that of Examples 2.4 and 2.5.

Remark 8.10. We have assumed that $\mathcal{V}$ is rational in order to give that shortest and best motivated definition of the algebra $B(\mathcal{V})$. In [BLPWa], however, we do not make this assumption. Although the toric varieties X_α and the hypertoric variety $\mathfrak{M}(\mathcal{V})$ are not defined when $\mathcal{V}$ is not rational, it is still possible to give combinatorial definitions of rings and localization algebras that specialize to the ordinary and equivariant cohomology rings of these spaces in the rational case. In this more general setting, Theorem 8.3, Proposition 8.5, Proposition 8.6, Theorem 8.7, and Corollary 8.8 go through exactly as stated. The rings $B(\mathcal{V})$ for $\mathcal{V}$ not rational are the only examples that we know of flexible algebras that are not associated with any algebraic variety.

9 Example: Category $\mathcal{O}$ and Spaltenstein varieties

In this section, we apply our deformation result to integral blocks of parabolic category $\mathcal{O}$ for the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. In particular, we show that the universal deformation of the endomorphism algebra of a minimal projective generator of such a block is malleable and flexible, and identify the associated localization algebra, which turns out to come from the equivariant cohomology ring of a Spaltenstein variety. We accomplish this by identifying modules over our universal deformation with objects in "deformed category $\mathcal{O}$" as considered by Soergel and Fiebig. Most of the results of this section should be true for $\mathfrak{g}$ an arbitrary reductive Lie algebra, but in our proofs we use Brundan's computation of centers of blocks of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_n$ and therefore restrict ourselves to $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

We mostly follow Brundan's notation to describe the blocks of $\mathcal{O}$. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, and let $\mathfrak{b}$ and $\mathfrak{h}$ be the Lie subalgebras of $\mathfrak{g}$ consisting of upper triangular and diagonal matrices, respectively. The Weyl group $W = W(\mathfrak{g}, \mathfrak{h})$ is the symmetric group S_n, which acts on $\mathfrak{h}^* \cong \mathbb{C}^n$ and on the weight lattice $\Lambda = X(T) \subset \mathfrak{h}^*$ by permuting coordinates. With these conventions, a weight $\lambda \in \Lambda$ is dominant if and only if $\lambda_i > \lambda_j$ for all $i < j$. Let $w_0 \in W$ denote the longest element.

A **composition** of n is a doubly-infinite sequence $\nu = (\dots, \nu_{-1}, \nu_0, \nu_1, \nu_2, \nu_3, \dots)$ of non-negative integers whose sum is n. Given such a ν, there is a unique dominant weight α_ν with ν_i entries equal to $-i$ for every integer $i \in \mathbb{Z}$. This gives a bijection between the W-orbits Λ/W of the weight lattice and the set of compositions of n.

For any composition ν of n, let $W_\nu = W_{\alpha_\nu} \subset W$ be the stabilizer of the dominant weight $\alpha_\nu \in \Lambda$. Elements of W_ν are permutations which preserve subsets of consecutive elements of $\{1, \dots, n\}$ of sizes $\dots \nu_{-1}, \nu_0, \nu_1, \dots$, in that order. We will refer to these subsets as *ν-blocks*. We also define another associated composition $\bar{\nu}$ by letting $\bar{\nu}_1 = \nu_i$ where i is the smallest index with $\nu_i \neq 0$, $\bar{\nu}_2 = \nu_{i'}$ where i' is the next smallest index where ν is nonzero, and so on, letting all other $\bar{\nu}_j$ be zero.

Associated to $\mathfrak{g} = \mathfrak{gl}_n$ we have the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of all finitely generated $\mathfrak{g}$-modules which are $\mathfrak{h}$-diagonalizable and locally finite over $\mathfrak{b}$. For simplicity we add the additional assumption that all weights lie in the lattice Λ. For $\alpha \in \Lambda$, there is a unique simple $\mathfrak{g}$-module $L(\alpha)$ with highest weight $\alpha - \rho$, where $\rho = (0, -1, -2, \dots, -n+1)$. These are the simple objects of $\mathcal{O}$.

For a composition ν of n, define $\mathcal{O}_\nu$ to be the Serre subcategory of $\mathcal{O}$ generated by all $L(w\alpha_\nu)$ for $w \in W$. Obviously the weight $w\alpha_\nu$ only depends on the image of w in W/W_ν. Define a composition ν^t by letting ν^t_j be the number of $i \in \mathbb{Z}$ for which $\nu_i \geq j$ if $j \geq 1$, and zero otherwise. It is a **partition**, meaning a composition that's supported on $\mathbb{N}$ and non-increasing. The partition $\nu^+ := (\nu^t)^t$ has the same parts as ν, sorted into non-increasing order.

Given another composition μ of n, let $\mathfrak{p} = \mathfrak{p}_\mu$ be the parabolic subalgebra of $\mathfrak{g}$ given by all block upper-triangular matrices where the blocks are the μ-blocks. Associated to μ we have the parabolic category $\mathcal{O}^\mu = \mathcal{O}^\mathfrak{p}$, the full subcategory of $\mathcal{O}$ of objects which are $\mathfrak{p}$-locally finite. Its simple objects are $\{L(\alpha) \mid \alpha \in \Lambda_\mu^+\}$, where

$$\Lambda_\mu^+ := \{\alpha \in \Lambda \mid \alpha_j > \alpha_k \text{ whenever } j < k \text{ and } j, k \text{ lie in the same } W_\mu\text{-orbit}\}.$$

Define $\mathcal{O}_\nu^\mu := \mathcal{O}^\mu \cap \mathcal{O}_\nu$; it is the Serre subcategory of $\mathcal{O}$ generated by the simple objects $L(w\alpha_\nu)$ for $w \in \mathcal{I}_\nu^\mu$, where

$$\mathcal{I}_\nu^\mu = \{w \in W/W_\nu \mid w\alpha_\nu \in \Lambda_\mu^+\}.$$

Lemma 9.1. *The category $\mathcal{O}_\nu^\mu$ is nonzero if and only if $\mu^+ \leq \nu^t$ in the dominance order on partitions. The map $w \mapsto W_\mu w$ defines a bijection between $\mathcal{I}_\nu^\mu$ and the set of double cosets in $W_\mu \backslash W / W_\nu$ of size $|W_\mu| \times |W_\nu|$.*

The category $\mathcal{O}_\nu^\mu$ has enough projectives. For $w \in \mathcal{I}_\nu^\mu$, let $P^\mu(w\alpha_\nu)$ be a projective cover of $L(w\alpha_\nu)$ in $\mathcal{O}_\nu^\mu$. (Note that although $L(w\alpha_\nu)$ can lie in $\mathcal{O}_\nu^\mu$ for many choices of μ, in general the projective covers in these categories will be different.) Let $P_\nu^\mu := \bigoplus_{w \in \mathcal{I}_\nu^\mu} P^\mu(w\alpha_\nu)$ be a minimal projective generator of $\mathcal{O}_\nu^\mu$, and let $A_\nu^\mu := \mathrm{End}(P_\nu^\mu)$, so that $M \mapsto \mathrm{Hom}(P_\nu^\mu, M)$ is an equivalence between $\mathcal{O}_\nu^\mu$ and the category of finitely generated right A_ν^μ-modules.

For a composition μ, let μ^o denote the reversed composition given by $\mu_i^o := \mu_{-i}$.

Proposition 9.2. *The ring A_ν^μ has a grading with respect to which it is standard Koszul. There is an isomorphism $(A_\nu^\mu)^! \cong A_{\mu^o}^\nu$ whose map on idempotents is induced by the map $\mathcal{I}_\nu^\mu \to \mathcal{I}_{\mu^o}^\nu$ taking wW_ν to $w^{-1}w_0W_{\mu^o}$, where w_0 has maximal length in the coset wW_ν.*

Proof. The construction of a Koszul grading and the identification of the Koszul dual is accomplished in Backelin [Bac99, 1.1]. The fact that A_ν^μ is quasi-hereditary follows from [RC80, Theorem 6.1]. Since both A_ν^μ and its dual are quasi-hereditary, and the associated partial orders on the idempotents are reversed, [ÁDL03, Theorem 3] implies that A_ν^μ is standard Koszul. □

Remark 9.3. Up to isomorphism the algebra A_ν^μ only depends on the subgroups $W_\nu, W_\mu \subset W$. This is obvious for μ, while for ν the required equivalences are given by translation functors. In addition, the rings A_ν^μ and $A_{\nu^o}^{\mu^o}$ are isomorphic, using the automorphism of $\mathfrak{g}$ given by the adjoint action by any representative for w_0 in $G = GL_n(\mathbb{C})$.

Since A_ν^μ is Koszul, we can consider our universal graded deformation $\tilde{A}_\nu^\mu$ and its associated localization algebra $\mathcal{Z}(\tilde{A}_\nu^\mu)$ as given by Corollary 5.7. We wish to relate this localization algebra with one arising from geometry, specifically, from the equivariant cohomology of a Spaltenstein variety.

As before, let μ, ν be compositions of n, and suppose that $\mu^+ \leq \nu^t$. Let $P_\nu \subset G := \mathrm{GL}_n(\mathbb{C})$ be the parabolic subgroup with Lie algebra $\mathfrak{p}_\nu$, and let

$$X_\nu = G/P_\nu = \{0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} F_i = \bar{\nu}_1 + \cdots + \bar{\nu}_i\}$$

be the associated partial flag variety. (Note that on the geometric side it is ν, not μ, which specifies the parabolic; this is related to Remark 9.4 below.) The cotangent bundle T^*X_ν may be identified with the variety of pairs

$$\{(F_\bullet, N) \in X_\nu \times \mathfrak{g} \mid NF_i \subset F_{i-1} \text{ for all } i > 0\}.$$

The moment map $\pi\colon T^*X_\nu \to \mathfrak{g}^*$ is a resolution of the closure of a nilpotent coadjoint orbit; the Spaltenstein variety X_ν^μ is the fiber of this map over a point in an orbit of type μ. More precisely, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the inner product $\langle A, B\rangle := \operatorname{tr}(AB)$, and we let $N_\mu \in \mathfrak{g}^*$ be the nilpotent matrix defined by $N_\mu(e_i) = e_{i+1}$ if i and $i+1$ are in the same μ-block, and $N_\mu(e_i) = 0$ otherwise, where e_i is the i^{th} standard basis element of $\mathbb{C}^n$. Then we have

$$X_\nu^\mu = \pi^{-1}(N_\mu) = \{F_\bullet \in X_\nu \mid N_\mu F_i \subset F_{i-1} \text{ for all } i > 0\}.$$

Let $T \subset G$ be the maximal torus consisting of diagonal matrices. It acts naturally on the flag variety X_ν, and the subtorus $T^\mu := T^{W_\mu} = Z_G(N_\mu) \cap T$ preserves the subvariety X_ν^μ.

Remark 9.4. The group G whose flag variety we have introduced is morally the Langlands dual of the group with Lie algebra $\mathfrak{g}$, whose representations we are studying. In particular, the Lie algebra $\mathfrak{t}$ of T should be identified with the dual of $\mathfrak{h}$. The Lie algebra $\mathfrak{t}^\mu$ is a subspace of $\mathfrak{t}$, so its dual $\mathfrak{h}^\mu$ should be thought of as a quotient of $\mathfrak{h}$.

This can be confusing, since the group $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to its own Langlands dual. In particular, $\mathfrak{h}^\mu$ is isomorphic to $\mathfrak{t}^\mu$, and the quotient map from $\mathfrak{h}$ to $\mathfrak{h}^\mu$ is given by averaging over W_μ. We will, however, be careful never to use this isomorphism; we will always distinguish between $\mathfrak{t}$ and $\mathfrak{h}$.

Recall that the T-fixed points in the flag variety X_ν are in bijection with W/W_ν by $w \mapsto p_w$, where p_w is the flag $F_\bullet(w)$ given by $F_i(w) = \operatorname{Span}\{e_{w(j)} \mid 1 \le j \le \bar\nu_1 + \cdots + \bar\nu_i\}$.

Proposition 9.5. *The set of T^μ-fixed points in X_ν^μ is $\{p_w \mid w \in \mathcal{I}_\nu^\mu\}$.*

Proof. First we show that a T^μ-fixed point in X_ν^μ must in fact be fixed by T. For a point $gP \in X_\nu$, we have $gP \in X_\nu^\mu$ if and only if $N_\mu \in \operatorname{Ad}(g)\mathfrak{p}$, and gP is fixed by T^μ if and only if $T^\mu \subset gPg^{-1}$. It is enough therefore to show that these two properties together imply that $T \subset gPg^{-1}$. Let $G' = Z_G(T^\mu)$; it is the subgroup of block diagonal matrices in G, and in particular contains both T and N_μ. By [Spr98, 6.4.7], $gPg^{-1} \cap G'$ is a parabolic subgroup of G', and since N_μ is a product of regular nilpotents in every simple factor of G', we have reduced the problem to the case where N_μ itself is regular, where it is obvious.

Thus all our fixed points are of the form p_w for some $w \in W/W_\nu$. In order to have $p_w \in X_\nu^\mu$, we must have $N_\mu(F_i(w)) \subset F_{i-1}(w)$ for all $i > 0$. This is equivalent to saying that for any $j < k$ which lie in the same μ-block, then $w^{-1}(j)$ lies in a later ν-block than $w^{-1}(k)$, or equivalently, $(w\alpha)_j > (w\alpha)_k$. Therefore $p_w \in X_\nu^\mu$ if and only if $w \in \mathcal{I}_\nu^\mu$. $\square$

This is the space whose equivariant cohomology will give the localization algebra of A_ν^μ. However, the torus T^μ is too large in general. For instance, take $n = 3$ and let $\mu = \nu$ be the partition

$(2,1)$. Then $\mathcal{I}^\mu_\nu$ has only one element, so $A^\mu_\nu \cong \mathbb{C}$ and its degree two part is zero, while T^μ is two-dimensional. The action of T^μ will factor through a quotient torus which we define as follows.

Let $\lambda = \nu^t$, the transpose partition to ν.

Definition 9.6. Let $\mathcal{J}$ be the collection of all subsets $J \subset \{1,\dots,n\}$ such that J is a union of μ-blocks and

$$|J| = \lambda_1 + \cdots + \lambda_k,$$

where k is the number of μ-blocks appearing in J.

Note that the existence of such a J other than $\{1,\dots,n\}$ implies that there is an index where the dominance inequality required for $\mu^+ \leq \lambda$ is an equality. Thus, the elements of J measure where μ^+ comes closest to not being less than λ.

For any $J \in \mathcal{J}$, let $\mathbf{1}_J := \sum_{i\in J} e_i \in \mathfrak{t}^\mu \subset \mathfrak{t} \cong \mathbb{C}^n$, and let T^μ_ν be the quotient of T^μ by the connected subtorus with Lie algebra spanned by $\{\mathbf{1}_J \mid J \in \mathcal{J}\}$, so

$$\operatorname{Lie} T^\mu_\nu = \mathfrak{t}^\mu_\nu := \mathfrak{t}^\mu / \operatorname{Span}\{\mathbf{1}_J \mid J \in \mathcal{J}\}.$$

The meaning of the sets $J \in \mathcal{J}$ is explained by the following combinatorial result. For any $J \subset \{1,\dots,n\}$, let $W_J := \{w \in W \mid w(J) = J\}$.

Lemma 9.7. *For any $J \in \mathcal{J}$, the set $\{w\alpha_\nu \mid w \in \mathcal{I}^\mu_\nu\}$ is contained in a single W_J-orbit. In particular, the inner product $\langle w\alpha_\nu, \mathbf{1}_J\rangle$ is independent of w for $w \in \mathcal{I}^\mu_\nu$.*

Proof. We can assume that $\mu^+ \leq \lambda$, since otherwise $\mathcal{I}^\mu_\nu = \emptyset$. Take an element $J \in \mathcal{J}$, and suppose that $|J| = \lambda_1 + \cdots + \lambda_k$. Fix a permutation $w \in W$. The vector $w\alpha_\nu$ has ν_i entries equal to $-i$ for all $i \in \mathbb{Z}$, so if we let $m_i = \#\{j \in J \mid (w\alpha_\nu)_j = -i\}$, we have $m_i \leq \nu_i$ for all i. If $w \in \mathcal{I}^\mu_\nu$ then $w\alpha_\nu$ lies in Λ^+_μ, so the entries in each μ-block are strictly decreasing. In particular, each μ-block has distinct entries, and since J is the union of exactly k μ-blocks, we must have $m_i \leq k$ for all $i \in \mathbb{Z}$. But then

$$|J| = \sum_{i\in\mathbb{Z}} m_i \leq \sum_{i\in\mathbb{Z}} \min(\nu_i, k) = \lambda_1 + \cdots + \lambda_k = |J|,$$

so we must have $m_i = \min(\nu_i, k)$ for all i. This means that the multiset of entries of $w\alpha_\nu$ in the places $j \in J$ is independent of $w \in \mathcal{I}^\mu_\nu$. □

Proposition 9.8. *The action of T^μ on X^μ_ν factors through the quotient $T^\mu \to T^\mu_\nu$.*

Proof. Take any $J \in \mathcal{J}$, and let $T_J \subset T^\mu$ be the connected subtorus with Lie algebra $\mathbb{C}\cdot\mathbf{1}_J$. We need to show that T_J acts trivially on X^μ_ν. Suppose not; then X^μ_ν must meet more than one connected component of the fixed point set $(X_\nu)^{T_J}$, and so the fixed point set $(X^\mu_\nu)^{T^\mu}$ must also meet more than one component. But two T-fixed points p_w, $p_{w'}$ lie in the same component of $(X_\nu)^{T_J}$ if and only if w and w' lie in the same W_J-orbit, contradicting Lemma 9.7. □

The following is our main result relating Spaltenstein varieties with category $\mathcal{O}$. The next section is devoted to its proof.

Theorem 9.9. *The universal deformation $\tilde{A}^\mu_\nu$ of A^μ_ν is malleable and flexible. There is an isomorphism of localization algebras between $Z(\tilde{A}^\mu_\nu)$ and $H^*_{T^\nu_\mu}(X^\mu_\nu)$.*

Remark 9.10. Theorem 9.9 implies that the rings $Z(A^\mu_\nu)$ and $H^*(X^\mu_\nu)$ are isomorphic. This was originally proved by Brundan in [Bru08a]. We use some of Brundan's results in the proof of Theorem 9.9, so although we have given a new point of view on this result, it is not an independent proof.

Note also that Theorems 9.2 and 9.9 imply that $\mathfrak{t}^\mu_\nu$ and $Z((A^\mu_\nu)^!)_2 \cong Z(A^\nu_{\mu^o})_2 \cong Z(A^{\nu^o}_\mu)$ must be isomorphic, as they can all be interpreted as the base of the universal deformation of A^μ_ν. This also follows from Brundan's work: in fact, our formula for $\mathfrak{t}^\mu_\nu$ is exactly the degree two part of the isomorphism $H^*(X^{\nu^o}_\mu) \cong Z(A^{\nu^o}_\mu)_2$ given in [Bru08a].

Remark 9.11. Like the algebras $B(\mathcal{V})$ considered in Section 8, the algebras A^μ_ν are related to the geometry of certain symplectic algebraic varieties. Let S^μ be the **Slodowy slice** to the nilpotent matrix N_μ, constructed explicitly in [Slo80, §7.4], and recall the moment map $\pi\colon T^*X_\nu \to \mathfrak{g}^*$. The preimage $\tilde{S}^\mu_\nu := \pi^{-1}(S^\mu)$ is a smooth symplectic algebraic variety, and the projection π is a symplectic resolution of singularities [Maf05, Theorem 12]. The variety $\tilde{S}^\mu_\nu$ deformation retracts onto the Spaltenstein variety X^μ_ν, and the irreducible components of X^μ_ν are Lagrangian in $\tilde{S}^\mu_\nu$. We conjecture that A^μ_ν is isomorphic to the Ext-algebra of a sum of Lagrangians in the Fukaya category of $\tilde{S}^\mu_\nu$. This conjecture is completely analogous to the one that we made above involving $B(\mathcal{V})$ and the hypertoric variety $\mathfrak{M}(\mathcal{V})$ in Remark 8.2. When N_μ is the zero matrix, so that $\tilde{S}^\mu_\nu$ is isomorphic to the cotangent bundle of the partial flag variety, our conjecture follows from the Beilinson-Bernstein localization theorem [BB81] and the work of Kapustin, Witten, Nadler, and Zaslow relating the Fukaya category of a cotangent bundle to the category of perverse sheaves on the base [KW07, NZ09].

10 Deformed category $\mathcal{O}$

In this section we prove Theorem 9.9. In order to understand the universal deformation $\tilde{A}^\mu_\nu$, we will compare it to a ring $\widehat{A}^\nu_\mu$ coming from the "deformed category $\mathcal{O}$" considered by Soergel [Soe90, Soe92] and Fiebig [Fie03, Fie06, Fie08]. Here the deformation comes from deforming the action of the Cartan subalgebra. Results of Fiebig and Soergel allow us to show that $\widehat{A}^\nu_\mu$ carries a formal grading in the sense of Remark 4.2, so it comes from the universal deformation by extension of scalars. It is easy to construct deformed standard objects in the deformed category $\mathcal{O}$, and to compute their central characters. We use this to show that the "formal localization algebra" $\mathcal{Z}(\widehat{A}^\nu_\mu)$ is isomorphic to the completion of the the equivariant cohomology ring of the Spaltenstein X^μ_ν, but for the larger torus $T^\mu \supset T^\mu_\nu$.

Most of the following material on deformed category $\mathcal{O}$ can be found in Fiebig's paper [Fie03]. However, he does not treat the parabolic case; when necessary, we will indicate how his arguments can be extended; see also [Str, §2]. Let μ, ν be compositions of n as in the previous section, and assume that $\mu^+ \leq \nu^t$, so the block $\mathcal{O}^\mu_\nu$ is nonzero. Let $S^\mu := \mathrm{Sym}(\mathfrak{t}^\mu)^* \cong \mathrm{Sym}\,\mathfrak{h}^\mu$. A ***μ*-deformation**

algebra is a commutative noetherian S^μ-algebra D with structure map $\tau\colon S^\mu \to D$. Given a μ-deformation algebra and a weight $\alpha \in \Lambda \subset \mathfrak{h}^*$, the **$\alpha$-weight space** of a left $U(\mathfrak{g}) \otimes_{\mathbb{C}} D$-module M is

$$M_\alpha := \{v \in M \mid Hv = (\alpha(H) + \tau(H^\mu))v \text{ for all } H \in \mathfrak{h}\},$$

where H^μ denotes the image of H in the quotient $\mathfrak{h}^\mu$ of $\mathfrak{h}$. Let $\mathfrak{p} = \mathfrak{p}_\mu$ be the parabolic subalgebra determined by μ, as defined in the previous section, and let D be a μ-deformation algebra.

Definition 10.1. The D-deformed μ-parabolic category $\mathcal{O}^\mu_D$ is the category of finitely generated $U(\mathfrak{g}) \otimes_{\mathbb{C}} D$-modules M such that

- $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$, and
- $(U(\mathfrak{p}) \otimes D)v$ is a finitely generated D-module for all $v \in M$.

For instance, if $D = \mathbb{C}$ and the map $S^\mu \to \mathbb{C}$ kills $\mathfrak{h}^\mu$, then $\mathcal{O}^\mu_{\mathbb{C}}$ is the usual parabolic category $\mathcal{O}^\mu$. If $D \to D'$ is a map to another μ-deformation algebra, then $M \mapsto M \otimes_D D'$ defines a base change functor $\mathcal{O}^\mu_D \to \mathcal{O}^\mu_{D'}$.

Deformed standard objects in $\mathcal{O}^\mu_D$, which we call **deformed Verma modules**, are defined in the following way. Let $\mathfrak{m} = Z_{\mathfrak{g}}(\mathfrak{t}^\mu) \subset \mathfrak{p}$ be the centralizer of $\mathfrak{t}^\mu$; it is the subalgebra of block diagonal matrices for the μ-blocks. For each $\alpha \in \Lambda^+_\mu$ there is a finite-dimensional irreducible $\mathfrak{m}$-module E_α with highest weight $\alpha - \rho$, and these give all finite-dimensional irreducible modules with integral weights. For $\alpha \in \Lambda^+_\mu$, we define a deformed Verma module

$$M^\mu_D(\alpha) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (E_\alpha \otimes_{\mathbb{C}} D),$$

where $U(\mathfrak{p})$ acts on E_α via the map $U(\mathfrak{p}) \to U(\mathfrak{m})$ obtained by projecting away the off-diagonal blocks of $\mathfrak{p}$, and on D via $U(\mathfrak{p}) \to S^\mu \xrightarrow{\tau} D$. It is an object of $\mathcal{O}^\mu_D$, where D acts only on the last factor. Since it is generated as a module over $U(\mathfrak{g}) \otimes D$ by $1 \otimes v \otimes 1$, where $v \in E_\alpha$ is a highest weight vector, the action of D induces an isomorphism $\operatorname{End}_{\widehat{\mathcal{O}}^\mu}(\widehat{M}^\mu_D(\alpha)) \cong D$. For any map $D \to D'$ of μ-deformation algebras, we have a natural isomorphism $M^\mu_D(\alpha) \otimes_D D' = M^\mu_{D'}(\alpha)$. The object $M^\mu_{\mathbb{C}}(\alpha)$ is the standard cover of the simple module $L(\alpha)$ in the usual parabolic category $\mathcal{O}^\mu$.

To apply Fiebig's results we need our deformation algebra to be local. Let $\widehat{S}^\mu = \prod_{i \geq 0} S^\mu_i$ be the completion of S^μ at the graded maximal ideal $S^\mu_{>0}$, and let $\widehat{\mathcal{O}}^\mu$ denote the corresponding deformed category $\mathcal{O}^\mu_{\widehat{S}^\mu}$. We denote the deformed Verma modules in this category by $\widehat{M}^\mu(\alpha) := M^\mu_{\widehat{S}^\mu}(\alpha)$.

Definition 10.2. A **Verma flag** for an object $M \in \widehat{\mathcal{O}}^\mu$ is a finite filtration with subquotients that are isomorphic to deformed Verma modules.

Theorem 10.3. [Fie03, §2] *The category $\widehat{\mathcal{O}}^\mu$ has enough projectives. The base change functor $\widehat{\mathcal{O}}^\mu \to \mathcal{O}^\mu_{\mathbb{C}} = \mathcal{O}^\mu$ induces bijections between isomorphism classes of simples in $\widehat{\mathcal{O}}^\mu$ and in $\mathcal{O}^\mu$, and between isomorphism classes of indecomposible projectives in both categories. All projective objects* *in $\widehat{\mathcal{O}}^\mu$ have Verma flags. If $P \in \widehat{\mathcal{O}}^\mu$ is projective and $M \in \widehat{\mathcal{O}}^\mu$ has a Verma flag, then* $\mathrm{Hom}_{\widehat{\mathcal{O}}^\mu}(P, M)$ *is a free $\widehat{S}^\mu$-module, and the natural map*

$$\mathrm{Hom}_{\widehat{\mathcal{O}}^\mu}(P, M) \otimes_{\widehat{S}^\mu} \mathbb{C} \to \mathrm{Hom}_{\mathcal{O}^\mu}(P \otimes_{\widehat{S}^\mu} \mathbb{C}, M \otimes_{\widehat{S}^\mu} \mathbb{C})$$

is an isomorphism.

Remark 10.4. Note that Fiebig does not treat the parabolic case. He assumes that $\mathfrak{p}$ is a Borel subgroup, meaning that $\mu_i \leq 1$ for all i, so $W_\mu = \{1\}$. We call such a composition **regular**. The main argument in [Fie03] that needs modifying when μ is not regular is Lemma 2.3, which constructs the projectives. He explains that arguments of Rocha-Caridi and Wallach [RCW82] can be adapted to the deformed situation. The arguments of [RCW82] do cover the parabolic case, so extending Fiebig's arguments is straightforward.

Theorem 10.3 gives us for each $\alpha \in \Lambda_\mu^+$ a projective object $\widehat{P}^\mu(\alpha)$ such that $\widehat{P}^\mu(\alpha) \otimes_{\widehat{S}^\mu} \mathbb{C} \cong P^\mu(\alpha)$. If ν is another composition of n, we define $\alpha = \alpha_\nu$ as before, and let $\widehat{\mathcal{O}}^\mu_\nu$ be the full subcategory of $\widehat{\mathcal{O}}^\mu$ whose objects are all quotients of direct sums of $\widehat{P}^\mu(w\alpha)$ for $w \in \mathcal{I}^\nu_\mu$. Then $\widehat{P}^\mu_\nu := \bigoplus_{w \in \mathcal{I}^\mu_\nu} \widehat{P}^\mu(w\alpha)$ is a projective generator of this category, so $\mathrm{Hom}_{\widehat{\mathcal{O}}^\mu_\nu}(\widehat{P}^\mu_\nu, -)$ defines an equivalence of categories between $\widehat{\mathcal{O}}^\mu_\nu$ and finitely generated right modules over $\widehat{A}^\mu_\nu := \mathrm{End}(\widehat{P}^\mu_\nu)$. Theorem 10.3 also implies that the image of $\widehat{M}^\mu(w\alpha_\nu)$ satisfies the formal analogues of the hypotheses of Corollary 5.5, so it is isomorphic to the deformed standard object in the category of $\widehat{A}^\mu_\nu$-modules which we defined in Section 5.

The base change functor $\widehat{\mathcal{O}}^\mu_\nu \to \mathcal{O}^\mu_\nu$ sends $\widehat{P}^\mu_\nu$ to P^μ_ν, so it induces a ring homomorphism $\widehat{A}^\mu_\nu \to A^\mu_\nu$. Theorem 10.3 implies that $\widehat{A}^\mu_\nu$ is a flat deformation of A^μ_ν over $\mathrm{Spec}\, \widehat{S}^\mu$. We wish to use Remark 4.2 to relate this deformation to the universal deformation $\tilde{A}^\mu_\nu$. To do this, we need to construct a formal grading on $\widehat{A}^\mu_\nu$.

When μ is regular, we use a geometric interpretation of deformed category $\mathcal{O}$ due to Soergel [Soe92] and Fiebig [Fie03, Fie08] to construct our formal grading; we then deduce the case when μ is general from this. So assume for the moment that μ is regular. To indicate this, we omit the superscript μ from our notations.

As in the last section, let X_ν denote the partial flag variety G/P_ν, and let $T \subset G$ be the diagonal subtorus acting on X_ν. Let $S := \mathrm{Sym}\, \mathfrak{t}^* \cong H^*_T(pt)$, and let $\widehat{S} := \prod_{i=1}^\infty S_i$ be the completion of S at the graded maximal ideal. For an element $w \in W/W_\nu$, let $C_w \subset X_\nu$ denote the B-orbit containing the T-fixed point p_w. Let $\widehat{Z}$ be the center of $\widehat{A}_\nu$.

Theorem 10.5. *We have $\widehat{S}$-algebra isomorphisms*

$$H^*_T(X_\nu) \otimes_S \widehat{S} \;\cong\; \widehat{Z} \;\cong\; \mathrm{End}_{\widehat{\mathcal{O}}_\nu}(\widehat{P}_0),$$

where $\widehat{P}_0 := \widehat{P}(w_0\alpha_\nu)$ is the antidominant projective. The functor

$$\mathbb{V} = \mathrm{Hom}_{\widehat{\mathcal{O}}_\nu}(\widehat{P}_0, -) \colon \widehat{\mathcal{O}}_\nu \to \widehat{Z} - \mathrm{mod}$$

is full and faithful on objects with a Verma flag (in particular, on projectives) and we have natural $\widehat{Z}$-module isomorphisms

$$\mathbb{V}\widehat{P}(w\alpha_\nu) \cong \mathrm{IH}^*_T(\overline{C_w}) \otimes_S \widehat{S} \quad \text{and} \quad \mathbb{V}\widehat{M}(w\alpha_\nu) \cong H^*_T(C_w) \otimes_S \widehat{S}$$

for all $w \in W/W_\nu$.

Proof. The identification of the center of $\widehat{A}_\nu$ is accomplished in [Soe90, Theorem 9] and [Fie03, 3.6]. Fiebig's proof is instructive from our point of view: he shows that the map

$$\widehat{Z} \to \bigoplus_{w \in W/W_\nu} \mathrm{End}_{\widehat{\mathcal{O}}_\nu}\left(\widehat{M}(w\alpha)\right) \;\cong\; \bigoplus_{w \in W/W_\nu} \widehat{S} \;\cong\; H^*_T(X^T_\nu) \otimes_S \widehat{S}$$

is an injection, and the relations cutting out the image are the same as those that describe the image $H^*_T(X_\nu) \to H^*_T(X^T_\nu)$ in terms of the T-invariant curves in X_ν [GKM98, 1.2.2]. Since tensoring with $\widehat{S}$ is exact for graded S-modules, this gives the first isomorphism in Theorem 10.5. It also gives the identification of $\mathbb{V}\widehat{M}(w\alpha_\nu)$ with $H^*_T(C_w) \otimes_S \widehat{S}$.

The full faithfulness of $\mathbb{V}$ is proven in [Fie08, 7.1]. Finally, [Fie08, 7.6] identifies $\mathbb{V}\widehat{P}(w\alpha_\nu)$ with the completion of sections of a sheaf on a "moment graph" constructed from the zero and one-dimensional orbits of X_ν. By [BM01, §2] this gives exactly $\mathrm{IH}^*_T(\overline{C_w}) \otimes_S \widehat{S}$. □

Corollary 10.6. *We have $\widehat{S}$-algebra isomorphisms*

$$\widehat{A}_\nu \;\cong\; \mathrm{End}_{\widehat{Z}}\left(\bigoplus_{w \in W/W_\nu} \mathbb{V}\widehat{P}(w\alpha_\nu)\right) \;\cong\; \mathrm{End}_{H^*_T(X)}\left(\bigoplus_{w \in W/W_\nu} \mathrm{IH}^*_T(\overline{C_w})\right) \otimes_S \widehat{S}.$$

This gives our formal grading of $\widehat{A}_\nu$ in the non-parabolic case: the i^{th} graded piece consists of maps which increase degree of the intersection cohomology groups on the right by i. The grading in the general case now arises from the following proposition.

Proposition 10.7. *Let μ, ν be arbitrary compositions of n. There is a surjective map from $\widehat{A}_\nu$ to $\widehat{A}^\mu_\nu$, with kernel generated by the idempotents $\{e_w \mid w \in \mathcal{I}_\nu \setminus \mathcal{I}^\mu_\nu\}$ and the kernel of the projection $\mathfrak{h} \to \mathfrak{h}^\mu$.*

Remark 10.8. In fact, the kernel is generated by the idempotents alone, but we do not need this, and will not prove it.

Proof. We construct a truncation functor $\hat{\tau}\colon \widehat{\mathcal{O}}_\nu \to \widehat{\mathcal{O}}^\mu_\nu$ which is the deformed analogue of the functor that takes the maximal $\mathfrak{p}$-locally finite quotient of objects in category $\mathcal{O}$. We do this in two steps. First, for an object $M \in \widehat{\mathcal{O}}_\nu$, define $M^\mu := M \otimes_{\widehat{S}} \widehat{S}^\mu$, the image of M under the functor $\widehat{\mathcal{O}} \to \mathcal{O}_{\widehat{S}^\mu}$. Next, define

$$Q \;:=\; \bigoplus_{w \in \mathcal{I}_\nu \setminus \mathcal{I}^\mu_\nu} \widehat{P}(w\alpha)^\mu,$$

and define $\hat{\tau}M$ to be the cokernel of the natural map $\mathrm{Hom}_{\mathcal{O}_{\widehat{S}^\mu}}(Q, M^\mu) \otimes_{\widehat{S}^\mu} Q \to M^\mu$. It is the largest quotient of M^μ which contains no subquotients isomorphic to any simple object $L(w\alpha)^\mu$ with $w \in \mathcal{I}_\nu \setminus \mathcal{I}_\nu^\mu$.

It follows that this functor does indeed send $\widehat{\mathcal{O}}_\nu$ to $\widehat{\mathcal{O}}_\nu^\mu$. It is not hard to see that $\hat{\tau}$ is right exact and left adjoint to the inclusion $\iota\colon \widehat{\mathcal{O}}_\nu^\mu \to \widehat{\mathcal{O}}_\nu$, so it sends projectives to projectives, and in fact sends a projective generator of $\widehat{\mathcal{O}}_\nu$ to a projective generator of $\widehat{\mathcal{O}}_\nu^\mu$. This gives a natural homomorphism $\widehat{A}_\nu \to \widehat{A}_\nu^\mu$, which clearly contains the ideal described in the statement of the theorem. It is also surjective, since the adjunction map $M \to \iota\hat{\tau}M$ is surjective for any M. Our description of the kernel now follows from the characterization of $\hat{\tau}M$ in the previous paragraph. □

Let $\tilde{A}_\nu^\mu$ be the universal deformation of A_ν^μ. Theorem 4.1, Remark 4.2, and Proposition 10.7 now imply the following result.

Theorem 10.9. *There exists a linear map $\psi_\nu^\mu\colon Z((A_\nu^\mu)^!)_2^* \to \mathfrak{h}^\mu$, inducing a graded ring homomorphism $S_\nu^\mu \to \widehat{S}^\mu$, such that we have an $\widehat{S}^\mu$-algebra isomorphism*

$$\widehat{A}_\nu^\mu \;\cong\; \tilde{A}_\nu^\mu \otimes_{S_\nu^\mu} \widehat{S}^\mu.$$

By itself, Theorem 10.9 doesn't help us understand the universal deformation. For instance, if it turned out that $\psi_\nu^\mu = 0$, then we would have $\widehat{A}_\nu^\mu \cong A_\nu^\mu \otimes \widehat{S}^\mu$. However, the following result implies that $\widehat{A}_\nu^\mu$ carries all the information of the universal deformation.

Proposition 10.10. *The map ψ_ν^μ of Theorem 10.9 is injective. As a result, we have an S_ν^μ-algebra isomorphism*

$$\tilde{A}_\nu^\mu \;\cong\; \bigoplus_{i \geq 0} \left(\widehat{A}_\nu^\mu\right)_i \otimes_{S^\mu} S_\nu^\mu,$$

where the map $S^\mu \to S_\nu^\mu$ comes from any left inverse of ψ_ν^μ.

We postpone the proof of Proposition 10.10 until we have established some further properties of the deformation $\widehat{A}_\nu^\mu$. Note that since the maps $\widehat{A}_\nu \to \widehat{A}_\nu^\mu \to A_\nu^\mu$ are surjective, they induce maps between the centers of these algebras.

Lemma 10.11. *The maps $Z(\widehat{A}_\nu) \to Z(\widehat{A}_\nu^\mu) \to Z(A_\nu^\mu)$ are surjective.*

Proof. By [Bru08b, Theorem 2], the action of the center of the enveloping algebra induces a surjection $Z(U(\mathfrak{g})) \to Z(A_\nu^\mu)$. It follows that $Z(U(\mathfrak{g})) \otimes \widehat{S}^\mu$ surjects onto $Z(\widehat{A}_\nu^\mu)$ and (as a special case) $Z(U(\mathfrak{g})) \otimes \widehat{S}$ surjects onto $Z(\widehat{A}_\nu)$. The result then follows from the surjectivity of the maps $Z(U(\mathfrak{g})) \otimes \widehat{S} \to Z(U(\mathfrak{g})) \otimes \widehat{S}^\mu \to Z(U(\mathfrak{g}))$. □

Remark 10.12. Proposition 10.10 and Lemma 10.11 together imply that the universal deformation $\tilde{A}_\nu^\mu$ is flexible in the sense of Section 6.

The center of $\widehat{A}_\nu^\mu$ acts on the deformed Verma module $\widehat{M}^\mu(w\alpha)$ by a character

$$h_w^\mu\colon Z(\widehat{A}_\nu^\mu) \to \mathrm{End}_{\mathcal{O}_\nu^\nu}(\widehat{M}^\mu(w\alpha)) \cong \widehat{S}^\mu.$$

When μ is regular, which we indicate as usual by omitting μ from the notation, Theorem 10.5 identifies h_w with the map $H_T^*(X_\nu) \otimes_S \widehat{S} \to H_T^*(p_w) \otimes_S \widehat{S} \cong \widehat{S}$ induced by restriction. In terms of the identification

$$H_T^*(X_\nu) \;\cong\; H_G^*(X_\nu) \otimes_{H_G^*(pt)} H_T^*(pt) \;\cong\; H_{P_\nu}^*(pt) \otimes_{H_G^*(pt)} H_T^*(pt) \;\cong\; S^{W_\nu} \otimes_{S^W} S,$$

we have $h_w(f \otimes g) = g \cdot w(f)$. In particular, in degree 2 we have

$$H_T^2(X) \cong \mathfrak{h}^\nu \oplus \mathfrak{h} \quad \text{and} \quad h_w(x, y) = y + w(x). \tag{9}$$

The characters of deformed parabolic Verma modules are determined by the non-parabolic characters via the following result.

Lemma 10.13. *The diagram*

$$\begin{tikzcd} Z(\widehat{A}_\nu) \arrow[r, "h_w"] \arrow[d] & \widehat{S} \arrow[d] \\ Z(\widehat{A}_\nu^\mu) \arrow[r, "h_w^\mu"] & \widehat{S}^\mu \end{tikzcd}$$

commutes, where the left vertical map is the natural projection.

Proof. Using the proof of Proposition 10.7, we see that the deformed parabolic Verma module $\widehat{M}^\mu(w\alpha) \cong \hat{\tau}\widehat{M}(w\alpha)$ is a quotient of $\widehat{M}(w\alpha) \otimes_{\widehat{S}} \widehat{S}^\mu$. □

Remark 10.14. Proposition 10.10, Lemma 10.13, and our formula for the characters h_w in Equation (9) together imply that the universal deformation $\tilde{A}_\nu^\mu$ is malleable in the sense of Section 5.

We now use these calculations to determine the center of $\widehat{A}_\nu^\mu$. This result, along with Proposition 10.10, will complete the proof of Theorem 9.9.

Theorem 10.15. *The formal localization algebra $\mathcal{Z}(\widehat{A}_\nu^\mu)$ is isomorphic to $H_{T^\mu}^*(X_\nu^\mu) \otimes \widehat{S}^\mu$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} Z(\widehat{A}_\nu) \arrow[r, "\oplus h_w"] \arrow[d] & \bigoplus_{w \in \mathcal{I}_\nu} \widehat{S} \arrow[d, "q"] \\ Z(\widehat{A}_\nu^\mu) \arrow[r, "\oplus h_w^\mu"] & \bigoplus_{w \in \mathcal{I}_\nu^\mu} \widehat{S}^\mu \end{tikzcd}$$

where q is the quotient map $\widehat{S} \to \widehat{S}^\mu$ for all $w \in \mathcal{I}_\nu^\mu$, and kills all terms for $w \notin \mathcal{I}_\nu^\mu$. The horizontal maps are injective by Equation (9) and Lemma 10.13, while the left vertical map is surjective by Lemma 10.11. Thus we have an isomorphism $Z(\widehat{A}_\nu^\mu) \cong q(\operatorname{Im} \oplus h_w)$.

This diagram has a topological analogue:

$$\begin{tikzcd} H^*_T(X_\nu)\otimes_S \widehat{S} \arrow[r, "\iota^*"] \arrow[d] & \bigoplus_{w\in\mathcal{I}_\nu} \widehat{S} \arrow[d, "q"] \\ H^*_{T^\mu}(X^\mu_\nu)\otimes_{S^\mu} \widehat{S}^\mu \arrow[r, "(\iota^\mu)^*"] & \bigoplus_{w\in\mathcal{I}^\mu_\nu} \widehat{S}^\mu \end{tikzcd}$$

The left vertical map is the composition of the restriction to the subtorus T^μ with the restriction to X^μ. The maps ι^* and $(\iota^\mu)^*$ are the restrictions to the fixed point sets, which are indexed as indicated by Proposition 9.5.

Both X_ν and X^μ_ν have vanishing odd cohomology, so they are equivariantly formal, which implies that the horizontal maps are injections. Brundan [Bru08a] shows that the restriction from $H^*(X)$ to $H^*(X^\mu)$ is surjective; this and equivariant formality imply that the left vertical map is surjective, so we have an isomorphism $H^*_{T^\mu}(X^\mu_\nu)\cong q(\operatorname{Im}\iota^*)$. We have already seen in the proof of Proposition 10.5 that ι^* is identified with $\oplus h_w$, so this proves the theorem. $\square$

By Lemma 10.13 the image of (x,y) under the composition

$$\mathfrak{h}^\nu\oplus\mathfrak{h}\cong Z(\widehat{A}_\nu)_2\to Z(\widehat{A}^\mu_\nu)_2\to \operatorname{End}(\widehat{M}^\mu(w\alpha))$$

is multiplication by $(y+w(x))^\mu\in\mathfrak{h}^\mu\subset\widehat{S}^\mu$. It follows from [Bru08b, Theorem 2] that any two simples in $\mathcal{O}^\mu_\nu$ can be connected by a chain of non-trivial Ext^1 groups, so by Lemma 6.5 we see that the image of ψ^μ_ν contains

$$\Sigma:=\{(w(\alpha_\nu)-v(\alpha_\nu))^\mu\mid v,w\in\mathcal{I}^\mu_\nu\}\subset\mathfrak{h}^\mu.$$

Using this, we can finally prove Proposition 10.10.

Proof of Proposition 10.10. We have $\dim\operatorname{Span}(\Sigma)\le\operatorname{rank}\psi^\mu_\nu\le\dim Z(A^{\nu^o}_\mu)_2$, so it will be enough to show that $\dim\operatorname{Span}(\Sigma)=\dim Z(A^{\nu^o}_\mu)_2$.

As we noted in Remark 9.10, Brundan [Bru08a] shows that $Z(A^{\nu^o}_\mu)_2\cong\mathfrak{t}^\mu_\nu$. Lemma 9.7 shows that the pairing between $\mathfrak{h}^\mu$ and $\mathfrak{t}^\mu$ induces a well-defined pairing between Σ (a subspace of $\mathfrak{h}^\mu$) and $\mathfrak{t}^\mu_\nu$ (a quotient of $\mathfrak{t}^\mu$). We will show that this pairing is non-degenerate.

As we saw in the proof of Lemma 9.7, the set $\{w(\alpha_\nu)\mid w\in\mathcal{I}^\mu_\nu\}$ is the set of all vectors in $\mathbb{Z}^n$ which have ν_{-i} entries equal to i for all $i\in\mathbb{Z}$, and for which the entries in each μ-block are strictly decreasing. It follows that if the μ-blocks are reordered, the effect on the set Σ is just to apply the appropriate permutation to each element. The same holds for $\mathfrak{t}^\mu_\nu$, so without loss of generality we can assume that $\mu=\mu^+$. It is also easy to see that we can take $\nu=\nu^+$.

We must show that every element in $\mathfrak{t}^\mu$ which pairs to zero with Σ must lie in the span of

$\{\mathbf{1}_J \mid J \in \mathcal{J}\}$. To do this, let $k_1 < \cdots < k_r$ be the solutions k to the equation

$$\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k,$$

and let J_i be the union of the first k_i μ-blocks. Then Lemma 9.7 says that for any vector $w\alpha_\nu$ and any i the multiset of entries indexed by $j \in J_i \setminus J_{i-1}$ is independent of w. It is also clear that the entries in each $J_i \setminus J_{i-1}$ can be chosen independently. It follows that without loss of generality we can assume that $\mathcal{J} = \{\{1, \dots, n\}\}$.

We make this assumption, and proceed by induction on the number of nonzero entries in μ. If there is only one, then $\mathfrak{t}_\nu^\mu = 0$ and we are done. Otherwise, consider filling the μ-blocks with entries of α_ν, starting with the left-most block first. If we fill the first block with the entries $1, \dots, \mu_1$, then the remaining blocks give an element of $\{w(\alpha_{\nu'}) \mid w \in \mathcal{I}_{\nu'}^{\mu'}\}$ for the pair

$$\mu' = (\mu_2, \mu_3, \dots) \text{ and } \nu' = (\nu_1 - 1, \dots, \nu_{\mu_1} - 1, \nu_{\mu_1+1}, \dots)^+.$$

We can describe the transpose partition $\lambda' := (\nu')^t$ as follows: if m is the unique integer such that $\lambda_m \geq \mu_1$ and $\lambda_{m+1} < \mu_1$, then

$$\lambda' = (\lambda_1, \dots, \lambda_{m-1}, \lambda_{m+1} + \lambda_m - \mu_1, \lambda_{m+2}, \dots).$$

Note that for these new partitions, we have $\mu' < \lambda'$ and $\mathcal{J} = \{\{1, \dots, n\}\}$

$$(\lambda_1' + \cdots + \lambda_k') = (\lambda_1 + \cdots + \lambda_k) > (\mu_1 + \cdots + \mu_k) \geq (\mu_2 + \cdots + \mu_{k+1}) = (\mu_1' + \cdots + \mu_k') \quad \text{if } k < m$$

and

$$(\lambda_1' + \cdots + \lambda_k') = (\lambda_1 + \cdots + \lambda_{k+1} - \mu_1) > (\mu_2 \cdots + \mu_{k+1}) = (\mu_1' + \cdots + \mu_k') \quad \text{if } k \geq m.$$

Only using permutations that keep the first block fixed, we obtain an inclusion $\Sigma_{\nu'}^{\mu'} \hookrightarrow \Sigma_\nu^\mu$, and by our inductive hypothesis, no element of the span of this subset is annihilated by $\mathfrak{t}_\nu^\mu$, which pairs via the surjective quotient map $\mathfrak{t}_\nu^\mu \to \mathfrak{t}_{\nu'}^{\mu'}$.

The kernel of this map is one dimensional, spanned by the element $\mathbf{1}_{[1,\mu_1]}$ which has all 1's on the first block and 0's elsewhere. So by the inductive hypothesis, we only need to find an element of Σ which pairs non-trivially with this vector. That is, we must find must find vectors $w(\alpha_\nu)$ and $v(\alpha_\nu)$ which have different entries in the first block. Rather than construct the whole vector, we note that a choice of entries in the first block can be extended to a vector of the form $w(\alpha_\nu)$ if and only if the remaining partitions μ'' and ν'' still satisfy the dominance condition $(\mu'')^+ \leq \lambda''$.

We have already noted that we can take the entries in our first block to be $1, \dots, \mu_1$. We claim that $1, \dots, \mu_1 - 1, \mu_1 + 1$ will also extend to a vector of the form $v(\alpha_\nu)$. This will finish the proof, since the difference of these vectors will pair non-trivially with $\mathbf{1}_{[1,\mu_1]}$, and thus establish non-degeneracy.

In this case, we have

$$\nu'' = (\nu_1 - 1, \ldots, \nu_{\mu_1 - 1} - 1, \nu_{\mu_1}, \nu_{\mu_1+1} - 1, \nu_{\mu_1+2}, \ldots)^+.$$

If $\nu_{\mu_1} = \nu_{\mu_1+1}$, then $\nu' = \nu''$, so $(\mu'')^+ \leq \lambda'' = \lambda'$ and we are done. Otherwise, we find that

$$(\nu'')^t = (\lambda_1, \ldots, \lambda_{m-2}, \lambda_{m-1} - 1, \lambda_{m+1} + \lambda_m - \mu_1 + 1, \lambda_{m+2}, \ldots).$$

Since $\lambda_1 + \cdots + \lambda_{m-1} > \mu_1 + \cdots + \mu_{m-1}$ by assumption, we have $\mu \leq (\nu'')^t$, and so it is possible to continue filling the remaining blocks. □

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