STABILITY OF VOLUME COMPARISON FOR COMPLEX CONVEX BODIES

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Abstract. We prove the stability of the affirmative part of the solution to the complex Busemann-Petty problem. Namely, if K and L are origin-symmetric convex bodies in $\mathbb{C}^n$, $n=2$ or $n=3$, $\varepsilon>0$ and $\mathrm{Vol}_{2n-2}(K\cap H)\le \mathrm{Vol}_{2n-2}(L\cap H)+\varepsilon$ for any complex hyperplane H in $\mathbb{C}^n$, then $(\mathrm{Vol}_{2n}(K))^{\frac{n-1}{n}} \le (\mathrm{Vol}_{2n}(L))^{\frac{n-1}{n}}+\varepsilon$, where Vol_{2n} is the volume in $\mathbb{C}^n$, which is identified with $\mathbb{R}^{2n}$ in the natural way.

1. Introduction

The Busemann-Petty problem, posed in 1956 (see [BP]), asks the following question. Suppose that K and L are origin symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{Vol}_{n-1}(K\cap H)\le \mathrm{Vol}_{n-1}(L\cap H)$$

for every hyperplane H in $\mathbb{R}^n$ containing the origin. Does it follow that

$$\mathrm{Vol}_n(K)\le \mathrm{Vol}_n(L)?$$

The answer is affirmative if $n\le 4$ and negative if $n\ge 5$. The solution was completed in the end of the 90's as the result of a sequence of papers [LR], [Ba], [Gi], [Bo], [L], [Pa], [G1], [G2], [Z1], [Z2], [K1], [K2], [Z3], [GKS] ; see [K3, p. 3] or [G3, p. 343] for the history of the solution.

The complex version of the Busemann-Petty problem was solved in [KKZ], the answer is affirmative for convex bodies in $\mathbb{C}^n$ when $n\le 3$, and it is negative for $n\ge 4$. To formulate the complex version, we need several definitions.

For $\xi\in\mathbb{C}^n,\ |\xi|=1$, denote by

$$H_\xi=\{z\in\mathbb{C}^n:\ (z,\xi)=\sum_{k=1}^n z_k\overline{\xi_k}=0\}$$

the complex hyperplane through the origin perpendicular to ξ.

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Origin symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K :

$$K = \{z \in \mathbb{C}^n : \ \|z\|_K \le 1\}.$$

In order to define volume, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the mapping

$$\xi = (\xi_1, ..., \xi_n) = (\xi_{11} + i\xi_{12}, ..., \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, ..., \xi_{n1}, \xi_{n2}).$$

Under this mapping the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$.

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda|\|z\|, \quad \forall z \in \mathbb{C}^n, \ \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $\xi = (\xi_{11}, \xi_{12}, ..., \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), ..., R_\theta(\xi_{n1}, \xi_{n2})\|_K, \tag{1}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We shall simply say that K *is invariant with respect to all* R_θ if it satisfies (1).

The complex Busemann-Petty problem can be formulated as follows: suppose K and L are origin symmetric invariant with respect to all R_θ convex bodies in $\mathbb{R}^{2n}$ such that

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) \le \mathrm{Vol}_{2n-2}(L \cap H_\xi)$$

for each ξ from the unit sphere S^{2n-1} of $\mathbb{R}^{2n}$. Does it follow that

$$\mathrm{Vol}_{2n}(K) \le \mathrm{Vol}_{2n}(L)?$$

As mentioned above, the answer is affirmative if and only if $n \le 3$. In this article we prove the stability of the affirmative part of the solution:

Theorem 1. *Suppose that $\varepsilon > 0$, K and L are origin-symmetric invariant with respect to all R_θ convex bodies bodies in $\mathbb{R}^{2n}$, $n = 2$ or $n = 3$. If for every $\xi \in S^{2n-1}$*

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) \le \mathrm{Vol}_{2n-2}(L \cap H_\xi) + \varepsilon, \tag{2}$$

then

$$\mathrm{Vol}_{2n}(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_{2n}(L)^{\frac{n-1}{n}} + \varepsilon.$$

The result does not hold for $n > 3$, simply because the answer to the complex Busemann-Petty problem in these dimensions is negative; see [KKZ].

It immediately follows from Theorem 1 that

Corollary 1. *If $n=2$ or $n=3$, then for any origin-symmetric invariant with respect to all R_θ convex bodies K, L in $\mathbb{R}^{2n}$,*

$$\left|\mathrm{Vol}_{2n}(K)^{\frac{n-1}{n}} - \mathrm{Vol}_{2n}(L)^{\frac{n-1}{n}}\right|$$
$$\leq \max_{\xi\in S^{2n-1}} \left|\mathrm{Vol}_{2n-2}(K\cap H_\xi) - \mathrm{Vol}_{2n-2}(L\cap H_\xi)\right|.$$

Note that stability in comparison problems for volumes of convex bodies was studied in [K5], where it was proved for the original (real) Busemann-Petty problem.

For other results related to the complex Busemann-Petty problem see [R], [Zy1], [Zy2].

2. Proofs

We use the techniques of the Fourier approach to sections of convex bodies; see [K3] and [KY] for details.

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$ for every test function ϕ from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$.

If K is a convex body and $0<p<n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on $\mathbb{R}^n$ and represents a distribution. Suppose that K is infinitely smooth, i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$ is an infinitely differentiable function on the sphere. Then by [K3, Lemma 3.16], the Fourier transform of $\|\cdot\|_K^{-p}$ is an extension of some function $g\in C^\infty(S^{n-1})$ to a homogeneous function of degree $-n+p$ on $\mathbb{R}^n$. When we write $\left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)$, we mean $g(\xi)$, $\xi\in S^{n-1}$. If K, L are infinitely smooth star bodies, the following spherical version of Parseval's formula was proved in [K4] (see [K3, Lemma 3.22]): for any $p\in(-n,0)$

$$\int_{S^{n-1}} \left(\|\cdot\|_K^{-p}\right)^\wedge(\xi)\left(\|\cdot\|_L^{-n+p}\right)^\wedge(\xi) = (2\pi)^n\int_{S^{n-1}} \|x\|_K^{-p}\|x\|_L^{-n+p}\,dx. \tag{3}$$

A distribution is called *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi\rangle \geq 0$ for every non-negative test function ϕ.

The Fourier transform formula for the volume of complex hyperplane sections was proved in [KKZ]:

Proposition 1. *Let K be an infinitely smooth origin symmetric invariant with respect to R_θ convex body in $\mathbb{R}^{2n}, n\geq 2$. For every $\xi\in S^{2n-1}$, we have*

$$\mathrm{Vol}_{2n-2}(K\cap H_\xi) = \frac{1}{4\pi(n-1)}\left(\|\cdot\|_K^{-2n+2}\right)^\wedge(\xi). \tag{4}$$

We also use the result of Theorem 3 from [KKZ]. It is formulated in [KKZ] in terms of embedding in L_{-p}, which is equivalent to our formulation below. However, the reader does not need to worry about embeddings in L_{-p}, because the proof of Theorem 3 in [KKZ] directly establishes the following:

Proposition 2. *Let $n \geq 3$. For every origin symmetric invariant with respect to R_θ convex body K in $\mathbb{R}^{2n}$, the function $\|\cdot\|_K^{-2n+4}$ represents a positive definite distribution.*

Let us formulate precisely what we are going to use later. The case $n = 2$ follows from Proposition 1 (obviously, the volume is positive), the case $n = 3$ is immediate from Proposition 2.

Corollary 2. *If $n = 2$ or $n = 3$, then for every origin symmetric infinitely smooth invariant with respect to R_θ convex body K in $\mathbb{R}^{2n}$, $\left(\|\cdot\|_K^{-2}\right)^\wedge$ is a non-negative infinitely smooth function on the sphere S^{2n-1}.*

We need the following simple fact:

Lemma 1. *For every $n \in \mathbb{N}$,*

$$(\Gamma(n))^{\frac{1}{n}} \leq n^{\frac{n-1}{n}}.$$

Proof : By log-convexity of the Γ-function (see [K3, p.30]),

$$\frac{\log(\Gamma(n+1)) - \log(\Gamma(1))}{n} \geq \frac{\log(\Gamma(n)) - \log(\Gamma(1))}{n-1},$$

so

$$(\Gamma(n+1))^{\frac{n-1}{n}} \geq \Gamma(n).$$

Now note that $\Gamma(n+1) = n\Gamma(n)$.

□

The polar formula for the volume of a convex body K in $\mathbb{R}^{2n}$ reads as follows (see [K3, p.16]):

$$\mathrm{Vol}_{2n}(K) = \frac{1}{2n} \int_{S^{2n-1}} \|x\|_K^{-2n} dx. \tag{5}$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. By the approximation argument of [S, Th. 3.3.1] (see also [GZ]), we may assume that the bodies K and L are infinitely smooth. Using [K3, Lemma 3.16] we get in this case that the Fourier transforms $\left(\|\cdot\|_K^{-2n+2}\right)^\wedge, \left(\|\cdot\|_L^{-2n+2}\right)^\wedge, \left(\|\cdot\|_K^{-2}\right)^\wedge$ are the extensions of infinitely differentiable functions on the sphere to homogeneous functions on $\mathbb{R}^{2n}$.

By (4), the condition (2) can be written as

$$\left(\|\cdot\|_K^{-2n+2}\right)^\wedge(\xi) \le \left(\|\cdot\|_L^{-2n+2}\right)^\wedge(\xi) + 4\pi(n-1)\varepsilon$$

for every $\xi \in S^{2n-1}$. Integrating both sides with respect to a non-negative (by Corollary 2) density, we get

$$\int_{S^{2n-1}} \left(\|\cdot\|_K^{-2n+2}\right)^\wedge(\xi) \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi$$
$$\le \int_{S^{2n-1}} \left(\|\cdot\|_L^{-2n+2}\right)^\wedge(\xi) \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi$$
$$+ 4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi.$$

By the Parseval formula (3) applied twice,

$$(2\pi)^n \int_{S^{2n-1}} \|x\|_K^{-2n}dx \le (2\pi)^n \int_{S^{2n-1}} \|x\|_L^{-2n+2}\|x\|_K^{-2}dx$$
$$+ 4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi.$$

Estimating the first summand in the right-hand side of the latter inequality by Hölder's inequality,

$$(2\pi)^n \int_{S^{2n-1}} \|x\|_K^{-2n}dx \le (2\pi)^n \left(\int_{S^{2n-1}} \|x\|_L^{-2n}dx\right)^{\frac{n-1}{n}} \left(\int_{S^{2n-1}} \|x\|_K^{-2n}dx\right)^{\frac{1}{n}}$$
$$+ 4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi.$$

and using the polar formula for the volume (5),

$$(2\pi)^n(2n)\mathrm{Vol}_{2n}(K) \le (2\pi)^n(2n)\left(\mathrm{Vol}_{2n}(L)\right)^{\frac{n-1}{n}}\left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}}$$
$$+ \ 4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi. \tag{6}$$

We now estimate the second summand in the right-hand side. First we use the formula for the Fourier transform (in the sense of distributions; see [GS, p.194])

$$\left(|\cdot|_2^{-2n+2}\right)^\wedge(\xi) = \frac{4\pi^n}{\Gamma(n-1)},$$

where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^{2n}$ and $\xi \in S^{2n-1}$. We get

$$4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)d\xi$$
$$= \frac{4\pi(n-1)\Gamma(n-1)\varepsilon}{4\pi^n} \int_{S^{2n-1}} \left(\|\cdot\|_K^{-2}\right)^\wedge(\xi)\left(|\cdot|_2^{-2n+2}\right)^\wedge(\xi)d\xi,$$

and by Parseval's formula (3) and Hölder's inequality,

$$= \frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \int_{S^{2n-1}} \|x\|_K^{-2} dx$$

$$\leq \frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \left(\int_{S^{2n-1}} \|x\|_K^{-2n} dx \right)^{\frac{1}{n}} \left| S^{2n-1} \right|^{\frac{n-1}{n}},$$

where $|S^{2n-1}| = (2\pi^n)/\Gamma(n)$ is the surface area of the unit sphere in $\mathbb{R}^{2n}$. By the polar formula for the volume, the latter is equal to

$$(2\pi)^n (2n) \varepsilon \left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}} \frac{(\Gamma(n))^{\frac{1}{n}}}{n^{\frac{n-1}{n}}} \leq (2\pi)^n (2n) \varepsilon \left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}}$$

by Lemma 1. Combining this with (6), we get the result. □

We finish with the following "separation" property (see [K5] for more results of this kind). Note that for any $x \in S^{2n-1}$, $\|x\|_K^{-1} = \rho_K(x)$ is the radius of K in the direction x, and denote by

$$r(K) = \frac{\min_{x \in S^{2n-1}} \rho_K(x)}{\left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{2n}}}$$

the normalized inradius of K. Clearly, for every $x \in S^{2n-1}$ we have

$$\|x\|_K^{-1} \geq r(K) \left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{2n}}.$$

Theorem 2. *Suppose that $\varepsilon > 0$, K and L are origin-symmetric invariant with respect to all R_θ convex bodies bodies in $\mathbb{R}^{2n}$, $n = 2$ or $n = 3$. If for every $\xi \in S^{2n-1}$*

$$\mathrm{Vol}_{2n-2}(K \cap H_\xi) \leq \mathrm{Vol}_{2n-2}(L \cap H_\xi) - \varepsilon,$$

then

$$\mathrm{Vol}_{2n}(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_{2n}(L)^{\frac{n-1}{n}} - \frac{\pi r^2(K)}{n} \varepsilon.$$

Proof : We follow the lines of the proof of Theorem 1 to get

$$(2\pi)^n (2n) \mathrm{Vol}_{2n}(K) \leq (2\pi)^n (2n) \left(\mathrm{Vol}_{2n}(L)\right)^{\frac{n-1}{n}} \left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}}$$

$$- \ 4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\| \cdot \|_K^{-2}\right)^\wedge (\xi) d\xi. \tag{7}$$

We now need a lower estimate for

$$4\pi(n-1)\varepsilon \int_{S^{2n-1}} \left(\| \cdot \|_K^{-2}\right)^\wedge (\xi) d\xi.$$

Similarly to how it was done in Theorem 1, we write the latter as

$$\frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \int_{S^{2n-1}} \|x\|_K^{-2} dx \geq \frac{(2\pi)^n \varepsilon \Gamma(n) r^2(K) \left(\mathrm{Vol}_{2n}(K)\right)^{\frac{1}{n}}}{\pi^{n-1}} \left| S^{2n-1} \right|. \quad \square$$

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