SCHUBERT CALCULUS FOR ALGEBRAIC COBORDISM

JENS HORNBOSTEL AND VALENTINA KIRITCHENKO

Abstract. We establish a Schubert calculus for Bott-Samelson resolutions in the algebraic cobordism ring of a complete flag variety G/B extending the results of Bressler–Evens [4] to the algebro-geometric setting .

1. Introduction

We fix a base field k of characteristic 0. Algebraic cobordism $\Omega^*(-)$ has been invented some years ago by Levine and Morel [14] as the universal oriented algebraic cohomology theory on smooth varieties over k. In particular, its coefficient ring $\Omega^*(k)$ is isomorphic to the Lazard ring $\mathbb{L}$ (introduced in [12]). In a recent article [15], Levine and Pandharipande show that algebraic cobordism $\Omega^n(X)$ allows a presentation with generators being projective morphisms $Y \to X$ of relative codimension $n(:= \dim(X) - \dim(Y))$ between smooth varieties and relations given by a refinement of the naive algebraic cobordism relation (involving double point relations). A recent result of Levine [13] which relies on unpublished work of Hopkins and Morel asserts an isomorphism $\Omega^n(-) \cong MGL^{2n,n}(-)$ between Levine-Morel and Voevodsky algebraic cobordism for smooth quasiprojective varieties. In particular, algebraic cobordism is representable in the motivic stable homotopy category.

In short, algebraic cobordism is to algebraic varieties what complex cobordism $MU^*(-)$ is to topological manifolds.

The above fundamental results being established, it is high time for computations, which have been carried out only in a very small number of cases (see e.g. [22] and [23]). The present article focuses on cellular varieties X, for which the additive structure of $\Omega^*(X)$ is easy to describe: it is the free $\mathbb{L}$-module generated by the cells (see the next section for more precise definitions, statements, proofs and references). So additively, algebraic cobordism for cellular varieties behaves exactly as Chow groups do. Of course, algebraic K-theory also behaves in a similar way, but we will restrict our comparisons here and below to Chow groups. There is a ring homomorphism $\Omega^*(X) \to MU^{2*}(X(\mathbb{C})^{an})$ which for cellular varieties is an isomorphism, see Section 2.2 and the appendix. However, computations in $\Omega^*(X)$ become more transparent and suitable for algebro-geometric applications if they are done by algebro-geometric methods rather than by a translation of the already existing results for $MU^{2*}(X(\mathbb{C})^{an})$ (e. g. those of Bressler and Evens, see [4] and below), especially if the latter were obtained by topological methods which do not have counterparts in algebraic geometry.

Let us concentrate on complete flag varieties $X = G/B$ where B is a Borel subgroup of a connected split reductive group G over k. In the case where $G = GL_n(k)$, the cobordism ring

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$\Omega^*(X)$ may be described as the quotient of a free polynomial ring over $\mathbb{L}$ with generators x_i being the first Chern classes of certain line bundles on X and explicit relations. More precisely, we show (see Theorem 2.6):

Theorem 1.1. *The cobordism ring $\Omega^*(X)$ is isomorphic to the graded ring $\mathbb{L}[x_1, \dots, x_n]$ of polynomials with coefficients in the Lazard ring $\mathbb{L}$ and* $\deg x_i = 1$*, quotient by the ideal S generated by the homogeneous symmetric polynomials of strictly positive degree:*

$$\Omega^*(X) \simeq \mathbb{L}[x_1, \dots, x_n]/S.$$

This generalizes a theorem of Borel [2] on the Chow ring (or equivalently the singular cohomology ring) of a flag variety to its algebraic cobordism ring.

The Chow ring of the flag variety has a natural basis given by the *Schubert cycles*. The central problem in Schubert calculus was to find polynomials (later called Schubert polynomials) representing the Schubert cycles in the Borel presentation. This problem was solved independently by Bernstein–Gelfand–Gelfand [1] and Demazure [10] using *divided difference operators* on the Chow ring (most of the ingredients were already contained in a manuscript of Chevalley [8], which for many years remained unpublished). Explicit formulas for Schubert polynomials give an algorithm for decomposing the product of any two Schubert cycles into a linear combination of other Schubert cycles with integer coefficients.

The complex (as well as the algebraic) cobordism ring of the flag variety also has a natural generating set given by the *Bott-Samelson resolutions* of the Schubert cycles (note that the latter are not always smooth and so, in general, do not define any cobordism classes). For the complex cobordism ring, Bressler and Evens described the cobordism classes of Bott-Samelson resolutions in the Borel presentation using *generalized divided difference operators* on the cobordism ring [3, 4] (we thank Burt Totaro from whom we first learned about this reference). Their formulas for these operators are not algebraic and involve a passage to the classifying space of a compact torus in G and homotopy theoretic considerations (see [3, Corollary-Definition 1.9, Remark 1.11] and [4, Proposition 3]). One of the goals of the present paper is to prove an algebraic formula for the generalized divided difference operators (see Definition 2.2 and Corollary 2.3). This formula in turn implies explicit purely algebraic formulas for the polynomials (now with coefficients in the Lazard ring $\mathbb{L}$) representing the classes of Bott-Samelson resolutions. Note that each such polynomial contains the respective Schubert polynomial as the lowest degree term (but in most cases also has non-trivial higher order terms). We also give an algorithm for decomposing the product of two Bott-Samelson resolutions into a linear combination of other Bott-Samelson resolutions with coefficients in $\mathbb{L}$.

We now formulate our main theorem (compare Theorem 3.2), which can be viewed as an algebro-geometric analogue of the results of Bressler-Evens [4, Corollary 1, Proposition 3]. Let $I = (\alpha_1, \dots, \alpha_l)$ be an l-tuple of simple roots of G, and R_I the corresponding Bott-Samelson resolution of the Schubert cycle X_I (see Section 3 for the precise definitions). Recall that there is an isomorphism between the Picard group of the flag variety and the weight lattice of G such that very ample line bundles map to strictly dominant weights (see, for instance, [5, 1.4.3]). We denote by $L(\lambda)$ the line bundle on X corresponding to a weight λ, and by $c_1(L(\lambda))$ its first Chern class in algebraic cobordism. For each α_i, we define the operator A_i on $\Omega^*(X)$ in a purely algebraic way (see Section 3.2 for the rigorous definition for arbitrary reductive groups).

Informally, the operator A_i can be defined in the case $G = GL_n$ by the formula

$$A_i = (1 + \sigma_{\alpha_i})\frac{1}{c_1(L(\alpha_i))},$$

where σ_{α_i} acts on the variables $(x_1, \dots, x_n)$ by the transposition corresponding to α_i. Here we use that the Weyl group of GL_n can be identified with the symmetric group S_n so that the simple reflections s_{α_i} correspond to elementary transpositions (see Section 2 for more details). Note that the $c_1(L(\alpha_i))$ can be written explicitly as polynomials in x_1,...,x_n using the formal group law (see Section 2).

Theorem 1.2. *For any complete flag variety $X = G/B$ and any tuple $I = (\alpha_1, \dots, \alpha_l)$ of simple roots of G, the class of the Bott-Samelson resolution R_I in the algebraic cobordism ring $\Omega^*(X)$ is equal to*

$$A_l \dots A_1 R_e,$$

where R_e is the class of a point.

This theorem reduces the computation of the products of the geometric Bott-Samelson classes to the products in the polynomial ring given by the previous theorem. Note that in the cohomology case analogously defined operators A_i coincide with the *divided differences operators* defined in [1, 10], so our theorem generalizes [1, Theorem 4.1] and [10, Theorem 4.1] for Schubert cycles in cohomology and Chow ring, respectively, to Bott-Samelson classes in algebraic cobordism.

Note that in the case of Chow ring, the theorem analogous to Theorem 1.2 has two different proofs. A more algebraic proof using the Chevalley-Pieri formula was given by Bernstein–Gelfand–Gelfand ([1, Theorem 4.1], see also Section 4 for a short overview). Demazure gave a more geometric proof by identifying the divided difference operators with the push-forward morphism for certain Chow rings ([10, Theorem 4.1], see also Section 3). At first glance, it seems that the former proof is easier to extend to the algebraic cobordism. Indeed, we were able to extend the main ingredient of this proof, namely, the algebraic Chevalley–Pieri formula (see Proposition 4.3). However, the rest of the Bernstein–Gelfand–Gelfand argument fails for cobordism (see Section 4 for more details) while the more geometric argument of Demazure can be extended to cobordism with some extra work. For the complex cobordism ring this was done by Bressler and Evens [3, 4]. To describe the push-forward morphism they used results from homotopy theory, which are not (yet) applicable to algebraic cobordism. In our article, we also follow Demazure's approach. A key ingredient for extending this approach to algebraic cobordism is a formula for the push-forward in algebraic cobordism for projective line fibrations due to Vishik, see Proposition 2.1. We provide a new proof of this formula using the *double point relation* in cobordism introduced by Levine and Pandharipande [15]. In general, push-forwards (sometimes also called "transfers" or "Gysin homomorphisms") for algebraic cobordism are considerably more intricate than the ones for Chow groups. Consequently, their computation, which applies to any orientable cohomology theory, is more complicated.

Using the ring isomorphism $\Omega^*(X) \simeq MU^{2*}(X(\mathbb{C})^{an})$ for cellular varieties, it seems possible to deduce our Theorem 1.2 from the results of Bressler–Evens [3, 4] on complex cobordism (the main task would be to compare our algebraically defined operators A_i with theirs). We will not exploit this approach. Instead, all our proofs are purely algebraic or algebro-geometric. Conversely, we note that all our proofs concerning algebraic cobordism ring of the flag variety

(such as the proof of Proposition 4.3) may be easily translated to proofs for the analogue statements concerning the complex cobordism ring.

The article [4] does not contain any computations. It would be interesting to do some computation using their algorithm and then compare them with our approach, which we consider to be the easier one due to our explicit formula for the product of a Bott-Samelson class with the first Chern class (see formula 5.1) based on our algebraic Chevalley-Pieri formula. (Note also that the notations of [4] are essentially consistent with [1], but not always with [16]. We rather stick to the former than to the latter.)

This paper is organized as follows. In the next section, we give some further background on algebraic cobordism, in particular, the formula for the push-forward mentioned above. In the case of the flag variety for GL_n, we describe the multiplicative structure of its algebraic cobordism ring. In the third section, we recall the definition of Bott-Samelson resolutions and then express the classes of Bott-Samelson resolutions as polynomials with coefficients in the Lazard ring. Section 4 contains an algebraic Chevalley-Pieri formula and a short discussion of why the proof of [1] for singular cohomology does not carry over to algebraic cobordism. The final section contains an algorithm for computing the products of Bott-Samelson classes in terms of other Bott-Samelson classes as well as some examples and explicit computations.

Our main results are valid for the flag variety of an arbitrary reductive group G, but can be made more explicit in the case $G = GL_n$ using Borel presentation given by Theorem 2.6. So we will use the flag variety for GL_n as the main illustrating example whenever possible. One might conjecture that the algebraic cobordism rings of flag varieties with respect to other reductive groups G also allow a Borel presentation as polynomial rings over $\mathbb{L}$ in certain first Chern classes modulo the polynomials fixed by the appropriate Weyl groups (at least when passing to rational coefficients), because the corresponding statement is valid for singular cohomology resp. Chow groups (compare [2] resp. [9]).

After most of our preprint was finished, we learned that Calmès, Petrov and Zainoulline are also working on Schubert calculus for algebraic cobordism. It will be interesting to compare their results and proofs to ours (their preprint is now available, see [7]).

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2. Algebraic cobordism groups, push-forwards and cellular varieties

We briefly recall the geometric definition of algebraic cobordism [15] and some of its basic properties as established in [14]. For more details see [14, 15]. Recall that (up to sign) any element in the algebraic cobordism group $\Omega^n(X)$ for a scheme X (separated, of finite type over k) may be represented by a projective morphism $Y \to X$ with Y smooth and $n = \dim(X) - \dim(Y)$, the relations being the "double point relations", which we explain further below. In particular, $\Omega^*(X)$ only lives in degrees $\leq \dim X$, which we will use several times throughout the paper. Similar to the Chow ring CH^*, algebraic cobordism Ω^* is a functor on the category of smooth varieties over k, covariant for projective and contravariant for smooth and more generally lci morphisms, which allows a theory of Chern classes. However, the map from the Picard group of a smooth variety X to $\Omega^1(X)$ given by the first Chern class is neither a bijection nor a homomorphism anymore (unlike the corresponding map in the Chow ring

case). Its failure of being a group homomorphism is encoded in a *formal group law* that can be constructed from Ω^*. More precisely, any algebraic *orientable* cohomology theory allows by definition a calculus of Chern classes, and consequently the construction of a formal group law. A formal group law is a formal power series $F(x, y)$ in two variables such that for any two line bundles L_1 and L_2 we have the following identity relating their first Chern classes:

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

E. g. the formal group law for CH^* is additive, that is, $F(x+y) = x+y$. Algebraic cobordism is the universal one among the algebraic orientable cohomology theories. In what follows, $F(x, y)$ will always denote the universal formal group law corresponding to algebraic cobordism unless stated otherwise.

In this and in many other ways - as the computations below will illustrate - algebraic cobordism is a refinement of Chow ring, and one has a natural isomorphism of functors $\Omega^*(-) \otimes_{\mathbb{L}} \mathbb{Z} \cong CH^*(-)$ (see [14] where all these results are proved). Here and in the sequel, $\mathbb{L}$ denotes the Lazard ring, which classifies one-dimensional commutative formal group laws and is isomorphic to the graded polynomial ring $\mathbb{Z}[a_1, a_2, \ldots]$ in countably many variables [12], where we put a_i in degree $-i$. When considering polynomials $p(x_1, ...x_n)$ over $\mathbb{L}$ with $\deg(x_i) = 1$, we will distinguish the (total) *degree* and the *polynomial degree* of $p(x_1, ..., x_n)$.

Note that the Lazard ring is isomorphic to the algebraic (as well as complex) cobordism ring of a point. In particular, its elements can be represented by the cobordism classes of smooth varieties. In what follows, we use this geometric interpretation.

We are also going to use a geometric interpretation of the formal group law, namely, the *double point relation*. This is an equality for elements in the algebraic cobordism ring established in [15]. We recall the definition for the reader's convenience.

Double point relation:

Assume that we have three smooth hypersurfaces A, B and C on a smooth variety Z such that the following conditions hold

(1) C is linearly equivalent to $A + B$
(2) A, B and C have transverse pairwise intersections
(3) C does not intersect $A \cap B$

Then we have the following *double point relation*. Denote by D the intersection $A \cap B$. We have

$$[C \to Z] = [A \to Z] + [B \to Z] - [\mathbb{P}_D \to Z]$$

in $\Omega^*(X)$, where $\mathbb{P}_D = \mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) = \mathbb{P}_D(N_{B/D} \oplus \mathcal{O}_D)$ and the map $\mathbb{P}_D \to Z$ is the composition of the natural projection $\mathbb{P}_D \to D$ with the embedding $D \subset Z$. Here $N_{A/D}$ and $N_{B/D}$ are the normal bundles to D in A and B, respectively. The second condition ensures that $\mathbb{P}(\mathcal{O}_D \oplus N_{A/D}) = \mathbb{P}_D(N_{B/D} \oplus \mathcal{O}_D)$ (since $L(C)|_D = (L(A) \otimes L(B))|_D = N_{A/D} \otimes N_{B/D}$ is trivial).

This formulation is a special case of the extended double point relation in [15, Lemma 5.2]. The double point relation allows to express geometrically the discrepancy between the additive formal group law and the universal one. Namely, since $C = F(A, B)$ by the first condition, we get

$$A + B - F(A, B) = [\mathbb{P}_D \to Z].$$

We will use this equation when proving Proposition 2.1.

We will also use repeatedly the projective bundle formula, which we recall below for the reader's convenience. For more details see [14, Section 1.1] and [16, 3.5.2].

Projective bundle formula: Let $E \to X$ be a vector bundle of rank r over X. Denote by $Y = \mathbb{P}(E^*)$ the variety of hyperplanes of E, and by π the natural projection $\pi : Y \to X$. The variety $\mathbb{P}(E^*)$ is a fibration over X with fibers isomorphic to $\mathbb{P}^{r-1}$. Note that equivalently $\mathbb{P}(E^*)$ can be defined as the variety of one-dimensional quotients of E since there is a canonical isomorphism between the variety of hyperplanes and the variety of quotients by hyperplanes in a vector space. This is how $\mathbb{P}(E^*)$ is defined in [14, Section 1.1] (where it is denoted by $\mathbb{P}(E)$). Let $A^*(-)$ be any oriented cohomology theory. Denote by ξ the first Chern class of the tautological quotient line bundle $\mathcal{O}_E(1)$ on Y whose restriction on each fiber of Y over X coincides with $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. The first Chern can be defined as $\xi = s^*s_*(1_Y)$ where $s : Y \to \mathcal{O}_E(1)$ is the zero section and $1_Y \in A^0(Y)$ is the multiplicative unit element. Then there is a ring isomorphism:

$$A^*(Y) = A^*(X)[\xi]/(\sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j}).$$

The isomorphism identifies a polynomial $b_0 + b_1\xi + \ldots + b_{n-1}\xi^{n-1}$ in $A^*(X)[\xi]$ with the element $\pi^*b_0 + (\pi^*b_1)\xi + \ldots + (\pi^*b_{n-1})\xi^{n-1}$ in $A^*(Y)$. In particular, $A^*(Y)$ splits into the direct sum $\pi^*A^*(X) \oplus \xi\pi^*A^*(X) \oplus \ldots \oplus \xi^{n-1}\pi^*A^*(X)$.

Note that the relation

$$\sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j} = 0$$

admits the following alternative description. Consider a short exact sequence of vector bundles on Y:

$$0 \to \tau_E \to \pi^*E \to \mathcal{O}_E(1) \to 0,$$

where τ_E is the tautological hyperplane bundle on Y. By the Whitney sum formula we have that the total Chern class $c(\pi^*E)$ is equal to the product $c(\tau_E)c(\mathcal{O}_E(1))$. Since $c(\mathcal{O}_E(1)) = 1+\xi$ we have $c(\pi^*E) = c(\tau_E)(1 + \xi)$. We now divide this identity by $(1 + \xi)$ (that is, multiply by $\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j$) and get that $c(\tau_E) = c(\pi^*E)(\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j)$. In particular,

$$c_r(\tau_E) = (-1)^r \sum_{j=0}^{r}(-1)^j c_j(\pi^*E)\xi^{r-j},$$

so we can interpret the relation above as the vanishing of the r-th Chern class of the bundle τ_E (which has rank $r - 1$).

2.1. **A formula for the push-forward.** Let X be a smooth algebraic variety, and $E \to X$ a vector bundle of rank two on X. Consider the projective line fibration $Y = \mathbb{P}(E)$ defined as the variety of all lines in E. We have a natural projection $\pi : Y \to X$ which is projective and hence induces a *push-forward* (or *transfer*, sometimes also called *Gysin map*) $\pi_* : \Omega^*(Y) \to \Omega^*(X)$. We now state a formula for this push-forward. Note that this formula is true not only for algebraic cobordism but for any orientable cohomology theory, as the proofs remain true in this more general case.

Consider the ring of formal power series in two variables y_1 and y_2 with coefficients in $\Omega^*(X)$. Define the operator A on this ring by the formula

$$A(f) = (1+\sigma)\frac{f}{F(y_1, \chi(y_2))},$$

where $[\sigma(f)](y_1, y_2) := f(y_2, y_1)$. Here F is the universal formal group law (or more generally, the one of the orientable cohomology theory one considers) and χ is the inverse for the formal group law F, that is, χ is uniquely determined by the equation $F(x, \chi(x)) = 0$ (we use notation from [14, 2.5]). The operator A is an analog of the *divided difference operator* introduced in [1, 10]. In the case of Chow rings, our definition coincides with the classical divided difference operator, since the formal group law for Chow ring is additive, that is, $F(x, y) = x + y$ and $\chi(x) = -x$. Though $A(f)$ is defined as a fraction, it is easy to write it as a formal power series as well (see Section 5). Such a power series is unique since $F(y_1, \chi(y_2)) = y_1 - y_2 + \dots$ is clearly not a zero divisor. E.g. we have

$$A(1) = \frac{x + \chi(x)}{x\chi(x)} = q(x, \chi(x)) = -a_{11} - a_{12}(x + \chi(x)) + \dots,$$

where $x = F(y_1, \chi(y_2))$, and $q(x, y)$ is the power series uniquely determined by the equation $F(x, y) = x + y - xyq(x, y)$. In particular, since $F(x, \chi(x)) = 0$ by definition of the power series $\chi(x)$, we have $x + \chi(x) - x\chi(x)q(x, \chi(x)) = 0$ which justifies the second equality. For the last equality, we used computation of the first few terms of $F(x, y)$ and $\chi(x)$ from [14, 2.5]. Here a_{11}, a_{12} etc. denote the coefficients of the universal formal group law, that is,

$$F(x, y) = x + y + a_{11}xy + a_{12}xy^2 + \dots.$$

The coefficients a_{ij} are the elements of the Lazard ring $\mathbb{L}^*$, e.g. $a_{11} = -[\mathbb{P}^1]$, $a_{12} = a_{21} = [\mathbb{P}^1]^2 - \mathbb{P}^2$ (see [14, 2.5]). We also have

$$A(y_1) = y_2 A(1) + \frac{F(x, y_2) - y_2}{x} = y_2 q(x, \chi(x)) - y_2 q(x, y_2) + 1 = 1 + a_{12}y_1y_2 + \dots.$$

The pull-back $\pi^* : \Omega^*(X) \to \Omega^*(Y)$ gives $\Omega^*(Y)$ the structure of an $\Omega^*(X)$-module. Recall that by the projective bundle formula we have an isomorphism of $\Omega^*(X)$-modules

$$\Omega^*(Y) \cong \pi^*\Omega^*(X) \oplus \xi\pi^*\Omega^*(X),$$

where $\xi = c_1(\mathcal{O}_E(1))$. Since the push-forward is a homomorphism of $\Omega^*(X)$-modules, it is enough to determine the action of π_* on 1_Y and on ξ. The following result is a special case of [21, Theorem 5.30], which gives an explicit formula for the push-forward π_* for vector bundles of arbitrary rank.

Proposition 2.1. [21, Theorem 5.30] *Let ξ_1 and ξ_2 be the Chern roots of E, that is, formal variables satisfying the conditions $\xi_1 + \xi_2 = c_1(E)$ and $\xi_1\xi_2 = c_2(E)$. Then the push-forward acts on 1_Y and ξ as follows:*

$$\pi_*(1_Y) = [A(1)](\xi_1, \xi_2),$$
$$\pi_*(\xi) = [A(y_1)](\xi_1, \xi_2),$$

where $A(1)$ and $A(y_1)$ are the formal power series in two variables defined above.

Since $A(1)$ and $A(y_1)$ are symmetric in y_1 and y_2, they can be written as power series in $y_1 + y_2$ and y_1y_2. Hence, the right hand sides are power series in $c_1(E)$ and $c_2(E)$ and even

polynomials (as all terms of degree greater than $\dim X$ *will vanish by* [15]*). So the right hand sides indeed define elements in* $\Omega^*(X)$*.*

For the Chow ring and K_0, analogous statements were proved in [10, Propositions 2.3,2.6] for certain morphisms $Y \to X$. Note that for both of these theories, the formula for $\pi_*(\xi)$ reduces to $\pi_*(\xi) = 1$ since the corresponding formal group laws do not contain terms of degree greater than two. As Vishik showed (see [21, Theorem 5.35]), his formula is equivalent to Quillens formula [18] for complex cobordism, as also proved by Shinder in the algebraic setting [19]. We give a new geometric proof of Proposition 2.1 (that is, of Vishik's formula for rank two bundles) based on the double point relation in algebraic cobordism.

Proof. First, note that replacing E with $E_M = E \otimes M$ for an arbitrary line bundle M on X does not change the variety Y and the map π. However, this does change the tautological quotient line $c_1(\mathcal{O}_E(1))$. More precisely, we have the following isomorphism of line bundles on Y (compare e.g. [15, Proof of Lemma 7.1]):

$$\pi^* M \otimes \mathcal{O}_E(1) = \mathcal{O}_{E_M}(1).$$

Let us denote by ξ_M the first Chern class of the tautological quotient line bundle $\mathcal{O}_{E_M}(1)$. The identity above implies that $\xi_M = F(\xi, \pi^* c_1(M))$ or equivalently $\xi = F(\xi_M, \pi^* c_1(M^*))$. Hence, to compute $\pi_*\xi$ it is enough to compute $\pi_* 1_Y$ and $\pi_* \xi_M$ for some M. It is convenient to choose $M = L_1^*$ so that E_M has a trivial summand, and hence one of the Chern roots of E_M is zero. Thus we can assume that $E = \mathcal{O}_X \oplus L$. In this case, the second formula of Proposition 2.1 reduces to $\pi_*\xi = 1$, which is easy to show by similar methods as for the Chow ring. Namely, consider the natural embedding $i : X = \mathbb{P}(\mathcal{O}_X) \to Y = \mathbb{P}(E)$. Then $\xi = i_* 1_X$ by [14, Lemma 5.1.11]. Hence, $\pi_*(\xi) = \pi_* i_* 1_X = 1_X$ since $\pi \circ i = id_X$.

It is more difficult to compute $\pi_* 1_Y$, which is the cobordism class of $[\pi : Y \to X]$. For the Chow ring, it is zero by degree reasons, but for cobordisms it is not. E.g. even for a trivial bundle E we have $[\pi : Y \to X] = [\pi : X \times \mathbb{P}^1 \to X] = -a_{11} 1_X$. We compute $[\pi : Y \to X]$ by representing it as one of the terms in a suitable double point relation. Namely, consider the variety $Z = Y \times_X Y$ and define three smooth hypersurfaces A, B and C on Z as follows: $A = \{y \times i(\pi(y)) : y \in Y)\}$, $B = \{i'(\pi(y)) \times y : y \in Y\}$ and $C = \{y \times y : y \in Y\}$. Here $i : X \to Y$ and $i' : X \to Y$ are the embeddings $\mathbb{P}(\mathcal{O}_X) \to \mathbb{P}(E)$ and $\mathbb{P}(L) \to \mathbb{P}(E)$, respectively. Then it is easy to check (using again [14, Lemma 5.1.11]) that $A = c_1(p_1^* \mathcal{O}_E(1))$, $B = c_1(p_2^* \mathcal{O}_{E \otimes L^*}(1))$ and $C = c_1(\mathcal{O}_E(1) \boxtimes \mathcal{O}_{E \otimes L^*}(1))$, where p_1, p_2 are the projections of Z onto the first and the second factor, respectively. Hence, we have the FGL identity $C = A + B - ABq(A, B)$ on Z, from which we can easily get the double point relation we need. Namely, apply $\pi_* p_{1*}$ to both sides and get that

$$[\pi : Y \to X] = \pi_* p_{1*} ABq(A, B),$$

because the other two terms cancel out. The right hand side can be computed by the projection formula using that $AB = \sigma_* 1_X$, where $\sigma : X \to Z$ sends x to $i(x) \times i'(x)$. □

If we identify $\Omega^*(Y)$ with the polynomial ring $\Omega^*(X)[\xi]/(\xi^2 - c_1(E)\xi + c_2(E))$ by the projective bundle formula, we can reformulate Proposition 2.1 as follows:

$$\pi_*(f(\xi)) = [A(f(y_1))](\xi_1, \xi_2)$$

for any polynomial f with coefficients in $\Omega^*(X)$ (where $f(y_1)$ in the right hand side is regarded as an element in $\Omega^*(X)[[y_1, y_2]]$). In this form, Proposition 2.1 is consistent with the classical

formula for the push-forward in the case of Chow ring (cf. [16, Remark 3.5.4]). Indeed, since the formal group law for Chow ring is additive we have $A(1) = \frac{1}{y_1-y_2} + \frac{1}{y_2-y_1} = 0$ and $A(y_1) = \frac{y_1}{y_1-y_2} + \frac{y_2}{y_2-y_1} = 1$.

Definition 2.2. *We define an $\Omega^*(X)$-linear operator A_π on $\Omega^*(Y)$ as follows. We have an isomorphism*

$$\Omega^*(X)[[y_1, y_2]]/(y_1 + y_2 - c_1(E), y_1y_2 - c_2(E)) \cong \Omega^*(Y)$$

given by $f(y_1, y_2) \mapsto f(\xi, c_1(E) - \xi))$. Then the operator A on $\Omega^*(X)[[y_1, y_2]]$ descends to an operator A_π on $\Omega^*(Y)$, which can be described using the above isomorphism as follows

$$A_\pi : f(\xi, c_1(E) - \xi) \to [A(f(y_1, y_2))](\xi, c_1(E) - \xi).$$

We also define a $\Omega^*(X)$-linear endomorphism σ_π of $\Omega^*(Y)$ by the formula:

$$\sigma_\pi : f(\xi, c_1(E) - \xi) = f(c_1(E) - \xi, \xi).$$

The operator A_π is well-defined since A preserves the ideal $(y_1 + y_2 - c_1(E), y_1y_2 - c_2(E))$. Indeed, for any power series $f(y_1, y_2)$ symmetric in y_1 and y_2 (in particular, for $y_1 + y_2 - c_1(E)$ and $y_1y_2 - c_2(E)$) and any power series $g(y_1, y_2)$ we have $A(fg) = fA(g)$. The operator A_π decreases degrees by one, and its image is contained in $\pi^*\Omega^*(X) \subset \Omega^*(Y)$, which can be identified using the above isomorphism for $\Omega^*(X)$ with the subring of symmetric polynomials in y_1 and y_2. Proposition 2.1 tells us that the push-forward $\pi_* : \Omega^*(Y) \to \Omega^*(X)$ is the composition of A_π with the isomorphism $\pi^*\Omega^*(X) \cong \Omega^*(X)$, which sends (under the above identifications) a symmetric polynomial $f(y_1, y_2)$ into the polynomial $g(c_1(E), c_2(E))$ such that $g(y_1 + y_2, y_1y_2) = f(y_1, y_2)$. Hence, we get the following corollary, which we will use in the sequel.

Corollary 2.3. *The composition $\pi^*\pi_* : \Omega^*(Y) \to \Omega^*(Y)$ is equal to the operator A_π:*

$$\pi^*\pi_* = A_\pi.$$

In the special case $Y = G/B$ and $X = G/P_i$ (and this is the main application we have, see Section 3.2), the topological analogue of this formula appeared in [3, Corollary-Definition 1.9] for a different definition of A_π.

2.2. **Algebraic cobordism groups of cellular varieties.** We start with the definition of a cellular variety. The following definition is taken from [11, Example 1.9.1], other authors sometimes consider slight variations.

Definition 2.4. *We say that a smooth variety X over k is "cellular" or "admits a cellular decomposition" if X has a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n = X$ by closed subvarieties such that the $X_i - X_{i-1}$ are isomorphic to a disjoint union of affine spaces $\mathbb{A}^{d_i}$ for all $i = 0, ..., n$, which are called the "cells" of X.*

Examples of cellular varieties include projective spaces and more general Grassmannians, and complete flag varieties G/B where G is a split reductive group and B is a Borel subgroup.

The following theorem is a corollary of [22, Corollary 2.9]. We thank Sascha Vishik for explaining to us how it can be deduced using the projective bundle formula. The main point is that for $d = \dim X$ and i an arbitrary integer, one has for $A = \Omega$ that $\Omega^i(X) =: \Omega_{d-i}(X)$ is isomorphic to $Hom(A(d-i)[2(d-i)], M(X))$ using that in the notation of loc. cit. $Hom(A(d-i)[2d-$

$2i], M(X))$ is a direct summand in $Hom(M(\mathbb{P}^{d-i}), M(X)) = A_{d-i}(\mathbb{P}^{d-i} \times X) = \oplus_{j=0}^{d-i} A_{d-i-j}(X)$, and it is not difficult to see that it corresponds to the summand with $j = 0$.

Theorem 2.5. *Let X be a variety with a cellular decomposition as in the definition above. Then we have an isomorphism of graded abelian groups (and even of $\mathbb{L}$-modules)*

$$\Omega^*(X) \cong \oplus_i \mathbb{L}[d_i]$$

where the sum is taken over the cells of X. There is a basis in $\Omega^*(X)$ given by resolutions of cell closures (choose one resolution for each cell).

The second statement of this theorem follows from the first one if we show that the cobordism classes of resolutions of the cell closures generate $\Omega^*(X)$. This can be deduced from the analogous statement for the Chow ring using [14, Theorem 1.2.19, Remark 4.5.6]. For complex cobordism of topological complex cellular spaces, the corresponding theorem simply follows from an iterated use of the long exact localization sequence which always splits as everything in sight has MU^*-groups concentrated in even degrees only. Note also that in the topological case, the Atiyah-Hirzebruch spectral sequence degenerates for these spaces, which allows to transport information from singular cohomology to complex cobordism. As Morel points out, the analogous motivic spectral sequence invented by Hopkins-Morel (unpublished) converging to algebraic cobordism does not in general degenerate even for the point $Spec(k)$, because the one converging to algebraic K-theory does not.

We now turn to the ring structure. First, we note that if $k = \mathbb{C}$, then there is a map of graded rings and even of $\mathbb{L}$-algebras $\Omega^*(X) \to MU^{2*}(X(\mathbb{C})^{an})$ by universality of algebraic cobordism [14, Example 1.2.10]. Using the geometric description of push-forwards both for Ω^* and MU^* and the fact that the above morphism respects push-forwards [14] as well as [15], we may describe this map explicitly by mapping an element $[Y \to X]$ of $\Omega^*(X)$ to $[Y(\mathbb{C})^{an} \to X(\mathbb{C})^{an}]$ in $MU^{2*}(X(\mathbb{C})^{an})$. As both product structures are defined by taking cartesian products of the geometric representatives and pulling it back along the diagonal of X resp. $X(\mathbb{C})^{an}$, we see that this map does indeed preserve the graded $\mathbb{L}$-algebra structure. Also, for any embedding $k \to \mathbb{C}$ we obtain a ring homomorphism from algebraic cobordism over k to algebraic cobordism over $\mathbb{C}$.

For the flag variety of GL_n, this is an isomorphism by Theorem 2.6 below which is also valid for MU^*, as both base change from k to $\mathbb{C}$ and complex topological realization respect products and first Chern classes. For general cellular varieties, it is still an isomorphism. This is probably known to the experts, we provide a proof in the appendix.

For some varieties X, the ring structure of $\Omega^*(X)$ can be completely determined using the projective bundle formula [14, Section 1.1]. This is the case for the variety of complete flags for $G = GL_n$ (see Theorem 2.6 below) and also for Bott-Samelson resolutions of Schubert cycles in a complete flag variety for any reductive group G (see Section 3).

2.3. **Borel presentation for the flag variety of GL_n.** We now turn to the case of the complete flag variety X for $G = GL_n(k)$. The points of X are identified with *complete flags* in k^n. A *complete flag* is a strictly increasing sequence of subspaces

$$F = \{\{0\} = F^0 \subset F^1 \subset F^2 \subset \ldots \subset F^n = k^n\}$$

with $\dim(F^k) = k$. The group G acts transitively on the set of all flags, and the stabilizer of a point is isomorphic to a Borel subgroup $B \subset G$, which makes $X = G/B$ into a homogeneous space under G. By this definition, X has structure of an algebraic variety.

Note that over $\mathbb{C}$, one may equivalently define the flag variety X to be the homogeneous space K/T under the maximal compact subgroup $K \subset G$, where T is a maximal compact torus in K (that is, the product of several copies of S^1) [2]. E. g., for $G = GL_n(\mathbb{C})$ (resp. $SL_n(\mathbb{C})$), the maximal compact subgroup is $U(n)$ (resp. $SU(n)$). This is the language in which many of the definitions and results in [1], [2] and [4] are stated. We sometimes allow ourselves to use those definitions and results which do carry over to the "algebraic" case (reductive groups over k) without mentioning explicitly the obvious changes that have to be carried out.

There are n natural line bundles $L_1,\dots,L_n$ on X, namely, the fiber of L_i at the point F is equal to F^i/F^{i-1}. Put $x_i = c_1(L_i)$, where the first Chern class c_1 with respect to algebraic cobordism is defined in [14]. Note that our definition of x_i differs by sign from the one in [16]. The following result on the algebraic cobordism ring is an analog of the Borel presentation for the singular cohomology ring of a flag variety. In fact, it holds for any orientable cohomology theory since its proof only uses the projective bundle formula.

Theorem 2.6. *Let $A^*(-)$ be any orientable cohomology theory (e.g. $CH^*(-)$ or $\Omega^*(-)$). Then the ring $A^*(X)$ is isomorphic as a graded ring to the ring of polynomials in $x_1,\dots, x_n$ with coefficients in the coefficient ring $A^*(pt)$ and $deg(x_i) = 1$, quotient by the ideal S generated by the symmetric polynomials of strictly positive polynomial degree:*

$$A^*(X) \simeq A^*(pt)[x_1, \dots, x_n]/S.$$

More generally, let E be a vector bundle of rank n over a smooth variety Y and $\mathbb{F}(E)$ be the flag variety relative to this bundle. Then we have an isomorphism of graded rings

$$A^*(\mathbb{F}(E)) \simeq A^*(pt)[x_1, ..., x_n]/I$$

where I is the ideal generated by the relations $e_k(x_1, .., x_n) = c_k(E)$ for $1 \leq k \leq n$ with e_k denoting the k-th elementary symmetric polynomial.

Proof. The proof of [16, Theorem 3.6.15] for the Chow ring case can be slightly modified so that it becomes applicable to any other orientable theory A^*. Namely, for an arbitrary oriented cohomology theory A^*, it is more convenient to dualize the geometric argument in [16, Theorem 3.6.15] because we can no longer use that $c_i(E) = (-1)^i c_i(E^*)$ for a vector bundle E (which is used implicitly several times in the proof of [16, Theorem 3.6.15]). That is, we start with the variety of partial flags $P_i = \{ F^{n-i} \subset F^{n-i+1} \subset \dots \subset F^n = k^n\}$ (e. g. P_1 is the variety of hyperplanes in k^n and $P_{n-1} = X$). The rest of the argument is completely analogous to the proof of [16, Theorem 3.6.15]. We give the details below for the reader's convenience.

Denote by $\mathcal{W}_j$ the corresponding tautological vector bundle of rank j over P_i, where $j \geq n-i$ (that is, the fiber of $\mathcal{W}_j$ over a point $\{ F^{n-i} \subset F^{n-i+1} \subset \dots \subset F^n\}$ is equal to F_j). In particular, L_i defined above is equal to $\mathcal{W}_i/\mathcal{W}_{i-1}$. Put $x_i = c_1(L_i)$. As $P_i = \mathbb{P}((\mathcal{W}_{n-i+1})^*)$ is the projective bundle over P_{i-1} and the line bundle $\mathcal{O}_{\mathcal{W}_{n-i+1}}(1)$ is isomorphic to L_{n-i+1}, the projective bundle formula for orientable cohomology theories [14, Section 1.1] yields

$$A^*(P_i) \cong A^*(P_{i-1})[x_{n-i+1}]/(\sum_{j=0}^{n-i+1} (-1)^j c_j(\mathcal{W}_{n-i+1}) x_{n-i+1}^{n-i+1-j}).$$

Or, using the above interpretation of the relation in the projective bundle formula and the short exact sequence of vector bundles on P_i

$$0 \to \mathcal{W}_{n-i} \to \mathcal{W}_{n-i+1} \to L_{n-i+1} \to 0$$

we get

$$A^*(P_i) \cong A^*(P_{i-1})[x_{n-i+1}]/(c_{n-i+1}(\mathcal{W}_{n-i})).$$

It remains to compute $c_{n-i+1}(\mathcal{W}_{n-i})$. This can be done by induction on i starting from $i = 0$ (in which case $\mathcal{W}_n$ is a trivial line bundle) and applying the Whitney sum formula to the short exact sequence of vector bundles above. We get $c(\mathcal{W}_{n-i}) = \prod_{j=n-i+1}^{n} c(L_j)^{-1} = \prod_{j=n-i+1}^{n}(1 + x_j)^{-1} = \sum_{k\geq 0}(-1)^k h_k(x_{n-i+1}, \ldots, x_n)$, where $h_k(x_{n-i+1}, \ldots, x_n)$ denotes the sum of all monomials of degree k in x_{n-i+1},...,x_n. In particular, $c_{n-i+1}(\mathcal{W}_{n-i}) = (-1)^{n-i+1} h_{n-i+1}(x_{n-i+1}, \ldots, x_n)$. From this we deduce that

$$A^*(P_i) \cong A^*(P_{i-1})[x_{n-i+1}]/(h_{n-i+1}(x_{n-i+1}, \ldots, x_n)),$$

and hence

$$A^*(P_n) \cong A^*(pt)[x_1, \ldots, x_n]/(h_n(x_n), h_{n-1}(x_{n-1}, x_n), \ldots, h_1(x_1, \ldots, x_n)).$$

The ideal generated by the relations $h_{n-i+1}(x_{n-i+1}, \ldots, x_n)$ is exactly S, which is easy to check starting with the recurrence relation

$$h_i(x_1, \ldots, x_n) = h_i(x_i, \ldots, x_n) + \sum_{j<i} x_j h_{i-1}(x_j, \ldots, x_n).$$

The proof of the more general case is completely analogous to the proof of [16, Proposition 3.8.1]. Note that the proof of [16, Proposition 3.8.1] does not really use [16, Theorem 3.6.15] as an induction base (despite the claim in the proof) and in fact gives another proof for [16, Theorem 3.6.15], which is also applicable to an arbitrary oriented cohomology theory. □

Remark 2.7. The proof immediately implies that the class of a point in $A^*(X)$ is equal to $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2$. Since $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2 = \frac{1}{n!} \prod_{i>j}(x_i - x_j) \mod S$ (which is easy to show by induction on n using that $(x_n - x_{n-1}) \cdots (x_n - x_1) = n x_n^{n-1} \mod S$) we also have that the class of a point can be represented by the polynomial $\Delta_n = \frac{1}{n!} \prod_{i>j}(x_i - x_j)$.

Note that the proof also gives an explicit formula for the classes of one-dimensional *Schubert cycles* $X_1 = X_{s_{\gamma_1}}, \ldots, X_{n-1} = X_{s_{\gamma_{n-1}}}$ in X corresponding to the simple roots γ_1,...,γ_{n-1} of GL_n (see the beginning of Section 3 for the definition of the Schubert cycles X_w for w in the Weyl group of G). The cycle X_k consists of flags $F = \{\{0\} = F^0 \subset F^1 \subset F^2 \subset \ldots \subset F^n = k^n\}$ such that all F^i except for F^k are fixed. Then the class of X_k is equal to the class of a point divided by x_{k+1}. Indeed, to get the class of $X_k \subset X$ we should take the point in P_{n-k-1} corresponding to the fixed partial flag $\{F^{k+1} \subset F^{k+2} \subset \ldots \subset F^n = k^n\}$ and then take a line in a fiber of the projective bundle $P_{n-k} \to P_{n-k-1}$ over this point. Namely, the line will consist of all hyperplanes in F^{k+1} that contain the fixed codimension two subspace F^{k-1}. Again it is easy to show by induction on n that the polynomial $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2 / x_k$ is equal to $2\Delta_n/(x_{k+1} - x_k)$ modulo the ideal S.

Note that the Borel presentation for singular cohomology implies, in particular, that Picard group of the flag variety is generated (as an abelian group) by the first Chern classes of the line bundles L_1,...,L_n the only nontrivial relation being $\sum c_1(L_i) = 0$. In what follows, we

will also use the following alternative description of the Picard group of X. Recall that each strictly dominant weight λ of G defines an irreducible representation $\pi_\lambda : G \to GL(V_\lambda)$ and an embedding $G/B \to \mathbb{P}(V_\lambda)$. Hence, to each strictly dominant weight λ of G we can assign a very ample line bundle $L(\lambda)$ on X by taking the pull-back of the line bundle $\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ on $\mathbb{P}(V_\lambda)$. The map $\lambda \mapsto L(\lambda)$ extended to non-dominant weights by linearity gives an isomorphism between the Picard group of X and the weight lattice of G [5, 1.4.3]. In particular, for the line bundles above we have $L_i = L(-e_i)$ where e_i is the weight of GL_n given by the i-th entry of the diagonal torus in GL_n.

We now compute $c_1(L(\alpha_i))$ as a polynomial in $x_1,\dots, x_n$. Let $\gamma_1,\dots,\gamma_{n-1}$ be the simple roots of G (that is, $\gamma_i = e_i - e_{i+1}$). We can express the line bundles $L(\gamma_i)$ in terms of the line bundles $L_1,\dots,L_n$. Since $L_i = L(-e_i)$ and $\gamma_i = e_i - e_{i+1}$, we have that the line bundle $L(\gamma_i)$ is isomorphic to $L_i^{-1} \otimes L_{i+1}$. In particular, we can compute

$$c_1(L(\gamma_i)) = c_1(L_i^{-1} \otimes L_{i+1}) = F(\chi(x_i), x_{i+1}).$$

E.g. by the formulas for $F(x, y)$ and $\chi(x)$ from [14, 2.5] the first few terms of $c_1(L(\gamma_i))$ look as follows

$$c_1(L(\gamma_i)) = -x_i + x_{i+1} + a_{11}x_i^2 - a_{11}x_ix_{i+1} + \dots,$$

where $a_{11} = -[\mathbb{P}^1]$.

In what follows, we will use the isomorphism $W \cong S^n$. The simple reflection s_α for any root $\alpha = e_i - e_j$ acts on the weight lattice (spanned by the weights $e_1,\dots,e_n$, which form an orthonormal basis) by the reflection in the plane perpendicular to $e_i - e_j$ and hence permutes the weights $e_1,\dots, e_n$ by the transposition $(i\ j)$.

3. Schubert calculus for algebraic cobordism of flag varieties

In this section, we assume that G is an arbitrary connected split reductive group unless we explicitly mention that $G = GL_n(k)$, and $X = G/B$ is the complete flag variety for G. We now investigate the ring structure of $\Omega^*(X)$ in more geometric terms.

3.1. **Schubert cycles and Bott-Samelson resolutions.** Recall that the flag variety X is cellular with the following cellular decomposition into *Bruhat cells*. Let us fix a Borel subgroup B. For each element $w \in W$ of the Weyl group of G, define the *Bruhat (or Schubert) cell* C_w as the B–orbit of the the point $wB \in G/B = X$ (we identify the Weyl group with $N(T)/T$ for a maximal torus T of G inside B). The *Schubert cycle* X_w is defined as the closure of C_w in X. The dimension of X_w is equal to the length of w [1]. Recall that the length of an element $w \in W$ is defined as the minimal number of factors in a decomposition of w into the product of simple reflections. Recall also that for each l-tuple $I = (\alpha_1, \dots, \alpha_l)$ of simple roots of G, one can define the *Bott-Samelson resolution* R_I (which has dimension l) together with the map $r_I : R_I \to X$. Bott-Samelson resolutions are smooth. Consequently, for any I the map $r_I : R_I \to X$ represents an element in $\Omega^*(X)$ which we denote by Z_I.

Denote by $s_\alpha \in W$ the reflection corresponding to a root α, and by s_I the product $s_{\alpha_1} \cdots s_{\alpha_l}$. If the decomposition $s_I = s_{\alpha_1} \cdots s_{\alpha_l}$ defined by I is reduced (that is, s_I can not be written as a product of less than l simple reflections, or equivalently, the length of s_I is equal to l), then the image $r_I(R_I)$ coincides with the Schubert cycle X_{s_I} (which we will also denote by X_I). The dimension of X_I in this case is also equal to l and the map $r_I : R_I \to X_I$ is a birational

isomorphism. In this case, the variety R_I is a resolution of singularities for the Schubert cycle X_I.

Bott-Samelson resolutions were introduced by Bott and Samelson in the case of compact Lie groups, and by Demazure in the case of algebraic semisimple groups [10]. There are several equivalent definitions, see e. g. [6, 10, 16]. We will use the definition below (which follows easily from [6, 2.2]), since it is most suited to our needs. Namely, R_I is defined by the following inductive procedure starting from $R_\emptyset = pt = Spec(k)$ (in what follows we will rather denote $R_\emptyset$ by R_e). For each j-tuple $J = (\alpha_1, \dots, \alpha_j)$ with $j < l$, denote by $J \cup \{j+1\}$ the $(j+1)$-tuple $(\alpha_1, \dots, \alpha_j, \alpha_{j+1})$. Define $R_{J\cup\{j+1\}}$ as the fiber product $R_J \times_{G/P_{j+1}} G/B$, where P_{j+1} is the minimal parabolic subgroup corresponding to the root α_{j+1}. Then the map $r_{J\cup\{j+1\}} : R_{J\cup\{j+1\}} \to X$ is defined as the projection to the second factor. In what follows, we will use that R_J can be embedded into $R_{J\cup\{j+1\}}$ by sending $x \in R_J$ to $(x, r_J(x)) \in R_J \times_{G/P_{j+1}} G/B$.

In particular, one-dimensional Bott-Samelson resolutions are isomorphic to the corresponding Schubert cycles. It is easy to show that any two-dimensional Bott-Samelson resolution R_I for a reduced I is also isomorphic to the corresponding Schubert cycle. More generally, R_I is isomorphic to X_I if and only if all simple roots in I are pairwise distinct (in particular, the length of I should not exceed the rank of G). The simplest example where R_I and X_I are not isomorphic for a reduced I is $G = GL_3$ and $I = (\gamma_1, \gamma_2, \gamma_1)$ (where γ_1, γ_2 are two simple roots of GL_3).

It is easy to show that $R_{J\cup\{j+1\}}$ is the projectivization of the bundle $r_J^*\pi_{j+1}^*E$, where E is the rank two vector bundle on G/P_{j+1} defined in the next subsection and $\pi_{j+1} : G/B \to G/P_{j+1}$ is the natural projection. This is the definition used in [4]. In the topological setting, the vector bundle $r_J^*\pi_{j+1}^*E$ splits into the sum of two line bundles [4] but in in the algebro-geometric setting this is no longer true (though $r_J^*\pi_{j+1}^*E$ still contains a line subbundle as follows from the proof of Lemma 3.4).

This definition of R_I allows to describe easily (by repeated use of the projective bundle formula) the ring structure of the cobordism ring $\Omega^*(R_I)$. It also implies that R_I is cellular with 2^l cells labeled by all subindices $J \subset I$.

The cobordism classes Z_I of Bott-Samelson resolutions generate $\Omega^*(X)$ but do not form a basis. The following proposition is an immediate corollary of Theorem 2.5. An analogous statement for complex cobordism is proved in [4, Proposition 1] by using the Atiyah-Hirzebruch spectral sequence (as mentioned in Section 2).

Proposition 3.1. *As an $\mathbb{L}$-module, the algebraic cobordism ring $\Omega^*(X)$ of the flag variety is freely generated by the Bott-Samelson classes $Z_{I(w)}$ where $w \in W$ and $I(w)$ defines a reduced decomposition for w (we choose exactly one $I(w)$ for each w).*

There is no canonical choice for a decomposition $I(w)$ of a given element w in the Weyl group. From the geometric viewpoint it is more natural to consider all Bott-Samelson classes at once (including those for non-reduced I) even though they are not linearly independent over $\mathbb{L}$. So throughout the rest of the paper we will not put any restrictions on the multiindex I.

3.2. **Schubert calculus.** We will now describe the cobordism classes Z_I as polynomials in the first Chern classes of line bundles on X. This allows us to compute products of Bott-Samelson resolutions and hence achieves the goal of a Schubert calculus for algebraic cobordism.

We first define operators A_i on $\Omega^*(X)$ following the approach of the previous section (see Definition 2.2). These operators generalize the *divided difference operators* on the Chow ring $CH^*(X)$ defined in [1, 10, 8] to algebraic cobordism.

We first define operators A_i for GL_n since in this case the Borel presentation allows to make them more explicit. We start with the subgroup B of upper triangular matrices and the diagonal torus, which yields an isomorphism $W \cong S_n$. Under this isomorphism, the reflection s_α with respect to a root $\alpha = e_i - e_j$ goes to the transposition $(i\ j)$ (see the end of Section 2). For each positive root α of G, we define the operators σ_α and $\hat{A}_\alpha$ on the ring of formal power series $\mathbb{L}[[x_1, \dots, x_n]]$ as follows:

$$(\sigma_\alpha f)(x_1, \dots, x_n) = f(x_{s_\alpha(1)}, \dots, x_{s_\alpha(n)}),$$

$$\hat{A}_\alpha = (1 + \sigma_\alpha)\frac{1}{F(x_{i+1}, \chi(x_i))}.$$

It is easy to check that $\hat{A}_\alpha$ is well-defined on the whole ring $\mathbb{L}[[x_1, \dots, x_n]]$ (see Section 5). Note also that under the homomorphism $\mathbb{L}[[x_1, \dots, x_n]] \to \mathbb{L}[x_1, \dots, x_n]/S \cong \Omega^*(X)$ the power series $F(x_{i+1}, \chi(x_i))$ maps to $c_1(L(\gamma_i))$ (see the end of Section 2), so our definition for additive formal group law reduces to the definition of divided difference operator on the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ (see [16, 2.3.1]). Finally, we define the operator $A_\alpha : \Omega^*(X) \to \Omega^*(X)$ using the Borel presentation by the formula

$$A_\alpha(f(x_1, \dots, x_n)) = \hat{A}_\alpha(f)(x_1, \dots, x_n)$$

for each polynomial $f \in \mathbb{L}[x_1, \dots, x_n]$. Again, by degree reasons the right hand side is a polynomial. The operator A_α is well defined (that is, does not depend on a choice of a polynomial f representing a given class in $\mathbb{L}[x_1, \dots, x_n]/S$) since for any polynomial h and any symmetric polynomial g we have $\hat{A}_\alpha(gh) = g\hat{A}_\alpha(h)$.

We now define $A_i = A_{\alpha_i}$ for an arbitrary reductive group G and a simple root α_i. Denote by $P_i \subset G$ the minimal parabolic subgroup corresponding to the root α_i. Then $X = G/B$ is a projective line fibration over G/P_i. Indeed, consider the projection $\pi_i : G/B \to G/P_i$. Take the line bundle $L(\rho)$ on G/B corresponding to the weight ρ, where ρ is the half-sum of all positive roots or equivalently the sum of all fundamental weights of G (the weight ρ is uniquely characterized by the property that $(\rho, \alpha) = 1$ for all simple roots α). Then it is easy to check that the vector bundle $E := \pi_{i*}L(\rho)$ on G/P_i has rank two and $G/B = \mathbb{P}(E)$. Note that tensoring E with any line bundle L on G/P_i does not change $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$ so the property $\mathbb{P}(E) = X$ does not uniquely define the bundle E. However, the choice $E = \pi_{i*}L(\rho)$ (suggested to us by Michel Brion) is the only uniform choice for all i, since $L(\rho)$ is the only line bundle on X with the property $\mathbb{P}(\pi_{i*}L(\rho)) = X$ for all i. We now use Definition 2.2 to define an $\Omega^*(G/P_i)$-linear operator $A_i := A_{\pi_i}$ on $\Omega^*(X)$. For $G = GL_n$, this definition coincides with the one given above. This is easy to show using that G/P_i for $\alpha_i = \gamma_i$ is the partial flag variety whose points are flags $F = \{\{0\} = F^0 \subset \ldots \subset F^{i-1} \subset F^{i+1} \subset \ldots \subset F^n = k^n\}$.

Let $I = (\alpha_1, \dots, \alpha_l)$ be an l-tuple of simple roots of G. Define the element $\mathcal{R}_I$ in $\Omega^*(X)$ by the formula

$$\mathcal{R}_I := A_l \dots A_1 Z_e.$$

In the case $G = GL_n$, we can also regard $\mathcal{R}_I$ as a polynomial in $\mathbb{L}[x_1, \dots, x_n]/S$. Similar to [1, Theorem 3.15] or [4, page 807], one may describe Z_e for general G using the formula

$$Z_e = \mathcal{R}_e := \frac{1}{|W|} \prod_{\alpha \in R^+} c_1(L(\alpha)),$$

where R^+ denotes the set of positive roots of G (recall that $|R^+| = \dim X =: d$). As in the Chow ring case, there is also the formula

$$Z_e = \frac{1}{d!} L(\rho)^d.$$

Both formulas immediately follow from the analogous formulas for the Chow ring [1, Theorem 3.15, Corollary 3.16] since $\Omega^d(X) \simeq CH^d(X)$ (as follows from [14, Theorem 1.2.19, Remark 4.5.6]).

Note that for GL_n the formula for $\mathcal{R}_e$ reduces to $\mathcal{R}_e = \Delta_n$ since $c_1(L(e_i - e_j)) = x_j - x_i +$higher order terms, and hence the equality $Z_e = \mathcal{R}_e$ follows from Remark 2.7. In particular, by the same remark $\mathcal{R}_e$ modulo S has a denominator-free expression $x_n^{n-1} x_{n-1}^{n-2} \cdots x_2$.

We now prove an algebro-geometric version of [4, Corollary 1, Proposition 3] using our algebraic operators A_i.

Theorem 3.2. *The cobordism class $Z_I = [r_I : R_I \to X]$ of the Bott-Samelson resolution R_I is equal to $\mathcal{R}_I$.*

Proof. The essential part of the proof is the formula for the push-forward as stated in Corollary 2.3. Once this formula is established it is not hard to show that $A_i Z_I = Z_{I \cup \{i\}}$ for all I by exactly the same methods as in the Chow ring case [16] and in the complex cobordism case [4]. Namely, we have the following cartesian square

$$\begin{array}{ccc} G/B \times_{G/P_i} G/B & \xrightarrow{p_2} & G/B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ G/B & \xrightarrow{\pi_i} & G/P_i. \end{array}$$

E.g., if $G = GL_n$ we get exactly the diagram of [16, proof of Lemma 3.6.20]. Using this commutative diagram and the definition of Bott-Samelson resolutions it is easy to show that ${\pi_i}^* \pi_{i*} Z_I = Z_{I \cup \{i\}}$ [4, proof of Proposition 2.1]. We now apply Corollary 2.3 and get that $A_i = {\pi_i}^* \pi_{i*}$. It follows by induction on the length of I that $Z_I = A_l \dots A_1 Z_e$. □

Remark 3.3. Note that if we apply the base change formula [14, Definition 1.1.2 (A2)] to the cartesian diagram from the proof of Theorem 3.2, we get $p_{1*} p_2^* = \pi_i^* \pi_{i*}$, where the right hand side is precisely the definition of the "geometric" operator denoted A_i in [4], while the left hand side is the operator denoted δ_i in [16, proof of Theorem 3.6.18]. Hence Manivel and Bressler–Evens consider the same operators.

We now compute the action of the operator A_i on polynomials in the first Chern classes (this computation will be used in Sections 4 and 5). Consider the operator $\sigma_i := \sigma_{\pi_i}$ again defined as in Definition 2.2. Note that σ_i corresponds to the simple reflection $s_i := s_{\alpha_i}$ in the following sense.

Lemma 3.4. *For any line bundle $L(\lambda)$ on X, we have*

$$\sigma_i(c_1(L(\lambda))) = c_1(L(s_i\lambda)).$$

Proof. Since $X = \mathbb{P}(E)$ (recall that $E = \pi_{i*}L(\rho)$), the bundle π_i^*E on X admits the usual short exact sequence

$$0 \to \tau_E \to \pi_i^*E \to \mathcal{O}_E(1) \to 0,$$

where τ_E is the tautological line bundle on X (that is, the fiber of τ_E at the point $x \in X = \mathbb{P}(E)$ is the line in E represented by x). Note that in our case $\mathbb{P}(E) = \mathbb{P}(E^{dual})$ since E is of rank two (thus hyperplanes in E are the same as lines in E). It is easy to show that there is an isomorphism of line bundles

$$\tau_E^{-1} \otimes \mathcal{O}_E(1) = L(\alpha_i).$$

(Moreover, one can show that $\tau_E = L(\rho - \alpha_i)$ and $\mathcal{O}_E(1) = L(\rho)$.) Indeed, $\tau_E^{-1} \otimes \mathcal{O}_E(1) \simeq Hom(\tau_E, \mathcal{O}_E(1))$ can be thought of as the bundle of tangents along the fibers of π_i. The latter is the line bundle associated with the B–module $\mathfrak{p}_i/\mathfrak{b}$, which has weight $-\alpha_i$ (see [5, Remark 1.4.2] for an alternative definition of the line bundles $L(\lambda)$ in terms of the one-dimensional B–modules). Here $\mathfrak{p}_i$ and $\mathfrak{b}$ denote the Lie algebras of P_i and B, respectively.

By definition, σ_i switches $c_1(\tau_E)$ and $c_1(\mathcal{O}_E(1))$. Hence, $\sigma_i(c_1(L(\alpha_i))) = c_1(L(-\alpha_i))$. Since the Picard group of G/P_i can be identified with with the sublattice $\{\lambda | (\lambda, \alpha_i) = 0\}$ of the weight lattice of G (this follows from [5, remark after Proposition 1.3.6] combined with [5, Proposition 1.4.3]) we also have $\sigma_i(c_1(L(\lambda))) = c_1(L(\lambda))$ for all λ perpendicular to α_i. These two identities imply the statement of the lemma. □

This lemma allows us to describe explicitly the action of σ_i and hence of A_i on any polynomial in the first Chern classes. Indeed, since for any weight λ we have $s_i\lambda = \lambda + k\alpha_i$ for some integer k, we can compute $c_1(L(\sigma_i\lambda)) = c_1(L(\lambda) \otimes L(\alpha_i)^k)$ as a power series in $c_1(L(\lambda))$ and $c_1(L(\alpha_i))$ using the formal group law. This will be used in the proof of Proposition 4.3 below and in Subsection 5.1.

4. Chevalley-Pieri formulas

A key ingredient for the classical Schubert calculus is the Chevalley-Pieri formula for the product of the Schubert cycle with the first Chern class of the line bundle on X, see e. g. [1, Proposition 4.1] and [10, Proposition 4.2]. We now establish analogous formulas for the products of Z_I and $\mathcal{R}_I$ with $c_1(L(\lambda))$ (without using that $Z_I = \mathcal{R}_I$). At the end of this section, we explain why in the case of algebraic cobordism this alone is not enough to show that $Z_I = \mathcal{R}_I$, hence justifying our different approach of the previous two sections.

By $L(D)$ denote the line bundle corresponding to the divisor D. For each l-tuple I as above, denote by I^j the $(l-1)$–tuple $(\alpha_1, \dots, \hat{\alpha}_j, \dots, \alpha_l)$. For each root α, define the linear function $(\cdot, \alpha)$ (that is, the *coroot*) on the weight lattice of G by the property $s_\alpha\lambda = \lambda - (\lambda, \alpha)\alpha$ for all weights λ. (The pairing (a, b) is often denoted by $\langle a, b^\vee \rangle$ or by $\langle a, b \rangle$.) Note that by definition $(\lambda, \alpha) = (w\lambda, w\alpha)$ for all elements w of the Weyl group.

Proposition 4.1. ***Geometric Chevalley-Pieri formula***

(1) (for Bott-Samelson resolutions) *In the Picard group of R_I we have*

$$r_I^*L(\lambda) = \otimes_{j=1}^{l} L(R_{I^j})^{(\lambda, \beta_j)}$$

where $\beta_j = s_l \cdots s_{j+1}\alpha_j$.

(2) (for Schubert cycles)[1, Proposition 4.1], [10, Proposition 4.4], [8] *In the Chow ring of* X *we have*

$$c_1(L(\lambda))X_I = \sum_j (\lambda, \beta_j) X_{I^j}$$

where the sum is taken over $j \in \{1, \ldots, l\}$ *for which the decomposition defined by* I^j *is reduced.*

The first part of this proposition was proved in [4, Proposition 4] in the topological setting (for flag varieties of compact Lie groups). It is not hard to check that the proof carries over to algebro-geometric setting. We instead provide a shorter proof along the same lines. Our proof is based on the following lemma.

Lemma 4.2. [10, Proposition 2.1] *Let* $p : R_I \to R_{I^l}$ *be the natural projection (coming from the fact that we defined* R_I *as a projective bundle over* R_{I^l}*). Then we have an isomorphism*

$$r_I^* L(\lambda) \cong p^* r_{I^l}^* L(s_l\lambda) \otimes L(R_{I^l})^{(\lambda, \alpha_l)}$$

of line bundles on R_I.

Proposition 4.1(1) now follows from Lemma 4.2 by induction on l. The base $l = 1$, that is $r_1^* L(\lambda) = \mathcal{O}_{\mathbb{P}^1}(1)^{(\lambda,\alpha_1)}$, follows from the fact that $r_1 : R_1 \to X$ maps R_1 isomorphically to $P_1/B \cong \mathbb{P}^1$, which can be regarded as the flag variety for SL_2. Then the weight λ restricted to SL_2 is equal to (λ, α_1) times the highest weight of the tautological representation of SL_2, which corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$. To prove the induction step plug in the induction hypothesis for $r_{I^l}^* L(s_l\lambda) = \otimes_{j=1}^{l-1} L(R_{I^{j,l}})^{(s_l\lambda, s_{l-1}\cdots s_{j+1}\alpha_j)}$ into the lemma and use that $(s_l\lambda, s_{l-1}\cdots s_{j+1}\alpha_j) = (\lambda, \beta_j)$ (since $s_l^2 = e$) and $p^* R_{I^{j,l}} = R_{I^j}$.

Proposition 4.1(1) was used in [4] to establish an algorithm for computing $c_1(L(\lambda))Z_I$ in $\Omega^*(X)$ [4]. We now briefly recall this algorithm. By the projection formula we have

$$c_1(L(\lambda))Z_I = (r_I)_*(c_1(r_I^* L(\lambda))).$$

Note that the usual projection formula with respect to smooth projective morphisms $f : X \to Y$ holds for algebraic cobordism as well. This follows from the definition of products via pull-backs along the diagonal and the base change axiom $(A2)$ of [14] applied to the cartesian square obtained from $Y \overset{diag}{\to} Y \times Y \overset{p\times id}{\leftarrow} X \times Y$.

One can now use Proposition 4.1(1) and the formal group law to compute $c_1(r_I^* L(\lambda))$ in terms of the Bott-Samelson classes in $\Omega^*(R_I)$ by an iterative procedure (since the multiplicative structure of $\Omega^*(R_I)$ can be determined by the projective bundle formula and the Chern classes arising this way again have form $c_1(L(\lambda))$ for some λ). After $c_1(r_I^* L(\lambda))$ is written as $\sum_{J\subset I} a_J [R_J]$ for some $a_J \in \mathbb{L}$ it is easy to find $(r_I)_*(c_1(r_I^* L(\lambda)))$ since $(r_I)_*[R_J] = Z_J$.

However, this procedure is rather lengthy, and we will not use it. Instead, we will prove a more explicit formula for $c_1(L(\lambda))Z_I$ (see formula 5.1 below) using our algebraic Chevalley-Pieri formula together with Theorem 3.2.

Proposition 4.3. ***Algebraic Chevalley-Pieri formula:***

(1) (cobordism version) *Let $A_1 = A_{\alpha_1}, \dots, A_l = A_{\alpha_l}$ be the operators on $\Omega^*(X)$ corresponding to $\alpha_1, \dots, \alpha_l$. Then we have*

$$c_1(L(\lambda))A_1 \dots A_l\mathcal{R}_e = \sum_{j=1}^{l} A_1 \dots A_{j-1}\frac{c_1(L(\lambda_j)) - c_1(L(s_j\lambda_j))}{c_1(L(\alpha_j))}A_{j+1}\dots A_l\mathcal{R}_e$$

in $\Omega^*(X)$, where $\lambda_j = s_{j-1}\cdots s_1\lambda$ and $s_j = s_{\alpha_j}$ is the reflection corresponding to the root α_j.

(2) (Chow ring version) [1, Corollary 3.7] *Let $A_1 = A_{\alpha_1}, \dots, A_l = A_{\alpha_l}$ be the operators on $CH^*(X)$ corresponding to $\alpha_1, \dots, \alpha_l$. Then*

$$c_1(L(\lambda))A_1 \dots A_l\mathcal{R}_e = \sum_{j=1}^{l}(\lambda, s_1\cdots s_{j-1}\alpha_j)A_1 \dots \hat{A}_j \dots A_l\mathcal{R}_e$$

in $CH^*(X)$.

Proof. First, note that $\frac{c_1(L(\lambda_j))-c_1(L(s_j\lambda_j))}{c_1(L(\alpha_j))}$ is a well-defined element in $\Omega^*(X)$ because $s_j\lambda = \lambda - (\lambda, \alpha_j)\alpha_j$ (and hence $L(\lambda) = L(s_j\lambda) \otimes L(\alpha_j)^{(\lambda,\alpha_j)}$) and the formal group law expansion for $c_1(L_1 \otimes L_2^k) - c_1(L_1)$ is divisible by $c_1(L_2)$ for any integer k [14, (2.5.1)]. Next we show that

$$c_1(L(\lambda))A_1 - A_1c_1(L(s_1\lambda)) = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\alpha_1))},$$

where both sides are regarded as operators on $\Omega^*(X)$. Indeed, by definition $A_1 = (1+\sigma_1)\frac{1}{c_1(L(\alpha_1))}$ and $c_1(L(\lambda))\sigma_1 = \sigma_1 c_1(L(s_1\lambda))$ by Lemma 3.4.

Hence, we can write

$$c_1(L(\lambda))A_1 \dots A_l\mathcal{R}_e = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\alpha_1))}A_2 \dots A_l\mathcal{R}_e + A_1c_1(L(s_1\lambda))A_2 \dots A_l\mathcal{R}_e,$$

and then continue moving $c_1(L(s_1\lambda))$ to the right until we are left with with the term $A_1 \dots A_l c_1(L(s_l \dots s_1\lambda))\mathcal{R}_e$. This term is equal to zero since $c_1(L(s_l \dots s_1\lambda))\mathcal{R}_e$ is the product of more than $\dim X$ first Chern classes, and hence its degree is greater than $\dim X$. The Chow ring case follows immediately from the cobordism case since

$$\frac{c_1(L(\lambda)) - c_1(L(s_j\lambda))}{c_1(L(\alpha_j))} = (\lambda, \alpha_j)$$

in the Chow ring. The last identity holds because the formal group law for the Chow ring is additive, and hence $c_1(L(\lambda)) - c_1(L(s_j\lambda)) = (\lambda, \alpha_j)c_1(L(\alpha_j))$. □

The second part of this proposition was proved in [1] by more involved calculations. A calculation similar to ours was used in [17] to deduce a combinatorial Chevalley-Pieri formula for K-theory. It would be interesting to find an analogous combinatorial interpretation of our Chevalley-Pieri formula in the cobordism case.

Note that in the case of Chow groups, the algebraic Chevalley-Pieri formula for $A_l \dots A_1\mathcal{R}_e$ is exactly the same as the geometric one for the Schubert cycle X_I. Together with the Borel presentation this easily implies that the polynomial $A_l \dots A_1\mathcal{R}_e$ represents the Schubert cycle X_I whenever I defines a reduced decomposition [1]. Indeed, we can proceed by the induction on l. Algebraic and geometric Chevalley-Pieri formulas allow to compute the intersection indices of $A_l \dots A_1\mathcal{R}_e$ and of X_I, respectively, with the product of k first Chern classes, and the result is

the same in both cases by the induction hypothesis (for all $k > 0$). By the Borel presentation we know that the products of first Chern classes span $\oplus_{i=1}^{d} CH^i(X)$. Hence, by the non-degeneracy of the intersection form on $CH^*(X)$ (that is, by Poincaré duality) we have that $A_l \dots A_1 \mathcal{R}_e - X_I$ must lie in $CH^d(X) = \mathbb{Z}[pt]$ (that is, in the orthogonal complement to $\oplus_{i=1}^{d} CH^i(X)$). This is only possible if $A_l \dots A_1 \mathcal{R}_e - X_I = 0$ (unless $l = 0$, which is the induction base). Note that the only geometric input in this proof is the geometric Chevalley-Pieri formula.

In the cobordism case, it is not immediately clear why geometric and algebraic Chevalley-Pieri formulas are the same (though, of course, it follows from Theorem 3.2). But even without using that $\mathcal{R}_I = Z_I$ it might be possible to show that both formulas have the same structure coefficients, that is, if $c_1(L(\lambda))Z_I = \sum_{J \subset I} a_J Z_J$ then necessarily $c_1(L(\lambda))\mathcal{R}_I = \sum_{J \subset I} a_J \mathcal{R}_J$ with the same coefficients $a_J \in \mathbb{L}$. However, this does not lead to the proof of $\mathcal{R}_I = Z_I$ as in the case of the Chow ring. The reason is that even though there is an analog of Poincaré duality for the cobordism rings of cellular varieties, this only yields an equality $\mathcal{R}_I = Z_I$ up to a multiple of $[pt]$ (as in Lemma 4.4 below in the case of GL_n/B), and this is not enough to carry out the desired induction argument. For the Chow ring, Poincaré duality also yields only an equality up to the class of a point, but unless $I = \emptyset$, the difference $\mathcal{R}_I - Z_I$ (where now Z_I means the Schubert cycle and not the Bott-Samelson class) can not be a non-zero multiple of $[pt]$ because the coefficient ring $CH^*([pt]) = CH^*(k) \cong \mathbb{Z}$ is concentrated in degree zero, hence has no nonzero elements in the corresponding degree $l - d$. However, for algebraic cobordism the coefficient ring $\Omega^*(k) \cong \mathbb{L}$ does contain plenty of elements of negative degree, so one can not deduce $\mathcal{R}_I = Z_I$.

Lemma 4.4. *Let $X = GL_n/B$ and let $c \in \Omega^*(X) \cong \mathbb{L}[x_1, \dots, x_n]/S$ be a homogeneous element of degree l such that the product of c with any non-constant monomial in $x_1, \dots, x_n$ is zero. Then c belongs to $\mathbb{L}^{l-d}[pt]$, where $d = n(n-1)/2$.*

Proof. We will use that the ideal S contains all homogeneous polynomials of degree greater than d with integer coefficients [16, Corollary 2.5.6]. Let $\hat{c}$ be a homogenous element in $\mathbb{L}[x_1, ..., x_n]$ that represents c. Recall that $\mathbb{L}$ is isomorphic to the graded polynomial ring $\mathbb{Z}[a_1, a_2, \dots]$ in countably many variables, where a_i has degree $-i$. Therefore $\hat{c}$ has a unique decomposition as a sum of integral polynomials with coefficients being monomials in the a_i, that is

$$\hat{c} = c_0 + a_1 c_1 + a_2 c_2 + a_1^2 c_{1,1} + a_3 c_3 + a_1 a_2 c_{1,2} + a_1^3 c_{1,1,1} + \dots$$

where $c_{i_1,\dots,i_s}$ is a polynomial of degree $l - \sum \deg(a_{i_j})$ with integer coefficients. Note that we might choose a $\hat{c}$ such that the sum is finite since $c_{i_1,\dots,i_s}$ vanishes modulo S if $l - \sum \deg(a_{i_j}) > d$. Now we multiply $\hat{c}$ with an arbitrary monomial m_{d-l} in the x_i of degree $d - l$. Since $m_{d-l} c = 0$ it follows that $m_{d-l} c_0$ is zero modulo S. By algebraic Poincaré duality [16, Proposition 2.5.7] it follows that $c_0 = 0$ modulo S. Next, we multiply with monomials of degree $d - l - 1$ to deduce that c_1 equals zero modulo S, and then deduce inductively that all the $c_{i_1,\dots,i_s}$ of degree strictly less than d are zero. It remains to note that each $c_{i_1,\dots,i_s}$ of degree d is equal to an integer multiple of $[pt]$ (since all homogeneous polynomials of degree d with integer coefficients are equal to a multiple of $\mathcal{R}_e$ modulo the ideal S [16, 2.5.2]) Hence $\hat{c} = a[pt]$ for some $a \in \mathbb{L}$, which must have degree $l - d$ by homogeneity of $\hat{c}$. □

5. Computations and examples

Until now, we used the formal group law of algebraic cobordism (i.e., the universal one) as little as possible in order to make our presentation simpler. In this section, we make the results of the previous section more explicit using this formal group law. In particular, we give an explicit formula for the products of a Bott-Samelson resolution with the first Chern class of a line bundle in terms of other Bott-Samelson resolutions (see formula 5.1 below). Using this formula, we give an algorithm for computing the product of two Bott-Samelson resolutions.

First, we show that the operator A from Section 2 and the operator $\hat{A}_\alpha$ from Section 3 are well-defined.

We use notation of Subsection 2.1, so $F(u,v)$ is the universal formal group law and $\chi(u)$ is the inverse for the universal formal group law defined by the identity $F(u,\chi(u))=0$. To show that the operator $A=(1+\sigma)\frac{1}{F(y_1,\chi(y_2))}$ is well defined on $\Omega^*(X)[[y_1,y_2]]$ it is enough to show that $A(m)$ is a formal power series for any monomial $m=y_1^{k_1}y_2^{k_2}$. We compute $A(y_1^{k_1}y_2^{k_2})$ using that $y_1=F(x,y_2)=y_2+\chi(x)p(x,y_2)$ and $y_2=F(\chi(x),y_1)=y_1+\chi(x)p(\chi(x),y_1)$ where $x=F(y_1,\chi(y_2))$ and $p(u,v)=\frac{F(u,v)-u}{v}$ is a well-defined power series (since $F(u,v)-u$ contains only terms u^iv^j for $j\geq 1$). We get

$$A(y_1^{k_1}y_2^{k_2})=(1+\sigma)\frac{y_1^{k_1}y_2^{k_2}}{x}=\frac{y_1^{k_1}y_2^{k_2}}{x}+\frac{y_1^{k_2}y_2^{k_1}}{\chi(x)}=\frac{(y_2+\chi(x)p(x,y_2))^{k_1}(y_1+\chi(x)p(\chi(x),y_1))^{k_2}}{x}+\frac{y_2^{k_1}y_1^{k_2}}{\chi(x)}=$$

$$=y_2^{k_1}y_1^{k_2}q(x,\chi(x))+\frac{\text{terms divisible by } x \text{ or by } \chi(x)}{x}.$$

The second term in the last expression is a power series since the formal group law expansion for $\chi(x)$ is divisible by x [14, (2.5.1)].

A similar argument shows that the operator $\hat{A}_\alpha$ from Section 3 is indeed well-defined on the whole ring $\mathbb{L}[[x_1,\dots,x_n]]$ for any root α. Indeed, by relabeling $x_1,\dots,x_n$ we can assume that $\alpha=e_1-e_2$. Then for any monomial $m=x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}$ we have

$$\hat{A}_\alpha(m)=x_3^{k_3}\dots x_n^{k_n}\hat{A}_\alpha(x_1^{k_1}x_2^{k_2}).$$

Then exactly the same argument as the one above for A shows that $\hat{A}_\alpha(x_1^{k_1}x_2^{k_2})$ is a power series in x_1 and x_2.

5.1. **Algorithm for computing the products of Bott-Samelson resolutions.** We now produce an explicit algorithm for computing the product of the Bott-Samelson classes Z_I in terms of other Bott-Samelson classes, where $I=(\alpha_1,...,\alpha_l)$. The key ingredient is our algebraic Chevalley-Pieri formula (Proposition 4.3) which can be reformulated as follows

$$c_1(L(\lambda))A_1\dots A_lZ_e=\sum_{j=1}^{l}A_1\dots A_{j-1}A_j^*(c_1(L(\lambda_j)))A_{j+1}\dots A_lZ_e,$$

where $\lambda_j=s_{j-1}\cdots s_1\lambda$ (in other words, $c_1(L(\lambda_j))=[\sigma_{j-1}\dots\sigma_1](c_1(L(\lambda))))$ and the operator A_j^* is defined as follows

$$A_j^*=A_{\alpha_j}^*=\frac{1}{c_1(L(\alpha_j))}(1-\sigma_{\alpha_j}).$$

We can compute A_j^* on any polynomial in the first Chern classes by the same methods as A_j (see the end of Section 3). Note that for the Chow ring $A_j = A_j^*$ (this follows from Lemma 3.4 and the fact that $s_j\alpha_j = -\alpha_j$ and $c_1(L(\alpha_j)) = -c_1(L(-\alpha_j))$ for the additive formal group law), but for the algebraic cobordism ring this is no longer true.

More generally, for any polynomial $f = f(c_1(L(\mu_1)), \dots, c_1(L(\mu_k)))$ in the first Chern classes of some line bundles on X, we can compute its product with $A_1 \dots A_l Z_e$ by exactly the same argument as in the proof of Proposition 4.3:

$$f \cdot A_1 \dots A_l Z_e = \sum_{j=1}^{l} A_1 \dots A_{j-1}[A_j^* \sigma_{j-1} \dots \sigma_1](f) A_{j+1} \dots A_l Z_e + A_1 \dots A_l [\sigma_l \dots \sigma_1](f) Z_e \quad (5.0)$$

Note that the last term on the right hand side is equal to the constant term of the polynomial $[\sigma_l \dots \sigma_1](f)$ (which is of course the same as the constant term of f) times $A_1 \dots A_l Z_e$. In particular, for $f = c_1(L(\lambda))$ this term vanishes modulo S. Here and below, by the "constant term" of a polynomial in $\mathbb{L}[x_1, \dots, x_n]$ we mean the term of polynomial degree zero (the total degree of such a constant term might be negative since the Lazard ring $\mathbb{L}$ contains elements of negative degree). Note that all elements of $\mathbb{L} \subset \mathbb{L}[x_1, \dots, x_n]$ are invariant under the operators σ_i, and hence commute with the operators A_i. For an arbitrary reductive group, the constant term of an element $f \in \Omega^*(X)$ is defined as the product of f with the class of a point.

It is now easy to show by induction on l that

$$f A_1 \dots A_l Z_e = \sum_{J \subset I} a_J(f) [\prod_{i \in I \setminus J} A_i] Z_e,$$

where $a_J(f)$ for the k-subtuple $J = (\alpha_{j_1}, \dots, \alpha_{j_k})$ of I is the constant term in the expansion for $[\sigma_l \dots \sigma_{j_k+1} A_{j_k}^* \sigma_{j_k-1} \dots \sigma_{j_1+1} A_{j_1}^* \sigma_{j_1-1} \dots \sigma_1] f$, which is invariant under σ_i (for all i) and hence equal to $[A_{j_k}^* \sigma_{j_k-1} \dots \sigma_{j_1+1} A_{j_1}^* \sigma_{j_1-1} \dots \sigma_1] f$. Indeed, we first use formula (5.0) above and then apply the induction hypothesis to all terms in the right hand side except for the last term, which already has form $a_J(f)[\prod_{i \in I \setminus J} A_i] Z_e$ for $J = \emptyset$. We get

$$A_1 \dots A_{j-1}[A_j^* \sigma_{j-1} \dots \sigma_1](f) A_{j+1} \dots A_l Z_e =$$

$$= A_1 \dots A_{j-1} \sum_{J \subset I \setminus \{1,\dots,j\}} a_J([A_j^* \sigma_{j-1} \dots \sigma_1](f)) [\prod_{i \in I \setminus (J \cup \{1,\dots,j\})} A_i] Z_e =$$

$$= \sum_{J' \subset I} a_{J'}(f) [\prod_{i \in I \setminus J'} A_i] Z_e,$$

where the last summation goes over all subsets J' of I that do contain j but do not contain $1,\dots,j-1$. Plugging this back into formula (5.0) we get the desired formula. Combining this with Theorem 3.2, we get the following formula in $\Omega^*(X)$ for the product of the Bott-Samelson class Z_I with the first Chern class $c_1(L(\lambda))$ in terms of other Bott-Samelson classes

$$c_1(L(\lambda)) Z_I = \sum_{J \subset I} b_J(\lambda) Z_{I \setminus J}, \quad (5.1)$$

where $b_J(\lambda)$ is the constant term in the expansion for

$$[A_{j_1}^* \sigma_{j_1+1} \dots \sigma_{j_k-1} A_{j_k}^* \sigma_{j_k+1} \dots \sigma_l](c_1(L(\lambda))).$$

We changed the order of the σ_i when passing from a_J to b_J since $Z_I = A_l \dots A_1 Z_e$. Note that for $J = \emptyset$ we have $b_J = 0$, and for $J = (\alpha_j)$ we have $b_J = (\lambda, \beta_j)$ since the constant term in $A_j^*(c_1(L(s_{j+1} \dots s_l \lambda))) = \frac{c_1(L(s_{j+1} \dots s_l \lambda)) - c_1(L(s_j s_{j+1} \dots s_l \lambda))}{c_1(L(\alpha_j))}$ is equal to $(s_{j+1} \dots s_l \lambda, \alpha_j)$ (see the proof of Proposition 4.3, and Proposition 4.1 for the definition of β_i), which is equal to (λ, β_j). So the lowest order terms (with respect to the polynomial grading) of this formula give an analogous formula for the Chow ring as expected.

We now have assembled all necessary tools for actually performing the desired Schubert calculus. Namely, to compute the product $Z_I Z_J$ we apply the following procedure (which is formally similar to the one for the Chow ring). We replace Z_J with the respective polynomial $\mathcal{R}_J$ in the first Chern classes (using Theorem 3.2 together with the formula for Z_e) and then compute the product of Z_I with each monomial in $\mathcal{R}_J$ using repeatedly formula (5.1). Note that formula (5.1) allows us to make this algorithm more explicit than the one given in [4] (see an example below).

The naive approach to represent both Z_I and Z_J as fractions of polynomials in first Chern classes and then computing their product is less useful. In particular translating the product of the fractions back into a linear combination of Bott-Samelson classes will be very hard, if possible at all.

5.2. **Examples.** We now compute the Bott-Samelson classes Z_I in terms of the Chern classes x_i for the example $X = SL_3/B$ where B is the subgroup of upper-triangular matrices. We then compute certain products of Bott-Samelson classes in two ways, by hand and then using the algorithm above together with formula (5.1). Note that only the second approach generalizes to higher dimensions.

In SL_3, there are two simple roots γ_1 and γ_2. In X, there are six Schubert cycles $X_e = pt$, X_1, X_2, X_{12}, X_{21} and $X_{121} = X$ (here 12 is a short hand notation for (γ_1, γ_2), etc.). Each X_I except for X_{121} coincides with its Bott-Samelson resolution R_I. Note that in general R_I and X_I do not coincide even when X_I is smooth. (By the way, for $G = GL_n$ the first non-smooth Schubert cycles show up for $n = 4$.)

Computing Z_I as a polynomial in the first Chern classes. We want to express Z_I as a polynomial in x_1, x_2, x_3 using the formulas

$$Z_{s_{i_1} \dots s_{i_l}} = A_{i_l} \dots A_{i_1} \mathcal{R}_e; \quad \mathcal{R}_e = \frac{1}{6} c_1(L(\gamma_1)) c_1(L(\gamma_2)) c_1(L(\gamma_1 + \gamma_2)).$$

Note that in computations involving the operators A_α it is more convenient not to replace $c_1(L(\alpha))$ with its expression in terms of x_i until the very end.

Let us for instance compute R_1 as a polynomial in x_1, x_2, x_3 modulo the ideal S generated by the symmetric polynomials of positive degree:

$$\mathcal{R}_1 = A_1 \mathcal{R}_e = \frac{1}{6}(1 + s_1) c_1(L(\gamma_2)) c_1(L(\gamma_1 + \gamma_2)) =$$

$$\frac{1}{3} c_1(L(\gamma_2)) c_1(L(\gamma_1 + \gamma_2)) = \frac{1}{3} F(\chi(x_2), x_3) F(\chi(x_1), x_3).$$

We have $\chi(u) = -u + a_{11}u^2 - a_{11}^2u^3$ and $F(u,v) = u + v + a_{11}uv + a_{12}u^2v + a_{21}uv^2$, where $a_{11} = -[\mathbb{P}^1]$ and $a_{12} = a_{21} = [\mathbb{P}^1]^2 - [\mathbb{P}^2]$ [14, 2.5]. Thus

$$\frac{1}{3}F(\chi(x_2),x_3)F(\chi(x_1),x_3) = \frac{1}{3}F(-x_2 + a_{11}x_2^2 - a_{11}x_2^3, x_3)F(-x_1 + a_{11}x_1^2 - a_{11}x_1^3, x_3) =$$

$$= \frac{1}{3}(-x_2 + x_3 + a_{11}x_2^2 - a_{11}x_2x_3)(-x_1 + x_3 + a_{11}x_1^2 - a_{11}x_1x_3) = x_3^2,$$

since $(x_3 - x_2)(x_3 - x_1) = 3x_3^2 \mod S$, and $(x_2 + x_1)(x_2 - x_3)(x_1 - x_3) = -3x_3^3 = 0 \mod S$. So the answer agrees with the one we got in Remark 2.7.

Here are the polynomials for the other Bott-Samelson resolutions:

$$\mathcal{R}_{212} = 1 + a_{12}x_1^2; \quad \mathcal{R}_{121} = 1 + a_{12}x_1x_2$$

$$\mathcal{R}_{12} = -x_1 - [\mathbb{P}^1]x_1^2 \quad \mathcal{R}_{21} = x_3 = -x_1 - x_2$$

$$\mathcal{R}_1 = x_3^2 = x_1x_2 \quad \mathcal{R}_2 = x_1^2$$

$$\mathcal{R}_e = -x_1^2x_2.$$

Note that the Bott-Samelson resolutions R_I in this list coincide with the Schubert cycles they resolve if I has length ≤ 2. The corresponding polynomials $\mathcal{R}_I$ are the classical Schubert polynomials (see e.g. [16] and keep in mind that his x_i is equal to our $-x_i$) except for the polynomial $\mathcal{R}_{12}$.

In general, polynomials $\mathcal{R}_I$ can be computed by induction on the length of I. E.g. to compute $\mathcal{R}_{212}$ we can use that $\mathcal{R}_{212} = A_2\mathcal{R}_{21}$ and $\mathcal{R}_{21} = x_3$. Hence,

$$\mathcal{R}_{212} = A_2(x_3) = 1 + a_{12}x_2x_3 = 1 + a_{12}x_1^2$$

The middle equation is obtained using the formula $A(y_1) = 1 + a_{12}y_1y_2 + \dots$ from Section 2.1 and the observation that all symmetric polynomials in x_2 and x_3 of degree greater than 2 vanish modulo S.

Computing products of the Bott-Samelson resolutions. Let us for instance compute $Z_{12}Z_{21}$. First, we do it by hand.

Denote by ω_1, ω_2 the fundamental weights of SL_3. Applying Proposition 4.1(2) to $X_{121} = X$ we get

$$L(\lambda) = L(X_{21})^{(\lambda,\gamma_2)} \otimes L(X_{12})^{(\lambda,\gamma_1)}.$$

Hence, $c_1(L(\omega_1)) = X_{12}$ and $c_1(L(\omega_2)) = X_{21}$. (Note that if we instead applied Proposition 4.1(1) to R_{121}, we would obtain the more complicated expression

$$r_{121}^*L(\lambda) = L(R_{21})^{(\lambda,\gamma_2)} \otimes L(R_{11})^{(\lambda,\gamma_1+\gamma_2)} \otimes L(R_{12})^{(\lambda,\gamma_1)},$$

which does not allow us to express $X_{12} = R_{12}$ as the Chern class of the line bundle $L(\lambda)$ on R_{121}.)

Hence,

$$Z_{12}Z_{21} = c_1(L(\omega_1))Z_{21} = r_{21*}c_1(r_{21}^*L(\omega_1))$$

by the projection formula: $c_1(L(\lambda)) \cdot Z_I = r_{I*}c_1(r_I{}^*L(\lambda))$. We now apply Proposition 4.1(1) to R_{21} and $L(\omega_1)$ and get $r_{21}^*L(\omega_1) = L(R_1) \otimes L(R_2)$. Using the formal group law we compute $c_1(L(R_1) \otimes L(R_2)) = R_1 + R_2 - [\mathbb{P}^1]R_e$. Finally, we use that $r_{J*}[R_J] = Z_J$ and get that

$$Z_{12}Z_{21} = Z_1 + Z_2 - [\mathbb{P}^1]Z_e.$$

Similarly, we can easily compute the following products:

$$Z_{12}Z_{12} = Z_2; \quad Z_{21}Z_{21} = Z_1$$
$$Z_{12}Z_1 = Z_{21}Z_2 = Z_e, \quad Z_{12}Z_2 = Z_{21}Z_1 = 0,$$

which in particular gives us another way to compute polynomials $\mathcal{R}_I$.

So the only product that differs from the analogous product in the Chow ring case is the product $Z_{12}Z_{21}$.

We now compute the product $Z_{12}Z_{21}$ using formula (5.1). We have $Z_{12} = c_1(L(\omega_1))$ by Proposition 4.1(1). Hence, according to formula (5.1)

$$Z_{12}Z_{21} = c_1(L(\omega_1))Z_{21} = b_1(\omega_1)Z_2 + b_2(\omega_1)Z_1 + b_{21}(\omega_1)Z_e,$$

where b_1, b_2 and b_{21} are the constant terms in $A_1^*(c_1(L(\omega_1)))$, $[A_2^*s_1](c_1(L(\omega_1)))$ and $[A_2^*A_1^*](c_1(L(\omega_1)))$, respectively. We already know that $b_1(\lambda) = (\lambda, \gamma_1)$ and $b_2(\lambda) = (\lambda, s_1\gamma_2)$. It remains to compute $b_{21}(\lambda)$. First, by using that $L(\lambda) = L(s_1\lambda) \otimes L(\gamma_1)^{(\lambda,\gamma_1)}$ and the formal group law we write

$$A_1^*(c_1(L(\lambda))) = \frac{c_1(L(\lambda)) - c_1(L(s_1\lambda))}{c_1(L(\gamma_1))} =$$
$$= (\lambda, \gamma_1) + a_{11}(\lambda, \gamma_1)[c_1(L(s_1\lambda)) + \frac{(\lambda, \gamma_1) - 1}{2}c_1(L(\gamma_1))] + \text{terms of deg} \geq 2$$

Hence,

$$[A_2^*A_1^*](c_1(L(\lambda))) = a_{11}(\lambda, \gamma_1)A_2^*[c_1(L(s_1\lambda)) + \frac{(\lambda, \gamma_1) - 1}{2}c_1(L(\gamma_1))] + \text{terms of deg} \geq 1 =$$
$$= a_{11}(\lambda, \gamma_1)[(\lambda, s_1\gamma_2) - \frac{(\lambda, \gamma_1) - 1}{2}] + \text{terms of deg} \geq 1,$$

and $b_{21} = a_{11}(\lambda, \gamma_1)[(\lambda, s_1\gamma_2) - \frac{(\lambda,\gamma_1)-1}{2}]$. We get

$$c_1(L(\lambda))Z_{21} = (\lambda, \gamma_1)Z_2 + (\lambda, s_1\gamma_2)Z_1 + a_{11}(\lambda, \gamma_1)[(\lambda, s_1\gamma_2) - \frac{(\lambda, \gamma_1) - 1}{2}]Z_e.$$

In particular, $c_1(L(\omega_1))Z_{21} = Z_2 + Z_1 + a_{11}Z_e$ (which coincides with the answer we have found above by hand).

Finally, note that it takes more work to compute $c_1(L(\lambda))Z_{21}$ using the algorithm in [4] because apart from certain formal group law calculations (which are more involved than the calculations we used to find b_{21}) one has also to compute the products R_1^2 and R_2^2 in $CH^*(R_{21})$.

6. Appendix: Complex realization for cellular varieties

We will now prove the following result stated in Section 2:

Theorem 6.1. *For any smooth cellular variety X over k and any embedding $k \to \mathbb{C}$, the complex geometric realization functor of $\mathbb{L}$-algebras $r : \Omega^*(X) \to MU^*(X(\mathbb{C})^{an})$ is an isomorphism.*

Proof. Recall (see above) that the geometric realization functor coincides with the map given by the universal property of Ω^*, and that both sides are freely generated by (resolutions of the closures of) the cells. Thus it suffices to show that it is an isomorphism if we pass to the induced morphism after taking $\otimes_{\mathbb{L}}\mathbb{Z}$ on both sides, which we denote by r'. Now by a theorem of Totaro [20, Theorem 3.1] (compare also [14, Remark 1.2.21]), for cellular varieties

the classical cycle class map $c : CH^*(X) \to H^*(X(\mathbb{C})^{an})$ (which is an isomorphism for cellular varieties X) defined using fundamental classes and Poincaré duality (see e. g. [16, section A.3]) factors as $CH^*(X) \to MU^*(X(\mathbb{C})^{an}) \otimes_{\mathbb{L}} \mathbb{Z} \cong H^*(X(\mathbb{C})^{an})$, and the left arrow in this factorization is given by first taking any resolution of singularities of the algebraic cycle and then applying $(\mathbb{C})^{an}$. We also have a morphism $q : \Omega^*(X) \to CH^*(X)$ which induces an isomorphism $q' : \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} \to CH^*(X)$ by Levine-Morel [14, Theorem 1.2.19] and corresponds to resolution of singularities [14, Section 4.5.1]. Putting everything together, we obtain a commutative square

$$\begin{array}{ccc} \Omega^*(X) \otimes_{\mathbb{L}} \mathbb{Z} & \xrightarrow{r'} & MU^*(X(\mathbb{C})^{an}) \otimes_{\mathbb{L}} \mathbb{Z} \\ {\scriptstyle q'}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f'} \\ CH^*(X) & \xrightarrow[c]{\cong} & H^*(X(\mathbb{C})^{an}) \end{array}$$

with the vertical maps and c being isomorphisms, which finishes the proof. □

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