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An effective version of a theorem of Kawamata on the Albanese map

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To any smooth complex projective variety X are associated an abelian variety $\mathrm{Alb}(X)$ of dimension $q(X) := h^1(X, \mathscr{O}_X)$, its *Albanese variety,* and a morphism $a_X : X \to \mathrm{Alb}(X)$, the *Albanese map,* which are very useful tools to study the geometry of X.

Kawamata proved in [K] that when the Kodaira dimension $\kappa(X)$ is zero, the Albanese map is an algebraic fiber space, which means that:

- a_X is surjective;
- the fibers of a_X are connected.

This kind of result (especially the second part) yields for example birational characterizations of abelian varieties: X is birational to an abelian variety if and only if $\kappa(X) = 0$ and $q(X) = \dim(X)$.

However, the vanishing of $\kappa(X)$ is not an effective condition (it means that the plurigenera $P_m(X) := h^0(X, \omega_X^m)$ are all 0 or 1 when $m > 0$ and that one of them is 1). It is therefore natural to try to prove the same result with weaker and effective assumptions on the plurigenera of X.

For the surjectivity of a_X, this was done in a series of articles initiated by Kollár ([Ko1]), followed by Ein and Lazarsfeld ([EL]) and later by Hacon and Pardini ([HP]) and Chen and Hacon ([CH4]), who proved that a_X is surjective if $0 < P_m(X) \leq 2m - 3$ for some $m \geq 2$, or if $P_3(X) = 4$. We put here the finishing touch to this series by proving the following optimal result (Theorem 2.8).

Theorem *Let X be a smooth complex projective variety. If*

$$0 < P_m(X) \leq 2m - 2$$

for some $m \geq 2$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is surjective.

When C is a smooth projective curve of genus 2, we have $P_m(C) = 2m-1$ for $m \geq 2$. However $a_C : C \to \mathrm{Alb}(C)$ is not surjective. This example shows that without other assumptions, our bound is optimal.

As far as connectedness of the fibers of the Albanese map is concerned, they were no previous results in that direction. The main purpose of this paper is to show that there exists a similar effective criterion for the Albanese morphism to be an algebraic fiber space. More pecisely, we prove the following optimal bound (Theorem 3.1 and Theorem 3.3).

Theorem *Let X be a smooth complex projective variety. If $P_1(X) = P_2(X) = 1$, or if*

$$0 < P_m(X) \leq m - 2$$

for some $m \geq 3$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space.

Hacon and Pardini show in [HP] that for varieties with $P_3(X) = 2$ and $q(X) = \dim(X)$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is a double covering. Hence a_X is surjective but does not have connected fibers. Furthermore, $P_m(X) = m - 1$ for any odd $m \geq 3$. From this example, we see that our result is optimal to a large extent.

As mentioned above, this criterion yields a numerical birational characterization of abelian varieties by adding $q(X) = \dim(X)$ to its hypotheses. The results and constructions developed here also lead to explicit descriptions of varieties with $q(X) = \dim(X)$ and small plurigenera, in the line of the series of papers [CH1], [CH4], [HP], and [H1]. For example, we can get a complete description of varieties with $P_2(X) = 2$ and $q(X) = \dim(X)$. We will come back to this in a future article.

1 Preliminaries

In this section we recall several theorems which will be used later. Throughout this article, we work over the fied of complex numbers and we denote numerical equivalence by $\equiv$.

Vanishing theorem. We state a result of Kollár ([Ko1], 10.15), which was generalized later by Esnault and Viehweg.

Theorem 1.1 (Kollár, Esnault-Viehweg) *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a normal variety Y. Let L be a line bundle on X such that $L \equiv f^*M + \Delta$, where M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y and (X, Δ) is klt. Then,*

a) $R^j f_*(\omega_X \otimes L)$ is torsion free for $j \geq 0$;

b) if in addition, M is big and nef, $H^i(Y, R^j f_*(\omega_X \otimes L)) = 0$ for all $i > 0$ and all $j \geq 0$.

Cohomological support loci. These were first studied by Green and Lazarsfeld for the canonical bundle in [GL1] and [GL2], through their generic vanishing theorems. Simpson also contributed to the subject ([S]).

Let X be a smooth projective variety and let $\mathscr{F}$ be a coherent sheaf on X. The cohomological support loci of $\mathscr{F}$ are defined as

$$V_i(X, \mathscr{F}) = \{P \in \mathrm{Pic}^0(X) \mid H^i(X, \mathscr{F} \otimes P) \neq 0\},$$

which we often write as $V_i(\mathscr{F})$.

GV-objects. These were first considered by Hacon in [H2] and systematically studied by Pareschi and Popa in [PP]. In this paper, we just need to consider GV-sheaves with respect to the universal Poincaré line bundle.

Definition 1.2 *A sheaf $\mathscr{F}$ on X is called a GV-sheaf if*

$$\mathrm{codim}_{\mathrm{Pic}^0(X)} V_i(\mathscr{F}) \geq i$$

for all $i \geq 0$.

Let $a_X : X \to A$ be the Albanese map of X; then $\mathrm{Pic}^0(X)$ is isomorphic to the dual abelian variety $\widehat{A}$. Let M be an ample line bundle on $\widehat{A}$. We denote by $\widehat{M}$ its Fourier-Mukai transform, which is a locally free sheaf on A (see [Mu]). Let $\phi_M : \widehat{A} \to A$ be the standard isogeny induced by M; then $\phi_M^* \widehat{M}^\vee \simeq H^0(M) \otimes M$. Consider the cartesian diagram:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\varphi_M} & X \\ {\scriptstyle a_{\widehat{X}}}\downarrow & & \downarrow{\scriptstyle a_X} \\ \widehat{A} & \xrightarrow{\phi_M} & A \end{array} \tag{1}$$

Hacon proved the following theorem in [H2] (it was later generalized by Pareschi and Popa in [PP] Theorem A):

Theorem 1.3 *Let $\mathscr{F}$ be a coherent sheaf on a smooth projective variety X. If $H^i(\widehat{X}, \varphi_M^* \mathscr{F} \otimes a_{\widehat{X}}^* M) = 0$, for all $i > 0$ and any sufficiently ample M, then $\mathscr{F}$ is a GV-sheaf.*

Finally, the following elementary lemma from [HP] will frequently be used.

Lemma 1.4 *Let X be a smooth projective variety, let L and M be line bundles on X, and let $T \subset \mathrm{Pic}^0(X)$ be a subvariety of dimension t. If for some positive integers a and b and all $P \in T$, we have $h^0(X, L \otimes P) \geq a$ and $h^0(X, M \otimes P^{-1}) \geq b$, then $h^0(X, L \otimes M) \geq a + b + t - 1$.*

2 When is the Albanese map surjective?

In this section I use the language of asymptotic multiplier ideal sheaves. However many of the ideas come from [Ko1], [HP], and [H2].

Lemma 2.1 *Suppose that $f : X \to Y$ is a surjective morphism between smooth projective varieties, L is a $\mathbb{Q}$-divisor on X, and the Iitaka model of (X, L) dominates Y. Assume that D is a nef $\mathbb{Q}$-divisor on Y such that $L + f^*D$ is a divisor on X. Then we have*

$$H^i(Y, R^j f_*(\mathscr{O}_X(K_X + L + f^*D) \otimes \mathscr{J}(||L||) \otimes Q)) = 0,$$

for all $i \geq 1$, $j \geq 0$, and all $Q \in \mathrm{Pic}^0(X)$.

Proof. Let $m > 0$ be such that mL is a divisor and $\mathscr{J}(||L||) = \mathscr{J}(\frac{1}{m}|mL|)$ ([L], §11.2). Let H be a very ample divisor on Y. By assumption there exists an integer $t > 0$ such that $|tmL - f^*H|$ is non-empty. Let $\mu : X' \to X$ be a log resolution such that:

$$\begin{aligned} \mu^*|tmL| &= |L_1| + \sum_i a_i F_i, \\ \mu^*|tmL - f^*H| &= |L_2| + \sum_i b_i F_i, \end{aligned}$$

where $|L_1|$ and $|L_2|$ are base-point-free, $\sum_i a_i F_i$ and $\sum_i b_i F_i$ are the fixed divisors, and $\sum_i F_i + \mathrm{Exc}(\mu)$ is a divisor with simple normal crossings (SNC) support. Since $\mathscr{J}(||L||) = \mathscr{J}(\frac{1}{m}|mL|)$, we also have $\mathscr{J}(||L||) = \mathscr{J}(\frac{1}{tm}|tmL|)$, hence

$$\mathscr{J}(||L||) = \mu_* \mathscr{O}_{X'}\Big(K_{X'/X} - \Big\lfloor \frac{\sum_i a_i F_i}{tm} \Big\rfloor\Big).$$

Take

$$\begin{aligned} B_1 &= D_1 + \sum_i a_i F_i \in \mu^*|tmL| \\ B_2 &= D_2 + \sum_i b_i F_i \in \mu^*|tmL - f^*H| \end{aligned}$$

where $D_1 \in |L_1|$ and $D_2 \in |L_2|$ are general elements, so that $B_1 + B_2$ is a divisor with SNC support. We then show that for $k > 0$ large enough,

$$\left\lfloor \frac{kB_1 + B_2}{(k+1)tm} \right\rfloor = \left\lfloor \frac{\sum_i a_i F_i}{tm} \right\rfloor. \tag{2}$$

It is obvious that $\left\lfloor \frac{kB_1 + B_2}{(k+1)tm} \right\rfloor = \left\lfloor \frac{\sum_i (ka_i + b_i) F_i}{(k+1)tm} \right\rfloor$. We write $\frac{a_i}{tm} = m_i + s_i$ with $m_i = \lfloor \frac{a_i}{tm} \rfloor$. Then,

$$\left\lfloor \frac{\sum_i a_i F_i}{tm} \right\rfloor = \sum_i m_i F_i.$$

Because H is very ample on Y, we have $b_i \geq a_i$. Write $b_i = a_i + c_i$, with $c_i \geq 0$. Then,

$$\left\lfloor \sum_i \frac{(ka_i + b_i)}{(k+1)tm} F_i \right\rfloor = \left\lfloor \sum_i \frac{((k+1)a_i + c_i)}{(k+1)tm} F_i \right\rfloor = \left\lfloor \sum_i (m_i + s_i + \frac{c_i}{(k+1)tm}) F_i \right\rfloor.$$

Since $0 \leq s_i < 1$, we can let $k \geq 0$ be large enough such that $s_i + \frac{c_i}{(k+1)tm} < 1$, and this implies (2). Then by local vanishing ([L], Theorem 9.4.1),

$$\begin{aligned} & R^j f_*(\mathscr{O}_X(K_X + L + f^*D) \otimes \mathscr{J}(||L||) \otimes Q) \\ = \;& R^j (f \circ \mu)_* \big(\mathscr{O}_{X'}\big(K_{X'} + \mu^*L + \mu^*f^*D - \left\lfloor \frac{kB_1 + B_2}{(k+1)tm} \right\rfloor + \mu^*Q\big)\big), \end{aligned} \tag{3}$$

for all $j \geq 0$. We also have

$$\begin{aligned} & \mu^*L + \mu^*f^*D - \left\lfloor \frac{kB_1 + B_2}{(k+1)tm} \right\rfloor + \mu^*Q \\ \equiv \;& \mu^*L + \mu^*f^*D - \mu^*\frac{kL}{k+1} - \mu^*\frac{L}{k+1} + \mu^*f^*\frac{H}{(k+1)tm} + \left\{ \frac{kB_1 + B_2}{(k+1)tm} \right\} \\ \equiv \;& \mu^*f^*\frac{H}{(k+1)tm} + \mu^*f^*D + \left\{ \frac{kB_1 + B_2}{(k+1)tm} \right\}. \end{aligned}$$

So Theorem 1.1 gives us that

$$H^i\big(Y, R^j(f\circ\mu)_*\mathscr{O}_{X'}(K_{X'}+\mu^*L+\mu^*f^*L-\left\lfloor\frac{kB_1+B_2}{(k+1)tm}\right\rfloor+\mu^*Q)\big)=0,$$

for all $i\geq 1$, all $j\geq 0$, and all $Q\in \mathrm{Pic}^0(X)$. By (3), this proves the lemma. □

The following lemma is essentially Proposition 2.12 in [HP]. I use Lemma 2.1 to make the proof a little bit simpler.

Lemma 2.2 *Let $f: X\to Y$ be a surjective morphism between smooth projective varieties and assume that the Iitaka model of X dominates Y. Fix a torsion element $Q\in \mathrm{Pic}^0(X)$ and an integer $m\geq 2$. Then $h^0(X,\omega_X^m\otimes Q\otimes f^*P)$ is constant for all $P\in\mathrm{Pic}^0(Y)$.*

Proof. We consider $h^0(X,\omega_X^m\otimes Q\otimes f^*P)$ as a function of $P\in\mathrm{Pic}^0(Y)$. Let $P_0\in\mathrm{Pic}^0(Y)$ be such that $h^0(X,\omega_X^m\otimes Q\otimes f^*P_0)=h$ is maximal. We are going to prove that

$$h^0(X,\omega_X^m\otimes Q\otimes f^*P_0\otimes f^*P)=h,$$

for any torsion $P\in\mathrm{Pic}^0(Y)$. Since $P_0+\{\text{torsion points}\}$ is dense in $\mathrm{Pic}^0(Y)$, we then deduce the lemma from semicontinuity.

Let P_1, P_2, and Q_1 be such that $P_1^m=P_0$, $P_2^m=P$ and $Q_1^m=Q$. From the properties of asymptotic multiplier ideal sheaves ([L], Theorem 11.1.8), we know that

$$\begin{aligned}
& H^0(X,\omega_X^m\otimes Q\otimes f^*P_0\otimes f^*P)\\
=\ & H^0\big(X,\omega_X^m\otimes Q\otimes f^*P_0\otimes f^*P\otimes\mathscr{J}(||\omega_X^m\otimes Q_1^m\otimes f^*P_1^m\otimes f^*P_2^m||)\big)\\
=\ & H^0\big(X,\omega_X^m\otimes Q\otimes f^*P_0\otimes f^*P\otimes\mathscr{J}(||\omega_X^{m-1}\otimes Q_1^{m-1}\otimes f^*P_1^{m-1}\otimes f^*P_2^{m-1}||)\big).
\end{aligned}$$

Since P is a torsion point, there exists $N>0$ such that $P^N=\mathscr{O}_Y$. For $k>0$ large enough and divisible, we have

$$\begin{aligned}
& \mathscr{J}(||\omega_X^{m-1}\otimes Q_1^{m-1}\otimes f^*P_1^{m-1}\otimes f^*P_2^i||)\\
=\ & \mathscr{J}(\frac{1}{kN}|(\omega_X^{m-1}\otimes Q_1^{m-1}\otimes f^*P_1^{m-1}\otimes f^*P_2^i)^{kN}|)\\
=\ & \mathscr{J}(||\omega_X^{m-1}\otimes Q_1^{m-1}\otimes f^*P_1^{m-1}||),
\end{aligned}$$

for all $i \geq 0$. Hence we have

$$\begin{aligned} & H^0(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P) \\ = \; & H^0\big(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||)\big) \\ = \; & H^0\big(Y, f_*\big(\omega_X^m \otimes Q_1^{m-1} \otimes f^*P_1^{m-1} \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||) \\ & \qquad \otimes Q_1 \otimes f^*P_1 \otimes f^*P\big)\big). \end{aligned}$$

We then apply Lemma 2.1 (the Iitaka model of $(X, \omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1})$ dominates Y by assumption) to get that

$$h^0(X, \omega_X^m \otimes Q \otimes f^*P_0 \otimes f^*P) = \chi\big(Y, f_*\big(\omega_X^m \otimes Q \otimes \mathscr{J}(||\omega_X^{m-1} \otimes Q_1^{m-1} \otimes f^*P_1^{m-1}||)\big)\big)$$

is the constant h. $\square$

Lemma 2.3 *Suppose that $f : X \to Z$ is an algebraic fiber space between smooth projective varieties. Assume that $P_m(X) \neq 0$, for some $m \geq 2$, that H is a big $\mathbb{Q}$-divisor on Z, and that K is a nef $\mathbb{Q}$-divisor on Z such that $H_1 \equiv H + K$ is a big and nef divisor. Then,*

1) we have

$$\begin{aligned} & H^i\big(Z, R^j f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z} + f^*H_1) \\ & \qquad \qquad \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big) \otimes P\big) = 0, \end{aligned}$$

for all $i \geq 1$, $j \geq 0$ and all $P \in \mathrm{Pic}^0(Z)$.

2) the sheaf

$$f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big)$$

has rank $P_m(X_z)$, where X_z is a general fiber of f.

Proof. The point here is the weak positivity of $f_*(\omega_{X/Z}^{m-1})$, due to Viehweg ([V2] Theorem 4.1 and Corollary 7.1, or [Ko1] Proposition 10.2). There are two conclusions:

A. the Iitaka model of $(X, (m-1)K_{X/Z} + f^*H)$ dominates Z and

B. there exists $k > 0$ sufficient big and divisible such that the restriction:

$$H^0(X, \mathscr{O}_X(km(m-1)K_{X/Z}+kmf^*H)) \to H^0(X_z, \mathscr{O}_{X_z}(km(m-1)K_{X_z}))$$

is surjective, where $z \in Z$ is a general point.

By A, we can directly apply Lemma 2.1 to deduce item 1) in the lemma.

We take a log resolution $\tau : X^{'} \to X$ such that the restriction $\tau_z : X^{'}_z \to X_z$ is also a log resolution for sufficiently general $z \in Z$ (see [L], Theorem 9.5.35) and fix such a point $z \in Z$. Set

- $\tau^*|km(m-1)K_{X/Z} + kmf^*H| = |L_1| + E_1$,
- $\tau_z^*|mK_{X_z}| = |L_2| + E_2$,

where $|L_1|$ and $|L_2|$ are base-point-free, E_1 and E_2 are the fixed divisors, and $E_1 + \mathrm{Exc}(\tau)$ has SNC support. We have

$$E_1|_{X_z'} \preceq k(m-1)E_2 \tag{4}$$

by B. Let $f^{'} : X^{'} \xrightarrow{\tau} X \xrightarrow{f} Z$ be the composition of morphisms. Then $f^{'}$ is flat over a dense Zariski open subset of Z. Hence the sheaf

$$f^{'}_*\big(\mathscr{O}_{X'}(K_{X'} + (m-1)\tau^*K_{X/Z} - \left\lfloor \frac{E_1}{km} \right\rfloor)\big)$$

has rank

$$h^0\big(X_z^{'}, \mathscr{O}_{X_z'}\big(mK_{X_z'} - \left\lfloor \frac{E_1}{km} \right\rfloor |_{X_z'}\big)\big) = P_m(X_z).$$

We have the following inclusions

$$\begin{aligned} & f_*\tau_*\mathscr{O}_{X'}\Big(K_{X'} + (m-1)\tau^*K_{X/Z} - \left\lfloor \frac{E_1}{km} \right\rfloor\Big) \\ \subset\ & f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big) \\ \subset\ & f_*(\mathscr{O}_X(mK_X)) \otimes \mathscr{O}_Z(-(m-1)K_Z). \end{aligned}$$

Since the latter sheaf has rank $P_m(X_z)$, the middle sheaf $f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + f^*H||)\big)$ also has rank $P_m(X_z)$. $\square$

Under the assumptions of Lemma 2.3, we fix a big and base-point-free divisor H. For $n > 0$, we set

$$\begin{aligned} \mathscr{J}_{m-1,n} &= \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n}f^*H||) \\ \mathscr{F}_{m-1,n} &= f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}\big). \end{aligned}$$

By Lemma 2.3, $\mathscr{F}_{m-1,n}$ has rank $P_m(X_z) > 0$. These sheaves were first considered by Hacon in [H2].

Lemma 2.4 *We have $\mathscr{J}_{m-1,n} \supset \mathscr{J}_{m-1,n+1}$ and there exists $N > 0$ such that for any $n \geq N$, one has $\mathscr{F}_{m-1,n} = \mathscr{F}_{m-1,N}$. We will denote by $\mathscr{F}_{m-1,H}$ the fixed sheaf $\mathscr{F}_{m-1,N}$.*

Proof. We may suppose that $k > 0$ is such that the linear series $|k(n+1)n((m-1)K_{X/Z} + \frac{1}{n}f^*H)|$ and $|k(n+1)n((m-1)K_{X/Z} + \frac{1}{n+1}f^*H)|$ compute $\mathscr{J}_{m-1,n}$ and $\mathscr{J}_{m-1,n+1}$, respectively. Let $\tau : X' \to X$ be a log resolution for both linear series. We can write

$$\begin{aligned} \tau^*|k(n+1)n(m-1)K_{X/Z} + k(n+1)f^*H| &= |L_1| + E_1, \\ \tau^*|k(n+1)n(m-1)K_{X/Z} + knf^*H| &= |L_2| + E_2, \end{aligned}$$

where L_1 and L_2 are base-point-free and E_1 and E_2 are fixed divisors. Since H is base-point-free, we have $E_2 \succeq E_1$. By the definition of asymptotic multiplier ideal sheaves, $\mathscr{J}_{m-1,n} \supset \mathscr{J}_{m-1,n+1}$.
Take H_1 very ample on Z such that $H_1 - H$ is a nef divisor. Then by Lemma 2.3, we have

$$H^i(Z, f_*\big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}\big) \otimes \mathscr{O}_Z(H_1)) = 0,$$

for $i \geq 1$. Using Hacon's argument in the proof of Proposition 5.1 in [H2], there exists $N > 0$ such that for $n \geq N$, the inclusion

$$\begin{aligned} & f_*(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,N}) \otimes \mathscr{O}_Z(H_1) \\ \supset \ & f_*(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}_{m-1,n}) \otimes \mathscr{O}_Z(H_1) \end{aligned}$$

is an equality. This implies that the inclusion

$$f_*(\mathscr{O}_X(K_X+(m-1)K_{X/Z})\otimes\mathscr{J}_{m-1,N}) \supset f_*(\mathscr{O}_X(K_X+(m-1)K_{X/Z})\otimes\mathscr{J}_{m-1,n})$$

is again an equality. □

Lemma 2.5 *Under the above assumptions, namely $f : X \to Z$ is an algebraic fiber space between smooth projective varieties and $P_m(X) \neq 0$ with $m \geq 2$, we suppose moreover that Z is of maximal Albanese dimension and that H is a big and base-point-free divisor on Z pulled back from $\mathrm{Alb}(Z)$. Then $\mathscr{F}_{m-1,H}$ is a nonzero GV-sheaf.*

PROOF. We apply Theorem 1.3. Let M be any ample divisor on $\mathrm{Pic}^0(Z)$. We have cartesian diagrams as in (1):

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\upsilon_M} & X \\ {\scriptstyle \widehat{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \widehat{Z} & \xrightarrow{\varphi_M} & Z \\ {\scriptstyle a_{\widehat{Z}}}\downarrow & & {\scriptstyle a_Z}\downarrow \\ \mathrm{Pic}^0(Z) & \xrightarrow{\phi_M} & \mathrm{Alb}(Z) \end{array}$$

where horizontal maps are étale. By Theorem 11.2.16 in [L], for any $n > 0$,

$$\upsilon_M^* \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n} f^* H||) = \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||),$$

hence by flat base change

$$\begin{aligned} & \varphi_M^* f_* \big(\mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes \mathscr{J}(||(m-1)K_{X/Z} + \frac{1}{n} f^* H||)\big) \\ = \ & \widehat{f}_* \big(\mathscr{O}_{\widehat{X}}(K_{\widehat{X}} + (m-1)K_{\widehat{X}/\widehat{Z}}) \otimes \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||)\big). \end{aligned}$$

It follows that

$$\varphi_M^* \mathscr{F}_{m-1,H} = \widehat{f}_* \big(\mathscr{O}_{\widehat{X}}(K_{\widehat{X}} + (m-1)K_{\widehat{X}/\widehat{Z}}) \otimes \mathscr{J}(||(m-1)K_{\widehat{X}/\widehat{Z}} + \frac{1}{n}\widehat{f}^* \varphi_M^* H||)\big)$$

for all $n \gg 0$. Since H is a divisor pulled back by a_Z, we can take n such that $n a_{\widehat{Z}}^* M - \varphi_M^* H$ is nef. Then Lemma 2.3 gives us the vanishing of

$$H^i(\widehat{Z}, \varphi_M^* \mathscr{F}_{m-1,H} \otimes a_{\widehat{Z}}^* M),$$

for all $i > 0$ and we are done. □

Lemma 2.6 *In the situation of Lemma 2.5, denoting by $a_Z : Z \to A$ the Albanese morphism of Z, we have $R^j a_{Z*}(\mathscr{F}_{m-1,H}) = 0$, for all $j > 0$. Hence*

$$V_i(\mathscr{F}_{m-1,H}) = V_i(a_{Z*}(\mathscr{F}_{m-1,H})),$$

for all $i \geq 0$.

Proof. Suppose that $R^t a_{Z*}(\mathscr{F}_{m-1,H}) \neq 0$ for some $t > 0$. Let H_1 be a ample divisor on A such that

$$H^k(A, R^j a_{Z*}(\mathscr{F}_{m-1,H}) \otimes \mathscr{O}_A(H_1)) = 0$$

for all $k \geq 1$ and $j \geq 0$ and

$$H^0(A, R^t a_{Z*}(\mathscr{F}_{m-1,H}) \otimes \mathscr{O}_A(H_1)) \neq 0.$$

By the Leray spectral sequence, we have

$$H^t(Z, \mathscr{F}_{m-1,H} \otimes \mathscr{O}_Z(a_Z^* H_1)) \neq 0.$$

Since H is pulled back from A, we may take H_1 such that $a_Z^* H_1 - H$ is big and nef, then by Lemma 2.3, we have $H^t(Z, \mathscr{F}_{m-1,H} \otimes \mathscr{O}_Z(a_Z^* H_1)) = 0$, which is a contradiction. Thus $R^j a_{Z*}(\mathscr{F}_{m-1,H}) = 0$ for all $j > 0$. For any $P \in \mathrm{Pic}^0(Z)$, we have $H^i(Z, \mathscr{F}_{m-1,H} \otimes a_Z^* P) \simeq H^i(A, a_{Z*}(\mathscr{F}_{m-1,H}) \otimes P)$, hence $V_i(\mathscr{F}_{m-1,H}) = V_i(a_{Z*}(\mathscr{F}_{m-1,H}))$ for all $i \geq 0$. $\square$

Corollary 2.7 *The cohomological support $V_0(\mathscr{F}_{m-1,H})$ is not empty.*

Proof. By Lemma 2.5, $\mathscr{F}_{m-1,H}$ is a GV-sheaf, hence ([H2], Corollary 3.2)

$$V_0(\mathscr{F}_{m-1,H}) \supset V_1(\mathscr{F}_{m-1,H}) \supset \cdots \supset V_d(\mathscr{F}_{m-1,H}).$$

If $V_0(\mathscr{F}_{m-1,H})$ is empty, $V_i(\mathscr{F}_{m-1,H})$ is empty for all $i \geq 0$, hence

$$H^i(Z, \mathscr{F}_{m-1,H} \otimes a_Z^* P) = H^i(A, a_{Z*}\mathscr{F}_{m-1,H} \otimes P) = 0,$$

for all $i \geq 0$. By the properties of the Fourier-Mukai transform on an abelian variety (see [Mu]), $a_{Z*}\mathscr{F}_{m-1,H} = 0$. However this is impossible since a_Z is generically finite and $\mathscr{F}_{m-1,H}$ is a sheaf with positive rank. $\square$

Theorem 2.8 *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq 2m-2,$$

for some $m \geq 2$, the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is surjective.

Proof. If a_X is not surjective, by Ueno's theorem ([M], Theorem (3.7)), upon replacing X by a birational model, there exists a surjective morphism $f_1 : X \to Z_1$ onto a smooth variety Z_1 of general type of dimension $d > 0$ such that $Z_1 \to \mathrm{Alb}(Z_1)$ is a birational map onto its image and $Z_1 \to \mathbb{P}(H^0(Z_1, \mathscr{O}_{Z_1}(K_{Z_1})))$ is a map generically finite onto its image. Obviously, $P_k(Z_1) \geq \binom{d+k}{d}$ for all $k \geq 1$. Taking the Stein factorization and making birational modifications, we may suppose that there is an algebraic fiber space $f : X \to Z$ such that Z is a smooth variety of general type and of maximal Albanese dimension d, and $P_k(Z) \geq \binom{d+k}{k}$ for all $k \geq 1$.

We let H be a big and base-point-free divisor pulled back by the Albanese morphism $a_Z : Z \to \mathrm{Alb}(Z)$. By Corollary 2.7, $V_0(\mathscr{F}_{m-1,H})$ is not empty thus there exists $P \in \mathrm{Pic}^0(Z)$ such that $h^0(Z, \mathscr{F}_{m-1,H} \otimes P) \geq 1$. Hence

$$h^0(X, \mathscr{O}_X(K_X + (m-1)K_{X/Z}) \otimes f^*P) \geq 1. \tag{5}$$

On the other hand, we have $h^0(X, \mathscr{O}_X((m-1)f^*K_Z)) \geq \binom{d+m-1}{m-1}$. We get

$$h^0(X, \mathscr{O}_X(mK_X) \otimes f^*P) \geq \binom{d+m-1}{m-1}. \tag{6}$$

Since Z is of general type, the Iitaka model of (X, K_X) dominates Z because of (5), hence we apply Lemma 2.2 to get $h^0(X, \mathscr{O}_X(mK_X)) \geq \binom{d+m-1}{m-1}$.

If $\dim(Z) = d \geq 2$, then $P_m(X) \geq \binom{m+1}{2} \geq 2m-1$, which is a contradiction.

If $\dim(Z) = 1$, $P_m(X) = h^0(Z, f_*(\omega_{X/Z}^m) \otimes \omega_Z^m)$. As in Corollary 3.6 in [V1], $f_*(\omega_{X/Z}^m)$ is a nonzero nef vector bundle on Z hence has nonnegative degree. By the Riemann-Roch theorem, we obtain $P_m(X) \geq 2m-1$, again a contradiction. $\square$

Remark 2.9 The proof follows ideas of Kollár's ([Ko1]), later improved by Hacon and Pardini. Briefly speaking, Kollár proved that $P_m(X) \geq P_{m-2}(Z)$ and Hacon and Pardini used the finite map

$$|(m-2)K_Z + P| \times |K_X + (m-1)K_{X/Z} + K_Z - f^*P| \to |mK_X|,$$

where $P \in \mathrm{Pic}^0(Z)$, to give a better estimate of $P_m(X)$. However, the dimension $h^0(Z, \mathscr{O}_Z(kK_Z))$ grows very fast with k, so my starting point was to prove $P_m(X) \geq P_{m-1}(Z)$ by applying the theory of GV-sheaves.

Corollary 2.10 *Suppose that $0 < P_m(X) < \binom{d+m}{m-1}$ for some $m \geq 2$ and $d \geq 1$. Then $\kappa(a_X(X)) \leq d$.*

Proof. It is just (6) in the proof of Theorem 2.8, where by Ueno's theorem d is the Kodaira dimension of $a_X(X)$. □

3 When does the Albanese map have connected fibers?

Ein and Lazarsfeld in [EL] gave another proof of Kawamata's theorem based on the generic vanishing theorem. Their proof is actually very close to an effective result. With the help of a proposition of Chen and Hacon, we prove the following:

Theorem 3.1 *Let X be a smooth projective variety with $P_1(X) = P_2(X) = 1$. The Albanese map $a_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space.*

Proof. Let A be the Albanese variety of X. The Albanese morphism is already surjective by [HP]. Suppose that it has non-connected fibers. We start with the Stein factorization of a_X and, resolving singularities and indeterminacies, we can assume that a_X admits a factorization

$$X \xrightarrow{g} V \xrightarrow{b} A,$$

where b is a generically finite non birational morphism, g is surjective with connected fibers, V is smooth and projective. Since a_X is the Albanese morphism of X, V is not birational to an abelian variety. Thus V is of maximal Albanese dimension and by Chen and Hacon's characterization of abelian varieties ([CH1], Theorem 3.2), we have $P_2(V) \geq 2$. We set $\dim(X) = n$ and $\dim(V) = \dim(A) = d$.

Since $P_1(X) = P_2(X) = 1$, $0 \in V_0(X, \omega_X)$ is an isolated point ([EL], Proposition 2.1). Hence $0 \in V_0(V, g_*\omega_X)$ is also an isolated point. By Proposition 2.5 in [CH3], for any $v \neq 0$ in $H^1(V, \mathscr{O}_V)$, the sequence

$$0 \to H^0(V, g_*\omega_X) \xrightarrow{\cup v} H^1(V, g_*\omega_X) \to \cdots \xrightarrow{\cup v} H^d(V, g_*\omega_X) \to 0$$

is exact. Since b is surjective, we may, through the map b^*, consider $H^1(A, \mathscr{O}_A)$ as a subspace of $H^1(V, \mathscr{O}_V)$. Then, as in the proof of Theorem 3 in [EL], we have an exact complex of vector bundles on $\mathbf{P} = \mathbf{P}(H^1(A, \mathscr{O}_A)) = \mathbf{P}^{d-1}$:

$$0 \to H^0(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}}(-d) \to H^1(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}}(-d+1) \to \cdots$$
$$\cdots \to H^d(V, g_*\omega_X) \otimes \mathscr{O}_{\mathbf{P}} \to 0.$$

Take $(v_1, \ldots, v_d)$ a basis for $H^1(A, \mathscr{O}_A)$. By chasing through the diagram, we obtain that $H^0(V, g_*\omega_X) \xrightarrow{\wedge v_1 \wedge \cdots \wedge v_d} H^d(V, g_*\omega_X)$ is an isomorphism.

By Theorem 3.4 in [Ko3],

$$H^d(X, \omega_X) \simeq \bigoplus_i H^i(V, R^{d-i} g_*\omega_X).$$

Hence we have

$$\begin{array}{ccc} H^0(V, g_*\omega_X) & \xrightarrow[\simeq]{\wedge v_1 \wedge \cdots \wedge v_d} & H^d(V, g_*\omega_X) \\ \downarrow \simeq & & \downarrow \\ H^0(X, \omega_X) & \xrightarrow{\wedge g^*(v_1 \wedge \cdots \wedge v_d)} & H^d(X, \omega_X) \end{array}$$

By Hodge conjugation and Serre duality $H^d(X, \omega_X) \simeq H^0(X, \Omega_X^{n-d})$. We will denote by $E \subset H^0(X, \Omega_X^{n-d})$ the nonzero subspace corresponding to $H^d(V, g_*\omega_X) \subset H^d(X, \omega_X)$. Let $(\eta_1, \ldots, \eta_d)$ in $H^0(A, \Omega_A)$ be the conjugate basis of $(v_1, \ldots, v_d)$. By Serre duality and Hodge conjugation, we get from the above diagram that

$$E \xrightarrow{\wedge g^*(\eta_1 \wedge \cdots \wedge \eta_d)} H^0(X, \omega_X)$$

is an isomorphism. Since $\eta_1 \wedge \cdots \wedge \eta_d$ is a nonzero section of K_V, we have $K_X \succeq g^*K_V$. We deduce $P_2(X) \geq P_2(V) \geq 2$, which is a contradiction. $\square$

The proof of Theorem 3.1 is closely related to Green and Lazarsfeld's generic vanishing theorem, which is Hodge-theoretic. Meanwhile Theorem 2.8 relies heavily on the weak positivity theorem of Viehweg. It is natural to ask whether we can use the ideas in section 2 to prove other criteria to tell when the Albanese map is an algebraic fiber space.

We again let A be $\mathrm{Alb}(X)$. Suppose that $a_X : X \to A$ is surjective but has non-connected fibers. We take the Stein factorization and obtain that a_X factors as $X \xrightarrow{g} V \xrightarrow{b} A$ where V is normal and finite over A with, again $P_2(V) \geq 2$. The problem here is that we cannot expect the image of the Iitaka fibration of V to be of general type.

Fortunately, a structure theorem for varieties of maximal Albanese dimension due to Kawamata (Theorem 13 in [K]) tells us that the situation is still manageable.

Theorem 3.2 (Kawamata) *Let $b : V \to A$ be a finite morphism from a projective normal algebraic variety to an abelian variety. Then $\kappa(V) \geq 0$ and there are an abelian subvariety K of A, étale covers $\widetilde{V}$ and $\widetilde{K}$ of V and K respectively, a projective normal variety $\widehat{W}$, and a finite abelian group G, which acts on $\widetilde{K}$ and faithfully on $\widehat{W}$, such that:*

(1) $\widehat{W}$ is finite over A/K, of general type and of dimension $\kappa(V)$,

(2) $\widetilde{V}$ is isomorphic to $\widetilde{K} \times \widehat{W}$,

(3) $V = \widetilde{V}/G = (\widetilde{K} \times \widehat{W})/G$, where G acts diagonally and freely on $\widetilde{V}$.

The construction of $\widehat{W}$ and $\widetilde{V}$ is crucial for our purpose so I will recall the proof of this theorem following Kawamata.

Let $\delta : V' \to V$ be a birational modification of V such that V' is smooth and there exists a morphism $h' : V' \to W'$ such that W' is also smooth and h' is a model of the Iitaka fibration of V. Then a general fiber $V'_{w'}$ of h' is smooth, of Kodaira dimension 0, and generically finite over an abelian variety, hence by Kawamata's theorem, $V'_{w'}$ is birational to an abelian variety and $(b \circ \delta)(V'_{w'})$ is then an abelian subvariety of A, denoted by $K_{w'}$. Since w' moves continuously, $K_{w'}$ is a translate of a fixed abelian subvariety $K \subset A$ for every $w' \in W'$. Let $\pi : A \to A/K$ be the quotient map.

Consider the Stein factorization

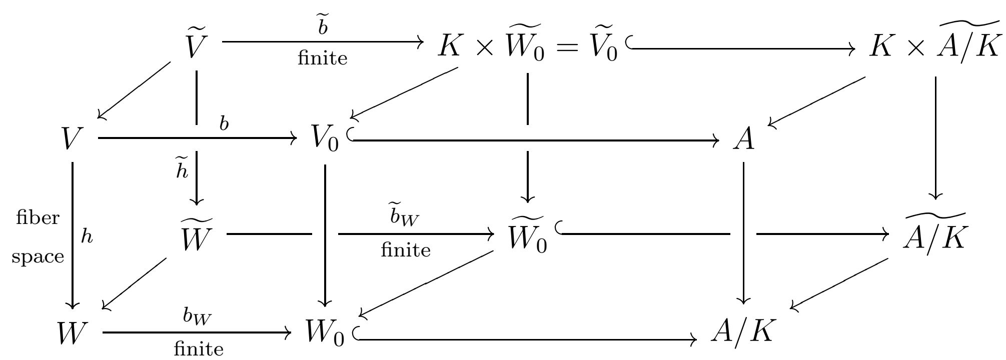
$$\pi \circ b : V \xrightarrow{h} W \xrightarrow{b_W} A/K.$$

Since general fibers of h' are contracted by $\pi \circ b \circ \delta$, hence by $h \circ \delta$, the map $h \circ \delta$ factors through h' by rigidity, and we get the following commutative diagram:

$$
\begin{array}{ccccccc}
V' & \xrightarrow{\delta} & V & \xrightarrow[\text{finite}]{b} & V_0 & \hookrightarrow & A \\
\downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle \pi} \\
W' & \xrightarrow{\delta'} & W & \xrightarrow[\text{finite}]{b_W} & W_0 & \hookrightarrow & A/K
\end{array}
\tag{7}
$$

where W is normal, b_W is finite, $h : V \to W$ has connected fibers, δ and δ' are birational, and V_0 and W_0 are the images of V and W in A and A/K respectively.

By Poincaré reducibility, there exists an isogeny $\widetilde{A/K} \to A/K$ such that $A \times_{A/K} \widetilde{A/K} \simeq K \times \widetilde{A/K}$. We then apply the étale base change $(\cdot) \times_{A/K} \widetilde{A/K} \to \cdot$ in the diagram (7) and get the following commutative diagram:



where $\widetilde{W_0}$ is some connected component of the inverse image of W_0 in $\widetilde{A/K}$, $\widetilde{V}$ is some connected component of $V \times_{V_0} \widetilde{V_0}$, $\widetilde{W}$ is some connected component of $W \times_{W_0} \widetilde{W_0}$, and all slanted arrows are étale.

Let us look at

$$
\begin{array}{ccc}
\widetilde{V} & \xrightarrow[\text{finite}]{\widetilde{b}} & K \times \widetilde{W_0} \\
\downarrow{\scriptstyle \widetilde{h}} & & \downarrow \\
\widetilde{W} & \xrightarrow[\text{finite}]{\widetilde{b}_W} & \widetilde{W_0}.
\end{array}
$$

A general fiber of $\widetilde{h}$ is an étale cover of a general fiber of h hence an étale cover of K, thus isomorphic to an abelian variety $\widetilde{K}$.

The morphism $\widetilde{b}$ is étale over a product $K \times U_0$ for U_0 a dense Zariski open subset of $\widetilde{W_0}$:

$$\begin{array}{ccc} \widetilde{h}^{-1}(U) & \xrightarrow{\widetilde{b}} & K \times U_0 \\ {\scriptstyle \text{smooth}}\downarrow & & \downarrow \\ U & \longrightarrow & U_0. \end{array}$$

The group K acts on $\widetilde{V_0} = K \times \widetilde{W_0}$, and on $K \times U_0$. The infinitesimal action corresponds to vector fields, which lift to $\widetilde{b}^{-1}(K \times U_0)$ because $\widetilde{b}$ is étale there.

This induces an action of $\widetilde{K}$ on $\widetilde{h}^{-1}(U) = \widetilde{b}^{-1}(K \times U_0)$ hence a rational action on $\widetilde{V}$. Let $\widetilde{k} \in \widetilde{K}$ and let $k \in K$ be its image. Let $\widetilde{\Gamma} \subset \widetilde{V} \times \widetilde{V}$ and $\Gamma \subset \widetilde{V_0} \times \widetilde{V_0}$ be the graphs of the actions of $\widetilde{k}$ and k respectively. We have

$$\begin{array}{ccccc} \widetilde{V} \times \widetilde{V} & \hookleftarrow & \widetilde{\Gamma} & \xrightarrow{\widetilde{pr_1}} & \widetilde{V} \\ \downarrow{\scriptstyle (\widetilde{b},\widetilde{b})} & & \downarrow & & \downarrow{\scriptstyle \widetilde{b}} \\ \widetilde{V_0} \times \widetilde{V_0} & \hookleftarrow & \Gamma & \xrightarrow{pr_1} & \widetilde{V_0}, \end{array} \tag{8}$$

where $(\widetilde{b}, \widetilde{b})$ is finite and pr_1 is an isomorphism. We see that $\widetilde{pr_1}$ is finite and birational hence an isomorphism because $\widetilde{V}$ is normal. Thus the action of $\widetilde{k}$ is an isomorphism. So $\widetilde{K}$ acts on $\widetilde{V}$ and $\widetilde{b}$ is equivariant for the $\widetilde{K}$-action on $\widetilde{V}$ and the K-action on $\widetilde{V_0}$.

Set $G_1 = \widetilde{K}/K$. For $y \in \widetilde{W_0}$ general, we have

$$\widetilde{h}^{-1}\widetilde{b_W}^{-1}(y) = \widetilde{b}_W^{-1}(y) \times \widetilde{K} = \widetilde{b}^{-1}(K \times \{y\}),$$

hence

$$\deg \widetilde{b} = \sharp G_1 \cdot \deg \widetilde{b}_W.$$

Set $\widehat{W_0} = \widetilde{b}^{-1}(k \times \widetilde{W_0})$ for $k \in K$ general. Then $\widehat{W_0}$ is normal and G_1 acts on $\widehat{W_0}$ ($\widehat{W_0}$ may be not connected). We have a diagram:

$$\begin{array}{ccc} \widehat{W_0} & \xrightarrow{\deg \widetilde{b}:1} & k \times \widetilde{W_0} \\ \downarrow{\scriptstyle \sharp G_1:1} & & \parallel \\ \widetilde{W} & \xrightarrow{\deg \widetilde{b}_W:1} & \widetilde{W_0}, \end{array}$$

hence $\widehat{W_0}/G_1 = \widetilde{W}$.

Note that G_1 acts on $\widetilde{K} \times \widehat{W_0}$ diagonally and freely (because the action is free on $\widetilde{K}$). By the $\widetilde{K}$-action, we have a morphism $\varphi : \widetilde{K} \times \widehat{W_0} \to \widetilde{V}$ and there is a commutative diagram:

$$\begin{array}{ccc} \widetilde{K} \times \widehat{W_0} & \xrightarrow{\varphi} & \widetilde{V} \\ \downarrow & & \downarrow \tilde{h} \\ \widehat{W_0} & \xrightarrow{\text{finite}} & \widetilde{W}. \end{array}$$

Thus φ is finite because any contracted curve is in some $\widetilde{K} \times \widetilde{w}$ but because of the $\widetilde{K}$-action, this is impossible.

From the diagram, we have a finite morphism $\widetilde{K} \times \widehat{W_0} \to \widetilde{V} \times_{\widetilde{W}} \widehat{W_0}$. Since it is birational over U, it is an isomorphism. Hence

$$\widetilde{V} = (\widetilde{V} \times_{\widetilde{W}} \widehat{W_0})/G_1 = (\widetilde{K} \times \widehat{W_0})/G_1.$$

We then let $\widehat{W}$ be a connected component of $\widehat{W_0}$ and let $\widetilde{\widetilde{V}} = \widetilde{K} \times \widehat{W}$. Then $\widetilde{\widetilde{V}}$ is still a Galois étale cover of $\widetilde{V}$. There exists a commutative diagram:

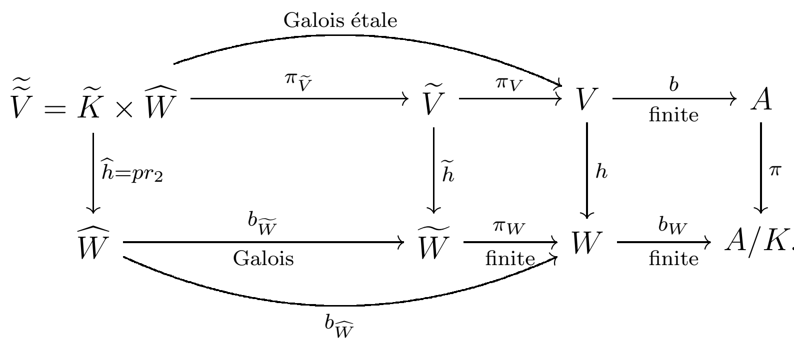
$$\begin{array}{ccc} \widetilde{\widetilde{V}} & \longrightarrow & \widetilde{K} \times \widetilde{A/K} \\ \downarrow & & \downarrow \\ \widetilde{V} & \longrightarrow & K \times \widetilde{A/K} \\ \downarrow & & \downarrow \\ V & \longrightarrow & A. \end{array}$$

We then conclude that $\widetilde{\widetilde{V}}$ is a connected component of $V \times_A (\widetilde{K} \times \widetilde{A/K})$. Let G_2 be the finite abelian group $(\widetilde{K} \times \widetilde{A/K})/A$. Then $V = \widetilde{\widetilde{V}}/G = (\widetilde{K} \times \widehat{W})/G$, for some quotient group G of G_2, where G acts diagonally. Since any quotient of $\widetilde{K}$ by a subgroup of G is still an abelian variety, we may assume that G acts faithfully on $\widehat{W}$.

A crucial fact is that $\widehat{W}$ is of general type because

$$\kappa(\widehat{W}) = \kappa(\widetilde{\widetilde{V}}) = \kappa(V) = \dim(W) = \dim(\widehat{W}).$$

We put everything in a commutative diagram:



(9)

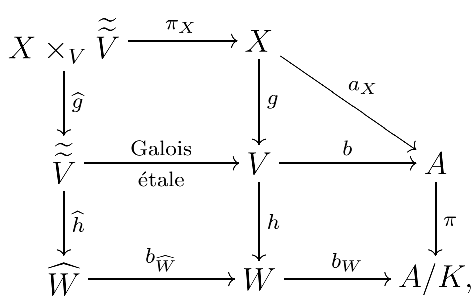
We are now ready to prove the main theorem.

Theorem 3.3 *Let X be a smooth projective variety. If*

$$0 < P_m(X) \leq m - 2,$$

for some $m \geq 3$, the Albanese map $a_X : X \to A$ is an algebraic fiber space.

PROOF. By Theorem 2.8, a_X is already surjective. Suppose that it has non-connected fibers. Again we have the Stein factorization $a_X : X \xrightarrow{g} V \xrightarrow{b} A$, where g has connected fibers, V is normal, and b is finite not birational. Applying the above description of the structure of V in (7) and (9), we get the following commutative diagram:

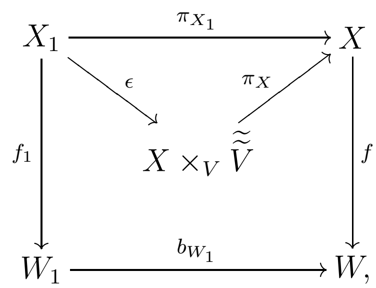


(10)

where π_X is étale Galois with Galois group G, $\widetilde{\widetilde{V}} = \widehat{W} \times \widetilde{K}$, and $\widehat{W}$ is of general type.

There exists a dense Zariski open subset U of W such that U and $b_{\widehat{W}}^{-1}(U)$ are smooth and $h \circ g$ and $\widehat{h} \circ \widehat{g}$ are smooth over U and $b_{\widehat{W}}^{-1}(U)$ respectively.

Through Hironaka's resolution of singularities, we can blow up W and X along smooth subvarieties of $W - U$ and $X - (h \circ g)^{-1}(U)$ respectively and assume that W is smooth. Similarly, let W_1 and X_1 be the smooth projective varieties obtained by blowing-up $\widehat{W}$ and $X \times_V \tilde{\tilde{\tilde{V}}}$ along subvarieties of $\widehat{W} - b_{\widehat{W}}^{-1}(U)$ and $X \times_V \tilde{\tilde{\tilde{V}}} - (b_{\widehat{W}} \circ \widehat{h} \circ \widehat{g})^{-1}(U)$ respectively such that we have the following commutative diagram:



(11)

where W_1 is of general type, b_{W_1} is generically finite and ϵ is the blow-up of $X \times_V \tilde{\tilde{\tilde{V}}}$. We write

$$K_{X_1} = \pi_{X_1}^* K_X + E,$$

where E is an effective exceptional divisor for π_{X_1}, $f_1(E)$ is a subvariety of $W_1 - b_{W_1}^{-1}(U)$, and

$$\pi_{X_1 *} \mathscr{O}_{X_1} = \pi_{X*} \epsilon_* \mathscr{O}_{X_1} = \pi_{X*} \mathscr{O}_{X \times_V \widetilde{V}} = \bigoplus_{\chi \in G^*} P_\chi,$$

where $P_\chi \in \mathrm{Pic}^0(X)$ is the torsion line bundle corresponding to $\chi \in G^*$.

In order to prove the theorem, we will need to treat two cases, $\kappa(W) > 0$ or $\kappa(W) = 0$. The strategies of the proofs are the same so I will treat the first case in detail and explain how very similar arguments work for the second case.

Lemma 3.4 *Let X be a smooth projective variety with $P_m(X) > 0$ for some $m \geq 2$. Let $f : X \to W$ be as above. The Iitaka model of $(X, (m-1)K_{X/W} + f^* K_W)$ dominates W.*

Proof. We use the same notation as above. In (11), we already know that W_1 is of general type so by Viehweg's result (see the proof of Lemma 2.3), the Iitaka model of $(X_1, (m-1)K_{X_1/W_1} + f_1^*K_{W_1})$ dominates W_1. On the other hand, we can write

$$\begin{aligned} & (m-1)K_{X_1/W_1} + f_1^*K_{W_1} \\ = \; & \pi_{X_1}^*((m-1)K_{X/W} + f^*K_W) - (m-2)f_1^*K_{W_1/W} + (m-1)E. \quad (12) \end{aligned}$$

Since $K_{W_1/W}$ is effective, the Iitaka model of $(X_1, \pi_{X_1}^*((m-1)K_{X/W}+f^*K_W)+(m-1)E)$ dominates W_1. Hence for any ample divisor H on W, there exists $N > 0$ such that $\pi_{X_1}^*\mathscr{O}_X(N((m-1)K_{X/W}+f^*K_W)-f^*H)\otimes\mathscr{O}_{X_1}(N(m-1)E)$ has a nonzero section. Since $\pi_{X_1*}\mathscr{O}_{X_1}(N(m-1)E) = \pi_{X_1*}\mathscr{O}_{X_1}$ is a direct sum of torsion line bundles, there exists $k > 0$ such that $kN((m-1)K_{X/W} + f^*K_W) - kf^*H$ is effective. Therefore the Iitaka model of $(X, (m-1)K_{X/W} + f^*K_W)$ dominates W. □

Since K_W is not necessarily big, we cannot directly apply Lemma 2.3. But we still have:

Lemma 3.5 *Under the assumptions of Lemma 3.4, the sheaf*

$$f_*(\mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||))$$

is nonzero, of rank $P_m(X_w)$*, where* X_w *is a general fiber of* f*.*

PROOF. We use the diagram (11). Since W_1 is of general type, as in Lemma 2.3, by Viehweg's result, there exists $k > 0$ such that for w_1 a general point of W_1 and $X_{w_1} \subset X_1$ the fiber of f_1, the restriction:

$$H^0(X_1, \mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1}+kmf_1^*K_{W_1})) \to H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}}))$$

is surjective. Since $K_{W_1/W} \succeq 0$, by (12), we have

$$\begin{aligned} & H^0(X_1, \mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1} + kmf_1^*K_{W_1})) \\ \subseteq \; & H^0(X_1, \mathscr{O}_{X_1}(km(m-1)\pi_{X_1}^*K_{X/W} + km\pi_{X_1}^*f^*K_W + km(m-1)E)). \end{aligned}$$

Since E is π_{X_1}-exceptional, we conclude that

$$\begin{aligned} & |km(m-1)\pi_{X_1}^*K_{X/W} + km\pi_{X_1}^*f^*K_W + km(m-1)E| \\ = \; & |km(m-1)\pi_{X_1}^*K_{X/W} + km\pi_{X_1}^*f^*K_W| + km(m-1)E. \end{aligned}$$

We also know that $f_1(E)$ is a proper subvariety of W_1. These imply that the restriction:

$$H^0(X_1, \mathscr{O}_{X_1}(km(m-1)\pi_{X_1}^* K_{X/W} + km\pi_{X_1}^* f^* K_W)) \\ \to H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}})) \quad (13)$$

is surjective.

Set $w = b_{W_1}(w_1)$, and let X_w be the fiber of f. In the following diagram

$$\begin{array}{ccc} \pi_{X_1}^{-1} f^{-1}(U) & \longrightarrow & f^{-1}(U) \\ \downarrow & & \downarrow \\ b_{W_1}^{-1}(U) & \longrightarrow & U, \end{array}$$

all the morphisms are smooth. Hence $\pi_{X_{w_1}} = \pi_{X_1}|_{X_{w_1}} : X_{w_1} \to X_w$ is étale and the pull-back of $H^0(X_w, \mathscr{O}_{X_w}(km(m-1)K_{X_w}))$ is a subspace of $H^0(X_{w_1}, \mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}}))$.

On the other side, we have

$$\begin{aligned} & H^0(X_1, \mathscr{O}_{X_1}(k(m-1)\pi_{X_1}^* K_{X/W} + k\pi_{X_1}^* f^* K_W)) \\ = & \bigoplus_{\chi \in G^*} \pi_{X_1}^* H^0(X, \mathscr{O}_X(k(m-1)K_{X/W} + kf^* K_W) \otimes P_\chi). \end{aligned} \quad (14)$$

Let M be the order of G. Take a resolution $\tau : X' \to X$ such that $\tau : X'_w \to X_w$ is also a resolution and

- $\tau^*|Mkm(m-1)K_{X/W} + Mkmf^* K_W| = |H| + E_M$,
- $\tau^*|\mathscr{O}_X(km(m-1)K_{X/W} + kmf^* K_W) \otimes P_\chi| = |H_\chi| + E_\chi$, for each $\chi \in G^*$,
- $\tau^*|km(m-1)K_{X_w}| = |H_w| + E_w$,
- $\tau^*|mK_{X_w}| = |H'_w| + E'_w$,

such that H, H_χ, H_w, and H'_w are base-point-free and E_M, E_χ, E_w, E'_w are the fixed divisors, with SNC supports.

Let X'_1 be a smooth model of the main component of $X_1 \times_X X'$ (the irreducible component that dominates X_1). We have the following commutative diagram:

$$\begin{CD}
X_1' @>{\pi_{X_1'}}>> X' \\
@V{\tau_1}VV @VV{\tau}V \\
X_1 @>{\pi_{X_1}}>> X \\
@V{f_1}VV @VV{f}V \\
W_1 @>{b_{W_1}}>> W.
\end{CD}$$

Let $U_1 = X_1 - E$. Then π_{X_1} is étale on U_1, hence $U_1 \times_X X'$ is irreducible and smooth. Since $f_1(E)$ is a proper subvariety of W_1, we can assume that there exists a divisor E' of X_1' such that $X_1' - E'$ is just $U_1 \times_X X'$ and $f_1\tau_1(E')$ is a proper subvariety of W_1. Let X_{w_1}' be the fiber of $f_1\tau_1$. Then $\pi_{X_{w_1}'} = \pi_{X_1'}|_{X_{w_1}'} : X_{w_1}' \to X_w'$ is Galois étale. We have another commutative diagram involving morphisms between fibers:

$$\begin{CD}
X_{w_1}' @>{\pi_{X_{w_1}'}}>{\text{étale}}> X_w' \\
@V{1:1}V{\tau_1}V @V{\tau}V{1:1}V \\
X_{w_1} @>{\pi_{X_{w_1}}}>{\text{étale}}> X_w.
\end{CD}$$

We then write

$$\begin{aligned}
& \tau_1^*|km(m-1)\pi_{X_1}^* K_{X/W} + km\pi_{X_1}^* f^* K_W| \\
= \ & |\pi_{X_1}'^{*}\tau^*(km(m-1)K_{X/W} + kmf^*K_W)| \\
= \ & |H'| + E_1',
\end{aligned}$$

where E_1' is the fixed divisor. Let F be the maximal divisor which is $\preceq E_\chi$ for all $\chi \in G^*$. By (14), $\pi_{X_1'}^* F \preceq E_1'$. Hence, by (13), we conclude that $\pi_{X_1'}^* F|_{X_{w_1}'}$ is fixed in $\tau_1^*|km(m-1)K_{X_{w_1}}|$ and in particular is fixed in $\pi_{X_{w_1}'}^* \tau^*|km(m-1)K_{X_w}|$, so $\pi_{X_1'}^* F|_{X_{w_1}'} \preceq \pi_{X_{w_1}'}^* E_w$. Since $\pi_{X_{w_1}'}$ is étale, we have

$$\pi_{X_{w_1}'}^*(F|_{X_w'}) \preceq \pi_{X_1'}^* F|_{X_{w_1}'} \preceq \pi_{X_{w_1}'}^* E_w.$$

We conclude that $F|_{X'_w} \preceq E_w$.

Since for any $\chi \in G^*$, we have the natural multiplication

$$H^0(X, \mathscr{O}_X(km(m-1)K_{X/W} + kmf^*K_W) \otimes P_\chi)^{\otimes M} \\ \to H^0(X, \mathscr{O}_X(Mkm(m-1)K_{X/W} + Mkmf^*K_W)),$$

we obtain $E_M \preceq MF$, hence $E_M|_{X'_w} \preceq ME_w \preceq Mk(m-1)E'_w$. This is just (4) in the proof of 2) of Lemma 2.3, and we can then finish the proof as there. $\square$

We may write Lemma 3.5 in a more general form:

Proposition 3.6 *Assume that we have the following commutative diagram between smooth projective varieties:*

$$\begin{CD} X_1 @>{\pi_{X_1}}>> X \\ @V{f_1}VV @VV{f}V \\ W_1 @>{b_{W_1}}>> W, \end{CD}$$

where $P_m(X) > 0$, the morphism π_{X_1} is birationally equivalent to an étale morphism and its exceptional divisor E is such that $f_1(E)$ is a proper subvariety of W_1, $\pi_{X_1*}\mathscr{O}_{X_1} = \bigoplus_\alpha P_\alpha$ is a direct sum of torsion line bundles on X, W_1 is of general type, and b_{W_1} is generically finite and surjective. Then the sheaf

$$f_*(\mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||))$$

is nonzero, of rank $P_m(X_w)$, where X_w is a general fiber of f.

According to Lemma 3.5,

$$\mathscr{F}_X = b_{W*}f_*(\mathscr{O}_X(K_X+(m-1)K_{X/W}+f^*K_W)\otimes\mathscr{J}(||(m-1)K_{X/W}+f^*K_W||)))$$

is a nonzero sheaf on A/K. By Lemma 2.1 and Lemma 3.4, it is an IT-sheaf of index 0.

Let $\widehat{\mathscr{F}_X}$ be the Fourier-Mukai transform of $\mathscr{F}_X$. By the properties of this transformation ([Mu], Theorem 2.2), we know that $\widehat{\mathscr{F}_X}$ is a W.I.T.-sheaf of index $\dim(A/K)$ and its Fourier-Mukai transform $\widehat{\widehat{\mathscr{F}_X}}$ is isomorphic to $(-1_{A/K})^*\mathscr{F}_X$. In particular, $\widehat{\mathscr{F}_X} \neq 0$. Therefore, by the Base Change Theorem and the definition of the Fourier-Mukai transform, there exists $P_0 \in \mathrm{Pic}^0(A/K)$ such that $h^0(A/K, \mathscr{F}_X \otimes P_0) \neq 0$. Thus for any $P \in \mathrm{Pic}^0(A/K)$,

$$h^0(A/K, \mathscr{F}_X \otimes P) = \chi(\mathscr{F}_X \otimes P) = \chi(\mathscr{F}_X \otimes P_0) = h^0(A/K, \mathscr{F}_X \otimes P_0) \geq 1.$$

Hence for any $P \in \mathrm{Pic}^0(A/K)$, we have

$$\begin{aligned} & h^0(X, \mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes f^*b_W^*P) \\ \geq\ & h^0\big(X, \mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \\ & \qquad \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||) \otimes f^*b_W^*P\big) \\ =\ & h^0\big(A/K, b_{W*}f_*(\mathscr{O}_X(K_X + (m-1)K_{X/W} + f^*K_W) \otimes \\ & \qquad \mathscr{J}(||(m-1)K_{X/W} + f^*K_W||)) \otimes P\big) \\ =\ & h^0(A/K, \mathscr{F}_X \otimes P) \\ \geq\ & 1. \end{aligned} \tag{15}$$

Lemma 3.7 *Let X and W be as in Lemma 3.4. Suppose $\kappa(W) > 0$. Then for any $r \geq 3$, there exists a translate $T \subset \mathrm{Pic}^0(A/K)$ of a positive-dimensional torus, such that*

$$h^0(W, \mathscr{O}_W((r-2)K_W) \otimes b_W^*P) \geq r-2,$$

for all $P \in T$.

Proof. Since $\kappa(W) > 0$, there exist a positive-dimensional abelian subvariety $T_0 \subset \mathrm{Pic}^0(A/K)$ and a torsion point $P_0 \in \mathrm{Pic}^0(A/K)$ such that $b_W^*(P_0 + T_0) \subset V_0(\omega_W)$ ([CH2], Corollary 2.4). Then we iterate Lemma 1.4 to get $h^0(W, \mathscr{O}_W((r-2)K_W) \otimes b_W^*P) \geq r-2$, for all $P \in (r-2)P_0 + T_0$. □

If $\kappa(W) > 0$, since $mK_X = K_X + (m-1)K_{X/W} + f^*K_W + (m-2)f^*K_W$, again by (15), Lemma 3.7 and Lemma 1.4, we obtain

$$P_m(X) \geq 1 + m - 2 + \dim(T) - 1 \geq m-1,$$

which contradicts our assumption. Hence we have finished the proof in the case $\kappa(W) > 0$.

If $\kappa(W) = 0$, in the diagram (10), b_W is surjective and finite and $\kappa(W) = 0$, hence W is an abelian variety by Kawamata's Theorem 3.2. We still have (15), however K_W is trivial, hence it is not enough for us to deduce a contradiction. We will need new versions of Lemma 3.4 and Lemma 3.5.

First we go back to diagrams (10) and (11):

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi_{X_1}} & X \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g} \\ V_1 & \xrightarrow{\pi_{V_1}} & V \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h} \\ W_1 & \xrightarrow{b_{W_1}} & W, \end{array} \qquad f_1 : X_1 \to W_1, \quad f : X \to W$$

where V_1 is birational to $\widetilde{K} \times W_1$.

Since $\pi_{V_1} : V_1 \to V$ is birationally equivalent to the étale cover $\widetilde{V} \to V$, we have $\pi_{V_1 *}\omega_{V_1} = \bigoplus_{\chi \in G^*}(\omega_V \otimes P_\chi)$. On the other hand, V_1 is birational to $\widetilde{K} \times W_1$, hence $h_{1*}\omega_{V_1} = \omega_{W_1}$. Therefore, we have

$$b_{W_1 *}\omega_{W_1} = \bigoplus_{\chi \in G^*} h_*(\omega_V \otimes P_\chi).$$

Since b_{W_1} is generically finite and W_1 is of general type, by Theorem 2.3 in [CH2], we know that the irreducible components of $V_0(b_{W_1 *}\omega_{W_1})$ generate $\mathrm{Pic}^0(W)$. Hence there exists a $\chi \in G^*$ such that $V_0(h_*(\omega_V \otimes P_\chi))$ is a translated positive-dimensional abelian subvariety of $\mathrm{Pic}^0(W)$. We denote $h_*(\omega_V \otimes P_\chi)$ by $\mathscr{F}_\chi$. Since a general fiber of h is an abelian variety, $\mathscr{F}_\chi$ is a rank-1 torsion-free sheaf.

We can again birationally modify X so that $f^*\mathscr{F}_\chi$ is a line bundle on X. We then have the following result similar to Lemma 3.4.

Lemma 3.8 *Under the assumptions of Lemma 3.4, assume moreover that* $\kappa(W) = 0$ *and let* $\mathscr{F}_\chi$ *be as above. Then the Iitaka model of* $(X, (m-1)K_X - (m-2)f^*\mathscr{F}_\chi)$ *dominates* W.

PROOF. The proof is analogue to that of Lemma 3.4. We have

$$\begin{aligned} & \pi_{X_1}^*((m-1)K_X - (m-2)f^*\mathscr{F}_\chi) + (m-1)E \qquad (16) \\ = \; & (m-1)K_{X_1/W_1} + f_1^*K_{W_1} + (m-2)f_1^*K_{W_1} - (m-2)\pi_{X_1}^* f^*\mathscr{F}_\chi. \end{aligned}$$

Since $\mathscr{F}_\chi \subset b_{W_1*}\omega_{W_1}$, we have an inclusion $b_{W_1}^*\mathscr{F}_\chi \hookrightarrow \omega_{W_1}$, hence an inclusion

$$(m-2)f_1^*b_{W_1}^*\mathscr{F}_\chi = (m-2)\pi_{X_1}^*f^*\mathscr{F}_\chi \hookrightarrow (m-2)f_1^*\omega_{W_1}.$$

Using Viehweg's result as in the proof of Lemma 3.4, we obtain that the Iitaka model of $\pi_{X_1}^*((m-1)K_X-(m-2)f^*\mathscr{F}_\chi)+(m-1)E$ dominates W_1. We finish the proof by the same argument as in Lemma 3.4. □

We also need an analogue of Lemma 3.5.

Lemma 3.9 *Under the same assumptions as in Lemma 3.8, the sheaf*

$$f_*(\mathscr{O}_X(mK_X-(m-2)f^*\mathscr{F}_\chi)\otimes \mathscr{J}(||(m-1)K_X-(m-2)f^*\mathscr{F}_\chi||))$$

is nonzero of rank $P_m(X_w)$, where X_w is a general fiber of f.

Proof. It is also parallel to the proof of Lemma 3.5. First, by Viehweg's result again, we have the surjectivity of the restriction map:

$$H^0(X_1,\mathscr{O}_{X_1}(km(m-1)K_{X_1/W_1}+kmf_1^*K_{W_1})) \\ \to H^0(X_{w_1},\mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}})).$$

Since E is π_{X_1}-exceptional and $(m-2)f_1^*K_{W_1}\succeq(m-2)\pi_{X_1}^*f^*\mathscr{F}_\chi$, by (16), we have the surjectivity of the restriction map:

$$H^0(X_1,\pi_{X_1}^*\mathscr{O}_{X_1}(km(m-1)K_X-km(m-2)f^*\mathscr{F}_\chi)) \\ \to H^0(X_{w_1},\mathscr{O}_{X_{w_1}}(km(m-1)K_{X_{w_1}})).$$

Then the rest of the proof is the same as the proof of Lemma 3.5. □

By Lemma 3.8 and Lemma 3.9, we again conclude as in (15) that

$$h^0(X,\mathscr{O}_X(mK_X-(m-2)f^*\mathscr{F}_\chi)\otimes f^*P)\geq 1,$$

for any $P\in \operatorname{Pic}^0(W)$.

As in the proof of Lemma 3.7, there exists a translate $T\subset \operatorname{Pic}^0(W)$ of a positive-dimensional abelian variety such that $h^0(X,\mathscr{O}_X((m-2)f^*\mathscr{F}_\chi)\otimes f^*P)\geq m-2$, for any $P\in T$. We again have $P_m(X)\geq m-1$, which is a contradiction. This finishes the proof of Theorem 3.3 in the case $\kappa(W)=0$.

In all, we have finished the proof of Theorem 3.3. □

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