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Noetherian loop spaces

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Abstract. The class of loop spaces of which the mod p cohomology is Noetherian is much larger than the class of p-compact groups (for which the mod p cohomology is required to be finite). It contains Eilenberg–Mac Lane spaces such as $\mathbb{C}P^\infty$ and 3-connected covers of compact Lie groups. We study the cohomology of the classifying space BX of such an object and prove it is as small as expected, that is, comparable to that of $B\mathbb{C}P^\infty$. We also show that BX differs basically from the classifying space of a p-compact group in a single homotopy group. This applies in particular to 4-connected covers of classifying spaces of compact Lie groups and sheds new light on how the cohomology of such an object looks like.

Introduction

From the point of view of homotopy theory compact Lie groups are finite loop spaces, i.e. triples (X, BX, e) where X is a finite complex, and $e : X \to \Omega BX$ is a homotopy equivalence. Most of their geometric features are captured p-locally in homotopy theory, where p is any prime, as shown by Dwyer and Wilkerson in [17]. They introduced the notion of p-compact group, replacing the finiteness condition by a cohomological one, namely that $H^*(X; \mathbb{F}_p)$ must be finite, and requiring the additional property that BX be local with respect to mod p homology, or equivalently p-complete ([5]). It is the "classifying space" BX that carries all of the information about the loop space. Amazingly enough, apart from compact Lie groups, there are only a few families of exotic p-compact groups and they have been recently completely classified by Andersen, Grodal, Møller, and Viruel: see [3] for the odd prime case and [2], [29], [30] for the prime 2 (the only exotic 2-compact group is basically the space $DI(4)$ constructed by Dwyer and Wilkerson [16]).

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If one aims at an understanding not only of finite loop spaces, but larger ones, the next natural step to take is to relax the cohomological finiteness condition. We thus define a *p-Noetherian group* to be a loop space (X, BX, e) where BX is p-complete and $H^*(X; \mathbb{F}_p)$ is a finitely generated (Noetherian) $\mathbb{F}_p$-algebra. Relying on Bousfield localization techniques ([4]), and Miller's solution to the Sullivan conjecture ([28]), more precisely on Lannes' T-functor technology ([26]), we describe the structure of p-Noetherian groups and their relation to p-compact groups. We compute qualitatively the cohomology of the classifying space BX and obtain new general results when X is the 3-connected cover of a compact Lie group (see Corollary 4.9).

Let us be more precise. In the case of p-compact groups, that is, when the mod p cohomology of the loop space $H^*(X; \mathbb{F}_p)$ is finite, Dwyer and Wilkerson's main theorem in [17] shows that there are severe restrictions on the cohomology of the classifying space: $H^*(BX; \mathbb{F}_p)$ is always a finitely generated (Noetherian) $\mathbb{F}_p$-algebra. Likewise the cohomology of the classifying space of a p-Noetherian group cannot be arbitrarily large.

Theorem 4.1. *Let (X, BX, e) be a p-Noetherian group. Then $H^*(BX; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

This information is not optimal as it does not permit one to decide for example when an Eilenberg–Mac Lane space of type $K(\mathbb{Z}/p, m)$ is a p-Noetherian group. The mod p cohomology of any of them is finitely generated as an algebra over $\mathcal{A}_p$, but the only classifying spaces of a p-Noetherian group are $K(\mathbb{Z}/p, 1)$ and $K(\mathbb{Z}/p, 2)$. Schwartz's Krull filtration of the category $\mathcal{U}$ of unstable modules is an established and convenient tool to measure how large an unstable algebra is ([33]). The Krull filtration $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots$ is defined inductively, starting with the full subcategory $\mathcal{U}_0$ of $\mathcal{U}$ of locally finite unstable modules (the span of every element under the action of the Steenrod algebra is finite). In fact the cohomology $H^*(X; \mathbb{F}_p)$ is locally finite if and only if the space X is $B\mathbb{Z}/p$-local, i.e. the evaluation map$(B\mathbb{Z}/p, X) \to X$ is a weak equivalence ([27, Théorème 0.14]).

There are many $B\mathbb{Z}/p$-local spaces, but there are none for which the cohomology lies in higher stages of the Krull filtration. This is the statement of Kuhn's non-realizability conjecture [25], which has been settled by Schwartz in [34] and [35], and proved in its full generality by Dehon and Gaudens in [13]. Thus the cohomology of a space lies in $\mathcal{U}_0$ or it does not lie in any $\mathcal{U}_n$. The cohomology $H^*(K(\mathbb{Z}/p, m); \mathbb{F}_p)$ for example does not lie in any $\mathcal{U}_n$.

The situation changes drastically if one looks at the quotient module of indecomposable elements instead. For example $QH^*(K(\mathbb{Z}/p, m); \mathbb{F}_p)$ lies in $\mathcal{U}_{m-1}$ for any $m \geq 1$ ([9, Example 2.2]). Moreover we observed in [9, Lemma 7.1] that if $H^*(BX; \mathbb{F}_p)$ is finitely generated as an algebra over $\mathcal{A}_p$, then $QH^*(BX; \mathbb{F}_p)$ must be finitely generated as a module over $\mathcal{A}_p$, and hence lies in $\mathcal{U}_k$ for some k. We remark that the condition that $H^*(X; \mathbb{F}_p)$ be a Noetherian $\mathbb{F}_p$-algebra is equivalent to saying that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over $\mathcal{A}_p$ and that the unstable module $QH^*(X; \mathbb{F}_p)$ of indecomposable elements lies in $\mathcal{U}_0$. Our second result shows that the cohomology of the classifying space of a p-Noetherian group is as small as expected in terms of the Krull filtration.

Theorem 2.6. *Let (X, BX, e) be a p-Noetherian group. Then $QH^*(BX; \mathbb{F}_p)$ belongs to $\mathcal{U}_1$.*

The arguments to prove these results are the following. We start in Section 1 with the study of the structure of p-Noetherian groups. The most basic examples of p-Noetherian groups are p-compact groups and Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p^r, 1)$ and, $K(\mathbb{Z}_p^\wedge, 2)$. In the spirit of our deconstruction results for H-spaces ([9]), we show that these are the basic building blocks for all p-Noetherian groups.

Theorem 1.9. *Let (X, BX, e) be any p-Noetherian group. There exists then a fibration*

$$K(P, 2)_p^\wedge \to BX \to BY,$$

where P is a p-group which is a finite direct sum of copies of cyclic groups and Prüfer groups and Y is a p-compact group.

We stress that this fibration is functorial in BX as the base space is obtained by Bousfield localization. The understanding of the cohomology of BX goes through the analysis of the Serre spectral sequence of this fibration. Note that, by Dwyer and Wilkerson's main theorem in [17], $H^*(BY; \mathbb{F}_p)$ is finitely generated as an algebra. Also, the mod p cohomology of $K(P, 2)_p^\wedge$ is finitely generated as an algebra over the Steenrod algebra by [36] and [7]. The spectral sequence is not nearly as nice as for H-spaces though (cf. [10]), and we must first tackle Theorem 2.6.

This we do in Section 2 by giving first a geometric interpretation to $\bar{T}QH^*(BX; \mathbb{F}_p)$, where $\bar{T}$ is Lannes' reduced T-functor. Recall that for "nice" spaces (such as BX) the unreduced T-functor $TH^*(BX; \mathbb{F}_p)$ computes the cohomology of the mapping space $\mathrm{map}(B\mathbb{Z}/p, BX)$ and we rely on Schwartz's characterization [33, Theorem 6.2.4] of the Krull filtration in terms of $\bar{T}$. In order to perform our calculation, we prove that the component $\mathrm{map}(B\mathbb{Z}/p, BX)_c$ of the constant map splits as a product $BX \times \mathrm{map}_*(B\mathbb{Z}/p, BX)_c$. By the properties of the T-functor, this splitting yields a geometric interpretation of the reduced T-functor in terms of the pointed mapping space, more precisely $\bar{T}QH^*(X; \mathbb{F}_p) \cong H^*(\mathrm{map}_*(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p)$.

We finally come back to Theorem 4.1, which we prove in two steps. First we investigate in Section 3 the Serre spectral sequence for fibrations over spaces with finite cohomology and fiber a finite product of Eilenberg–Mac Lane spaces. We show that in this situation the cohomology of the total space is finitely generated as an algebra over $\mathcal{A}_p$. Secondly, we reduce to the situation in which the base space of the fibration is a p-compact toral group.

Let us conclude the introduction with a remark. The results in this article show that for a compact simply-connected Lie group G, the module $QH^*((BG)\langle 4\rangle; \mathbb{F}_p)$ is finitely generated over $\mathcal{A}_p$ and belongs to $\mathcal{U}_1$. This puts into context the calculations made by Harada and Kono ([22]) and sheds new light on how the cohomology will look like even in the cases where an explicit description has not been obtained (cf. Example 4.10).

1. The structure of p-Noetherian groups

This first section is devoted to the description of the classifying space of p-Noetherian groups and the relation to p-compact groups. Let us start with the definition and basic examples. The statements about the action of the Steenrod algebra and the Krull filtration will be explained and developed in Section 2.

Definition 1.1. A *p-Noetherian group* is a triple (X, BX, e) where $H^*(X;\mathbb{F}_p)$ is a Noetherian algebra (it is finitely generated as an algebra), BX is a p-complete space, and $e : X \to \Omega BX$ is a weak equivalence.

We will often use the word p-Noetherian group for the loop space X and refer to BX as the classifying space. Hence, we will say that a p-Noetherian group is n-connected if so is the loop space X, or equivalently if the classifying space is $(n+1)$-connected. Note that if the *integral* cohomology of a space is Noetherian, as an algebra, then so is the mod p cohomology (just like finite loop spaces have finite mod p cohomology); this follows from Evens [19] (see also [11, Lemma 1.5]).

Remark 1.2. Since $\pi_1(BX) \cong \pi_0(X)$ and $H^*(X;\mathbb{F}_p)$ is of finite type, it follows that $\pi_1(BX)$ is finite and therefore BX is a p-good space by [5, VII, Proposition 5.1]. In fact, $\pi_1(BX)$ is a p-group by [5, VII, Proposition 4.3].

Example 1.3. In [17], Dwyer and Wilkerson introduced the notion of a p-compact group. A *p-compact group* is a loop space (X, BX, e) such that BX is p-complete and $H^*(X;\mathbb{F}_p)$ is a finite $\mathbb{F}_p$-vector space. It is clear from the definition that p-compact groups are p-Noetherian groups.

The most basic example of a p-compact group is given by the p-completed circle and its classifying space $K(\mathbb{Z}_p^\wedge, 2)$. Our definition of p-Noetherian group allows us to include not only all p-compact groups but also the following Eilenberg–Mac Lane spaces.

Example 1.4. Let $X = K(\mathbb{Z}_p^\wedge, 2)$, $BX = K(\mathbb{Z}_p^\wedge, 3)$, and e the obvious homotopy equivalence between $\Omega K(\mathbb{Z}_p^\wedge, 3)$ and $K(\mathbb{Z}_p^\wedge, 2)$. This is a p-Noetherian group since $H^*(K(\mathbb{Z}_p^\wedge, 2);\mathbb{F}_p) \cong \mathbb{F}_p[u]$ is finitely generated as an algebra. Let us point out here that $H^*(K(\mathbb{Z}_p^\wedge, 3);\mathbb{F}_p)$ is finitely generated as an algebra over $\mathcal{A}_p$ and that the module of indecomposable elements $QH^*(K(\mathbb{Z}_p^\wedge, 3);\mathbb{F}_p)$ lives in $\mathcal{U}_1$. For example $QH^*(K(\mathbb{Z}_2^\wedge, 3);\mathbb{F}_2) \cong \Sigma F(1)$, where $F(1)$ is the free unstable module on one generator in degree 1.

In fact, this is basically the only 1-connected p-Noetherian group X such that ΩX is $\mathbb{F}_p$-finite.

Proposition 1.5. *Let (X, BX, e) be a p-Noetherian group such that $H^*(\Omega X;\mathbb{F}_p)$ is finite. Then BX is 2-connected if and only if it is a product of a finite number of copies of $K(\mathbb{Z}_p^\wedge, 3)$.*

Proof. If BX is 2-connected the loop space $\Omega X \simeq \Omega^2 BX$ is a connected homotopy commutative mod p finite H-space. Thus, by the mod p version of Hubbuck's Torus Theorem ([23] and [1]), we see that ΩX is equivalent to a p-completed torus. □

Example 1.6. Let us consider the compact Lie group S^3 and its 3-connected cover $S^3\langle 3\rangle$. Identifying $B(S^3\langle 3\rangle)$ with $(BS^3)\langle 4\rangle$, we have a fibration $K(\mathbb{Z}, 3) \to (BS^3)\langle 4\rangle \to BS^3$. The triple $(S^3\langle 3\rangle^\wedge_p, B(S^3)\langle 4\rangle^\wedge_p, e)$ is then a p-Noetherian group since $H^*(S^3\langle 3\rangle; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$ where x has degree $2p$, y has degree $2p+1$, and a Bockstein connects x and y, $\beta(x) = y$. This p-Noetherian group is an extension of the p-compact group $(S^3)^\wedge_p$ and the Eilenberg–Mac Lane space from Example 1.4.

It is not difficult to compute the mod p cohomology of $(BS^3)\langle 4\rangle$ using the Serre spectral sequence. For example, $H^*(B(S^3\langle 3\rangle); \mathbb{F}_2) \cong \mathbb{F}_2[z, \mathrm{Sq}^1 z, \mathrm{Sq}^4 z, \mathrm{Sq}^{8,4} z, \dots]$, where z has degree 5. This is a subalgebra of the mod 2 cohomology of $K(\mathbb{Z}, 3)$ and z corresponds to $\mathrm{Sq}^2 \iota_3$, where ι_3 is the fundamental class. Again, we see that the module of indecomposable elements belongs to $\mathcal{U}_1$, as it differs from $\Sigma F(1)$ by only a few classes in low degrees.

This last example fits into a more general picture. One can consider 4-connected covers of classifying spaces of compact Lie groups.

Example 1.7. Let G be a simply connected compact Lie group and consider the 4-connected cover of its classifying space, $(BG)\langle 4\rangle$. Since the mod p cohomology of $G\langle 3\rangle$ is Noetherian, this provides an infinite number of examples of p-Noetherian groups.

Harada and Kono studied in [22] and [21] the fibration $K(\mathbb{Z}, 3) \to (BG)\langle 4\rangle \to BG$. They were able to compute explicitly, as an algebra, the cohomology of the total space at odd primes and a few cases at the prime 2. In all of these computations the result is the tensor product of a quotient of $H^*(BG; \mathbb{F}_p)$ and a certain subalgebra of $H^*(K(\mathbb{Z}, 3); \mathbb{F}_p)$, which turns out to be always finitely generated as an algebra over the Steenrod algebra.

In fact, p-Noetherian groups are closed under fibrations in the following sense (and this explains why the 3-connected cover $G\langle 3\rangle$ of a compact Lie group G defines a p-Noetherian group).

Proposition 1.8. *Let $BX \to E \to BZ$ be a fibration of connected spaces where BX and BZ are classifying spaces of p-Noetherian groups. Then E is also the classifying space of a p-Noetherian group.*

Proof. Since $\pi_1(BZ)$ is a finite p-group and both BX and BZ are p-complete and p-good (see Remark 1.2), the fiber lemma [5, II.5.1] shows that E is also p-complete. It remains to show that the mod p cohomology of ΩE is a finitely generated algebra. Since $\pi_1 E$ is a finite (p-) group it is enough to prove this for the connected component $\Omega_0 E$ of the base point (the constant loop).

Looping the fibration, we obtain an H-fibration $X \to \Omega E \to Z$ where both X and Z have finitely generated mod p cohomology. By [10, Theorem 4.1], the cohomology of ΩE is finitely generated as an algebra over the Steenrod algebra, in other words

$QH^*(\Omega E; \mathbb{F}_p)$ is finitely generated as an $\mathcal{A}_p$-module. This unstable module, which is isomorphic to the module of indecomposable elements of $\Omega_0 E$, is thus finite if and only if it is locally finite, and this last condition is equivalent to the pointed mapping space $\mathrm{map}_*(B\mathbb{Z}/p, \Omega\Omega_0 E)$ being contractible. For simply connected spaces this was shown in [15, Theorem 3.2], and it is easy to remove the connectedness assumption; for H-spaces this was done in [9, Proposition 1.2] and for arbitrary spaces in [8, Lemma 1.1]. These results tell us that both $\mathrm{map}_*(B\mathbb{Z}/p, \Omega X)$ and $\mathrm{map}_*(B\mathbb{Z}/p, \Omega Z)$ are contractible and the result follows. □

Let us analyze the structure of an arbitrary connected p-Noetherian group. The following theorem tells us that it always differs from a p-compact group in a single p-completed Eilenberg–Mac Lane space.

Theorem 1.9. *Let (X, BX, e) be any p-Noetherian group. There exists then a fibration*

$$K(P,2)^\wedge_p \to BX \to BY,$$

where P is a finite direct sum of copies of cyclic groups and Prüfer groups and Y is a p-compact group.

Proof. By assumption, the mod p cohomology of X is finitely generated as an algebra. In other words, the module of indecomposable elements $QH^*(X; \mathbb{F}_p)$ is finite. Therefore, by [9, Proposition 1.2], the loop space ΩX is $B\mathbb{Z}/p$-local, or equivalently the classifying space BX is $\Sigma^2 B\mathbb{Z}/p$-local ([20, Theorem 3.A.1]). The analysis in [4] by Bousfield of the Postnikov-like nullification tower shows then that the homotopy fiber of the nullification map $BX \to P_{\Sigma B\mathbb{Z}/p} BX$ is a single Eilenberg–Mac Lane space $K(P, 2)$, where P is an abelian p-torsion group. Moreover, he also shows that the corresponding fibration is principal. In particular, this implies that $P_{\Sigma B\mathbb{Z}/p} BX$ is a p-good space, and $(P_{\Sigma B\mathbb{Z}/p} BX)^\wedge_p$ is p-complete by the fiber lemma [5, II.5.1].

From the equivalence $P_{B\mathbb{Z}/p} X \simeq \Omega P_{\Sigma B\mathbb{Z}/p} BX$ ([20, Theorem 3.A.1]), we obtain a loop fibration $K(P, 1) \to X \to P_{B\mathbb{Z}/p} X$. Since X is a loop space with finitely generated mod p cohomology, we know from [9, Theorem 7.3], or directly from [12], that P is a finite direct sum of copies of cyclic groups and Prüfer groups and that $H^*(P_{B\mathbb{Z}/p} X; \mathbb{F}_p)$ is finite.

Let us consider the loop space $P_{B\mathbb{Z}/p} X$. Notice that $\pi_1 P_{\Sigma B\mathbb{Z}/p} BX \cong \pi_1 BX$, which must be a finite p-group by Remark 1.2. By p-completing we hence obtain the classifying space of a p-compact group $BY = (P_{\Sigma B\mathbb{Z}/p} BX)^\wedge_p$. □

The fibration we have obtained allows us to give a precise description of the component of the constant in the pointed mapping space $\mathrm{map}_*(B\mathbb{Z}/p, BX)$.

Corollary 1.10. *Let (X, BX, e) be a p-Noetherian group. Then $\mathrm{map}_*(B\mathbb{Z}/p, BX)_c$ is the classifying space of a finite elementary abelian p-group.*

Proof. Consider the fibration $K(P, 2)^\wedge_p \to BX \to BY$ from Theorem 1.9. Since Y is a p-compact group, $H^*(\Omega BY; \mathbb{F}_p)$ is finite and $\mathrm{map}_*(B\mathbb{Z}/p, \Omega BY) \simeq *$ by [28]. Therefore the component $\mathrm{map}_*(B\mathbb{Z}/p, BY)_c$ is contractible and $\mathrm{map}_*(B\mathbb{Z}/p, BX)_c$

$\simeq \operatorname{map}_*(B\mathbb{Z}/p, K(P,2)^\wedge_p)_c$. By [28, Theorem 1.5], $\operatorname{map}_*(B\mathbb{Z}/p, K(P,2)^\wedge_p) \simeq \operatorname{map}_*(B\mathbb{Z}/p, K(P,2))$, which has trivial homotopy groups in degrees ≥ 2. The component of the constant map is thus the classifying space of a finite elementary abelian p-group $V = \operatorname{Hom}(\mathbb{Z}/p, P)$. □

From Theorem 1.9 we deduce that many p-Noetherian groups are 3-connected covers of p-compact groups.

Corollary 1.11. *Let (X, BX, e) be a p-Noetherian group. Then X is 3-connected if and only if BX is the 4-connected cover of the classifying space of a p-compact group.*

Proof. One implication is obvious. Let us hence assume that X is 3-connected and consider the fibration $K(P,2)^\wedge_p \to BX \to BY$ from Theorem 1.9. We see that BY is 2-connected, hence 3-connected ([6, Theorem 6.10]). This shows that P must be a divisible abelian p-group, or equivalently that $K(P,2)^\wedge_p \simeq K(\oplus\mathbb{Z}^\wedge_p, 3)$. □

To the Lie group Example 1.7 we can now add new examples of p-Noetherian groups, namely those given by 3-connected covers of exotic p-compact groups.

Example 1.12. Let X be a p-compact group such that BX is 3-connected and $\pi_4(BX) \cong \mathbb{Z}^\wedge_p$. By looking at the classification of p-compact groups, we observe that there are only two sporadic examples, namely numbers 23 and 30 in the Shephard–Todd list ([37]), and one infinite family, number $2b$, corresponding to the dihedral groups D_{2m}. The triple $(X\langle 3\rangle, (BX)\langle 4\rangle, e)$ is a p-Noetherian group by Corollary 1.11.

The two sporadic examples are defined at primes $p \equiv 1, 4 \bmod 5$, and they are non-modular since the only primes which divide the order of their Weyl group are 2, 3 and 5. The family of p-compact groups corresponding to the dihedral groups D_{2m} is defined for primes $p \equiv \pm 1 \bmod m$. Note that $p = 2$ occurs when $m = 3$ and corresponds to the exceptional Lie group G_2.

Remark 1.13. From Corollary 1.11 we obtain a classification of 3-connected p-Noetherian groups. They are given by the 3-connected covers of simply connected p-compact groups, which are known from the recent classification results ([29], [30], [2], [3]). A general classification will be more difficult to obtain, even in the 2-connected case, as there are p-Noetherian groups fibering over a product of p-compact groups which do not split themselves as a product. Consider indeed the homotopy fiber of the composite map

$$f : BS^3 \times BS^3 \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4),$$

where the first map is the fourth Postnikov section and the second is given by the sum. Let us complete this fiber at the prime 7 for example and call it BX. Even though X splits as a product $(S^3)^\wedge_7 \times (S^3)^\wedge_7\langle 3\rangle$, the classifying space BX does not split as we show next.

Assume that BX splits as a product $(BS^3)^\wedge_7 \times (BS^3)^\wedge_7\langle 4\rangle$. There exists then an essential map $g : (BS^3)^\wedge_7 \to (BS^3)^\wedge_7 \times (BS^3)^\wedge_7$ such that $f \circ g \simeq *$ and $p_1 \circ g$ is an equivalence, where p_1 denotes the projection on the first factor. But, on the fourth homology group g induces a morphism of degree $n \neq 0$ on the first copy of $(BS^3)^\wedge_7$ and of degree m on the second. The composite $f \circ g$ will thus have degree $n+m$ on H_4. We claim that this cannot be zero. Both m and n must be squares in $\mathbb{Z}_7^\wedge$ as a self-map of $(BS^3)_7^\wedge$ is induced by a self-map on the maximal torus $(BS^1)_7^\wedge$. But the sum of two 7-adic squares is null if and only if both are. Therefore BX cannot split as a product. We refer the reader to Dwyer and Mislin's article [14] for a complete study of self-maps of BS^3.

2. Indecomposable elements and the Krull filtration

As mentioned in the introduction, a good way to understand the cohomology of a space as an algebra over the Steenrod algebra is to look at the module of indecomposable elements $QH^*(X;\mathbb{F}_p) = \tilde{H}^*(X;\mathbb{F}_p)/\tilde{H}^*(X;\mathbb{F}_p)\cdot\tilde{H}^*(X;\mathbb{F}_p)$. An important observation here is that this definition depends on the choice of a base point, or more exactly on the choice of a component X_0 if X is not connected. Since $H^0(X;\mathbb{F}_p)$ is a p-Boolean algebra, it follows that $QH^*(X;\mathbb{F}_p)$ is isomorphic to $QH^*(X_0;\mathbb{F}_p)$.

There is a (Krull) filtration of the category $\mathcal{U}$ of unstable modules, $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots$, such that $\mathcal{U}_0$ consists of the locally finite unstable modules. Schwartz established in [33, Theorem 6.2.4] a criterion to check whether (and where) an unstable module lives in the Krull filtration, namely $M \in \mathcal{U}_n$ if and only if $\bar{T}^{n+1}M = 0$, where $\bar{T}$ is Lannes' reduced T-functor.

Therefore our objective in this section is to prove that $\bar{T}^2 QH^*(BX;\mathbb{F}_p) = 0$ for a p-Noetherian group. To do so, we need first to find a geometrical interpretation of the reduced T-functor.

Recall that "under some mild assumptions", $TH^*(Z;\mathbb{F}_p) \cong H^*(\mathrm{map}(B\mathbb{Z}/p, Z);\mathbb{F}_p)$. Lannes' standard mild assumptions on Z are that $TH^*(Z;\mathbb{F}_p)$ is of finite type (or $H^*(\mathrm{map}(B\mathbb{Z}/p, Z);\mathbb{F}_p)$ is of finite type), and that $\mathrm{map}(B\mathbb{Z}/p, Z)$ is p-good ([26, Proposition 3.4.4]). We will not need to understand globally the mapping space, but restrict our attention to the component $\mathrm{map}(B\mathbb{Z}/p, Z)_c$ of the constant map, the natural choice of base point in the full mapping space. We thus only consider the component $T_c(H^*(Z;\mathbb{F}_p))$ of Lannes' T-functor. Our first goal is to prove that the finite type conditions are satisfied in our situation.

Lemma 2.1. *Let $K(G,1) \to E \to B$ be a fibration of connected spaces with a section where G is a discrete group. Then $E \simeq K(G,1) \times B$ if and only if the induced action $\pi_1(B) \to \mathrm{Aut}(G)$ is trivial.*

Proof. Since $\mathrm{aut}_*(K(G,1)) \simeq \mathrm{Aut}(G)$, the map which classifies this split fibration is of the form $B \to B\,\mathrm{Aut}(G)$ and it factors through $B\pi_1(B) \to B\,\mathrm{Aut}(G)$. □

Let us have a closer look at the mapping space $\mathrm{map}(B\mathbb{Z}/p, BX)_c$ for a p-Noetherian group X.

Proposition 2.2. *Let Z be a connected space such that $\Omega^2 Z$ is $B\mathbb{Z}/p$-local. Then the component of the mapping space $\mathrm{map}(B\mathbb{Z}/p, Z)_c$ splits as a product $Z \times \mathrm{map}_*(B\mathbb{Z}/p, Z)_c$ and $\mathrm{map}_*(B\mathbb{Z}/p, Z)_c$ is the classifying space of an elementary abelian p-group (not necessarily finite).*

Proof. By the work of Bousfield ([4, Theorem 7.2]), the homotopy fiber of the nullification map $Z \to P_{\Sigma B\mathbb{Z}/p}Z$ is a single Eilenberg–Mac Lane space $K(P, 2)$, where P is an abelian p-torsion group. He also shows that the fibration $K(P, 2) \to Z \to P_{\Sigma B\mathbb{Z}/p}Z$ is principal. By adjunction, the component $\mathrm{map}_*(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p}Z)_c$ is contractible. Therefore $\mathrm{map}_*(B\mathbb{Z}/p, Z)_c \simeq \mathrm{map}_*(B\mathbb{Z}/p, K(P, 2))_c$, which is the classifying space of the elementary abelian p-group $W = \mathrm{Hom}(\mathbb{Z}/p, P)$.

Now, by Lemma 2.1, we only need to check that the action of $\pi = \pi_1(Z)$ on W is trivial. By taking mapping spaces at the component of the constant map and evaluation, we obtain the following diagram of fibrations (since $\mathrm{map}_*(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p}Z)_c$ is contractible):

$$\begin{tikzcd}
BW \arrow[r, equal] \arrow[d] & BW \arrow[d] & \\
\mathrm{map}(B\mathbb{Z}/p, K(P,2))_c \arrow[r] \arrow[d] & \mathrm{map}(B\mathbb{Z}/p, Z)_c \arrow[r] \arrow[d] & \mathrm{map}(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p}Z)_c \arrow[d, "\simeq"] \\
K(P,2) \arrow[r] & Z \arrow[r] & P_{\Sigma B\mathbb{Z}/p}Z
\end{tikzcd}$$

The bottom and middle horizontal fibrations are principal, therefore the action of the fundamental group of the base space, $\pi_1 P_{\Sigma B\mathbb{Z}/p}Z \cong \pi$, is trivial on all homotopy groups of the fiber, in particular on the fundamental group of the fiber. This action can be seen as conjugation in the fundamental group of the total space $\mathrm{map}(B\mathbb{Z}/p, Z)_c$, but now it does not matter whether we look at the vertical fibration or the horizontal one (in both cases the induced morphism is surjective on the fundamental group). □

Corollary 2.3. *Let X be a p-Noetherian group. Then $\mathrm{map}(B\mathbb{Z}/p, BX)_c$ splits as a product $BX \times \mathrm{map}_*(B\mathbb{Z}/p, BX)_c$ where $\mathrm{map}_*(B\mathbb{Z}/p, BX)_c$ is the classifying space of a finite elementary abelian p-group. In particular, $\mathrm{map}(B\mathbb{Z}/p, BX)_c$ is p-good, p-complete and $H^*(\mathrm{map}(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p)$ is of finite type.*

Proof. The finiteness of the elementary abelian p-group follows from Corollary 1.10. □

In particular, we see that $\mathrm{map}(B\mathbb{Z}/p, BX)_c$ is again the classifying space of a p-Noetherian group.

Theorem 2.4. *Let Z be a connected p-complete space with $H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ and $H^*(Z; \mathbb{F}_p)$ of finite type. If $\Omega^2 Z$ is $B\mathbb{Z}/p$-local, then*

$$\bar{T} QH^*(Z; \mathbb{F}_p) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p).$$

In particular the unstable module $QH^*(Z; \mathbb{F}_p)$ lies in $\mathcal{U}_1$.

Proof. In Proposition 2.2 we obtained a splitting $\mathrm{map}(B\mathbb{Z}/p, Z)_c \simeq \mathrm{map}_*(B\mathbb{Z}/p, Z)_c \times Z$ and an equivalence $\mathrm{map}_*(B\mathbb{Z}/p, Z)_c \simeq BW$ where W is an elementary abelian p-group. With the hypothesis of the theorem this splitting shows that $H^*(BW; \mathbb{F}_p)$ is of finite type and therefore W is finite. Therefore $\mathrm{map}(B\mathbb{Z}/p, Z)_c$ is p-good. Since moreover $H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ is of finite type by assumption, we can apply Lannes' result [26, Proposition 3.4.4] to deduce that the T-functor computes what it should: $T_c H^*(Z; \mathbb{F}_p) \cong H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$.

Notice also that $QH^*(\mathrm{map}(B\mathbb{Z}/p, Z); \mathbb{F}_p) \cong QH^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$. Since Lannes' T-functor commutes with taking the module of indecomposable elements, $TQH^*(Z; \mathbb{F}_p) \cong QTH^*(Z; \mathbb{F}_p)$. But in degree zero $TH^*(Z; \mathbb{F}_p)$ is a Boolean algebra ([33, Section 3.8]), so that $QTH^*(Z; \mathbb{F}_p) \cong Q(T_c H^*(Z; \mathbb{F}_p))$, which is isomorphic to $QH^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$. The splitting yields next an isomorphism

$$TQH^*(Z; \mathbb{F}_p) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p) \oplus QH^*(Z; \mathbb{F}_p)$$

so that we have finally identified $\bar{T}QH^*(Z; \mathbb{F}_p) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$. This proves the first part of the theorem. For the second claim, use the fact that $\mathrm{map}_*(B\mathbb{Z}/p, Z)_c \simeq BW$, the classifying space of a finite elementary abelian group. The cohomology of W is finitely generated as an algebra, so $QH^*(BW; \mathbb{F}_p)$ is finite and lies in $\mathcal{U}_0$. Therefore $\bar{T}QH^*(BW; \mathbb{F}_p) = 0$, or equivalently $\bar{T}^2 QH^*(Z; \mathbb{F}_p) = 0$, and so $QH^*(Z; \mathbb{F}_p)$ lies in $\mathcal{U}_1$. □

Let us now turn to an even finer analysis of the module of indecomposable elements. Let us denote by Q_1 the unstable module $QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$ of the cohomology of a cyclic group of order p. At the prime $p = 2$, the unstable module Q_1 is isomorphic to $\Sigma\mathbb{F}_2 = \Sigma F(0)$. At an odd prime, Q_1 is an unstable module with one generator t in degree 1 and its Bockstein βt in degree 2.

Proposition 2.5. *Let Z be a connected p-complete space with $H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ and $H^*(Z; \mathbb{F}_p)$ of finite type. Assume $\Omega^2 Z$ is $B\mathbb{Z}/p$-local. Define $Q_1 = QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$. Then there exists a morphism $QH^*(Z; \mathbb{F}_p) \to F(1) \otimes (Q_1^{\oplus k})$ with finite cokernel and locally finite kernel.*

Proof. Schwartz characterizes in [35, Proposition 2.3] the unstable modules M in $\mathcal{U}_1$ as those sitting in an exact sequence $0 \to K \to M \to F(1) \otimes L \to N \to 0$, where K, L, and N are locally finite (i.e. in $\mathcal{U}_0$). In particular $\bar{T}M \cong L$ since $\bar{T}F(1) = F(0)$ and T commutes with tensor products ([33, Theorem 3.5.1]). In our case we know from the previous theorem that $\bar{T}QH^*(Z; \mathbb{F}_p) \cong QH^*(BW; \mathbb{F}_p)$ where W is an abelian elementary group, say of rank k. Thus $L = Q_1^{\oplus k}$. The quotient N of $F(1) \otimes (Q_1^{\oplus k})$ will be finitely generated. As it is locally finite it must be finite. □

We finally come back to p-Noetherian groups and prove that the module of indecomposable elements $QH^*(BX; \mathbb{F}_p)$ is as small as expected.

Theorem 2.6. *Let X be a p-Noetherian group. Then*

$$\bar{T}QH^*(BX; \mathbb{F}_p) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p).$$

In particular the unstable module $QH^*(BX; \mathbb{F}_p)$ lies in $\mathcal{U}_1$.

Proof. The assumptions in Theorem 2.4 are satisfied by Corollary 2.3. □

3. Fibrations over spaces with finite cohomology

In our study of $H^*(BX;\mathbb{F}_p)$, we have already managed to prove that $QH^*(BX;\mathbb{F}_p)$ lives in $\mathcal{U}_1$, that is, only one stage higher than where $QH^*(X;\mathbb{F}_p)$ lives. What is left to prove is that $H^*(BX;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Therefore we analyze the fibration $K(P,2)^\wedge_p \to BX \to BY$ of Theorem 1.9.

Let $F \to E \to B$ be a fibration where both $H^*(B;\mathbb{F}_p)$ and $H^*(F;\mathbb{F}_p)$ are finitely generated $\mathcal{A}_p$-algebras. In this situation, we ask whether the same finiteness condition holds for $H^*(E;\mathbb{F}_p)$. When the fibration is one of H-spaces and H-maps we proved in [10] that this is true. But in general some restrictions have to be imposed, even when the fiber is a single Eilenberg–Mac Lane space, as shown by the following example.

Example 3.1. Consider the folding map $K(\mathbb{Z},3) \vee K(\mathbb{Z},3) \to K(\mathbb{Z},3)$. An easy application of Puppe's theorem ([32]), shows that the homotopy fiber is $\Sigma\Omega K(\mathbb{Z},3) \simeq \Sigma K(\mathbb{Z},2)$. Therefore there exists a fibration

$$K(\mathbb{Z},2) \to \Sigma K(\mathbb{Z},2) \to K(\mathbb{Z},3) \vee K(\mathbb{Z},3).$$

The mod 2 cohomology of the fiber is finitely generated as an algebra, the cohomology of the base space is generated over $\mathcal{A}_2$ by the two fundamental classes in degree 3. However the cohomology of $\Sigma K(\mathbb{Z},2)$ is not finitely generated over $\mathcal{A}_2$ (as it is a suspension it would be finitely generated as an unstable module, and therefore would belong to some stage of the Krull filtration; by Schwartz's solution [34] to Kuhn's non-realizability conjecture this would imply that the cohomology were locally finite, which it is not).

This example indicates that we must impose stronger conditions on the base space of the fibration to make sure that the cohomology of the total space is finitely generated as an algebra over $\mathcal{A}_p$. In this section we study fibrations $F \to E \to B$ where $\pi_1 B$ acts trivially on the cohomology of the fiber. We will assume that $H^*(B;\mathbb{F}_p)$ is *finite* and the fiber F is a finite product of Eilenberg–Mac Lane spaces $\prod_{i=1}^q K(A_i,n_i)$ where A_i is a finitely generated abelian group for all i. Both assumptions will play an essential role in the analysis of the cohomology of the total space. The finiteness of the base forces the Serre spectral sequence to collapse at some finite stage, and the hypothesis on A_i implies that the cohomology of $K(A_i,n_i)$ is generated, as an algebra over the Steenrod algebra $\mathcal{A}_p$, by a finite number of fundamental classes $\iota_1,\dots,\iota_m$ of degree n, and possibly certain higher Bockstein on these classes. It is a free algebra by work of Serre at the prime 2 ([36]), and Cartan at odd primes ([7]).

Lemma 3.2. *There exists a splitting $H^*(\prod K(A_i,n_i);\mathbb{F}_p) \cong F^*\otimes G^*$ of algebras where F^* is finitely generated as an algebra, and G^* consists of permanent cycles in the Serre spectral sequence. Moreover G^* is finitely generated as an algebra over $\mathcal{A}_p$.*

Proof. By Kudo's transgression theorem, all classes obtained by applying Steenrod operations to transgressive operations are transgressive. Let us choose therefore an integer r larger than the dimension of the cohomology of the base. If $\{x_1,\dots,x_k\}$ is a set of generators of $H^*(F;\mathbb{F}_p)$ as an $\mathcal{A}_p$-algebra, the elements $\mathcal{P}^I x_k$ generate freely $H^*(F;\mathbb{F}_p)$ as an algebra, where I are certain admissible sequences ([33, p. 184]).

The elements $1 \otimes \mathcal{P}^I x_k$ are permanent cycles for any sequence I of degree larger than $r-n-1$ and any k. We will say that such generators $\mathcal{P}^I x_k$ have *large* degree, and the others, of which there are only a finite number, have *small* degree. We now define F^* to be the subalgebra generated by the generators of small degree, and G^* by all other large degree generators. Then $H^*(F;\mathbb{F}_p) \cong F^* \otimes G^*$.

The only claim left to prove is the finite generation of G^*. Let us look at the inclusion of algebras $G^* \subset H^*(F;\mathbb{F}_p)$. At the level of modules of indecomposable elements it induces an inclusion $QG^* \subset QH^*(F;\mathbb{F}_p)$, because of the freeness of $H^*(F;\mathbb{F}_p)$ and our choices of generators. Since the category $\mathcal{U}$ of unstable modules is locally Noetherian ([33, Theorem 1.8.1]), the unstable module QG^* is finitely generated. Therefore, G^* is a finitely generated $\mathcal{A}_p$-algebra. $\square$

The proof of the next proposition follows the lines of the Dwyer–Wilkerson result [17, Proposition 12.4]; see also Evens [18]. We use the notation from Lemma 3.2.

Proposition 3.3. *The cohomology of the total space $H^*(E;\mathbb{F}_p)$ is finitely generated as a module over $H^*(B;\mathbb{F}_p)[z_1,\dots,z_k] \otimes G^*$.*

Proof. The free algebra F^* in Lemma 3.2 is finitely generated and we first consider all polynomial generators $a_1,\dots,a_k$. We define $z_i = (a_i)^{p^{n_i}}$ where n_i is the smallest integer such that this power of a_i is a permanent cycle (it exists since these powers are transgressive, cf. the proof of Lemma 3.2). Then F^* is a finitely generated module over $\mathbb{F}_p[z_1,\dots,z_k]$. Choose now a finite set of generators $g_1,\dots,g_r$ of G^* as an algebra over the Steenrod algebra.

The elements z_i and the elements $g_1,\dots,g_r$ are permanent cycles in the vertical axis of the Serre spectral sequence, one can thus choose elements z_i' and g_j' in $H^*(E;\mathbb{F}_p)$ whose images in $H^*\big(\prod K(A_i,n_i);\mathbb{F}_p\big)$ are the z_i's and the g_j's. Better said, we can choose an algebra monomorphism $s\colon \mathbb{F}_p[z_i] \otimes G^* \hookrightarrow H^*(E;\mathbb{F}_p)$ since both $\mathbb{F}_p[z_i]$ and G^* are free algebras. The elements in $H^*(B;\mathbb{F}_p)$ act on $H^*(E;\mathbb{F}_p)$ via the morphism $p^*\colon H^*(X;\mathbb{F}_p) \to H^*(E;\mathbb{F}_p)$. This explains the module structure.

Thus $E_\infty = E_r$ is finitely generated as a module over $H^*(B;\mathbb{F}_p)[z_1,\dots,z_k] \otimes G^*$. Therefore so is $H^*(E;\mathbb{F}_p)$ by [38, Corollary VII.3.3]. $\square$

The difficulty to infer information about the $\mathcal{A}_p$-algebra structure from the module structure comes from the fact that the algebra morphism s is not a morphism of $\mathcal{A}_p$-algebras. To circumvent this problem we will appeal to the algebraic result proved in Appendix A.

Theorem 3.4. *Consider a fibration $\prod_{i=1}^q K(A_i,n_i) \to E \to B$ where $H^*(B;\mathbb{F}_p)$ is finite and A_i is a finitely generated abelian group for all i. The cohomology $H^*(E;\mathbb{F}_p)$ is then finitely generated as an algebra over $\mathcal{A}_p$.*

Proof. In the notation of the appendix, $H^*(B;\mathbb{F}_p)[z_1,\dots,z_k]$ is the connected and commutative finitely generated algebra C^*, and $B^* = H^*(E;\mathbb{F}_p)$. By Proposition 3.3, $H^*(E;\mathbb{F}_p)$ is a finitely generated $C^* \otimes G^*$-module. The action of $C^* \otimes G^*$ on B^* has been defined in the previous proof via an algebra morphism (constructed from a section $s : G^* \to B^*$), thus B^* is a $C^* \otimes G^*$-algebra. Define now $\pi : H^*(\prod K(A_i,n_i);\mathbb{F}_p) \cong F^* \otimes G^* \to G^*$ to be the projection and $p : B^* \to G^*$ to be the composite $\pi \circ i^*$. This is a morphism of G^*-modules so that Proposition A.2 applies. Hence $H^*(E;\mathbb{F}_p)$ is finitely generated as an algebra over $\mathcal{A}_p$. $\square$

Remark 3.5. The nature of Theorem 3.4 is purely cohomological. Therefore the same statement remains true if we relax the assumption on the fiber in the following way: The fiber F should be homotopic, up to p-completion, to $\prod_{i=1}^q K(A_i, n_i)$ where each A_i is a finitely generated abelian group. This will allow us to include summands of the form $\mathbb{Z}_{p^\infty}$ or $\mathbb{Z}_p^\wedge$. In fact the same proof goes through with the assumption that $H^*(F;\mathbb{F}_p)$ is a free algebra, finitely generated as an algebra over $\mathcal{A}_p$.

4. The cohomology of p-Noetherian groups

We are about to conclude our study of the cohomology of classifying spaces of p-Noetherian groups. We have seen in Theorem 1.9 that any p-Noetherian group is the total space of a fibration over a p-compact group with fiber an Eilenberg–Mac Lane space. Recall from the main theorem in [17] that the mod p cohomology of the classifying space of a p-compact group is finitely generated as an algebra. Our objective is to prove the following theorem, which, together with Theorem 2.6, gives a very accurate description for the cohomology of classifying spaces of p-Noetherian groups.

Theorem 4.1. *Let (X, BX, e) be a p-Noetherian group. Then $H^*(BX;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

The strategy is to use the fibration $K(P,2)_p^\wedge \to BX \to BY$ of Theorem 1.9. As we do not know whether the Serre spectral sequence collapses at some finite stage, we reduce the problem in several steps to the study of a spectral sequence over a finite base (in order to apply our results from the previous section).

A *p-compact toral group* P is a p-compact group which is an extension of a p-compact torus $((S^1)^n)_p^\wedge$ by a finite p-group. Dwyer and Wilkerson show in [17] that any p-compact group Y admits a maximal p-compact toral subgroup $N \le Y$ such that the homotopy fiber Y/N of the map $Bi\colon BN \to BY$ has finite mod p cohomology and Euler characteristic prime to p (see [17, proof of Theorem 2.4]). An important property of Y/N for us is that it is connected. This is obvious when Y is connected and follows in general from the analysis of the relationship between the Weyl groups of Y and a connected component, undertaken by Møller and Notbohm in [31, Corollary 3.9]. A transfer argument (see [17, Theorem 9.13]) then shows that Bi induces a monomorphism in mod p cohomology. Consider now the pull-back diagram

$$\begin{array}{ccc} \widetilde{BN} & \xrightarrow{B\tilde{i}} & BX \\ \downarrow & & \downarrow \\ BN & \xrightarrow{Bi} & BY \end{array}$$

First note that, by Proposition 1.8, $\widetilde{BN}$ is the classifying space of a p-Noetherian group because we have a fibration $K(P,2)_p^\wedge \to \widetilde{BN} \to BN$. We will first show that

$H^*(\widetilde{BN};\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra by using this fibration, and then we will show that the cohomology $H^*(BX;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra by using the fibration $Y/N \to \widetilde{BN} \to BX$. We start with a technical result which will be used in both steps of the proof.

Proposition 4.2. *Let $F \to E \xrightarrow{q} B$ be a fibration such that $\pi_1(B)$ acts nilpotently on $H_*(F;\mathbb{F}_p)$. Assume that q induces an isomorphism $\bar{T}QH^*B \cong \bar{T}QH^*E$, $H^*(F;\mathbb{F}_p)$ is locally finite, and $H^*(E;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra. Then, if $\operatorname{Ker} q^*$ is a finitely generated ideal then $H^*(B;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra.*

Proof. Let A be the algebra which is the quotient of $H^*(E;\mathbb{F}_p)$ by the ideal generated by the image of q^*, that is, $\mathbb{F}_p \otimes_{H^*(B;\mathbb{F}_p)} H^*(E;\mathbb{F}_p)$. There is a coexact sequence

$$H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^* \xrightarrow{q^*} H^*(E;\mathbb{F}_p) \to A.$$

Since $H^*(E;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra, the same is true for A. Therefore QA is a finitely generated $\mathcal{A}_p$-module, which is the cokernel of Qq^* by right-exactness of the functor Q. Moreover, since by assumption $\bar{T}QH^*B \cong \bar{T}QH^*E$, it follows from exactness of the reduced T-functor that $\bar{T}QA = 0$. This means that QA belongs to $\mathcal{U}_0$, i.e. it is locally finite, hence finite. Equivalently A is a finitely generated algebra.

We want to show that A is in fact a finite algebra. The fiber inclusion $F \to E$ of the fibration q induces a morphism $\iota^*\colon A \to H^*(F;\mathbb{F}_p)$. A careful study of the Eilenberg–Moore spectral sequence ([27, Theorem 0.5] (for the prime 2) or [33, Theorem 8.7.8]) shows that this morphism is an F-monomorphism. Take now any element $a \in A$. Because $H^*(F;\mathbb{F}_p)$ is locally finite, there exists $M > 0$ such that $a^{p^M} \in \operatorname{Ker}\iota^*$, which is nilpotent. Thus there exists $N > 0$ such that $a^{p^{N+M}} = 0$. Since all elements of A are nilpotent and it is a finitely generated algebra, A must be finite.

Note next that, as an $H^*(B;\mathbb{F}_p)$-module, the cohomology $H^*(E;\mathbb{F}_p)$ is isomorphic to the module $(H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*)\{b_1,\dots,b_k\}$, where the b_i's are the generators of A as a (graded) $\mathbb{F}_p$-vector space. This description shows that the morphism $Q(H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*) \to QH^*(E;\mathbb{F}_p)$ is an isomorphism in high degrees. That is, there exists $K > 0$ such that

$$(Q(H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*))^{>K} \cong (QH^*(E;\mathbb{F}_p))^{>K},$$

which is an unstable submodule of $QH^*(E;\mathbb{F}_p)$. Since $H^*(E;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra, $QH^*(E;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-module. But then, as the category of unstable modules over the Steenrod algebra is locally Noetherian ([33, Theorem 1.8.1]), the same holds for $(Q(H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*))^{>K}$. In particular, $Q(H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*)$ is a finitely generated $\mathcal{A}_p$-module since it is of finite type, that is, $H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*$ is a finitely generated $\mathcal{A}_p$-algebra.

Finally, $H^*(B;\mathbb{F}_p)$ is generated by $H^*(B;\mathbb{F}_p)/\!/\operatorname{Ker} q^*$ and $\operatorname{Ker} q^*$, it is therefore also finitely generated as an algebra over the Steenrod algebra since $\operatorname{Ker} q^*$ is a finitely generated ideal . ◻

We next apply this result to the fibration $Y/N \to \widetilde{BN} \to BX$.

Corollary 4.3. *$H^*(BX;\mathbb{F}_p)$ is a finitely generated algebra if $H^*(\widetilde{BN};\mathbb{F}_p)$ is so.*

Proof. Consider the fibration $Y/N \to \widetilde{BN} \xrightarrow{B\tilde{i}} BX$. The fiber Y/N is $\mathbb{F}_p$-finite and p-complete by [17, Proposition 5.8] so that Miller's Theorem of [28] applies: The mapping space $\mathrm{map}(B\mathbb{Z}/p, Y/N)$ is weakly equivalent to Y/N, in particular it is connected. We thus have a diagram of horizontal fibrations

$$\begin{tikzcd}
\mathrm{map}(B\mathbb{Z}/p, Y/N)_c \arrow[r] \arrow[d, "\mathrm{ev}", "\simeq"'] & \mathrm{map}(B\mathbb{Z}/p, \widetilde{BN})_c \arrow[r] \arrow[d, "\mathrm{ev}_r"] & \mathrm{map}(B\mathbb{Z}/p, BX)_c \arrow[d, "\mathrm{ev}_n"] \\
Y/N \arrow[r] & \widetilde{BN} \arrow[r] & BX
\end{tikzcd}$$

where the left vertical arrow is an equivalence. The connectedness of Y/N also implies that the right hand square is a homotopy pull-back square and thus $B\tilde{i}$ induces an equivalence $\mathrm{map}_*(B\mathbb{Z}/p, \widetilde{BN})_c \to \mathrm{map}_*(B\mathbb{Z}/p, BX)_c$. Both BX and $\widetilde{BN}$ are p-Noetherian groups and Theorem 2.6 then implies that $\bar{T}QH^*BX \to \bar{T}QH^*\widetilde{BN}$ is an isomorphism.

We conclude now by Proposition 4.2, because $\pi_1(BX)$ is a finite p-group and q^* is injective by a transfer argument of [17, Theorem 9.13] (the Euler characteristic $\chi(Y/N)$ is prime to p). □

The key fact in the next argument is that any p-compact toral group satisfies the Peter–Weyl theorem. That is, it admits a homotopy monomorphism into $U(n)^\wedge_p$ for some n. This is shown for example in [24, Proposition 2.2]. Let us then choose such a map $\rho: BN \to BU(n)^\wedge_p$ for the maximal p-compact toral group BN. The mod p cohomology of the fiber F is hence $\mathbb{F}_p$-finite.

Proposition 4.4. *Let E_0 be the homotopy pull-back of the diagram $F \to BN \leftarrow \widetilde{BN}$. Then $H^*(E_0;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

Proof. The space E_0 fits by construction into a fibration $K(P,2)^\wedge_p \to E_0 \to F$. Since the mod p cohomology of F is finite, Theorem 3.4 shows that $H^*(E_0;\mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra. □

Let us now concentrate on the map $E_0 \to \widetilde{BN}$. We will filter it by using the top left block diagonal inclusions $U(r-1) \subset U(r)$ for $1 \le r \le n$. At the level of classifying spaces these inclusions induce oriented spherical fibrations

$$S^{2r-1} \to BU(r-1) \to BU(r).$$

Let $BU(r) \to BU(n)$ be the appropriate composite and define E_r to be the homotopy pull-back of $\widetilde{BN} \xrightarrow{\rho} BU(n)^\wedge_p \leftarrow BU(r)^\wedge_p$. Thus $E_n = \widetilde{BN}$ and, for $1 \le r \le n$, we have spherical fibrations $(S^{2r-1})^\wedge_p \to E_{r-1} \to E_r$. We next summarize some important properties of these spaces.

Proposition 4.5. *The component $\mathrm{map}(B\mathbb{Z}/p, E_r)_c$ splits as $E_r \times \mathrm{map}_*(B\mathbb{Z}/p, E_r)_c$ for any $0 \le r \le n$.*

Proof. Let us denote by $\mathrm{ev}_r : \operatorname{map}(B\mathbb{Z}/p, E_r)_c \to E_r$ the evaluation at the component of the constant map and consider the following commutative diagram of horizontal fibrations of connected spaces:

$$\begin{tikzcd}
\operatorname{map}(B\mathbb{Z}/p, H)_c \arrow[r] \arrow[d, "\simeq"', "\mathrm{ev}"] & \operatorname{map}(B\mathbb{Z}/p, E_r)_c \arrow[r] \arrow[d, "\mathrm{ev}_r"] & \operatorname{map}(B\mathbb{Z}/p, \widetilde{BN})_c \arrow[d, "\mathrm{ev}_n"] \\
H \arrow[r] & E_r \arrow[r] & \widetilde{BN}
\end{tikzcd}$$

The homotopy fiber H has finite cohomology, and is thus equivalent via the evaluation map to the homotopy fiber $\operatorname{map}(B\mathbb{Z}/p, H)_c$ of the top fibration. Since H is connected (because $U(n)$ is), this shows that the right hand square is a pull-back square. But ev_n is a trivial fibration by Proposition 2.3. Hence so is ev_r and $\operatorname{map}(B\mathbb{Z}/p, E_r)_c$ must split as a product $E_r \times \operatorname{map}_*(B\mathbb{Z}/p, E_r)_c$. □

Proposition 4.6. *The morphism of $\mathcal{A}_p$-modules $\bar{T} QH^*(\widetilde{BN}; \mathbb{F}_p) \to \bar{T} QH^*(E_r; \mathbb{F}_p)$ induced by $E_r \to \widetilde{BN}$ is an isomorphism.*

Proof. We have seen in the previous proposition that the map $E_r \to BX$ induces a weak equivalence on the connected component of the constant map in the pointed mapping space $\operatorname{map}_*(B\mathbb{Z}/p, -)_c$. From this point on, the same argument as in the proof of Theorem 2.6 goes through. □

Proof of Theorem 4.1. We know that $H^*(E_0; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. It is thus sufficient to prove that $H^*(E_r; \mathbb{F}_p)$ is a finitely generated $\mathcal{A}_p$-algebra if so is $H^*(E_{r-1}; \mathbb{F}_p)$. Denote by q_r the map $E_{r-1} \to E_r$ turned into a fibration. Since $\operatorname{Ker} q_r^*$ is generated by a single element, namely the Euler class, Proposition 4.2 applies. □

Remark 4.7. In general, an unstable algebra which is finitely generated as an algebra over $\mathcal{A}_p$ may contain unstable subalgebras which are not finitely generated over $\mathcal{A}_p$. Such an example appears in [10, Remark 2.2] as an unstable subalgebra B of $H^*(BS^1 \times S^2; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$, which is not a finitely generated B-module. This B is the ideal generated by y turned into an unstable algebra by adding 1. The quotient is a polynomial algebra, in particular it is not finite.

Remark 4.8. To prove Theorem 4.1 one could also use the fact that p-compact groups satisfy the Peter–Weyl theorem (see [3, Theorem 1.6] and [2, Remark 7.3]). This would slightly shorten the proof and avoid the use of the maximal p-compact toral subgroup. But the Peter–Weyl theorem for p-compact groups is proved using the classification of p-compact groups and we want to emphasize that Theorem 4.1 does not depend on the classification.

Corollary 4.9. *Let G be a simply connected, simple, compact Lie group. Then $H^*((BG)\langle 4\rangle; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra and $QH^*(BG\langle 4\rangle; \mathbb{F}_p)$ belongs to $\mathcal{U}_1$. In fact there exists a morphism $QH^*((BG)\langle 4\rangle; \mathbb{F}_p) \to F(1) \otimes Q_1$ with finite kernel and cokernel.*

Proof. Recall that $Q_1 = QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$. Since BG is 3-connected and $\pi_4 BG \cong \mathbb{Z}$, Proposition 2.5 yields a morphism $QH^*((BG)\langle 4\rangle; \mathbb{F}_p) \to F(1) \otimes Q_1$ (the elementary abelian group W appearing there has rank one). The cokernel is finite and the kernel locally finite. But since $QH^*((BG)\langle 4\rangle; \mathbb{F}_p)$ is a finitely generated module over $\mathcal{A}_p$, the kernel must be finite. □

Example 4.10. Let X be either any simply connected compact Lie group (such as *Spin*(10), which is one of the smallest examples Harada and Kono could not handle at the prime 2) or one of the p-compact groups numbered $2b$, 23 or 30 in the Shephard–Todd list. For odd primes, the exotic p-compact groups arising from this construction are again non-modular. Hence, in all the cases, $H^*(X; \mathbb{Z}_p^\wedge)$ is torsion free. The mod p cohomology is given by $H^*(BX_{23}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{12}, x_{20}]$, $H^*(BX_{30}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{24}, x_{40}, x_{60}]$, and $H^*(BX_{2b,m}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2m}]$. Since all examples are torsion free, the same techniques used by Harada and Kono in [22] and [21] show that $H^*(BX\langle 4\rangle; \mathbb{F}_p) \cong H^*(BX; \mathbb{F}_p)/J \otimes R_h$ where J is the ideal generated by $x_4, \mathcal{P}^1 x_4, \ldots, \mathcal{P}^h x_4$ for a certain h, and R_h is an unstable subalgebra of $H^*(K(\mathbb{Z}_p^\wedge, 3); \mathbb{F}_p)$ which is finitely generated over $\mathcal{A}_p$. In fact Theorems 4.1 and 2.6 show directly that $QH^*((BX)\langle 4\rangle; \mathbb{F}_p)$ is finitely generated as a module over $\mathcal{A}_p$ and belongs to $\mathcal{U}_1$. From the last corollary we see for example that $QH^*((BSpin(10))\langle 4\rangle; \mathbb{F}_p)$ differs from $\Sigma F(1)$ in only a finite number of "low dimensional" classes.

Appendix A. Modules and algebras

Let us consider an unstable algebra B^*. Assume there is another unstable algebra G^* which is finitely generated as an algebra over $\mathcal{A}_p$ and which acts on B^*. Assume also that B^* is finitely generated as a module over G^*. When can we conclude that B^* is finitely generated as an algebra over $\mathcal{A}_p$? If the action of G^* on B^* is compatible with the action of the Steenrod algebra, this is obvious, but it is not true in general as illustrated by the following example. Set $G^* = H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ and let B^* be isomorphic to G^* as an algebra, but define the action of $\mathcal{A}_2$ to be trivial. Then B^* is finitely generated as a G^*-module, but not as an $\mathcal{A}_2$-algebra. We propose a weak notion of compatibility between the G^*-module and $\mathcal{A}_p$-algebra structures.

Proposition A.1. *Let G^* be a connected and commutative finitely generated $\mathcal{A}_p$-algebra, and let B^* be a connected commutative G^*-algebra which is finitely generated as a G^*-module. Assume there exists a morphism $p\colon B^* \to G^*$ of G^*-modules such that $p(\theta(g \cdot 1)) = \theta g$ for all $\theta \in \mathcal{A}_p$ and $g \in G^*$. Then B^* is also a finitely generated $\mathcal{A}_p$-algebra.*

Proof. Since B^* is finitely generated as a G^*-module, let us choose a finite set of generators $b_1 = 1, \ldots, b_n$, where the degree of b_i is positive for $i \geq 2$ by the connectedness assumption. In fact we will assume that b_i belongs to the kernel of p for $i \geq 2$: replace b_i by $b_i - p(b_i) \cdot 1$ if necessary.

Let x be an element in B^*, so x can be written as a G^*-linear combination $x = \sum_{i=1}^n \lambda_i \cdot b_i$. If $\{z_1, \ldots, z_m\}$ denote the $\mathcal{A}_p$-algebra generators of G^*, then each λ_i can be expressed as a polynomial in Steenrod operations θ_i applied to the generators, $\theta_i z_i$ (use the Cartan formula). Our claim is that $b_1, \dots, b_n$ and $z_1 \cdot 1, \dots, z_m \cdot 1$ generate B^* as an algebra over $\mathcal{A}_p$. Since B^* is a unital G^*-algebra, we need to prove that an element of the form $\theta z \cdot 1$, for $\theta \in \mathcal{A}_p$ and $z \in \{z_1, \dots, z_m\}$, can be written as a polynomial in the $\theta(z_i \cdot 1)$'s and the b_k's.

If the action of the Steenrod algebra were compatible with the module action, one would have $\theta z \cdot 1 = \theta(z \cdot 1)$. This is not the case, but it is sufficient to deal with the element $\xi = \theta z \cdot 1 - \theta(z \cdot 1)$. Note that $\xi \in \operatorname{Ker} p$ since $p(\theta z \cdot 1) = \theta z$. We will proceed by induction on the degree. If ξ is in degree zero, then the statement is clear since the algebras are connected. Assume that the statement is true for degrees $< |\xi|$ and write $\xi = \lambda_1 \cdot 1 + \lambda_2 \cdot b_2 + \cdots + \lambda_n \cdot b_n$. By the induction hypothesis we know that the elements $\lambda_i \cdot 1$ can be expressed as polynomials in the $\theta(z_i \cdot 1)$'s and the b_k's. So we only need to deal with $\lambda_1 \cdot 1$. But $\lambda_1 = p(\lambda_1 \cdot 1) = p(\xi - \lambda_2 \cdot b_2 - \cdots - \lambda_n \cdot b_n)$, which is zero because p is a morphism of G^*-modules and $b_i \in \operatorname{Ker} p$ for $i \geq 2$. This concludes the proof. □

In Section 3 we need a slight generalization of this proposition.

Proposition A.2. *Let G^* be a connected and commutative finitely generated $\mathcal{A}_p$-algebra, C^* be a connected and finitely generated algebra, and let B^* be a connected commutative $G^* \otimes C^*$-algebra which is finitely generated as a $G^* \otimes C^*$-module. Assume there exists a morphism $p\colon B^* \to G^*$ of G^*-modules such that $p(\theta(g \cdot 1)) = \theta g$ for all $\theta \in \mathcal{A}_p$ and $g \in G^*$. Then B^* is also a finitely generated $\mathcal{A}_p$-algebra.*

Proof. Just as in the previous proof, the only problem is to write an element of the form $(\theta z) \cdot 1$, with $\theta \in \mathcal{A}_p$ and $z \in G^*$, in terms of the generators z_i, b_k, and generators c_m of the algebra C^*. The proof is then basically the same. □

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