THE INCIDENCE CORRESPONDENCE AND ITS ASSOCIATED MAPS IN HOMOTOPY

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Abstract. The incidence correspondence in the grassmannian which determines the tautological bundle defines a map between cycle spaces on grassmannians. These cycle spaces decompose canonically into a product of Eilenberg-MacLane spaces. These decompositions and the associated maps are calculated up to homotopy.

1. Introduction

Let $\mathcal{C}^p_d(\mathbb{P}^n)$ denote the space of algebraic cycles of codimension p and degree d in $\mathbb{P}^n$. This set can be given the structure of an algebraic variety via the Cayley-Chow-Van der Waerden embedding which takes an irreducible cycle X into $\Psi(X)$ where the Chow Form Ψ is obtained from the following incidence correspondence:

$$\begin{tikzcd} & \{(x,\Lambda) \in \mathbb{P}^n \times \mathcal{G}_{p-1}(\mathbb{P}^n) \mid x \in \Lambda\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathcal{G}_{p-1}(\mathbb{P}^n) \end{tikzcd}$$

$$\Psi(X) = \pi_2 \pi_1^{-1}(X)$$

This map is then extended additively to the topological monoid $\mathcal{C}^p(\mathbb{P}^n)$ of all cycles and to its naive group completion $\mathcal{Z}^p(\mathbb{P}^n)$. The cycle $\Psi(X)$ has codimension 1 in the grassmannian, therefore Ψ defines a map

$$\Psi : \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right)$$

Moreover, if $B = \bigoplus B_d$ denotes the coordinate ring of the grassmannian in the Plücker embedding, then every irreducible hypersurface Z of degree d in the grassmannian is given by an element $f \in B_d$ defined uniquely up to a constant factor (see [5]), this defines a grading on the space of hypersurfaces in the grassmannian and with respect to this grading the map above preserves the degree.

The topology thus inherited defines in turn a topology in the space $\mathcal{C}^p(\mathbb{P}^n)$ of all codimension p algebraic cycles in $\mathbb{P}^n$ (c.f. [3] for this and other equivalent definitions) The results presented here are of the following type:

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Theorem 1.1. *The Chow Form map*

$$\Psi : \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right)$$

can be represented with respect to canonical decompositions of the corresponding spaces in the following way:

$$\prod_{j=0}^p K(\mathbb{Z}, 2j) \xrightarrow{\ p\ } K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2)$$

where p is the projection into the first two factors of the product

hence, the Chow Form map can be interpreted as the classifying map of a (non trivial) line bundle in the space of algebraic cycles in $\mathbb{P}^n$, the class of this line bundle generates $H^2\left(\mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right)\right)$.

An equivalent way of looking at the theorem is via the chain of inclusions

$$\mathcal{G}^p(\mathbb{P}^n) \subset \mathcal{Z}^p(\mathbb{P}^n) \subset \mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right)$$

The first space classifies p-dimensional bundles (not all of them), the second space classifies all total integer cohomology classes in $\bigoplus_{j=0}^p H^{2j}(-;\mathbb{Z})$ and the third space classifies all integer cohomology classes in $H^0(-;\mathbb{Z}) \times H^2(-;\mathbb{Z})$. The corresponding maps associated to the inclusions are the total chern class map (see [9]) and the projection into the first two factors.

2. The Chow Form Map

In this section we prove the theorem regarding the chow form map using an explicit description of the map. In order to state the results and its proofs we recall some definitions and facts about Lawson Homology. A survey of these and other related results is given in [6].

Definition 2.1. Let X be a projective variety. The *Lawson Homology groups* $L_pH_n(X)$ of X are defined by

$$L_pH_n(X) := \pi_{n-2p}\left(\mathcal{Z}_p(X)\right)$$

where $\mathcal{Z}_p(X)$ is the naive group completion of the Chow monoid $\mathcal{C}_p(X)$ of all p-dimensional effective algebraic cycles in X

These groups stand between the group of algebraic cycles modulo algebraic equivalence $\mathcal{A}_p(X) = L_pH_{2p}(X)$ and the singular homology group $H_n(X) = L_0H_n(X)$. Friedlander and Mazur defined a *cycle map* between the Lawson Homology groups and the singular homology groups

$$s_X^{(p)} : L_pH_n(X) \to H_n(X;\mathbb{Z})$$

this map will be referred to as the F-M map.

The following results of Lima-Filho [7] will be used throughout the section.

Theorem 2.2 (Lima-Filho, [7])**.** *The F-M cycle map coincides with the composition*

$$L_pH_n(X) = \pi_{n-2p}(\mathcal{Z}_{n-p}(X)) \xrightarrow{e_*} \pi_{n-2p}(\mathfrak{Z}_{2p}(X)) \xrightarrow{\mathcal{A}} H_n(X;\mathbb{Z})$$

where e_* is the map induced by the inclusion

$$e : \mathcal{Z}_m(X) \to \mathfrak{Z}_m(X)$$

of the space of algebraic cycles into the space of all integral currents and $\mathcal{A}$ is the Almgren isomorphism defined in [1]. *In particular, the F-M is functorial and is compatible with proper push-forwards and flat-pullbacks of cycles.*

Theorem 2.3 (Lima-Filho, [7])**.** *If X is a projective variety with a cellular decomposition in the sense of Fulton, (i.e., X is an algebraic cellular extension of $\emptyset$) then the inclusion*

$$\mathcal{Z}_p(X) \hookrightarrow \mathfrak{Z}_{2p}(X)$$

into the space $\mathfrak{Z}_{2p}(X)$ of integral currents is a homotopy equivalence

We also recall some facts about the cohomology of the Grassmannian. We follow the notation of [4] Chapter 14.

Definition 2.4. Let $\mathcal{G}_k(\mathbb{P}^n)$ be the Grassmann variety of k-dimensional linear spaces in $\mathbb{P}^n$. $\mathcal{G}_k(\mathbb{P}^n)$ is a smooth algebraic variety of complex dimension $d := (k+1)(n-k)$. The *special Schubert classes* are the homology classes $\sigma_m \in H_{2(d-m)}(\mathcal{G}_k(\mathbb{P}^n))$ defined by the cycle class of

$$\sigma_m := \{L \in \mathcal{G}_k(\mathbb{P}^n) \mid L \cap A \neq \emptyset\}$$

where A is any linear subspace of $\mathbb{P}^n$ of codimension $k+m$.

Theorem 2.5. *The integral cohomology of the grassmannian $\mathcal{G}_k(\mathbb{P}^n)$ is generated by the Poincare duals $c_m(Q)$ of the special Schubert classes σ_m. These Poincare duals are the chern classes of the universal quotient bundle Q.*

Theorem 2.6. *The Chow Form map*

$$\Psi : \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right)$$

can be represented with respect to canonical decompositions of the corresponding spaces in the following way:

$$\textstyle\prod_{j=0}^{p} K(\mathbb{Z}, 2j) \xrightarrow{\;p\;} K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2)$$

where p is the projection into the first two factors of the product

Proof. The homotopy equivalences

$$\mathcal{Z}^p(\mathbb{P}^n) \simeq \prod_{j=1}^{p} K(\mathbb{Z}, 2j)$$

and

$$\mathcal{Z}^1\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right) \simeq K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2)$$

are a consequence of theorem 2.3 and Almgren's theorem which asserts that

$$\pi_i(\mathfrak{Z}_k X) \cong H_{i+k}(X)$$

for the second homotopy equivalence, if $d = \dim_{\mathbb{C}} \mathcal{G}_{p-1}(\mathbb{P}^n)$ then

$$\pi_i\left(\mathcal{Z}_{d-1}\mathcal{G}_{p-1}(\mathbb{P}^n)\right) \cong \begin{cases} H_{2d-1}\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right) \cong \mathbb{Z} & \text{if } i = 0 \\ H_{2d}\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right) \cong \mathbb{Z} & \text{if } i = 2 \\ H_{d-1+i}\left(\mathcal{G}_{p-1}(\mathbb{P}^n)\right) = 0 & \text{otherwise} \end{cases} \tag{1}$$

this last calculation being well known for grassmannians (see [2]). Now we use theorem 2.2 to calculate the induced maps in homotopy. We have the following commutative diagram:

$$\begin{tikzcd}
\pi_m(\mathcal{Z}_{n-p}(\mathbb{P}^n)) = L_{n-p}H_{m+2(n-p)}(\mathbb{P}^n) \arrow[r, "\Psi"] \arrow[d, "s"'] & \pi_m(\mathcal{Z}_{d-1}\mathcal{G}_{p-1}(\mathbb{P}^n)) \cong L_{d-1}H_{m+2(g-1)}(\mathcal{G}_{p-1}(\mathbb{P}^n)) \arrow[d, "s"] \\
H_{m+2(n-p)}(\mathbb{P}^n) \arrow[r, "\tilde{\Psi}"] & H_{m+2(d-1)}(\mathcal{G}_{p-1}(\mathbb{P}^n))
\end{tikzcd}$$

where Ψ is the Chow Form map and $\tilde{\Psi}$ is the corresponding map in the space of integral currents.

Theorem 2.3 implies that the vertical maps are isomorphisms. Since we know the homology of the grassmannian, the only non-zero dimensions in the lower right correspond to the cases $m = 0$ and $m = 2$. We will describe explicitly the morphism in these two cases.

$m = 0$ In this case $H_{2(n-p)}(\mathbb{P}^n)$ is generated by the class of an $(n-p)$-plane Λ in $\mathbb{P}^n$. The cycle $\tilde{\Psi}(\Lambda)$ is then

$$\tilde{\Psi}(\Lambda) = \{P \in \mathcal{G}_{p-1}(\mathbb{P}^n) \mid \Lambda \cap P \neq \emptyset\}$$

this is precisely definition 2.4 of the special Schubert cycle σ_1.

$m = 2$ In this case $H_{2+2(n-p)}(\mathbb{P}^n)$ is generated by the class of an $(n-(p-1))$-plane Λ in $\mathbb{P}^n$. The cycle $\tilde{\Psi}(\Lambda)$ is then

$$\tilde{\Psi}(\Lambda) = \{P \in \mathcal{G}_{p-1}(\mathbb{P}^n) \mid \Lambda \cap P \neq \emptyset\}$$

but by a dimension count this is the whole variety $\mathcal{G}_{p-1}(\mathbb{P}^n)$ again, this corresponds to the special Schubert cycle σ_0

□

The results of Lima-Filho about the F-M cycle class map may also be used to prove the following theorem of Lawson and Michelsohn which appeared in [9].

Theorem 2.7. *The complex join pairing in the cycle spaces*

$$\# : \mathcal{Z}^q(\mathbb{P}^n) \wedge \mathcal{Z}^{q'}(\mathbb{P}^m) \to \mathcal{Z}^{q+q'}(\mathbb{P}^{n+m+1})$$

represents the cup product pairing in the canonical decompositions

$$\cup : \prod_{s=0}^{q} K(\mathbb{Z}, 2s) \wedge \prod_{t=0}^{q'} K(\mathbb{Z}, 2t) \to \prod_{r=0}^{q+q'} K(\mathbb{Z}, 2r)$$

i.e. if i_{2c} represents the generator of $H^{2c}(K(\mathbb{Z}, 2c); \mathbb{Z}) \cong \mathbb{Z}$ then

$$(\#)^*(i_{2c}) = \sum_{\substack{a+b=c \\ a>0 \\ b>0}} i_{2a} \otimes i_{2b}$$

Proof. The linear join of two irreducible varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ with defining ideals $\langle F_i \rangle \in \mathbb{C}[x_0, \ldots, x_n]$ and $\langle \mathcal{G}_j \rangle \in \mathbb{C}[y_0, \ldots, y_m]$ is the variety in $\mathbb{P}^{n+m+1}$ defined by the ideal $\langle F_i, G_j \rangle \in \mathbb{C}[x_0, \ldots, x_n, y_0, \ldots, y_m]$. A synthetic construction of this pairing is given in [9]. The join pairing $\#$ is obtained by extending bi-additively to all cycles and taking the induced pairing in the smash product. The join pairing induces a bilinear pairing

$$\pi_s(\mathcal{Z}^q(\mathbb{P}^n)) \times \pi_r(\mathcal{Z}^{q'}(\mathbb{P}^m)) \to \pi_{r+s}(\mathcal{Z}^{q+q'}(\mathbb{P}^{n+m+1}))$$

which corresponds to a homomorphism

$$\#_* : L_{n-q}H_{s+2(n-q)}(\mathbb{P}^n) \otimes L_{m-q'}H_{r+2(m-q')}(\mathbb{P}^m) \to L_{(n-q)+(m-q')+1}H_{s+2(n-q)+r+2(m-q')+2}(\mathbb{P}^{n+m+1})$$

This homomorphism is non-zero only when s and r are even, so we may assume $s = 2a$ and $r = 2b$. Now we take the F-M map and we get the following commutative diagram

$$\begin{tikzcd} L_{n-q}H_{s+2(n-q)}(\mathbb{P}^n) \otimes L_{m-q'}H_{r+2(m-q')}(\mathbb{P}^m) \arrow[r, "\#_*"] \arrow[d] & L_{(n-q)+(m-q')+1}H_{s+2(n-q)+r+2(m-q')+2}(\mathbb{P}^{n+m+1}) \arrow[d] \\ H_{s+2(n-q)}(\mathbb{P}^n) \otimes H_{r+2(m-q')}(\mathbb{P}^m) \arrow[r, "\phi"] & H_{s+2(n-q)+r+2(m-q')+2}(\mathbb{P}^{n+m+1}) \end{tikzcd}$$

where we define ϕ on the generators and we extend it bilinearly, namely, if the cycle classes of the planes $\Lambda_1 \in H_{2a+2(n-q)}(\mathbb{P}^n)$ and $\Lambda_2 \in H_{2b+2(m-q')}(\mathbb{P}^m)$ are generators of the corresponding groups, then $\phi([\Lambda_1] \otimes [\Lambda_2]) = [\Lambda_1 \# \Lambda_2]$, this last class is again a generator of $H_{2(a+b)+2(n-q)+2(m-q')+2}(\mathbb{P}^{n+m+1})$. The vertical F-M maps are isomorphisms by theorem 2.3, so we get that the generator i_{2c} gets pulled back precisely to the sum of the generators $i_{2a} \otimes i_{2b}$ with $a + b = c$.

$\square$

3. Generalizations

The proof of theorem 2.6 suggests a generalization of the result. Instead of taking the Chow form map we will look at general correspondences.

Definition 3.1. Let Σ_k denote the *incidence correspondence* defined by

$$\begin{tikzcd} & \Sigma_k := \{(x, \Lambda) \in \mathbb{P}^n \times \mathcal{G}_k(\mathbb{P}^n) \mid x \in \Lambda\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & \mathcal{G}_k(\mathbb{P}^n) \end{tikzcd}$$

$$\Psi_k(X) = \pi_2 \pi_1^{-1}(X)$$

The map π_1 is flat and the map π_2 is proper (see [8]). Therefore Ψ_k is a well defined map of cycle spaces

$$\Psi_k : \mathcal{C}^p(\mathbb{P}^n) \to \mathcal{C}^{p-k}(\mathcal{G}_k(\mathbb{P}^n)).$$

Notice that the incidence correspondence which defines the universal quotient bundle is Σ_{p-1}.

With this notation we have the following

Theorem 3.2. *The map*

$$\Psi_k : \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^{p-k}\left(\mathcal{G}_k(\mathbb{P}^n)\right)$$

can be represented with respect to canonical decompositions of the corresponding spaces in the following way

$$\prod_{j=0}^{p} K(\mathbb{Z},2j) \xrightarrow{p} \prod_{r=0}^{p-k} \prod_{\alpha\in H_{2r}(\mathcal{G}_k(\mathbb{P}^n))} K(\mathbb{Z},2r)_\alpha$$

where p *is the projection into the factors corresponding to the classes* $\sigma_0, \sigma_1, \dots, \sigma_{(p-k)}$

Proof. The argument follows the proof of theorem 2.6. We take the homotopy groups and then we use the cycle map. In this case we have the following commutative diagram:

$$\begin{array}{ccc} \pi_m(\mathcal{Z}_{n-p}(\mathbb{P}^n)) = L_{n-p}H_{m+2(n-p)}(\mathbb{P}^n) & \xrightarrow{\Psi_k} & \pi_m(\mathcal{Z}_m\mathcal{G}_k(\mathbb{P}^n)) \cong L_{d-(p-k)}H_{d+2[d-(p-k)]}(\mathcal{G}_k(\mathbb{P}^n)) \\ \downarrow s & & \downarrow s \\ H_{m+2(n-p)}(\mathbb{P}^n) & \xrightarrow{\tilde{\Psi}_k} & H_{m+2[d-(p-k)]}(\mathcal{G}_k(\mathbb{P}^n)) \end{array}$$

Since the homology of the grassmannian is zero in odd degrees we are only concerned with the case $m = 2r$. The lower right corner of the diagram imposes the condition $0 \le 2r \le 2(p-k)$. Once again, the definition of the special Schubert cycles implies that if $\Lambda_{2r+2(n-p)}$ is a plane of dimension $2r+2(n-p)$ (i.e. the class of a generator of $H_{2r+2(n-p)}(\mathbb{P}^n)$) then $\tilde{\Psi}_k(\Lambda_{2r+2(n-p)}) = \sigma_{p-(k+r)}$ □

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