REIDEMEISTER SPECTRUM FOR METABELIAN GROUPS OF THE FORM $Q^n \rtimes \mathbb{Z}$ AND $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$, p PRIME.

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ABSTRACT. In this note we study the Reidemeister spectrum for metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$. Particular attention is given to the R_∞ property of a subfamily of these groups. We also define a Nielsen spectrum of a space and discuss some examples.

1. INTRODUCTION

Let $\phi : G \to G$ be an automorphism of a group G. A class of equivalence defined by the relation $x \sim gx\phi(g^{-1})$ for $x, g \in G$ is called the *Reidemeister class of ϕ* (or the *ϕ-conjugacy class* or the *twisted conjugacy class of ϕ*). The number of Reidemeister classes, denoted by $R(\phi)$, is called the *Reidemeister number* of ϕ. The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see [24, 4]), in Selberg theory (see [29, 1]), and Algebraic Geometry (see [21]). A current important problem of the field concerns obtaining a twisted analogue of the Burnside-Frobenius theorem [8, 4, 13, 14, 32, 12, 11], i.e., to show the equality of the Reidemeister number of ϕ and the number of fixed points of the induced homeomorphism of an appropriate dual object. One step in this process is to describe the class of groups G such that $R(\phi) = \infty$ for any automorphism $\phi : G \to G$.

The work of discovering which groups belong to the mentioned class of groups was begun by Fel'shtyn and Hill in [8]. Later, it was shown by various authors that the following groups belong to this class: (1) non-elementary Gromov hyperbolic groups [27, 5]; (2) Baumslag-Solitar groups $BS(m, n) = \langle a, b | ba^m b^{-1} = a^n \rangle$ except for $BS(1, 1)$ [6]; (3) generalized Baumslag-Solitar

groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers being infinite cyclic [26], (4) lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ iff $2|n$ or $3|n$ [20]; (5) the solvable generalization Γ of $BS(1,n)$ given by the short exact sequence $1 \to \mathbb{Z}\left[\frac{1}{n}\right] \to \Gamma \to \mathbb{Z}^k \to 1$, as well as any group quasi-isometric to Γ [31], groups which are quasi-isometric to $BS(1,n)$ [30] (while this property is not a quasi-isometry invariant); (6) the R. Thompson group F [2]; (7) saturated weakly branch groups including the Grigorchuk group and the Gupta-Sidki group [15]; (8) mapping class groups, symplectic groups and braids groups [7]; (9) relatively hyperbolic groups(in particular the free products of finitely many finitely generated groups) [16]; (10) some classes of finitely generated free nilpotent groups [19, 28] and some classes of finitely generated free solvable groups [25]; (11) some classes of crystallographic groups [3].

The paper [31] suggests a terminology for this property, which we would like to follow. Namely, "a group G has *property* R_∞ or is a R_∞ group", if all of its automorphisms ϕ have $R(\phi) = \infty$.

For an immediate consequences of the R_∞ property for the topological fixed point theory see, e.g., [30].

Following [25], we define the *Reidemeister spectrum of a group* G, denoted by $Spec(G)$, as the set of natural numbers k such that there is an automorphism $\phi \in Aut(G)$ with $R(\phi) = k$ (k can be infinite). In terms of the spectrum, the R_∞-property of the group G simply means that $Spec(G)$ contains only one element which is the infinity.

It is easy to see that $Spec(\mathbb{Z}) = \{2\} \cup \{\infty\}$, and, for $n \geq 2$, the spectrum is full, i.e. $Spec(\mathbb{Z}^n) = \mathbb{N} \cup \{\infty\}$. Let $N = N_{rl}$ be the free nilpotent group of rank r and class l. Then for N_{22} (also known as discrete Heisenberg group) $Spec(N_{22}) = 2\mathbb{N} \cup \{\infty\}$ [22, 10, 25]. It is also known that $Spec(N_{23}) = \{2k^2|k \in \mathbb{N}\} \cup \{\infty\}$ [25] and $Spec(N_{32}) = \{2n-1|n \in \mathbf{N}\} \cup \{4n|n \in \mathbf{N}\} \cup \{\infty\}$ [25].

Let $X = L(m, q_1, \ldots, q_r)$ be a generalized lens space, and let $f : X \to X$ be a continuous map of degree d, where $|d| \neq 1$. Let $f_* : \pi_1(X) \to \pi_1(X)$ be induced homomorphism on the fundamental group $\pi_1(X) = \mathbb{Z}/m\mathbb{Z}$. In 1943 Franz [17] has observed that $N(f) = R(f) = R(f_*) = \#\mathrm{Coker}\,(1 - f_*) = \#(\mathbb{Z}/m\mathbb{Z})/(1 - d)(\mathbb{Z}/m\mathbb{Z}) = (|1 - d|, m)$, where $N(f)$ and $R(f)$ are Nielsen

and Reidemeister numbers of the map f and where $(|1 - d|, m)$ denotes the gcd of $|1 - d|$ and m. This gives a strong arithmetic restriction on the Reidemeister spectrum for endomorphisms of the group $\mathbb{Z}/m\mathbb{Z}$. We observe that the knowledge of the Reidemeister spectrum of a group can be quite useful for fixed point theory.

In this paper we study the Reidemeister spectrum and the R_∞-property for a subfamily of the family of the metabelian groups of the form $\mathbb{Q}^n \rtimes \mathbb{Z}$ and $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$ for p a prime.

We also define a Nielsen spectrum of a space and discuss some examples.

Acknowledgement. We would like to thank the Max-Planck-Institut für

Mathematik (MPIM) in Bonn for its kind support and hospitality while most of this work was done.

2. Preliminaries

In this section we show that for certain short exact sequences of groups the kernel is characteristic. Then we compute $Aut(\mathbb{Q})$, $Aut(\mathbb{Z}[1/p])$, and the Reidemeister spectrum of the groups $\mathbb{Q}$ and $\mathbb{Z}[1/p]$ where $\mathbb{Q}$ denote the rational numbers and p a prime.

Let us consider a short exact sequence of the form $1 \to \mathbb{K} \to G \to \mathbb{Z} \to 1$, which of course splits.

Lemma 2.1. *Suppose the group $\mathbb{K}$ has the property that for every $x \in \mathbb{K}$ there is a natural number $s > 1$ such that x is divisable by s (i.e. there is $y \in \mathbb{K}$ such that $y^s = x$). Then $\mathbb{K}$ is characteristic in G.*

Proof. Let $\phi : G \to G$ be an automorphism. We know that $G \approx K \rtimes_\theta \mathbb{Z}$ for some homomorphism $\theta : \mathbb{Z} \to Aut(\mathbb{K})$. Let $x \in \mathbb{K}$ and $\phi(x) = (z, r)$. We will show that $r = 0$. It follows from the definition of the operation on the semi-direct product that, if s divides x, then s also divides $\phi(x) = (z, r)$. Again, by the definition of the operation on the semi-direct product, s divides r. From the hypothesis it follows that there is an infinite sequence of positive integers such that for each integer s of the sequence, r is divisible by this integer. Therefore, r has an infinite number of divisors and must be zero. $\square$

For the rational numbers regarded as an abelian group with the additive operation we have:

Lemma 2.2. *The group of automorphims of $\mathbb{Q}$ is isomorphic to the multiplicative group of the rationals different from zero, denoted by $\mathbb{Q}^*$.*

Proof. First, we can observe that an automorphism $\phi : \mathbb{Q} \to \mathbb{Q}$ is determined if we know its value at 1. To see this let us consider an arbitrary element $p/q \in \mathbb{Q}$. Then $\phi(p/q) = p\phi(1/q)$ and it suffices to determine $\phi(1/q)$. Since $\mathbb{Q}$ is torsion free, it follows that the divisibility in $\mathbb{Q}$ is unique and hence $\phi(1/q)$ is uniquely determined. Conversely, a multiplication by any rational number different from zero provides an automorphism, since the multiplication by the inverse number provides the inverse homomorphism. $\square$

Now let $\mathbb{Z}[1/p]$ for p a prime be the subring of the rationals and also by the same notation we denote the abelian additive group. By $Aut(\mathbb{Z}[1/p])$ we mean the automorphism group of $\mathbb{Z}[1/p]$ as abelian group.

Lemma 2.3. *The group $Aut(\mathbb{Z}[1/p])$ is isomorphic to the multiplicative set of the elements of $\mathbb{Z}[1/p]$ generated by $\{\pm p\}$, which is isomorphic to the group $\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$.*

Proof. The first part is similar to the proof of the previous Lemma. Namely, a homomorphism ϕ is determined by the value of the homomorphism at 1, and it is multiplication by this value. In order to have ϕ be an automorphism we need that $\phi(1)$ is invertible. Let $r/s \in \mathbb{Z}[1/p]$ where r/s is written in the reduced form. If $r/s \in \mathbb{Z}[1/p]$, then we have either $r = 1$ and $s = p^t$ or $s = 1$ and $r = p^t$, and the result follows. $\square$

Now we determine the Reidemeister spectrum of $\mathbb{Q}$ and $\mathbb{Z}[1/p]$. We need a Lemma which will also be used for the computation of the Reidemeister spectrum of other groups. Given any $x \in \mathbb{Z}[1/p]$, x can be written uniquely in the form $\pm q/p^n$ where $n \in \mathbb{Z}$, $q \in \mathbb{N}$(the natural numbers) and p, q relatively prime. Denote this number q by $v_P(x)$. Here P is the set of all primes relatively prime with p. Also let us observe that the abelian group $\mathbb{Z}[1/p]^n$ is a free $\mathbb{Z}[1/p]-$module. Then a group homomorphism of the abelian group $\mathbb{Z}[1/p]^n$ is always a homomorphism of $\mathbb{Z}[1/p]-$module.
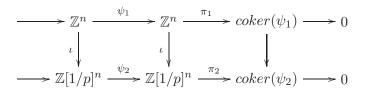
Lemma 2.4. *Let $\psi : A^n \to A^n$ be a homomorphism where A is either $\mathbb{Q}$ or $\mathbb{Z}[1/p]$. Then ψ is invertible if, and only if, the determinant of the matrix of ψ is invertible. Furthermore, for $A = \mathbb{Q}$ the cardinality of the cokernel*

of ψ is infinite if $\det(\phi) = 0$, and it is 1 if $\det(\psi) \neq 0$. For $A = \mathbb{Z}[1/p]$ the cardinality of the cokernel of ψ is the natural number $v_P(x)$, defined above, for $x = \det(\psi)$, if $x \neq 0$. If $\det(\psi) = 0$ then the cardinality of the cokernel is infinite.

Proof. That ψ is invertible if and only if $det(\psi)$ is invertible is a classical fact since A is a commutative ring. Therefore if $A = \mathbb{Q}$ then $det(\psi) \neq 0$ implies ψ invertible and we have the cardinality of the cokernel 1. If $det(\psi) = 0$ then the result follows promptly from the fact that $\mathbb{Q}$ is an infinite field.

Let $A = \mathbb{Z}[1/p]$. The matrix M of ψ has entries in A. Multiplication by p^l on A^n, denoted by $R_{p^l} : A^n \to A^n$, is an automorphism for any l. So the cokernel of ψ and the cokernel of the composite of ψ with the multiplication by p^l are isomorphic. For a sufficiently large l we can assume that the matrix of the composite has entries of the form xp^i for x an integer(possibly negative) relatively prime with p and $i \geq 0$. Observe that the determinant of the new matrix M_1 is $p^l det(\psi)$ and v_P of the two determinants are the same. Now consider the homomorphism $\psi_1 : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by M_1 and $\psi_2 = R_{p^l} \circ \psi : A^n \to A^n$ the homomorphism defined by the same matrix.

Then we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\longrightarrow & \mathbb{Z}^n & \xrightarrow{\psi_1} & \mathbb{Z}^n & \xrightarrow{\pi_1} & coker(\psi_1) & \longrightarrow 0 \\
& \iota\downarrow & & \iota\downarrow & & \downarrow & \\
\longrightarrow & \mathbb{Z}[1/p]^n & \xrightarrow{\psi_2} & \mathbb{Z}[1/p]^n & \xrightarrow{\pi_2} & coker(\psi_2) & \longrightarrow 0
\end{array}
$$

where $coker(\psi_2)$ has no p torsion, as result of the $p-$divisibility of the group $\mathbb{Z}[1/p]$. After we take the tensor product with $\mathbb{Z}[1/p]$ as $\mathbb{Z}-$module the two first vertical homomoprhisms becomes isomorphisms and the horizontal lines are exact. So we have an isomorphism between the tensor product of the cokernels. The cokernel $coker(\psi_1)$ is a finitely generated abelian group. Now we look at the two possible cases. If $coker(\psi_1)$ is infinite then it contains a copy of $\mathbb{Z}$ and it follows that $coker(\psi_2)$ is also infinite. If $coker(\psi_1)$ is finite then it is the direct product of two finite groups where the first has order a power of p and the other has order relatively prime to p. After taking the tensor product we obtain only the finite subgroup of order relatively prime

to p which is simultaneously the order of the cokernel of ψ_2 and $v_P(det(\psi_1))$. But $v_P(det(\psi_1)) = v_P(det(\phi_2)) = v_P(det(\psi))$ and the result follows.

The case when $A = \mathbb{Q}$ is simpler and we leave to the reader.   □

Proposition 2.5. a) The Reidemeister spectrum of $\mathbb{Q}$ is

$Spec(\mathbb{Q}) = \{1\} \cup \{\infty\}$.

b) The Reidemeister spectrum of $\mathbb{Z}[1/p]$ for p an odd prime is $Spec(\mathbb{Z}[1/p]) = \{p^l + 1,\ p^{l+1} - 1 | l \geq 0\} \cup \{\infty\}$

and $Spec(\mathbb{Z}[1/2]) = \{2^l + 1,\ 2^l - 1 | l \geq 1\} \cup \{\infty\}$

Proof. Part a)- Given an automorphism $\phi : \mathbb{Q} \to \mathbb{Q}$ we know that it is multiplication by a rational number r. If $r = 1$ then the cokernel of multiplication of $r - 1 = 0$ is $\mathbb{Q}$ so we obtain Reidemeister number infinite. Otherwise we have multiplication by $r - 1 \neq 0$ which is surjective. Therefore the Reidemeister number is 1 and the result follows.

Let us consider $Z[1/2]$ and $P = \mathbb{P} - \{2\}$. An automorphism is multiplication by a rational number r such that $v_P(r)$ is 1. So we have $r = \pm 2^l$, $l \in Z$. If $r = 1$ then we obtain that the Reidemeiter number is infinite, and for $r = -1$ we obtain multiplication by -2 so we obtain Reidemeister number 1. So let $l \neq 0$. Then the numbers $2^l \pm 1$ are always odd and the result follows. The case p an odd prime is similar and simpler. We leave it to the reader.   □

3. THE SEMI-DIRECT PRODUCT $\mathbb{Q}^n \rtimes \mathbb{Z}$, $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$, $n \leq 2$

We begin by recall some basic facts. Given any automorphism ϕ of one of the groups $\mathbb{Q}^n \rtimes \mathbb{Z}$, $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$, we know from section 2 that the subgroup either $\mathbb{Q}^n$ or $\mathbb{Z}[1/p]^n$ is characteristic. So we obtain a homomorphism of short exact sequence. Whenever the induced homomorphism $\bar{\phi}$ on the quotient is the identity then by well known facts, see [18] this implies that the Reidemeister number is infinite. Also from [18] in the case where $\bar{\phi} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by -1(the only other possibility) then the Reidemeister number is computed as the sum of the Reidemeister number of ϕ' and the Reidemeister number of $\theta(1) \circ \phi'$. We will use the above procedure for the calculation which follows.

3.1. **The case** $n = 1$. In this subsection we have an action $\theta : \mathbb{Z} \to A$ where A is either $\mathbb{Q}$ or $\mathbb{Z}[1/p]$. The homomorphism θ is completely determined by $\theta(1)$ which, in turn is determined by its values at $1 \in A$. So we identify $\theta(1)$ with its value at $1 \in A$.

Proposition 3.1. a) The Reidemeister spectrum of $\mathbb{Q} \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Q} \rtimes_\theta \mathbb{Z}) = \{\infty\}$ if $\theta(1)$ is a non zero rational number different from ± 1 and it is $Spec(\mathbb{Q} \rtimes \mathbb{Z}) = \{2\} \cup \{\infty\}$ otherwise.
b) For p an odd prime the Reidemeister spectrum of $\mathbb{Z}[1/p] \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/p] \rtimes_\theta \mathbb{Z}) = \{\infty\}$ if $\theta(1) \in \mathbb{Z}[1/p]$ is a non zero invertible element different from ± 1. If $\theta(1) = 1$ then $Spec(\mathbb{Z}[1/p] \rtimes_\theta \mathbb{Z}) = \{p^l + 1, \ p^{l+1} - 1 | l \geq 0\} \cup \{\infty\}$. If $\theta(1) = -1$ then $Spec(\mathbb{Z}[1/p] \rtimes_\theta \mathbb{Z}) = \{2p^{l+1} | l \geq 0\} \cup \{\infty\}$.
   c) The Reidemeister spectrum of $\mathbb{Z}[1/2] \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/2] \rtimes_\theta \mathbb{Z}) = \{\infty\}$ if $\theta(1) \in \mathbb{Z}[1/2]$ is a non zero invertible element different from ± 1. If $\theta(1) = 1$ then $Spec(\mathbb{Z}[1/2] \rtimes_\theta \mathbb{Z}) = \{2(2^l + 1), \ 2(2^l - 1) | l \geq 1\} \cup \{\infty\}$.
If $\theta(1) = -1$ then $Spec(\mathbb{Z}[1/2] \rtimes_\theta \mathbb{Z})) = \{2^{l+1} | l \geq 1\} \cup \{\infty\}$.

Proof. Let $\phi : Q \rtimes_\theta \mathbb{Z}$ be a automorphism. From the discussion in the beginning of the section we know that ∞ belongs to the spectrum and we will look at automorpisms such that $\bar{\phi}$ is multiplication by -1. We compute the automorphisms $\phi' : \mathbb{Q} \to \mathbb{Q}$ which arises as restriction of such automorphisms. In order to an automorphsm ϕ' be the restriction of an automorphism of the large group we must have the relation $\phi' \circ \theta(1) = \theta(-1) \circ \phi'$. This implies that $kr = r^{-1}k$ where $\phi'(1) = k$(so different from 0) and $\theta(1) = r$. This implies that $r^2 = 1$. So if $r \neq \pm 1$ then there is no such automorphism and the Reidemeister spectrum of the group is $\{\infty\}$. If $r = \pm 1$ then k can assume any non zero value. If $r = 1$ then the two automorphisms on the fibers are multiplication by k and each one has Reidemeister number 1 if $k \neq 1$. If $r = -1$ then the two automorphisms on the fibers are multiplication by k, $-k$ respectively. So for $k \neq \pm 1$ the Reidemeister number is 2 and part a) follows.

Part b) The infinity certainly belongs to the spectrum because the Reidemeister number of the identity is infinite. The element $\theta(1)$ is invertible since it is an isomorphism. As in case a) we have that if $\theta(1) \neq \pm 1$ there is no such automorphism and follows that the group has spectrum $\{\infty\}$. Again as in part a) for $\theta(1) = \pm 1$ we have ϕ' can be an arbitrary automorphism

of $\mathbb{Z}[1/p]$. For $r = 1$ the two automorpisms are the same and we obtain as Reidmeister number $2(p^l \pm 1)$, $l > 0$ and 4. If $r = -1$ then we have to look at $k - 1$ and $k + 1$ for k an invertible, so of the form ϵp^t. If $t = 0$ then we get Reidemeister infinite. If $t \neq 0$ then we get as Reidemeister number $2p^l$. So the result follows.

Part c). The proof is similar to the proof of case b) where the only difference is because $2^0 + 1$ is not a Reidemeister number of a homomorphism for case c) but $p^0 + 1$ it is for case b). This justifies why the formulas are slight different. $\qquad\square$

3.2. **The case $n = 2$.** First we analyze the case where the kernel is $\mathbb{Q}^2$. Let $G = \mathbb{Q}^2 \rtimes \mathbb{Z}$ and for an automorphism $\phi : G \to G$ let ϕ' be the restriction of ϕ to $\mathbb{Q}^2$. If M is the matrix of ϕ' then we call $det(\phi')$ the determinant of M.

Proposition 3.2. The Reidemeister spectrum of $\mathbb{Q}^2 \rtimes_\theta \mathbb{Z}$ is either $\{\infty\}$ or $\{2\} \cup \{\infty\}$. The $Spec(\mathbb{Q}^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$ if there is no an invertible matrix N over $\mathbb{Q}$ such that $N = MNM$, $det(Id - N) \neq 0$ and $det(Id - MN) \neq 0$, where $\theta(1) = M$. Otherwise the Reidemeister spectrum is $\{2\} \cup \{\infty\}$.

Proof. The result follows from the considerations on the beginning of this section and basic facts about Reidemeister numbers for homomorphisms of $\mathbb{Q}^2$. $\qquad\square$

A more explict description of the matrices N in terms of the matrix M which satisfies the conditions of the Proposition above is still in progress. But we give an example of a family of groups which have Reidemeister spectrum $\{\infty\}$ and also an example of another family of groups which have Reidemeister spectrum $\{2\} \cup \{\infty\}$.

Example 1- Let $\theta(1) = M$ be the automorphism given by

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

with $rs \neq 0, 1$ and either $r^2 \neq 1$ or $s^2 \neq 1$. There is no automorphism such that the induced homomorphism on the quotient is multiplication by -1. This can be proven by showing that the only solutions for the matrices N which satisfy $N = MNM$ has the property that $det(N) = 0$.

Example 2- Let $\theta(1)$ be the automorphism given by

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

$r^2 \neq 1$, $rs = 1$. By direct calculation we can find all matrices N and they are of the form

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

for arbitrary b, c. The matrix MN is

$$\begin{pmatrix} 0 & rb \\ cs & 0 \end{pmatrix}$$

Then $det(Id - N) = det(Id - MN) = 1 - bc$ and whenever $1 - bc \neq 0$ we obtain an example where the Reidemeister number is 2. Therefore this provides a family of examples of groups where the Reidemeister spectrum is $\{2\} \cup \{\infty\}$.

Example 3- Let $\theta(1)$ be the automorphism given by

$$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

Let us consider two cases. Suppose that $|uv| \neq 1$. Then follows that there is no invertible N such that $N = MNM$ and follows that the Reidemeister spectrum of such groups is $\{\infty\}$.

For the second case let $u = v = 1$. Then by direct calculation we can find all matrices N which turns out to be of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

and the matrix MN is of the form

$$\begin{pmatrix} b & a \\ a & b \end{pmatrix}.$$

The first matrix has determinant $a^2 - b^2$. The determinant of $Id - N$ and $Id - MN$ are respectively $1 - 2a + a^2 - b^2$ and $1 - 2b + b^2 - a^2$. There are plenty of rational values of a and b such that these 3 values are different from zero and consequently we obtain for each such values one example of a group which admits an automorphism which has Reidemeister number 2.

Now we consider the case $\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}$ for p a prime. We will compute the Reidemeister spectrum of $\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}$ for two families of actions θ.

The Proposition 3.2 holds partially in this case.

Proposition 3.3. The Reidemeister spectrum of $\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}$ is $\{\infty\}$ if and only if there is no an invertible matrix N over $\mathbb{Z}[1/p]$ such that $N = MNM$ and both $det(Id - N)$ and $det(Id - MN)$ are non zero. Here $\theta(1) = M$.

Proof. The proof is similar to the proof of Proposition 3.2. $\square$

The Reidemeister spectrum for the groups where there exist a solution N is not very simple in general but we can answer for two large families of actions θ. For example if $\theta(1)$ is the identity then the Reidemeister spectrum of this group is obtained multiplying by 2 the numbers(including infinite) which belong to the spectrum of the first factor. So the first step is to compute the Reidemeister spectrum of $\mathbb{Z}[1/p]^2$.

Proposition 3.4. The Reidemeister spectrum of $Z[1/p]^2$ is $Spec(\mathbb{Z}[1/p]^2) = \{n | n \in \mathbb{N} \ and \ (n,p) = 1\} \cup \{\infty\}$ where (n,p) denote the gcd of n and p.

Proof. Any matrix $N \in Gl(2,\mathbb{Z})$ can be regarded as an automorphism of $\mathbb{Z}[1/p]^2$. It is well known that the Reidemeister spectrum of $\mathbb{Z} + \mathbb{Z}$ is $\mathbb{N}$. So the Reidemeister spectrum of $\mathbb{Z}[1/p]^2$ contains $v_p(n)$ for every natural number by Lemma 2.4. So it contains all positive numbers relatively prime to p. But again by Lemma 2.4 an element of the spectrum has to be a positive integer relatively prime to p. So the result follows. $\square$

An immediate consequence of the Proposition above is that $Spec(\mathbb{Z}[1/p]^2 \times \mathbb{Z}) = \{2n | (n,p) = 1 \ n \in \mathbb{N}\} \cup \{\infty\}$ where (n,p) denote the gcd of n and p.

Let us start with the group $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z})$.

Proposition 3.5. Let $\theta(1)$ be of the form:

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

Then we have the following cases:

a) If $r = s = \pm 1$ then $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{2n | (n,2) = 1 \ n \in \mathbb{N}\}$ where $(n,2)$ denote the gcd of n and 2.

b) If $r = -s = \pm 1$ then the Reidemeister spectrum of the group $\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{2^{l+1}, 2^k(2^l \pm 1)| \, l \geq 1, k \geq 2\} \cup \{\infty\}$.

c) If $rs = 1$ and $|r| \neq 1$ then $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{2(2^k + 1), 2(2^l - 1)| \, k \geq 0, \, l \geq 1\} \cup \{\infty\}$

d) If either r or s does not have module equal to one, and $rs \neq 1$ then there is no automorphism of the group such that the induced on the quotient is multiplication by -1 and follows that $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$.

Proof. Let N be the matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
The equation $N = MNM$ corresponds to the system $a = ar^2$, $b = brs$, $c = crs$ and $d = ds^2$. Suppose that $|r| \neq 1$ and $rs \neq 1$. This implies that $a = b = c = 0$ and so the system has no solution for a matrix N invertible. Similarly if we assume $|s| \neq 1$. So part d) follows.

For the part a) we have that all invertible matrices N are solutions. So the result follows from the Proposition 3.4 above.

For the part b) we have that the matrix N is diagonal. By straightforward calculation we can assume that the elements of the diagonal of N are of the form 2^i for $i \geq 0$. A direct calculation shows the values for the spectrum.

For the part c) we have that the matrix M is of the form

$$\begin{pmatrix} \epsilon 2^\ell & 0 \\ 0 & \epsilon 2^{-\ell} \end{pmatrix}.$$

and N of the form

$$\begin{pmatrix} 0 & \delta_1 2^u \\ \delta_2 2^v & 0 \end{pmatrix}.$$

The product MN is given by

$$\begin{pmatrix} 0 & \epsilon \delta_1 2^{u+\ell} \\ \epsilon \delta_2 2^{v-\ell} & 0 \end{pmatrix}.$$

Follows the matrices of $Id - N$ and $Id - MN$:

$$\begin{pmatrix} 1 & -\delta_1 2^u \\ -\delta_2 2^v & 1 \end{pmatrix}.$$

and

$$\begin{pmatrix} 1 & -\epsilon\delta_1 2^{u+\ell} \\ -\epsilon\delta_2 2^{v-\ell} & 1 \end{pmatrix}$$

respectively. The result follows by straightforward calculation.

$\square$

Proposition 3.6. Let $\theta(1)$ be of the form:

$$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

Then we have the following cases:

a) If $uv = 1$ then the Reidemeister spectrum of the group $\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{2^k(2^l \pm 1)| \ k \geq 2, l \geq 1\} \cup \{\infty\}$.

b) If $uv = -1$ then the Reidemeister spectrum of the group $\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{2(2^{2m} - 1)| \ m > 0\} \cup \{\infty\}$.

c) If $u^2 v^2 \neq 1$ then there is no automorphism of the group such that the induced on the quotient is multiplication by -1 and follows that $Spec(\mathbb{Z}[1/2]^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$.

Proof. In this case the entries of the matrix N

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

must satisfy the equations $uvd = a, u^2 c = b, bv^2 = c, auv = d$. It follows that $a = (uv)^2 a, \ d = (uv)^2 d, \ b = (uv)^2 b, \ c = (uv)^2 c$. So part c) follows promptly from these equations.

Let us consider the case a). In this case the system of equations provide $a = d$. So the determinant of N becomes $a^2 - bc = a^2 - b^2 v^2 = (a+bv)(a-bv)$ which is an invertible element, so $(a + bv)$ and $(a - bv)$ are also invertible. Since $uv = 1$ follows that $u = \delta 2^{-t}$ and $v = \delta 2^t$ for some integer t and $\delta \in \{1, -1\}$.

The matrix N is of the form

$$\begin{pmatrix} a & b \\ bv^2 & a \end{pmatrix}$$

and the matrix MN is

$$\begin{pmatrix} bv & a/v \\ av & bv \end{pmatrix}.$$

Our task is to compute v_P of $det(Id - N)$ and $det(Id - MN)$. We have $det(Id - N) = 1 - 2a + a^2 - b^2v^2$ and $det(Id - MN) = 1 - 2bv + b^2v^2 - a^2$ so $-det(Id - MN) = -1 + 2bv + a^2 - b^2v^2$. We compute v_P of $-det(Id - MN)$.

Since $a + bv$ and $a - bv$ are invertible they are of the form $a + bv = \epsilon_1 2^{l_1}$ and $a - bv = \epsilon_2 2^{l_2}$. Therefore $a = \epsilon_1 2^{l_1-1} + \epsilon_2 2^{l_2-1}$ and $b = \delta(\epsilon_1 2^{l_1-t-1} - \epsilon_2 2^{l_2-t-1})$. Also $det(N) = a^2 - b^2v^2 = \epsilon_1\epsilon_2 2^{l_1+l_2}$.

So we obtain $det(Id - N) = \epsilon_1\epsilon_2 2^{l_1+l_2} - \epsilon_1 2^{l_1} - \epsilon_2 2^{l_2} + 1$ and $-det(Id - MN) = \epsilon_1\epsilon_2 2^{l_1+l_2} + \epsilon_1 2^{l_1} - \epsilon_2 2^{l_2} - 1$. It is not difficult to see that the rational numbers, up to multiplication by a power of 2(positive or negative) are the integers $2^{|l_1|+|l_2|} - \epsilon_2 2^{|l_1|} - \epsilon_1 2^{|l_2|} + 1$ and $2^{|l_1|+|l_2|} + \epsilon_2 2^{|l_1|} - \epsilon_1 2^{|l_2|} - 1$, respectively. Whenever these positive integers are odd they are the Reidemeister number of the correspondent matrices, which happens for $|l_1|, |l_2| > 0$. Then in this case the Reidemeister number is the sum equals to $2^{|l_2|+1}(2^{|l_1|} - \epsilon_1)$. In order to find the complete spectrum we have to analyze the particular cases. Suppose that $l_1 = l_2 = 0$. Then in this case we have four possibilities for the pair ϵ_1, ϵ_2. By straightforward calculation for each case either $det(Id - N)$ or $det(Id - MN)$ is zero (if not both) and we obtain Reidemeister infinite. In details, for $(\epsilon_1, \epsilon_2) = (1, 1)$ then $(det(Id - N), -det(Id - MN)) = (0, 0)$, for $(\epsilon_1, \epsilon_2) = (1, -1)$ then $(det(Id - N), -det(Id - MN)) = (0, 0)$, for $(\epsilon_1, \epsilon_2) = (-1, 1)$ then $(det(Id - N), -det(Id - MN)) = (0, -4)$ for $(\epsilon_1, \epsilon_2) = (-1, -1)$ then $(det(Id - N), -det(Id - MN)) = (4, 0)$

Now let $l_2 = 0$ and $l_1 \neq 0$. By direct inspection for $\epsilon_2 = 1$ we obtain $det(N) = 0$ and for $\epsilon_2 = -1$ we obtain $det(MN) = 0$, hence we obtain Reidemeister infinite.

Finally let $l_1 = 0$ and $l_2 \neq 0$. If $\epsilon_1 = 1$ then $det(Id-N) = det(Id-MN) = 0$ and Reidemeister is infinite. If $\epsilon_1 = -1$ we get $det(Id-N) = -2(\epsilon_2 2^{|l_2|}-1)$ and $det(Id-MN) = -2(\epsilon_2 2^{|l_2|}+1)$. After we compute v_P of these numbers and add them up we obtain the Reidemeister number $2^{|l_2|+1}$ for $|l_2| \geq 1$ or 2^k $k \geq 2$. But these numbers were obtained already in previous cases and the result follows.

Let us consider the case b). Some of the calculations are similar and in this case we do not give all details. In this case the system of equations provide $a = -d$. So the determinant of N becomes $-a^2 - bc = -a^2 - b^2 v^2 =$ which is an invertible element. Since $uv = -1$ follows that $u = -\delta 2^{-t}$ and $v = \delta 2^t$ for some integer t and $\delta \in \{1, -1\}$.

The matrix N is of the form

$$\begin{pmatrix} a & b \\ bv^2 & -a \end{pmatrix}$$

and the matrix MN is

$$\begin{pmatrix} -bv & a/v \\ av & bv \end{pmatrix}.$$

Our task is to commute v_P of $det(Id - N)$ and $det(Id - MN)$. We have $det(Id - N) = 1 - a^2 - b^2 v^2$ and $det(Id - MN) = 1 - b^2 v^2 - a^2$. So we compute v_P of $det(Id - N)$.

Let $a = r2^m$ and $bv = s2^n$ where r, s are odd numbers (possible negative) and m, n integers. Because $a^2 + (bv)^2$ is invertible then we obtain that $r^2 + s^2$ is necessarily 1 and we obtain as possible solutions $a = \epsilon_1 2^m$ and $bv = 0$ (or $b = 0$ since $v \neq 0$) $a = 0$ and $bv = \epsilon_2 2^n$.

For the case $b = 0$ we obtain $det(Id - N) = 1 - a^2 = 1 - 2^{2m}$. For $m = 0$ we obtain Reidemeister infinite otherwise we obtain the total Reidemeister number $2(2^{2m} - 1)$ for $m > 0$. For $a = 0$ we obtain $det(Id - N) = 1 - b^2 v^2 = 1 - 2^{2n}$. Then we obtain the same numbers as above and the result follows. $\square$

The example studied by Jabara in [23] is included in part a) above. Moreover, part a) above computes the Reidemeister spectrum of the group.

For an arbitrary prime $p \neq 2$ we will have similar results.

Proposition 3.7. Let $\theta(1)$ be of the form:

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$$

Then we have the following cases:

a) If $r = s = \pm 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2n | n \in \mathbb{N}, (n, p) = 1\} \cup \{\infty\}$ where (n, p) denote the gcd of n and p.

b) If $r = -s = \pm 1$ then the Reidemeister spectrum of the group $\mathbb{Z}[1/p]^2 \rtimes_\theta$ $\mathbb{Z}$ is $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2p^l(p^k \pm 1), 4p^l | l, k > 0\} \cup \{\infty\}$

c) If $rs = 1$ and $|r| \neq 1$ then $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2(p^l \pm 1), 4 | l > 0\} \cup \{\infty\}$

d) If either r or s does not have module equal to one, and $rs \neq 1$ then there is no automorphism of the group such that the induced on the quotient is multiplication by -1 and follows that $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$.

Proof. As in Proposition 3.5 we have the system $a = ar^2$, $b = brs$, $c = crs$ and $d = ds^2$. Part a) and d) follows as in Proposition 3.5.

For the part b) from the equations $a = ar^2$, $b = brs$, $c = crs$ and $d = ds^2$ follows that the matrix N is diagonal. Let $a = \epsilon_1 p^{l_1}$ and $d = \epsilon_2 p^{l_2}$ since these elements are invertible. The $det(Id - N) = 1 - a - d + ad = \epsilon_1\epsilon_2 p^{l_1+l_2} - \epsilon_1 p^{l_1} - \epsilon_2 p^{l_2} + 1$ and $-det(Id - MN) = ad + a - d - 1 = \epsilon_1\epsilon_2 p^{l_1+l_2} + \epsilon_1 p^{l_1} - \epsilon_2 p^{l_2} - 1$.

Without loss of generality let us assume that $l_1, l_2 \geq 0$. First let $l_1 = l_2 = 0$. Then one of the two determinants is zero and the Reidemeister number is infinite. Now let $l_1 = 0$ and $l_2 \neq 0$. We have $det(Id - N) = \epsilon_1\epsilon_2 p^{l_2} - \epsilon_1 - \epsilon_2 p^{l_2} + 1$ and $-det(Id - MN) = \epsilon_1\epsilon_2 p^{l_2} + \epsilon_1 - \epsilon_2 p^{l_2} - 1$. If $\epsilon_1 = 1$ then $det = 0$ in both cases and we have Reidemeister infinite. If $\epsilon_1 = -1$ then we obtain $det(Id - N) = -2\epsilon_2 p^{l_2} + 2$ and $-det(Id - MN) = -2\epsilon_2 p^{l_2} - 2$. Both numbers are not divisible by p and the Reidemeister number is the module of $4\epsilon_2 p^{l_2}$, $l_2 > 0$. Now let $l_2 = 0$ and $l_1 \neq 0$. We have $det(Id - N) = \epsilon_1\epsilon_2 p^{l_1} - \epsilon_2 - \epsilon_1 p^{l_1} + 1$ and $-det(Id - MN) = \epsilon_1\epsilon_2 p^{l_1} + \epsilon_1 p^{l_1} - \epsilon_2 - 1$. If $\epsilon_2 = 1$ then $det(Id - N) = 0$ and we have Reidemeister infinite. If $\epsilon_2 = -1$ then $det(Id - MN) = 0$ and we have Reidemeister infinite. Finally if $l_1, l_2 > 0$ then the two numbers $\epsilon_1\epsilon_2 p^{l_1+l_2} - \epsilon_1 p^{l_1} - \epsilon_2 p^{l_2} + 1$ and $\epsilon_1\epsilon_2 p^{l_1+l_2} + \epsilon_1 p^{l_1} - \epsilon_2 p^{l_2} - 1$ are not divisible by p and the Reidemeister number is the module of $2\epsilon_1\epsilon_2 p^{l_1+l_2} - 2\epsilon_2 p^{l_2} = 2\epsilon_2 p^{l_2}(\epsilon_1 p^{l_1} - 1)$. So the Reidemeister numbers are of the form $2p^l(p^k \pm 1)$, $l, k > 0$, and the result follows.

For the part c) from the equations $a = ar^2$, $b = brs$, $c = crs$ and $d = ds^2$ follows $a = d = 0$. So the matrix N is of the form

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

and MN is

$$\begin{pmatrix} 0 & rb \\ sc & 0 \end{pmatrix}.$$

By direct calculation, using that $rs = 1$, follows that $det(Id - N) = det(Id - MN) = 1 - bc$. Since bc is invertible then we can write $b = \epsilon_1 p^{l_1}$ and $c = \epsilon_2 p^{l_2}$ and follows that $det(Id - N) = 1 - \epsilon_1\epsilon_2 p^{l_1+l_2}$. If $l_1 + l_2 = 0$ then we obtain for the determinant 0 or 2. So we obtain for Reidemeister number infinite and 4. If $l_1 + l_2 \neq 0$ then we obtain as Reidemeister numbers the numbers of the form $2(p^l \pm 1), l > 0$ and the result follows.

$\square$

Proposition 3.8. Let $\theta(1)$ be of the form:

$$\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

Then we have the following cases:

a) If $uv = 1$ then the Reidemeister spectrum of the group $\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2p^l(p^l \pm 1)|l > 0\} \cup \{\infty\}$.

b) If $uv = -1$ and then the Reidemeister spectrum of the group $\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}$ is $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{2(p^l \pm 1)|l > 0\} \cup \{\infty\}$.

c) If $u^2v^2 \neq 1$ then there is no automorphism of the group such that the induced on the quotient is multiplication by -1 and follows that $Spec(\mathbb{Z}[1/p]^2 \rtimes_\theta \mathbb{Z}) = \{\infty\}$.

Proof. The proof follows the same steps as the proof of Proposition 3.6 and it is simpler. The matrix N

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

must satisfy the equations $uvd = a, u^2c = b, bv^2 = c, auv = d$. It follows that $a = (uv)^2a, \ d = (uv)^2d, \ b = (uv)^2b, \ c = (uv)^2c$. So part c) follows promptly from these equations.

Let us consider the case a). In this case the system of equations provide $a = d$. So the determinant of N becomes $a^2 - bc = a^2 - b^2v^2 = (a+bv)(a-bv)$ which is an invertible element, so $(a + bv)$ and $(a - bv)$ are also invertible.

Since $uv = 1$ follows that $u = \delta p^{-t}$ and $v = \delta p^t$ for some integer t and $\delta \in \{1, -1\}$.

The matrix N is of the form

$$\begin{pmatrix} a & b \\ bv^2 & a \end{pmatrix}$$

and the matrix MN is

$$\begin{pmatrix} bv & a/v \\ av & bv \end{pmatrix}.$$

Our task is to compute v_P of $det(Id - N)$ and $det(Id - MN)$. We have $det(Id - N) = 1 - 2a + a^2 - b^2v^2$ and $det(Id - MN) = 1 - 2bv + b^2v^2 - a^2$ so $-det(Id - MN) = -1 + 2bv + a^2 - b^2v^2$. We compute v_P of $-det(Id - MN)$ which is the same as v_P of $det(Id - MN)$.

Since $a + bv$ and $a - bv$ are invertible they are of the form $a + bv = \epsilon_1 p^{l_1}$ and $a - bv = \epsilon_2 p^{l_2}$. Therefore $2a = \epsilon_1 p^{l_1} + \epsilon_2 p^{l_2}$ and $2bv = \epsilon_1 p^{l_1} - \epsilon_2 p^{l_2}$. In order to have $\epsilon_1 p^{l_1} + \epsilon_2 p^{l_2}$ divisible by 2 we need to have $l_1 = l_2 = l$ and either $\epsilon_1 = \epsilon_2$ or $\epsilon_1 = -\epsilon_2$. In the first case we have $a = \epsilon_1 p^l$ and $2bv = 0$ so $b = 0$. In the latter case we have $a = 0$ and $bv = \epsilon_1 s^l$. Now we compute the Reidemeister for each of these two cases. Let $a = \epsilon_1 p^l$ and $b = 0$. Then $det(Id - N) = 1 - 2a + a^2$ and $-det(Id - MN) = a^2 - 1$. If $l = 0$ then $det(Id - MN) = 0$ and we get Reidemeister infinite. If $l \neq 0$ then the determinants are not divisible by p and we get as Reidemeister number the module of $2a^2 - 2a = 2a(a - 2) = 2\epsilon p^l(\epsilon p^l - 1)$. So the Reidemeister numbers are of the form $2p^l(p^l \pm 1)$, $l \neq 0$. So the result follows.

Let us consider the case b). Some of the calculations are similar to the corresponding case ofthe Proposition 3.6 and in this case we do not give the details. The system of equations provide $a = -d$. So the determinant of N becomes $-a^2 - bc = -a^2 - b^2v^2$ which is an invertible element. Since $uv = -1$ follows that $u = -\delta 2^{-t}$ and $v = \delta 2^t$ for some integer t and $\delta \in \{1, -1\}$.

The matrix N is of the form

$$\begin{pmatrix} a & b \\ bv^2 & -a \end{pmatrix}$$

and the matrix MN is

$$\begin{pmatrix} -bv & a/v \\ av & bv \end{pmatrix}.$$

Our task is to commute v_P of $det(Id - N)$ and $det(Id - MN)$. We have $det(Id - N) = 1 - a^2 - b^2v^2$ and $det(Id - MN) = 1 - b^2v^2 - a^2$. So we compute v_P of $det(Id - N)$.

Let $a = rp^m$ and $bv = sp^n$ where r, s are relatively prime with p. Because $a^2 + (bv)^2$ is invertible then we obtain that $r^2 + s^2$ is necessarily 1 and we obtain as possible solution $a = \epsilon_1 p^m$ and $bv = 0$ (or $b = 0$ since $v \neq 0$), or $a = 0$ and $bv = \epsilon_2 p^n$.

For the case $b = 0$ we obtain $det(Id - N) = 1 - a^2 = 1 - p^{2m}$. For $m = 0$ we obtain Reidemeister infinite otherwise we obtain the total Reidemeister number $2(p^{2m} - 1), m > 0$. For $a = 0$ we obtain $det(Id - N) = 1 - b^2v^2 = 1 - p^{2n}$. Then we obtain the same numbers as above and the result follows.

$\square$

4. THE SEMI-DIRECT PRODUCTS $\mathbb{Q}^n \rtimes Z$, $\mathbb{Z}[1/p]^n \rtimes Z$, $n > 2$

There are some of the above results that extend easily to the groups $\mathbb{Q}^n \rtimes Z$, $\mathbb{Z}[1/p]^n \rtimes Z$, $n > 2$. One of the results refer to the group $\mathbb{Q}^n \rtimes \mathbb{Z}$. The abelian group $\mathbb{Q}^n$ has the property that it does not have a subgroup of finite index. Then an immediate consequence of this fact is that the Reidemeister number of any homomorphism is either 1 or infinite. So the following result holds.

Proposition 4.1. The Reidemeister spectrum of $\mathbb{Q}^n \rtimes_\theta \mathbb{Z}$ is either $\{\infty\}$ or $\{2\} \cup \{\infty\}$. The $Spec(\mathbb{Q}^n \rtimes_\theta \mathbb{Z}) = \{\infty\}$ if there is no an invertible matrix N over $\mathbb{Q}$ such that $N = MNM$ and $det(Id - N)$ and $det(Id - MN)$ are non zero. Here $\theta(1) = M$. Otherwise the Reidemeister spectrum is $\{2\} \cup \{\infty\}$.

Proof. (sketch) The only possible automorphism which can have Reidemeister finite is one such that the induced homomorphism on the quotient $\mathbb{Z}$ is multipliction by -1. From the procedure described in the begin of section 3 we have to compute the Reidemeister of the homomorphism given by N and MN. But a homomorphism of $\mathbb{Q}^n$ is either surjective or has cokernel infinite. So the sum of the Reidemeister of the two homomorphisms is either infinite or 2. The case where both homomorphisms have Reidemeister

1 corresponds to say that $det(Id - N)$ and $det(Id - MN)$ are non zero and the result follows. □

It is easy to construct examples which illustrated both cases, i.e. when the Reidemeister spectrum of $\mathbb{Q}^n \rtimes_\theta \mathbb{Z}$ is $\{\infty\}$ and when it is $\{2\} \cup \{\infty\}$.

Now let p be an arbitrary prime. It is not difficult to construct action $\theta(1)$ which has diagonal matrix such that there is no automorphism of the group such that the induced automorphims on the quotient $\mathbb{Z}$ is $-id$. This gives the examples of groups with the R_∞ property.

4.1. **Final comments.** Further calculation of the Reidemeister spectrum for some of the groups of the form $A^n \rtimes_\theta Z$, with A either $\mathbb{Q}$ or $\mathbb{Z}[1/p]$, not studied above, is in progress.

It is interesting to investigate $Spec_N C(X) = \{N(f)|f \in Map(X)\}$ and $Spec_N H(X) = \{N(f)|f \in Homeo(X)\}$ i.e., the *Nielsen spectrum for continious maps of the space X* and *Nielsen spectrum for homeomorphisms of the space X*.

Example 4.2. For selfmaps f of nilmanifold, either $N(f) = 0$ or $N(f) = R(f)$ [9]. It is easy to see that $Spec_N H(S^1) = \{2\} \cup \{0\}$, and, for $n \geq 2$, the spectrum is of T^n is full, i.e. $Spec_N H(T^n) = \mathbb{N} \cup \{0\}$. Let Nil_{rl} be the nilmanifold with the fundamental group being a free nilpotent group of rank r and class l. It follows from Introduction that for Nil_{22} we have $Spec_N H(Nil_{22}) = 2\mathbb{N} \cup \{0\}$; for Nil_{23} we have $Spec_N H(Nil_{23}) = \{2k^2|k \in \mathbb{N}\} \cup \{0\}$ and for Nil_{32} we have $Spec_N H(Nil_{32}) = \{2n - 1|n \in \mathbf{N}\} \cup \{4n|n \in \mathbf{N}\} \cup \{0\}$.

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