On q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function III

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Abstract. We identify q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions with a specialization of Macdonald polynomials. This provides a representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions in terms of Demazure characters of affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$. We also define a system of dual Hamiltonians for q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chains and give a new integral representation for q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions. Finally an expression of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a matrix element of a quantum torus algebra is derived.

Introduction

In [GLO1] an explicit expression for a q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function was proposed. This expression provides a q-version of the Casselman-Shalika-Shintani formula [Sh], [CS]. More precisely the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is given by a character of an infinite-dimensional $GL(\ell+1,\mathbb{C})\times\mathbb{C}^*$-module. It was remarked in [GLO1] that multiplied by a simple factor the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions have a representation as character of a *finite-dimensional* $GL(\ell+1,\mathbb{C})\times\mathbb{C}^*$-modules. In this note we identify these modules as particular Demazure modules of affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$ (see Theorem 3.2). This easily follows from two interpretations of Macdonald polynomials $P_\lambda(x;q,t)$ specialized at $t=0$. Below we express q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions in terms of $P_\lambda(x;q,t=0)$. On the other hand a relation between characters of $\widehat{\mathfrak{gl}}_{\ell+1}$ Demazure modules and $P_\lambda(x;q,t=0)$ was established previously by Sanderson [San1]. Note that the results of [San1] were generalized to simply-laced semisimple Lie algebras in [I]. We are going to consider the generalization of the constructions of this note to the simply-laced case elsewhere.

The explicit expression for q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function was derived in [GLO1] by considering a limit $t\to\infty$ of the Macdonald polynomials $P_\lambda(x;q,t)$. In this paper using the same limit we derive a set of dual Hamiltonian operators of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. The Whittaker function constructed in [GLO1] is a common eigenfunction of these dual Hamiltonian operators as well as standard Hamiltonian operators of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. We also consider a limit $t\to 0$ of Macdonald polynomials and relate it with q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function. However in this interpretation of Whittaker function the role of standard Hamiltonian Toda operators and the dual ones is reversed. This leads to a new integral representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function which is an analog of Mellin-Banes integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker function [KL]. In some sense this representation of q-deformed Whittaker function is dual to the one considered in [GLO1].

According to Kostant [Ko], $\mathfrak{g}$-Whittaker functions naturally arise as matrix elements of infinite-dimensional representations of $\mathcal{U}(\mathfrak{g})$. Using an embedding of $\mathcal{U}(\mathfrak{g})$ into a tensor product of several copies of Heisenberg algebras one obtains a realization of $\mathfrak{g}$-Whittaker functions as matrix elements of several copies of Heisenberg algebras. In this paper we construct analogous representation of

q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a particular matrix element of several copies of quantum torus algebras. We demonstrate that this representation is compatible with a q-version of Kostant representation.

Finally note that we realize a q-deformed Whittaker function multiplied by simple factor as a character of a finite-dimensional Demazure module of affine Lie algebra. As for q-deformed Whittaker function *per se* we describe a representation of q-deformed $\mathfrak{gl}_2$-Whittaker function as a character of a certain infinite-dimensional representation in the cohomology of line bundles over a semi-infinite manifold [GLO1]. This character can be considered as a proper substitute of a semi-infinite Demazure character of $\widehat{\mathfrak{gl}}_2$ [GLO2]. We are going to discuss this interpretation (and its generalization to $\mathfrak{gl}_{\ell+1}$) in [GLO3].

The paper is organized as follows. In Section 1 we describe basic properties of Macdonald polynomials. In particular, using the self-duality of Macdonald polynomials we define a dual system of Macdonald operators. In Section 2 we propose two explicit expressions for q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions as common eigenfunctions of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. We also construct a system of dual Hamiltonians for q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. In Section 3 the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions are identified with Demazure characters for affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$. Finally in Section 4 a representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a matrix element of a quantum torus algebra is derived.

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1 Macdonald polynomials

In this section we recall the standard facts about Macdonald polynomials. The basic reference is [Mac] (see also [Ch] for details and further developments).

Consider symmetric polynomials in variables $(x_1,\ldots,x_{\ell+1})$ over the field $\mathbb{Q}(q,t)$ of rational functions in q,t. Given a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\ell+1})$, that is the set of non-negative integers such that $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{\ell+1}$. Let $\preceq$ be the partial ordering on the set of partitions; precisely, given two partitions λ', λ we write $\lambda'\preceq\lambda$ when $\lambda'_k\leq\lambda_k$ for $k=1,\ldots,\ell+1$.

Let m_λ and π_λ be polynomial basis of the space of symmetric polynomials indexed by partitions λ:

$$m_\lambda=\sum_{\sigma\in\mathfrak{S}_{\ell+1}}x_{\sigma(1)}^{\lambda_1}x_{\sigma(2)}^{\lambda_2}\cdot\ldots\cdot x_{\sigma(\ell+1)}^{\lambda_{\ell+1}},$$

$$\pi_\lambda\,=\,\pi_{\lambda_1}\pi_{\lambda_2}\cdot\ldots\cdot\pi_{\lambda_{\ell+1}},\qquad\qquad \pi_n=\sum_{k=1}^{\ell+1}x_k^n,$$

where $\mathfrak{S}_{\ell+1}$ is the permutation group of $\ell+1$ elements. Define a scalar product $\langle\,,\,\rangle_{q,t}$ on the space of symmetric functions over $\mathbb{Q}(q,t)$ as follows

$$\langle\pi_\lambda,\,\pi_{\lambda'}\rangle_{q,t}\,=\,\delta_{\lambda,\lambda'}\cdot z_\lambda(q,t),$$

where

$$z_\lambda(q,t) \,=\, \prod_{n\geq 1} n^{m_n} m_n! \cdot \prod_{\lambda_k\neq 0} \frac{1-q^{\lambda_k}}{1-t^{\lambda_k}}, \qquad\qquad m_n = \big|\{k|\,\lambda_k = n\}\big|.$$

In the following we always imply $q<1$.

Definition 1.1 *Macdonald polynomials $P_\lambda(x;q,t)$ are symmetric polynomial functions over $\mathbb{Q}(q,t)$ such that*

$$P_\lambda \,=\, m_\lambda + \sum_{\lambda'\preceq\lambda} u_{\lambda\lambda'} m_{\lambda'},$$

with $u_{\lambda\lambda'}\in\mathbb{Q}(q,t)$, and for $\lambda\neq\lambda'$

$$\big\langle P_\lambda,\, P_{\lambda'}\big\rangle_{q,t} \,=\, 0.$$

In the following we slightly extend the notion of Macdonald polynomials $P_\lambda(x;q,t)$ to the case of generalized partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{\ell+1})$, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{\ell+1}$, $\lambda_i\in\mathbb{Z}$ using the relation

$$P_{(\lambda_1,\lambda_2,\ldots,\lambda_{\ell+1})}(x;q,t) = \Big(\prod_{j=1}^{\ell+1} z_j^{\lambda_{\ell+1}}\Big)\, P_{(\lambda_1-\lambda_{\ell+1},\lambda_2-\lambda_{\ell+1},\ldots,\lambda_\ell-\lambda_{\ell+1},0)}(x;q,t)$$

Although now $P_\lambda(x;q,t)$ are not necessary polynomials we use the term 'Macdonald polynomial' for thus defined $P_\lambda(x;q,t)$.

Macdonald polynomials can be equivalently characterized as common eigenfunctions of a set of Hamiltonians H_r

$$H_r\, P_\lambda(x;q,t) = c_r(q^\lambda)\, P_\lambda(x;q,t), \tag{1.1}$$

$$c_r(q^\lambda) = \chi_r(q^\lambda t^\varrho) = \sum_{I_r}\prod_{i\in I_r} q^{\lambda_i}\ t^{\varrho_i}, \tag{1.2}$$

where the eigenvalues $\chi_r(z)$ are characters of fundamental representations $\bigwedge^r\mathbb{C}^{\ell+1}$ of $\mathfrak{gl}_{\ell+1}$, $\varrho_i=\ell+1-i$ and we define $q^\lambda t^\varrho=(q^{\lambda_1}t^{\varrho_1},\ldots,q^{\lambda_{\ell+1}}t^{\varrho_{\ell+1}})$. Here the sum is over ordered subsets

$$I_r=\{i_1<i_2<\ldots<i_r\}\subset\{1,2,\ldots,\ell+1\}.$$

Explicitly H_r are given by

$$H_r = \sum_{I_r} t^{r(r-1)/2} \prod_{i\in I_r,\, j\notin I_r} \frac{tx_i-x_j}{x_i-x_j}\ \prod_{i\in I_r} T_{x_i}, \qquad r=1,\ldots,\ell+1, \tag{1.3}$$

and difference operators T_{x_i} are defined as

$$T_{x_i}f(x_1,\ldots,x_i,\ldots,x_{\ell+1}) = f(x_1,\ldots,qx_i,\ldots,x_{\ell+1}),$$

for $i=1,\ldots,\ell+1$. The simplest operator is given by

$$H_1 = \sum_{i=1}^{\ell+1}\prod_{i\neq j}\frac{tx_i-x_j}{x_i-x_j}\,T_{x_i}. \tag{1.4}$$

Let $t<1$ and

$$\Delta(x|q,t)=\prod_{i\neq j}\prod_{n=0}^{\infty}\frac{1-x_ix_j^{-1}q^n}{1-tx_ix_j^{-1}q^n}.$$

Define another scalar product on symmetric functions of $(\ell+1)$-variables $x_1,\ldots,x_{\ell+1}$ as follows

$$\langle f,g\rangle'_{q,t}=\frac{1}{(\ell+1)!}\oint_{\Gamma}\prod_{i=1}^{\ell+1}\frac{dx_i}{2\pi\imath x_i}\,f(x^{-1})\,g(x)\,\Delta(x|q,t), \tag{1.5}$$

where the integration domain Γ is such that each x_i goes around $x_i=0$ and is in the region defined by inequalities $t<|x_i/x_j|<t^{-1}$. Difference operators $H_r^{\mathfrak{gl}_{\ell+1}}$ are self-adjoint with respect to $\langle\,,\,\rangle'_{q,t}$:

$$\langle f,H_r^{\mathfrak{gl}_{\ell+1}}\,g\rangle'_{q,t}\;=\;\langle H_r^{\mathfrak{gl}_{\ell+1}}\,f,g\rangle'_{q,t}\;.$$

The following statement was proved in [AOS].

Proposition 1.1 *The following relations hold*

1.

$$\begin{aligned}P_{\lambda}^{\mathfrak{gl}_{\ell+1}}(x;q,t)\;&=\;\frac{1}{\ell!}\,\frac{\langle P_{\lambda}^{\mathfrak{gl}_{\ell}},P_{\lambda}^{\mathfrak{gl}_{\ell}}\rangle_{q,t}}{\langle P_{\lambda}^{\mathfrak{gl}_{\ell}},P_{\lambda}^{\mathfrak{gl}_{\ell}}\rangle'_{q,t}}\times\\ \times\int_{\Gamma}\prod_{i=1}^{\ell}\frac{dy_i}{2\pi\imath y_i}&\;C_{\ell+1,\ell}(x,y^{-1}|q,t)P_{\lambda}^{\mathfrak{gl}_{\ell}}(y;q,t)\,\Delta(y|q,t),\end{aligned} \tag{1.6}$$

where the integration domain Γ is as in (1.5) with the additional conditions $|x_iy_j^{-1}|<1,\ i=1,\ldots\ell+1,\ j=1,\ldots,\ell$.

2.

$$P_{\lambda+(\ell+1)^k}^{\mathfrak{gl}_{\ell+1}}(x;q,t)=\Big(\prod_{j=1}^{\ell+1}x_j^k\Big)\,P_{\lambda}^{\mathfrak{gl}_{\ell+1}}(x;q,t). \tag{1.7}$$

Here $\lambda+(\ell+1)^k=(\lambda_1+k,\ldots,\lambda_\ell+k,k)$ is a partition obtained from λ by a substitution $\lambda_j\to\lambda_j+k$, $j=1,\ldots,\ell+1$ and

$$C_{\ell+1,\ell}(x,y|q,t)=\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell}\prod_{n=0}^{\infty}\frac{1-tx_iy_jq^n}{1-x_iy_jq^n},$$

$$\langle P_{\lambda}^{\mathfrak{gl}_{\ell}},P_{\lambda}^{\mathfrak{gl}_{\ell}}\rangle'_{q,t}\;=\;\prod_{1\le i<j\le\ell}\prod_{n=0}^{\infty}\frac{1-t^{j-i}q^{\lambda_i-\lambda_j+n}}{1-t^{j-i+1}q^{\lambda_i-\lambda_j+n}}\cdot\frac{1-t^{j-i}q^{\lambda_i-\lambda_j+n+1}}{1-t^{j-i-1}q^{\lambda_i-\lambda_j+n+1}},$$

$$\langle P_{\lambda}^{\mathfrak{gl}_{\ell}},\,P_{\lambda}^{\mathfrak{gl}_{\ell}}\rangle_{q,t}\;=\;\prod_{i=1}^{\ell}\prod_{k=i}^{\ell}\prod_{n=1}^{\lambda_k-\lambda_{k+1}}\frac{1-t^{k-i}\,q^{\lambda_i-\lambda_{k+1}+1-n}}{1-t^{k+1-i}\,q^{\lambda_i-\lambda_{k+1}-n}},$$

where $\lambda_{\ell+1}=0$ is assumed in the last formula.

These relations provide a recursive construction of Macdonald polynomials corresponding to arbitrary partitions.

Macdonald polynomials respect a remarkable symmetry (see e.g. [Ch]). Let us define the normalized Macdonald polynomial $\Phi_\lambda(x;q,t)$ as follows

$$\Phi_\lambda(x;q,t) \;=\; t^{\sum_{i=1}^{\ell+1}\lambda_i\rho_i}\prod_{n=0}^{\infty}\prod_{1\le i<j\le \ell}\frac{1-t^2q^{\lambda_i-\lambda_j+n}}{1-tq^{\lambda_i-\lambda_j+n}}\;P_\lambda(x;q,t), \tag{1.8}$$

where $\rho_i=\varrho_i-\ell/2=1-i+\ell/2$ for $i=1,\dots,\ell+1$.

In the following we will always imply that $t=q^{-k}$, $k\in\mathbb{Z}$ and $q<1$. Then for any partitions λ and μ we have:

$$\Phi_\lambda(q^{\mu-k\rho};q,t) \;=\; \Phi_\mu(q^{\lambda-k\rho};q,t). \tag{1.9}$$

Define dual Macdonald Hamiltonians by

$$H_r^\vee(q^\lambda) \;=\; H_r(q^\lambda\, t^\rho), \qquad r=1,\dots,\ell+1. \tag{1.10}$$

Normalized Macdonald polynomials satisfy the following eigenvalue problems.

Proposition 1.2 *For any partitions λ and μ the normalized Macdonald polynomials satisfy the following system of equations*

$$\begin{cases} H_r(x)\,\Phi_\lambda(x;q,t) \;=\; c_r(q^\lambda)\,\Phi_\lambda(x;q,t), \\ H_r^\vee(q^\lambda)\,\Phi_\lambda(x;q,t) \;=\; c_r^\vee(x)\,\Phi_\lambda(x;q,t), \end{cases} \tag{1.11}$$

where

$$\begin{aligned} c_r(q^\lambda) \;&=\; \chi_r(q^\lambda t^\varrho) \;=\; \sum_{I_r}\prod_{i\in I_r} q^{\lambda_i}\,t^{\varrho_i}, \\ c_r^\vee(x) \;&=\; \chi_r(x\,t^{\ell/2}) \;=\; t^{r\ell/2}\sum_{I_r}\prod_{i\in I_r} x_i. \end{aligned} \tag{1.12}$$

Proof: Let μ be any partition and let $x=q^\mu$, then

$$H_r(q^\mu)\,\Phi_\lambda(q^\mu;q,t) \;= t^{\frac{r\ell}{2}}\sum_{I_r}\Big(\prod_{i\in I_r} q^{\lambda_i}\,t^{\rho_i}\Big)\,\Phi_\lambda(q^\mu;q,t).$$

Let us make a change variables $\mu\to\mu-k\rho$. Then using self-duality (1.9) of Macdonald polynomials one obtains

$$H_r(q^\mu t^\rho)\,\Phi_\mu(q^\lambda t^\rho;q,t) \;=\; t^{\frac{r\ell}{2}}\sum_{I_r}\Big(\prod_{i\in I_r} q^{\lambda_i}\,t^{\rho_i}\Big)\,\Phi_\mu(q^\lambda t^\rho;q,t). \tag{1.13}$$

Shifting variables $\lambda\to\lambda+k\rho$ we have

$$H_r(q^\mu t^\rho)\,\Phi_\mu(q^\lambda;q,t) \;=\; t^{\frac{r\ell}{2}}\sum_{I_r}\Big(\prod_{i\in I_r} q^{\lambda_i}\Big)\,\Phi_\mu(q^\lambda;q,t). \tag{1.14}$$

Note that $\Phi_\lambda(x;q,t)$ are polynomials in x and thus can be characterized by its values at $x=q^\mu$, $\mu\in\mathbb{Z}^{\ell+1}$. Interchanging variables $\lambda\leftrightarrow\mu$ and denoting $H_r^\vee(q^\lambda) \;=\; H_r(q^\lambda t^\rho)$ we obtain the statement of the proposition □

2 q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

In [GLO1] an explicit construction of a q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{z}}(\underline{p}_{\ell+1})$ on the lattice $\underline{p}_{\ell+1}=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1})\in\mathbb{Z}^{\ell+1}$ was given. The construction is based on a particular degeneration of the defining relations for Macdonald polynomials. In this section using the same degeneration we define dual Hamiltonians for q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. We also consider another degeneration procedure which also leads to q-deformed Toda chain but the role of the Hamiltonians and the dual Hamiltonians is interchanged. This leads to the second explicit expression for q-deformed Whittaker functions considered as common eigenfunctions of (dual) Hamiltonians of q-deformed Toda chain.

2.1 First explicit formula

The q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions are a common eigenfunction of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\,=\,\sum_{I_r}\big(\widetilde{X}_{i_1}^{1-\delta_{i_2-i_1,\,1}}\cdot\ldots\cdot\widetilde{X}_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},\,1}}\cdot\widetilde{X}_{i_r}^{1-\delta_{i_{r+1}-i_r,\,1}}\big)T_{i_1}\cdot\ldots\cdot T_{i_r}, \tag{2.1}$$

where we assume $i_{r+1}=\ell+2$. We use here the following notations

$$T_i f(\underline{p}_{\ell+1})=f(\underline{\widetilde{p}}_{\ell+1})\qquad\qquad \widetilde{p}_{\ell+1,k}=p_{\ell+1,k}+\delta_{k,i},$$

and

$$\widetilde{X}_i=1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1},\qquad i=1,\ldots,\ell\qquad\qquad \widetilde{X}_{\ell+1}=1.$$

The first nontrivial Hamiltonian is given by:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_1(\underline{p}_{\ell+1})\;=\;\sum_{i=1}^{\ell}(1-q^{p_{\ell+1,i}-p_{\ell+1,i+1}+1})T_i\;+\;T_{\ell+1}. \tag{2.2}$$

The corresponding eigenvalue problem can be written in the following form:

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_r(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\;=\;\Big(\sum_{I_r}\prod_{i\in I_r}z_i\Big)\;\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}). \tag{2.3}$$

The main result of [GLO1] can be formulated as follows. Denote by $\mathcal{P}^{(\ell+1)}\subset\mathbb{Z}^{\ell(\ell+1)/2}$ a subset of parameters $p_{k,i},\ k=1,\ldots,\ell,\ i=1,\ldots,k$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i}\geq p_{k,i}\geq p_{k+1,i+1}$. Let $\mathcal{P}_{\ell+1,\ell}\subset\mathcal{P}^{(\ell+1)}$ be a set of $\underline{p}_\ell=(p_{\ell,1},\ldots,p_{\ell,\ell})$ satisfying the conditions $p_{\ell+1,i}\geq p_{\ell,i}\geq p_{\ell+1,i+1}$.

Theorem 2.1 *The common solution of the eigenvalue problem (2.3) can be written in the following form. For $\underline{p}_{\ell+1}$ being in the dominant domain $p_{\ell+1,1}\geq\ldots\geq p_{\ell+1,\ell+1}$ the solution is given by*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\;=\;\sum_{p_{k,i}\in\mathcal{P}^{(\ell+1)}}\;\prod_{k=1}^{\ell+1}z_k^{\sum_i p_{k,i}-\sum_i p_{k-1,i}}$$
$$\times\frac{\prod\limits_{k=2}^{\ell}\prod\limits_{i=1}^{k-1}(p_{k,i}-p_{k,i+1})_q!}{\prod\limits_{k=1}^{\ell}\prod\limits_{i=1}^{k}(p_{k+1,i}-p_{k,i})_q!\;(p_{k,i}-p_{k+1,i+1})_q!}, \tag{2.4}$$

where we use the notation $(n)_q! = (1-q)...(1-q^n)$*. When* $\underline{p}_{\ell+1}$ *is outside the dominant domain we set*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_{\ell+1,1},\dots,p_{\ell+1,\ell+1}) \,=\, 0.$$

Formula (2.4) can be written in the recursive form.

Corollary 2.1 *The following recursive relation holds*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}} \Delta(\underline{p}_\ell)\; z_{\ell+1}^{\sum_i p_{\ell+1,i}-\sum_i p_{\ell,i}}\; Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell}(\underline{p}_\ell),$$

where

$$\begin{aligned} Q_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q) \,&=\, \frac{1}{\prod\limits_{i=1}^{\ell}(p_{\ell+1,i}-p_{\ell,i})_q!\;(p_{\ell,i}-p_{\ell+1,i+1})_q!}, \\ \Delta(\underline{p}_\ell) &= \prod_{i=1}^{\ell-1}(p_{\ell,i}-p_{\ell,i+1})_q!\;\;. \end{aligned} \tag{2.5}$$

Lemma 2.1 *The* q*-deformed* $\mathfrak{gl}_{\ell+1}$*-Whittaker function at* $p_{\ell+1,i}=k+1$ *for* $i\leq r$*,* $p_{\ell+1,i}=k$*, for* $i>r$ *is proportional to the character* $\chi_r(z)$ *of the fundamental representation* $\Lambda^r\mathbb{C}$ *of* $\mathfrak{gl}_{\ell+1}$

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(k+1,\dots k+1,k,\dots k) = \Big(\prod_{i=1}^{\ell+1} z_i^k\Big)\,\chi_r(z) = \Big(\prod_{i=1}^{\ell+1} z_i^k\Big)\,\sum_{I_r}\prod_{i\in I_r} z_i\,.$$

Proof: Directly follows from the general expression (2.4) □

Example 2.1 *Let* $\mathfrak{g}=\mathfrak{gl}_2$*,* $p_{2,1}:=p_1\in\mathbb{Z}$*,* $p_{2,2}:=p_2\in\mathbb{Z}$ *and* $p_{1,1}:=p\in\mathbb{Z}$*. The function*

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) = \sum_{p_2\leq p\leq p_1}\frac{z_1^p z_2^{p_1+p_2-p}}{(p_1-p)_q!(p-p_2)_q!}, \qquad p_1\geq p_2\,,$$

$$\Psi^{\mathfrak{gl}_2}_{z_1,z_2}(p_1,p_2) = 0, \qquad p_1<p_2\,,$$

is a common eigenfunction of mutually commuting Hamiltonians

$$\mathcal{H}_1^{\mathfrak{gl}_2} \,=\, (1-q^{p_1-p_2+1})T_1+T_2, \qquad \mathcal{H}_2^{\mathfrak{gl}_2}=T_1T_2.$$

2.2 Dual Hamiltonians for $\mathfrak{gl}_{\ell+1}$-Toda chain

The Hamiltonian operators of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain can be obtained by a degeneration of Macdonald operators discussed in the previous section (see e.g. [GLO1]). Similarly the degeneration of dual Macdonald operators leads to a set of dual Hamiltonians of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain.

Proposition 2.1 *1. Let* $t=q^{-k}$*,* $q<1$*. Define the limit* $k\to\infty$ *of the Macdonald (dual) operators*

$$\begin{aligned} \mathcal{H}_r(x) \,&=\, \lim_{k\to\infty} D(x)\,H_r(xq^{-k\rho})\,D(x)^{-1} \,=\\ &= \sum_{I_r}\big(X_{i_1}^{1-\delta_{i_1,1}}\cdot X_{i_2}^{1-\delta_{i_2-i_1,1}}\cdot\ldots\cdot X_{i_r}^{1-\delta_{i_r-i_{r-1},1}}\big)T_{x_{i_1}}\cdot\ldots\cdot T_{x_{i_r}}, \end{aligned} \tag{2.6}$$

$$\mathcal{H}_r^\vee(q^\lambda) \,=\, \lim_{k\to\infty} q^{kr(2\ell+1-r)/2}\, G(q^\lambda)\, H_r^\vee(q^{\lambda+k\varrho})\, G(q^\lambda)^{-1} \,= \\ =\, q^{r(r-1)/2} \sum_{I_r} \prod_{i\in I_r,\, j\notin I_r} \frac{q^{\lambda_j}}{q^{\lambda_j}-q^{\lambda_i}} \prod_{i\in I_r} T_{\lambda_i}, \tag{2.7}$$

here $r=1,\ldots,\ell+1$ *and we set* $X_i = 1-x_{i-1}^{-1}x_i$, $X_1=1$, $T_{\lambda_i}\lambda_j = \lambda_j T_{\lambda_i}+\delta_{ij}$ *and we assume*

$$D(x) = \prod_{i=1}^{\ell+1} x_i^{-k\varrho_i},$$

$$G(q^\lambda) \,=\, (-1)^{\ell\sum_{i=1}^{\ell+1}\lambda_i}\; q^{-\ell\sum_{i=1}^{\ell+1}\lambda_i/2} \prod_{i<j} q^{(\lambda_i-\lambda_j)^2/2}. \tag{2.8}$$

2. Let

$$\Psi_\lambda(x) = \lim_{k\to\infty}\; G(q^\lambda)D(x)\;\Phi_{\lambda+k\varrho}(xq^{-k\rho};q,t), \tag{2.9}$$

then the following relations hold

$$\mathcal{H}_r(x)\,\Psi_\lambda(x) \,=\, \chi_r(q^\lambda)\,\Psi_\lambda(x), \\ \mathcal{H}_r^\vee(q^\lambda)\,\Psi_\lambda(x) \,=\, \Big(q^{r(r-1)/2}\prod_{i=1}^r x_i\Big)\,\Psi_\lambda(x), \tag{2.10}$$

for $r=1,\ldots,\ell+1$.

Proof: Direct calculations □

Observe that the following relation between (2.6) and (2.1) holds

$$\mathcal{H}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \mathcal{H}_r(x), \qquad x_i = q^{p_{\ell+1,\,\ell+2-i}+\varrho_{\ell+2-i}}, \qquad i=1,\ldots,\ell+1,$$

for $r=1,\ldots,\ell+1$.

The limit $t=q^{-k}\to\infty$, $k\to\infty$ was used in [GLO1] to obtain Hamiltonians of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. There is a "dual" limit $t=q^{-k}$, $k\to-\infty$ which also leads to (dual) Hamiltonians of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain but the Hamiltonians and the dual Hamiltonians are interchanged.

Proposition 2.2 *1. Let* $t=q^{-k}$, $q<1$. *Define the limit* $k\to-\infty$ *of the Macdonald (dual) operators and their common eigenfunction as follows*

$$\widehat{\mathcal{H}}_r(x) \,=\, \lim_{k\to-\infty} q^{kr(r-1)/2}\, H_r(x) \,=\, \sum_{I_r} \prod_{i\in I_r,\, j\notin I_r} \frac{x_j}{x_j-x_i} \prod_{i\in I_r} T_{x_i}, \tag{2.11}$$

$$\widehat{\mathcal{H}}_r^\vee(q^\lambda) \,=\, \lim_{k\to-\infty} q^{kr\ell/2}\, \widehat{D}(q^\lambda)\, H_r^\vee(q^\lambda)\, \widehat{D}(q^\lambda)^{-1} \,= \\ =\sum_{I_r} \big(\widehat{X}_{i_1}^{1-\delta_{i_1,1}}\cdot \widehat{X}_{i_2}^{1-\delta_{i_2-i_1,1}}\cdot\ldots\cdot \widehat{X}_{i_r}^{1-\delta_{i_r-i_{r-1},1}}\big) T_{\lambda_{i_1}}\cdot\ldots\cdot T_{\lambda_{i_r}}, \tag{2.12}$$

$\widehat{X}_i \,=\, 1 - q^{\lambda_i-\lambda_{i+1}}$ *and* $\widehat{X}_1 = 1$. *We assume here* $\widehat{D}(q^\lambda) = \prod_{i=1}^{\ell+1} q^{k\lambda_i\rho_i}$.

2. Define

$$\widehat{\Psi}_\lambda(x) \,=\, \lim_{k\to-\infty} \widehat{D}(q^\lambda)\,\Phi_\lambda(x;q,t). \tag{2.13}$$

Then the following equations hold

$$\begin{aligned} \widehat{\mathcal{H}}_r(x)\,\widehat{\Psi}_\lambda(x) \,&=\, q^{\lambda_{\ell+2-r}+\ldots+\lambda_{\ell+1}}\,\widehat{\Psi}_\lambda(x), \\ \widehat{\mathcal{H}}_r^\vee(q^\lambda)\,\widehat{\Psi}_\lambda(x) \,&=\, \chi_r(x)\,\widehat{\Psi}_\lambda(x). \end{aligned} \tag{2.14}$$

Proof. 1. The formula for $\widehat{\mathcal{H}}_r$ follows straightforwardly. 2. For $t = q^{-k}$ we obtain

$$\widehat{D}(q^\lambda)\,H_r^\vee(q^\lambda)\,\widehat{D}(q^\lambda)^{-1} \,=\, t^{r\ell/2}\sum_{I_r}\prod_{i\in I_r,\,j\notin I_r}\prod_{i<j}\frac{t^{j+1-i}q^{\lambda_i}-q^{\lambda_j}}{t^{j-i}q^{\lambda_i}-q^{\lambda_j}}\prod_{i>j}\frac{q^{\lambda_i}-t^{i-1-j}q^{\lambda_j}}{q^{\lambda_i}-t^{i-j}q^{\lambda_j}}\prod_{i\in I_r}T_{\lambda_i},$$

due to the following identity

$$\frac{r(r-1)}{2} + \sum_{i\in I_r}\big(\rho_i + b_{i,\,I_r}\big) \,=\, \frac{r\ell}{2},$$

where $b_{i,I_r} = \big|\{j\notin I_r \,|\, j<i\}\big|$.

Thus under the limit $t\to 0$ one gets the following.

$$\frac{t^{j+1-i}q^{\lambda_i}-q^{\lambda_j}}{t^{j-i}q^{\lambda_i}-q^{\lambda_j}} \longrightarrow 1,\ i<j, \qquad \frac{q^{\lambda_{i+1}}-q^{\lambda_i}}{q^{\lambda_{i+1}}-tq^{\lambda_i}} \longrightarrow 1-q^{\lambda_i-\lambda_{i+1}}, \qquad \frac{q^{\lambda_i}-t^{i-1-j}q^{\lambda_j}}{q^{\lambda_i}-t^{i-j}q^{\lambda_j}} \longrightarrow 1,\ i>j+1,$$

□

Remark 2.1 *Let* λ *be a partition, then*

1.

$$\begin{aligned} \widehat{\mathcal{H}}_r(x_1,\ldots,x_{\ell+1}) \,&=\, q^{-\frac{r(r-1)}{2}}\mathcal{H}_r^\vee(x_1,\ldots,x_{\ell+1}), \\ \widehat{\mathcal{H}}_r^\vee(q^{\lambda_1},\ldots,q^{\lambda_{\ell+1}}) \,&=\, \Delta(q^\lambda)\,\mathcal{H}_r(q^{\lambda_{\ell+1}+\varrho_{\ell+1}},\ldots,q^{\lambda_1+\varrho_1})\,\Delta(q^\lambda)^{-1}, \end{aligned} \tag{2.15}$$

for $r = 1,\ldots,\ell+1$ *and*

$$\Delta(q^\lambda) \,=\, \prod_{i=1}^{\ell}\,(\lambda_i-\lambda_{i+1})_q! \tag{2.16}$$

2. The specialization of Macdonald polynomial at $t=0$

$$\widehat{\Psi}_\lambda(x) = P_\lambda(x;q,t=0), \tag{2.17}$$

satisfies equations (2.14).

Proof: Proof of (1) is straightforward and the statement of (2) easily follows from (2.13) and (1.8) □

2.3 Second explicit formula

Now we construct an integral representation for q-deformed Whittaker functions by taking $t \to 0$ limit of the recursive construction of Macdonald polynomials.

In the limit $t \to 0$ the Macdonald scalar product on symmetric functions of $(\ell+1)$-variables $x_1, \ldots, x_{\ell+1}$ is reduced to

$$\langle f, g\rangle'_{q,t=0} = \frac{1}{(\ell+1)!} \oint_{\Gamma_0} \prod_{i=1}^{\ell+1} \frac{dx_i}{2\pi\imath x_i}\, f(x^{-1})\, g(x)\, \Delta(x|q, t=0), \tag{2.17}$$

where

$$\Delta(x|q,0) = \prod_{i\neq j} \prod_{n=0}^{\infty} (1 - x_i x_j^{-1} q^n).$$

and the integration domain Γ_0 is such that each x_i goes over a small circle around $x_i = 0$.

The limit $t \to 0$ of the recursive kernel $C_{\ell+1,\ell}$ is given by

$$C_{\ell+1,\ell}(x, y|q, t=0) = \prod_{i=1}^{\ell+1} \prod_{j=1}^{\ell} \prod_{n=0}^{\infty} \frac{1}{1 - x_i y_j q^n}.$$

Proposition 2.3 *1. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ the following recursive relation holds*

$$\begin{gathered} P_\lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t=0) \;=\; \frac{A_\ell}{\ell!} \int_{\Gamma_0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi\imath y_i}\; C_{\ell+1,\ell}(x, y^{-1}|q, t=0) \times \\ \times P_\lambda^{\mathfrak{gl}_\ell}(y; q, t=0)\, \Delta(y|q, t=0), \end{gathered} \tag{2.18}$$

where

$$A_\ell = \lim_{t\to 0} \frac{\langle P_\lambda^{\mathfrak{gl}_\ell}, P_\lambda^{\mathfrak{gl}_\ell}\rangle_{q,t}}{\langle P_\lambda^{\mathfrak{gl}_\ell}, P_\lambda^{\mathfrak{gl}_\ell}\rangle'_{q,t}} \;=\; \prod_{m=1}^{\infty} (1-q^m)^{\ell-1} \cdot (\lambda_\ell)_q!\,,$$

and the contour of integration Γ_0 is as in (2.17) with additionally conditions $x_i y_j^{-1} < 1$.

2. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

$$P^{\mathfrak{gl}_{\ell+1}}_{\lambda+(\ell+1)^k}(x; q, t=0) = \Big(\prod_{j=1}^{\ell+1} x_j^k\Big)\, P_\lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t=0), \tag{2.19}$$

where $\lambda + (\ell+1)^k = (\lambda_1 + k, \ldots, \lambda_\ell + k, k)$.

Proof: We have

$$\langle P_\lambda^{\mathfrak{gl}_\ell},\, P_\lambda^{\mathfrak{gl}_\ell}\rangle'_{q,t=0} \;=\; \prod_{i=1}^{\ell-1} \prod_{m=1}^{\infty} \frac{1}{(1 - q^{\lambda_i - \lambda_{i+1} + m})},$$

$$\langle P_\lambda^{\mathfrak{gl}_\ell},\, P_\lambda^{\mathfrak{gl}_\ell}\rangle_{q,t=0} \;=\; \prod_{i=1}^{\ell-1} (\lambda_i - \lambda_{i+1})_q! \times (\lambda_\ell)_q!$$

where $\prod_{m=1}^{0}(1-q^m) = 1$ is assumed. Thus we obtain the recursive relation (2.18) □

These relations provide a recursive construction of a q-deformed Whittaker function corresponding to an arbitrary partition. Note that the property of Macdonald polynomial being symmetric function of variables $z_1, \ldots, z_{\ell+1}$ remains true in the limit $t \to 0$.

Proposition 2.4 *Let $z_i := x_{\ell+1,i}$ for $i=1,\dots,\ell+1$. Define the function ${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1})$ given for the dominant domain $p_{\ell+1,1}\geq\dots\geq p_{\ell,1,\ell+1}$ by an integral expression*

$$\begin{aligned} {}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, (q,q)_\infty^{\ell(\ell-1)/2} \int_{\mathcal{S}} \prod_{n=1;j\leq n}^{\ell} \frac{dx_{nj}}{2\pi\imath x_{nj}} \\ \times \prod_{n=1}^{\ell+1}\prod_{j=1}^{n}\Big(\frac{x_{n,j}}{x_{n-1,j}}\Big)^{p_{\ell+1,\,n}} \prod_{n=1}^{\ell} \frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1}\Gamma_q(x_{nk}^{-1}x_{n+1,m})}{n!\prod\limits_{s\neq p}\Gamma_q(x_{ns}x_{np}^{-1})}, \end{aligned} \tag{2.19}$$

where the contour $\mathcal{S}$ is obtained by induction from the contours Γ_0 defined in the Proposition 2.3 and outside of the dominant domain by

$${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(p_{\ell+1,1},\dots,p_{\ell+1,\ell+1}) \,=\, 0.$$

Then the function ${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1})$ possess the following properties
1. It is $\mathfrak{S}_{\ell+1}$-symmetric:

$${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_{\sigma(1)},\dots,z_{\sigma(\ell+1)}}(\underline{p}_{\ell+1}) = {}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}), \qquad \sigma\in\mathfrak{S}_{\ell+1},$$

2. It is a common eigenfunction of (dual) Hamiltonians $\mathcal{H}_r$, $\mathcal{H}_r^\vee$:

$$\begin{aligned} \mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})\,{}^{MB}\Psi_z(\underline{p}_{\ell+1}) \,&=\, \chi_r(z)\,{}^{MB}\Psi_z(\underline{p}_{\ell+1}), \\ q^{-r(r-1)/2}\mathcal{H}_r^\vee(z)\,{}^{MB}\Psi_z(\underline{p}_{\ell+1}) \,&=\, \Big(\prod_{i=1}^{r} q^{p_{\ell+1,i}}\Big)\,{}^{MB}\Psi_z(\underline{p}_{\ell+1}), \end{aligned} \tag{2.20}$$

for $r=1,\dots,\ell+1$.

This integral representation is a q-version of Mellin-Barnes integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker functions introduced in [KL]. Let us compare ${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_z$ with the function $\Psi^{\mathfrak{gl}_{\ell+1}}_z$ given by (2.4).

Proposition 2.5 *q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function given by (2.4) is a $\mathfrak{S}_{\ell+1}$-symmetric function*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_{\sigma(1)},\dots,z_{\sigma(\ell+1)}}(\underline{p}_{\ell+1}) = \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\dots,z_{\ell+1}}(\underline{p}_{\ell+1}), \qquad \sigma\in\mathfrak{S}_{\ell+1}.$$

Proof: We prove this statement by the induction. Given a $\mathfrak{gl}_\ell$-Whittaker function which is symmetric

$$\Psi^{\mathfrak{gl}_\ell}_{z_{\sigma(1)},\dots z_{\sigma(\ell)}}(\underline{p}_\ell) = \Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell}(\underline{p}_\ell), \qquad \sigma\in\mathfrak{S}_\ell.$$

The function $\Psi^{\mathfrak{gl}_{\ell+1}}$ then given by

$$\Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell,z_{\ell+1}}(\underline{p}_{\ell+1}) = \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}} C_{\ell+1,\ell}(q)\; z_{\ell+1}^{\sum_{j=1}^{\ell+1}p_{\ell+1,j}-\sum_{j=1}^{\ell}p_{\ell,j}}\;\Psi^{\mathfrak{gl}_\ell}_{z_1,\dots,z_\ell}(\underline{p}_\ell).$$

The space of solutions of q-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain invariant with respect to $\mathfrak{S}_\ell\subset\mathfrak{S}_{\ell+1}$ is $(\ell+1)$-dimensional. Thus to verify that (2.4) is $\mathfrak{S}_{\ell+1}$-invariant one should check that it is invariant at

$\ell+1$ particular values of $\underline{p}_{\ell+1}$. Let us take $\underline{p}_{\ell+1}$ corresponding to fundamental representations. By Lemma 2.1 the corresponding q-Whittaker functions are given by characters of $\mathfrak{gl}_{\ell+1}$-fundamental representations and thus explicitly $\mathfrak{S}_{\ell+1}$-invariant $\Box$

The function ${}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})$ satisfies the full set of equations (i.e. including dual Hamiltonians) and the function $\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})$ satisfies the original q-deformed Toda equations. Thus one has

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})=C(z_1,\ldots,z_{\ell+1})\ {}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}).$$

Proposition 2.6

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\;=\;{}^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}).$$

Proof: Denote

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})\;=\;\Delta(\underline{p}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}),$$

$$\Delta(\underline{p}_{\ell+1})^{MB}\Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})=P_{\underline{p}_{\ell+1}}(z;q,t=0).$$

Then $\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1})|_{p_{\ell+1,i=0}}\;=\;1$ and $P_{(0,0,\cdots,0)}(z;q,t=0)=1$ by definition of Macdonald polynomials. Thus $C(z_1,\ldots,z_{\ell+1})=1$ $\Box$

Remark 2.2 *The normalized q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function coincides with a $t=0$ specialization of Macdonald polynomial*

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z}(\underline{p}_{\ell+1})=P_\lambda(z;q,t=0),\qquad \lambda=(p_{\ell+1,1},\ldots,p_{\ell+1,\ell+1}). \tag{2.20}$$

3 q-Whitaker functions as characters of affine Demazure modules

In this Section we identify the normalized q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z}(\underline{p}_{\ell+1})$ with characters of affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$ Demazure modules. This straightforwardly follows from a characterization of the normalized q-deformed $\widehat{\mathfrak{gl}}_{\ell+1}$-Whittaker function as a specialization of Macdonald polynomials $P_\lambda(z;q,t)$ at $t=0$ (see Remark 2.2) and a relation of $P_\lambda(z;q,t=0)$ with characters of Demazure modules of affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$ established in [San1].

To state precisely the relation between Whittaker functions and Demazure modules let us start recalling the notion of a Demazure module [De] (see [Ku], [M] for a general case of Kac-Moody algebras). Let $\mathfrak{g}$ be a Kac-Moody algebra with Cartan matrix $\|a_{ij}\|$, $\mathfrak{h}\subset\mathfrak{b}\subset\mathfrak{g}$ be a Cartan and Borel subalgebras. Let $R\subset\mathfrak{h}^*$ be a corresponding root system, $R_+\subset R$ be a subset of positive roots corresponding to the Borel subalgebra $\mathfrak{b}$, $\alpha_1,\ldots,\alpha_r\in R_+$ be a set of simple roots. Denote (λ,μ) the scalar product on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}$. Given a root α let $\alpha^\vee=2\alpha/(\alpha,\alpha)$ be the corresponding coroot where we identify $\mathfrak{h}\equiv\mathfrak{h}^*$ using quadratic form $(\,,\,)$. The weight lattice P is given by $P=\{\lambda\in\mathfrak{h}^*\,:\,(\lambda,\alpha^\vee),\in\mathbb{Z}\ \ \alpha\in R\}$. The weight lattice is generated by fundamental weights $\omega_1,\ldots,\omega_r$ defined by the conditions $(\omega_i,\alpha_j^\vee)=\delta_{ij}$. The set of dominant weights is given by $P^+=\{\lambda\in P\,:\,(\lambda,\alpha^\vee)\geq 0,\ \ \alpha\in R\}$. The Weyl group W is defined as a group of reflections $s_\alpha:\mathfrak{h}^*\to\mathfrak{h}^*$, $\alpha\in R$

$$s_\alpha\,:\,\lambda\;\longrightarrow\lambda-(\lambda,\alpha^\vee)\alpha,$$

and is generated by reflections s_i corresponding to simple roots α_i. An expression of a Weyl group element w as a product $w = s_{i_1} \cdots s_{i_l}$ which has minimal length is called reduced decomposition for w and its length $l(w) = l$ is called a length of w. Let T be a Cartan torus $\mathrm{Lie}(T) = \mathfrak{h}$. The group of characters $X = X(T)$ of T is isomorphic to the weight lattice P of $\mathfrak{g}$. Its group algebra $\mathbb{Z}[T] = R(T)$ is the representation ring of T and is generated by formal exponents $\{e^\mu : \mu \in P\}$, with the multiplication $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

Let ω be a dominant weight of $\mathfrak{g}$ and $V(\omega)$ be an integrable irreducible highest weight representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ with the highest weight ω. For any $w \in W$ the weight $w(\omega)$ subspace $V^{[w(\omega)]}(\omega)$ in $V(\omega)$ is one-dimensional. Let $V_w(\omega) \subseteq V(\omega)$ be a $\mathcal{U}(\mathfrak{b})$-submodule generated by enveloping algebra $\mathcal{U}(\mathfrak{b})$ of the Borel subalgebra $\mathfrak{b}$ acting on $V^{[w(\omega)]}(\omega)$. The $\mathcal{U}(\mathfrak{b})$-module $V_w(\omega)$ is called Demazure module. Characters of $V_w(\omega)$ are defined as

$$\mathrm{ch}_{V_w(\omega)} = \sum_{\mu \in P} (dim V_w^{[\mu]}(\omega)) e^\mu,$$

and can be calculated using Demazure operators as follows. Define Demazure operators corresponding to simple root α_i as

$$\mathcal{D}_{s_i} e^\mu = \frac{e^\mu - e^{-\alpha_i} e^{s_i(\mu)}}{1 - e^{-\alpha_i}},$$

where $s_i \in W$ is a simple reflection corresponding to α_i. Demazure operators commute with W-invariant elements in $\mathbb{Z}[T]$ and satisfy the following relations

$$\mathcal{D}_{s_i}^2 = \mathcal{D}_{s_i}, \qquad (\mathcal{D}_{s_i} \mathcal{D}_{s_j})^{m_{ij}} = 1,$$

where m_{ij} are equal to

$$m_{ij} = 2,\ 3,\ 4,\ 6,\ \infty,$$

for the values of entries of Cartan matrix $\|a_{ij}\|$ satisfying

$$a_{ij} a_{ji} = 0,\ 1,\ 2,\ 3,\ \geq 4,$$

respectively. Here we imply $x^\infty := 1$. These relations provide a correctly defined map $w \mapsto \mathcal{D}_w$:

$$w = s_{i_1} s_{i_2} \cdots s_{i_j} \longmapsto \mathcal{D}_w = \mathcal{D}_{s_{i_1}} \mathcal{D}_{s_{i_2}} \cdots \mathcal{D}_{s_{i_j}}.$$

Given a reduced (minimal length) decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_j}$ of an element $w \in W$ we have for the character of $V_w(\omega)$

$$\mathrm{ch}_{V_w(\omega)} = \mathcal{D}_{s_{i_1}} \mathcal{D}_{s_{i_2}} \cdots \mathcal{D}_{s_{i_j}} e^\omega. \tag{3.1}$$

Now let $\mathfrak{g}$ be the affine Lie algebra $\widehat{\mathfrak{gl}}_{\ell+1}$. The corresponding root system can be realized as a set of vectors in $\mathbb{R}^{\ell+2,1}$ supplied with a bi-linear symmetric form defined in the bases $\{e_1, \ldots, e_{\ell+1}, e_+, e_-\}$ by

$$(e_i, e_j) = \delta_{ij}, \qquad (e_\pm, e_i) = (e_\pm, e_\pm) = 0, \qquad (e_+, e_-) = 1.$$

Simple roots of $\widehat{\mathfrak{gl}}_{\ell+1}$ are given by

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_2 - e_3, \qquad \ldots \qquad \alpha_\ell = e_\ell - e_{\ell+1}, \qquad \alpha_0 = e_+ - (e_1 - e_{\ell+1}).$$

The fundamental weights $\omega_0, \omega_1 \ldots, \omega_{\ell+1}$ are defined by the conditions $(\omega_i, \alpha_j^\vee) = \delta_{ij}$

$$\omega_1 = e_1 + e_-, \qquad \omega_2 = e_1 + e_2 + e_-, \qquad \ldots \qquad \omega_{\ell+1} = \sum_{j=1}^{\ell+1} e_j + e_-, \qquad \omega_0 = e_-.$$

In the following we will also use the standard notation $\delta = \alpha_0 + \sum_{i=1}^{\ell} \alpha_i = e_+$. The Weyl group W has natural decomposition $W = \dot{W} \times Q$ where Q is a lattice generated by simple coroots and $\dot{W}$ is the Weyl group of the finite-dimensional Lie algebra $\mathfrak{gl}_{\ell+1}$. Define a projection of the weight lattice P of $\widehat{\mathfrak{gl}}_{\ell+1}$ onto the weight lattice $\dot{P}$ of the finite-dimensional algebra $\mathfrak{gl}_{\ell+1}$

$$\omega = \lambda_1 e_1 + \cdots + \lambda_{\ell+1} e_{\ell+1} + k e_- + r e_+ \longrightarrow \dot{\omega} = \lambda_1 e_1 + \cdots + \lambda_{\ell+1} e_{\ell+1}.$$

The projection on the lattice $\dot{P}$ of the action of the generators of W on $e_- + \sum \lambda_i e_i$ is given by

$$s_i\,(\lambda_1, \ldots, \lambda_{\ell+1}) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_{\ell+1}),$$

$$s_0 \cdot (\lambda_1, \ldots, \lambda_{\ell+1}) = (\lambda_{\ell+1} + 1, \lambda_2, \ldots, \lambda_\ell, \lambda_1 - 1).$$

Note that $|\omega| = \sum_{i=1}^{\ell+1} \lambda_i$ is invariant under the action of W.

Lemma 3.1 *A set of orbits of W acting on the weight lattice $\dot{P}$ of $\mathfrak{gl}_{\ell+1}$ can be identified with $\mathbb{Z} \times (\mathbb{Z}/\mathbb{Z}_{\ell+1})$ and a set of representatives can be chosen as follows*

$$\dot{\lambda}_{k,i} = (k+1, \ldots, k+1, k, \ldots, k) = k\dot{\mathbf{1}} + \dot{\omega}_i,$$

where $\mathbf{1} = (1, \ldots, 1)$ and $\dot{\omega}_i$ are fundamental weights of $\mathfrak{gl}_{\ell+1}$.

Proof: Using $\dot{W}$ one can transform any weight of $\mathfrak{gl}_{\ell+1}$ to a dominant one $\dot{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell+1})$. Now using elements W transforming dominant weights to dominant we can change weights in such a way that the difference $\lambda_i - \lambda_{i+1}$ is either 1 or 0 □

Define homomorphism

$$\pi \;:\; \mathbb{Z}[T] \to \mathbb{Z}[z_1, \ldots, z_{\ell+1}, q]$$

$$\pi(e^{\omega_i}) = z_1 \cdots z_i, \qquad \pi(e^{\omega_0}) = 1, \qquad \pi(e^{\delta}) = q.$$

The following result was proved by Sanderson [San1].

Theorem 3.1 *Let $\lambda_{k,i} = \omega_0 + \dot{\lambda}_{k,i}$ and $\dot{\lambda}_{k,i} \in \dot{P}^+$ is given by*

$$\dot{\lambda}_{k,i} = (k+1, \ldots, k+1, k, \ldots k) = k \cdot \mathbf{1} + \dot{\omega}_i.$$

Let $w \in W$ be such that for $\lambda = w \cdot \lambda_{k,i}$ its projection $\dot{\lambda}$ be antidominant weight i.e. $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{\ell+1}$. Define $\dot{\lambda}' = w_0 \dot{\lambda}$, where $w_0 \in \dot{W}$ is an element having a reduce decomposition of maximal length. Then the character of the Demazure module $V_w(\lambda_{k,i})$ satisfy the following relation

$$\pi\Big(ch_{V_w(\lambda_{k,i})}\Big) = q^{\frac{1}{2}(\dot{\lambda},\dot{\lambda}) - \frac{1}{2}(\dot{\lambda}_{k,i},\dot{\lambda}_{k,i})}\, P_{\dot{\lambda}'}(z; q, t=0),$$

where $P_{\dot{\lambda}'}(z; q, t)$ is a Macdonald polynomial corresponding to dominant partition $\dot{\lambda}'$ (see Definition 1.1).

The modified q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is given by

$$\widetilde{\Psi}_z^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \;=\; \Delta(\underline{p}_{\ell+1}) \Psi_z^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}),$$

where

$$\Delta(\underline{p}_{\ell+1}) \;=\; \prod_{i=1}^{\ell} (p_{\ell+1,i} - p_{\ell+1,i+1})_q!$$

and $\Psi_z^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})$ is defined by (2.4).

Theorem 3.2 *The following representation for the modified q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function holds*

$$\widetilde{\Psi}_z^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})=q^{\frac{1}{2}(\dot{\lambda}_{k,i},\dot{\lambda}_{k,i})-\frac{1}{2}(\dot{\lambda},\dot{\lambda})}\pi\Big(ch_{V_w(\lambda_{k,i})}\Big),\qquad p_{\ell+1,i}=(\dot{\lambda}')_i.$$

Thus the finite sum (2.4) up to a simple multiplier provides expression for a characters of affine Lie algebra Demazure module.

Example. Let us consider as an example the case of $\ell=1$. We have

$$ch_{V_{(s_1s_0)^m(\omega_0)}}=\mathcal{D}_{(s_1s_0)^m}\,e^{\omega_0},\qquad ch_{V_{s_1(s_0s_1)^m(\omega_1)}}=\mathcal{D}_{s_1(s_0s_1)^m}\,e^{\omega_1},$$

where $\omega_0=e_-$ and $\omega_1=e_-+e_1$. Note that due to the identity $\mathcal{D}_1^2=\mathcal{D}_1$ both characters are $W_1=S_2$ invariant and thus are given by linear combination of $\mathfrak{gl}_2$-characters.

$$\dot{\lambda}_{0,0}=(0,0),\qquad \lambda=(s_1s_0)^m\omega_0,\qquad \dot{\lambda}=(-m,m),\qquad \dot{\lambda}'=(m,-m),$$

$$\dot{\lambda}_{0,1}=(1,0),\qquad \lambda=s_1(s_0s_1)^m\omega_1,\qquad \dot{\lambda}=(-m,m+1),\qquad \dot{\lambda}'=(m+1,-m),$$

and thus

$$\pi\Big(ch_{V_{(s_1s_0)^m(\omega_0)}}\Big)=q^{m^2}\,P_{m,-m}(z_1,z_2;q,t=0),$$

$$\pi\Big(ch_{V_{s_1(s_0s_1)^m(\omega_1)}}\Big)=q^{m(m+1)}\,P_{m+1,-m}(z_1,z_2;q,t=0).$$

Let us note that there exists a generalization of the results in [San1] to the case of simply-laced affine Lie algebras [I]. Also the structure of Demazure modules for arbitrary simply-laced affine Lie group was clarified in [FL]. It was shown that as a module over corresponding finite-dimensional Lie algebra it is a finite tensor product of finite-dimensional irreducible representations. In the special case of $\widehat{\mathfrak{g}}=\widehat{\mathfrak{gl}}_{\ell+1}$ this is in complete agreement with the Proposition 3.4 in [GLO1]. Note that the case $\ell=1,2$ was considered before in [San2]. All this seems implies that the connection between q-deformed Whittaker functions, specialization of Macdonald polynomials and Demazure modules discussed above can be rather straightforwardly generalized at least to the simply-laced case. We conjecture that this indeed so and are going to discuss the details elsewhere.

4 q-deformed Whittaker function as a matrix element

According to Kostant [Ko] $\mathfrak{g}$-Whittaker functions can be understood as matrix elements of infinite-dimensional representations of universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ with action of Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ integrated to the action of the corresponding Cartan subgroup $H\subset G$. This interpretation can be generalized to the case of q-deformed $\mathfrak{g}$-Whittaker functions considered as matrix elements of infinite-dimensional representations of quantum groups $\mathcal{U}_q(\mathfrak{g})$ (see e.g. [Et]). In this Section we derive a representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions given explicitly by (2.4) as a matrix element of an infinite-dimensional representation of multidimensional quantum torus. Our construction is based on an iterative application of the following standard identity (see e.g. [CK])

$$(X+T)^n=\sum_{m=0}^{n}\binom{n}{m}_q X^mT^{n-m},\qquad TX=qXT$$

where $\binom{n}{m}_q=(n)_q!/(m)_q!(n-m)_q!$ is a q-binomial coefficient. This representation should arise in the Kostant framework using a realization of $\mathcal{U}_q(\mathfrak{gl}_{\ell+1})$ by difference operators generalizing Gauss-Givental realization of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ proposed in [GKLO]. We check this directly for $\mathfrak{g}=\mathfrak{sl}_2$ leaving the general case to another occasion.

Let $\mathcal{A}^{(\ell)}$ be an associative algebra, $\{X^{\pm 1}_{k,i}, T^{\pm 1}_{k,i}\}$, $k=1,\ldots,\ell$; $i=1,\ldots,k$ be a complete set of generators satisfying relations

$$T_{k,i}X_{m,j} \,=\, q^{\delta_{k,m}\delta_{i,j}}\cdot X_{m,j}T_{k,i}. \tag{4.1}$$

Introduce a set of polynomials $f_{n,i}(z)\in\mathcal{A}^{(\ell)}[z_1,\ldots,z_{\ell+1}]$

$$f_{n,i}=f_{n,i}(\underline{z};X_{k,j},T_{k,j}),\qquad n=1,\ldots,\ell+1;\quad i=1,\ldots,n,$$

of degree $\deg f_{n,\,i}=i$ in variables $\underline{z}=(z_1,\ldots,z_{\ell+1})$, defined by the following recursive relations:

$$f_{n,\,i} \,=\, f_{n-1,\,i}X_{n-1,\,i} \,+\, z_n\,f_{n-1,\,i-1}T_{n-1,\,i},\qquad\qquad i<n, \tag{4.2}$$

where $f_{n,0}=f_{00}=1$ and $f_{n,n}=z_1\cdots z_n$ with

$$f_{n,n} \,=\, z_n\cdot f_{n-1,n-1}\qquad\qquad n=1,\ldots,\ell+1.$$

In particular, we have $f_{11}=z_1$ and $f_{21}=f_{11}X_{11}+z_2f_{10}T_{11}=z_1X_{11}+z_2T_{11}$.

Let $\mathcal{V}$ be a representation of $\mathcal{A}^{(\ell)}$, and $\mathcal{V}^*$ be its dual. Consider $|v_+\rangle\in\mathcal{V}$, $\langle v_-|\in\mathcal{V}^*$ such that $\langle v_-|v_+\rangle=1$ and satisfying the conditions

$$T_{i,k}|v_+\rangle=|v_+\rangle,\qquad\qquad \langle v_-|X_{i,k}=\langle v_-|,\qquad\qquad i=1,\ldots,\ell,\ \ k=1,\ldots,i.$$

Let us introduce normalized q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \prod_{k=1}^{\ell}(p_{\ell+1,\,k}-p_{\ell+1,\,k+1})_q!\ \Psi^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}).$$

Theorem 4.1 *The following representation of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function holds*

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \Big\langle v_-\,\Big|\,\prod_{k=1}^{\ell+1}f_{\ell+1,\,k}(\underline{z};X_{n,i},T_{n,i})^{p_{\ell+1,\,k}-p_{\ell+1,\,k+1}}\,\Big|\,v_+\Big\rangle, \tag{4.3}$$

where we assume $p_{\ell+1,\,\ell+2}=0$.

Proof: Let us prove the Theorem by induction. We assume that the representation (4.3) for $\mathfrak{gl}_{\ell}$ holds

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell}}_{z_1,\ldots,z_{\ell}}(\underline{p}_{\ell}) \,=\, \langle v_-\,|\,\prod_{k=1}^{\ell}f_{\ell,\,k}(\underline{z}')^{p_{\ell,\,k}-p_{\ell,\,k+1}}\,|\,v_+\,\rangle,$$

where $\underline{z}'=(z_1,\ldots,z_\ell)$. The following recursive relation follows from (2.5)

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{z_1,\ldots,z_{\ell+1}}(\underline{p}_{\ell+1}) \,=\, \sum_{\underline{p}_\ell\in\mathcal{P}_{\ell+1,\ell}} z_{\ell+1}^{\sum p_{\ell+1,i}-\sum p_{\ell,k}}\,\widetilde{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q)\,\widetilde{\Psi}^{\mathfrak{gl}_{\ell}}_{z_1,\ldots,z_\ell}(\underline{p}_\ell), \tag{4.4}$$

where

$$\widetilde{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1},\underline{p}_\ell|q) \,=\, \prod_{k=1}^{\ell}\binom{p_{\ell+1,k}-p_{\ell+1,k+1}}{p_{\ell,k}-p_{\ell+1,k+1}}_q$$

Then (4.3) for $\mathfrak{gl}_{\ell+1}$ is obtained by repeated application of the identities

$$\sum_{p_{\ell,k}=p_{\ell+1,k+1}}^{p_{\ell+1,k}} z_{\ell+1}^{p_{\ell+1,k}-p_{\ell,k}} \binom{p_{\ell+1,k}-p_{\ell+1,k+1}}{p_{\ell,k}-p_{\ell+1,k+1}}_q \cdot$$

$$\cdot\,(f_{\ell,k-1})^{p_{\ell,k-1}-p_{\ell,k}}(f_{\ell,k})^{p_{\ell,k}-p_{\ell+1,k+1}}\; X_{\ell,k}^{p_{\ell,k}-p_{\ell+1,k+1}}\; T_{\ell,k}^{p_{\ell+1,k}-p_{\ell,k}} =$$

$$= (f_{\ell,k-1})^{p_{\ell,k-1}-p_{\ell+1,k}}\Big(f_{\ell,k}X_{\ell,k} + z_{\ell+1}f_{\ell,k-1}T_{\ell,k}\Big)^{p_{\ell+1,k}-p_{\ell+1,k+1}},$$

to convert q-binomial factors $\widetilde{Q}_{\ell+1,\ell}$ in (4.4) □

The representation (4.3) can be understood as a particular realization of q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a matrix element of an infinite-dimensional representation of $\mathcal{U}_q(\mathfrak{gl}_{\ell+1})$. In the following we demonstrate this for the simplest case q-deformed $\mathfrak{sl}_2$-Whittaker function.

Quantum deformed universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by generators E, F, K satisfying the relations

$$KE = qEK, \qquad KF = q^{-1}FK, \qquad EF - FE = -\frac{K - K^{-1}q}{1-q}.$$

The center of $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by a Casimir element

$$C = K + K^{-1} + (q + q^{-1} - 2)FE.$$

In irreducible representations the image of C is proportional to unite operator and we parametrize the corresponding eigenvalue c as follows

$$c = -(z + z^{-1}).$$

Consider a realization of $\mathcal{U}_q(\mathfrak{sl}_2)$ (see e.g. [KLS])

$$K = -zu^{-1}, \qquad E = \frac{v^{-1}(1-u^{-1})}{1-q}, \qquad F = -\frac{v(z - qz^{-1}u)}{1-q},$$

where $uv = qvu$. The general q-deformed $\mathfrak{sl}_2$-Whittaker function is given by (compare with (2.17) in [KLS] with $\omega_1 = 1$, $q = \exp(2\imath\pi\omega_1/\omega_2)$)

$$\Phi_z^{(\alpha_1,\alpha_2)}(x) = e^{-\pi x}\, q^{\imath x/2}\, \langle\psi_L^{(\alpha_1)}|\, q^{\imath\frac{x}{2}H}\, |\psi_R^{(\alpha_2)}\rangle, \tag{4.4}$$

where $K = q^{H/2}$ and $\psi_L^{(\alpha)}/\psi_R^{(\alpha)}$ are left/right Whittaker vectors defined by

$$E\psi_L^{(\alpha)} = q^{1-\alpha}\, e^{\imath\pi\alpha}\frac{K^{\alpha}}{1-q}\psi_L^{(\alpha)}, \qquad F\psi_R^{(\alpha)} = e^{\imath\pi\alpha}\frac{K^{-\alpha}}{1-q}\psi_R^{(\alpha)},$$

$$(\mathcal{T}^{-1} + \mathcal{T} - q^{(\alpha_1-\alpha_2+1)}q^{-\imath x}\mathcal{T}^{\alpha_1-\alpha_2})\Phi_z^{(\alpha_1,\alpha_2)}(x) = (z + z^{-1})\Phi_z^{(\alpha_1,\alpha_2)}(x), \tag{4.5}$$

where $\mathcal{T}\, f(x) = f(x+\imath)$.

We would like to compare this representation with a representation given in the previous section. The representation for $\ell = 1$ adopted to the case of $\mathfrak{sl}_2$ is given by

$$\Psi_z(n) = \frac{1}{(n)_q!}\langle v_-|\,(zX + z^{-1}T)^n\,|v_+\rangle, \qquad \widetilde{\Psi}_z(n) = \langle v_-|\,(zX + z^{-1}T)^n\,|v_+\rangle,$$

where $T|v_+\rangle = |v_+ >$ and $\langle v_-|X = \langle v_-|$. The functions $\Psi_z(n)$ and $\widetilde{\Psi}_z(n)$ satisfy the following equation

$$\begin{aligned}(\mathcal{T}^{-1} + (1-q^{n+1})\mathcal{T})\Psi_z(n) = (z+z^{-1})\Psi_z(n),\\ ((1-q^n)\mathcal{T}^{-1} + \mathcal{T})\widetilde{\Psi}_z(n) = (z+z^{-1})\widetilde{\Psi}_z(n),\end{aligned} \tag{4.6}$$

where $\mathcal{T}\,f(n) = f(n+1)$. To reconcile the last equation in (4.6) and the equation (4.5) we take

$$\alpha_1 = 1, \quad \alpha_2 = 2, \qquad x = \imath n.$$

Then one has

$$\Phi_z^{(1,2)}(\imath n) = \langle\psi_L^{(1)}|\,(-K)^{-n}\,|\psi_R^{(2)}\rangle,$$

and one would like to have the following relation for the q-deformed Whittaker function

$$\widetilde{\Psi}_z(n) = \Phi_z^{(1,2)}(\imath n). \tag{4.7}$$

To make this identification one should should have the following relation

$$-K^{-1} = z^{-1}u = z^{-1}T + zX.$$

and demonstrate that $\langle v_-|$ and $|v_+\rangle$ provide as representation for left and right Whittaker vectors $\langle\psi_L^{(1)}|$ and $|\psi_R^{(2)}\rangle$.

Consider the following unitary transformation

$$U^{-1}(u,v)uU(u,v) = u + z^2 v, \qquad U(u,v) = \prod_{n=0}^{\infty}(1+z^2vu^{-1}q^n)^{-1} \;=\Gamma_q(-z^2vu^{-1}),$$

where

$$\Gamma_q(x) = \frac{1}{\prod_{j=0}^{\infty}(1-xq^j)}.$$

Thus we have

$$\begin{aligned}U^{-1}(u,v)\,v\,U(u,v) &= (1+z^2vu^{-1})v,\\ U^{-1}(u,v)\,u\,U(u,v) &= (1+z^2vu^{-1})u,\\ U^{-1}(u,v)\,v^{-1}\,U(u,v) &= (1+q^{-1}z^2vu^{-1})^{-1}v^{-1},\\ U^{-1}(u,v)\,u^{-1}\,U(u,v) &= (1+q^{-1}z^2vu^{-1})^{-1}u^{-1}.\end{aligned}$$

The conjugated generators are then given by

$$\begin{aligned}\widehat{K} &= U^{-1}(u,v)\,K\,U(u,v) = -z(1+q^{-1}z^2vu^{-1})^{-1}u^{-1},\\ \widehat{E} &= U^{-1}(u,v)\,E\,U(u,v) = \frac{1}{1-q}\,(1+q^{-1}z^2vu^{-1})^{-1}v^{-1}(1-(1+q^{-1}z^2vu^{-1})^{-1}u^{-1}),\\ \widehat{F} &= U^{-1}(u,v)\,F\,U(u,v) = -\frac{1}{1-q}(1+z^2vu^{-1})v(z-z^{-1}q(1+z^2vu^{-1})u).\end{aligned}$$

Proposition 4.1 *The following identities hold*

$$\widehat{E}|u=1\rangle = -\frac{1}{1-q}\,\widehat{K}^2|u=1\rangle, \qquad \widehat{F}|v=1\rangle = \frac{1}{1-q}\,\widehat{K}^{-1}|v=1\rangle,$$

where $v|v=1 >= |v=1>$, $u|u=1>=|u=1>$.

Proof: Direct calculations □

Thus one can identify $|v_-\rangle \equiv |u=1\rangle = |\psi_L^{(2)}\rangle$, $|v_+\rangle \equiv |v=1\rangle = |\psi_R^{(1)}\rangle$ in the U-rotated bases. This also provides an identification (4.7).

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