Deformations of nilpotent cones and Springer correspondences

Syu KATO [*][†][‡]

April 26, 2010

Dedicated to the long-standing friendship of Ken-ichi Shinoda and Toshiaki Shoji on the occasion of their 60th birthdays

Abstract

Let $G = Sp(2n)$ be the symplectic group over $\mathbb{Z}$. We present a certain kind of deformation of the nilpotent cone of G with G-action. This enables us to make direct links between the Springer correspondence of $\mathfrak{sp}_{2n}$ over $\mathbb{C}$, that over characteristic two, and our exotic Springer correspondence. As a by-product, we obtain a complete description of our exotic Springer correspondence.

Introduction

Let $G = Sp(2n)$ be the symplectic group over $\mathbb{Z}$. Let $\Bbbk$ be an algebraically closed field. Let $\mathfrak{g}$ be the Lie algebra of G defined over $\mathbb{Z}$. Let $\mathcal{N}$ denote the subscheme of nilpotent elements of $\mathfrak{g}$. Let $G_{\Bbbk}$, $\mathfrak{g}_{\Bbbk}$, and $\mathcal{N}_{\Bbbk}$ denote the specializations of G, $\mathfrak{g}$, and $\mathcal{N}$ to $\Bbbk$, respectively.

Springer [Spr76] defines a correspondence between the set of $G_{\Bbbk}$-orbits in $\mathcal{N}_{\Bbbk}$ and a certain set of Weyl group representations (with a basis) when $\mathsf{char}\,\Bbbk$ is good (i.e. not equal to 2). This correspondence, together with the so-called "A-group data", lifts to a one-to-one correspondence.

This story is later deepened in two ways. One is Lusztig's generalized Springer correspondence [Lus84], which serves as a basis of his theories on Chevalley groups. The other is Joseph's realization [Jos83], which serves a model of the structure of the primitive spectrum of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

In our previous paper [K09], we found that a certain Hilbert nilcone $\mathfrak{N}$ gives a variant of one aspect of the above mentioned Lusztig's theory (c.f. [KL87] and [Lus88]). Quite unexpectedly, our correspondence gives a one-to-one correspondence without the "A-group data", which is needed in the original Springer

[*]Research Institute for Mathematical Sciences, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502, Japan. E-mail:kato@kurims.kyoto-u.ac.jp

[†]Current Address: Department of Mathematics, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502, Japan. E-mail:syuchan@math.kyoto-u.ac.jp

[‡]The author was partially supported by JSPS Grant-in-Aid for Young Scientists (B) 20-740011 during this research.

correspondence for Weyl groups of type C. Therefore, it seems natural to seek some meaning of $\mathfrak{N}$.

The main theme of this paper is to give one explanation of $\mathfrak{N}$. Roughly speaking, our conclusion is that $\mathfrak{N}$ is a model of $\mathcal{N}_{\mathbb{F}_2}$ over $\mathbb{Z}$, which is "better" than $\mathcal{N}$ in a certain sense.

To see what we mean by this, we need a more precise formulation: Let T be a maximal torus of G. We define the Weyl group of (G, T) as $W := N_G(T)/T$. We denote the set of isomorphism classes of irreducible representations of W by $W^\vee$. Let V_1 be the vector representation. Put $V_2 := \wedge^2 V_1$. We denote $V_1 \oplus V_2$ by $\mathbb{V}$. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard choice of T-weight basis of G (see eg. Bourbaki [Bou02]). We denote the "positive part" of $\mathfrak{g}$ and $\mathbb{V}$ by $\mathfrak{n}$ and $\mathbb{V}^+$, respectively (c.f. §1.2). Let $\mathfrak{N}$ be the Hilbert nilcone of $(G, \mathbb{V})$ over $\mathbb{Z}$. We have a natural map

$$\nu : G \times^B \mathbb{V}^+ \longrightarrow \mathfrak{N},$$

which we regard as a counter-part of the Springer resolution.

Theorem A. *The variety $\mathfrak{N}$ is normal and flat over $\mathbb{Z}$. Moreover, the number of $G_\mathbb{k}$-orbits of $\mathfrak{N}_\mathbb{k}$ is independent of the characteristic of $\mathbb{k}$.*

Theorem B. *Let $\mathbb{k} = \overline{\mathbb{F}}_2$. There exists a $G_\mathbb{k}$-equivariant flat family $\pi : \mathcal{N}_S \longrightarrow \mathbb{A}^1_\mathbb{k}$ with the following properties:*

1. *We have $\pi^{-1}(t) \cong \mathcal{N}_\mathbb{k}$ for $t \neq 0$;*

2. *There exists an isogeny $\mathsf{F}_1 : \mathfrak{N}_\mathbb{k} \longrightarrow \pi^{-1}(0)$, which is an endomorphism as varieties.*

Moreover, for a $G_\mathbb{k}$-orbit $\mathcal{O}_\mathbb{k} \subset \mathfrak{N}_\mathbb{k}$, there exists a flat subfamily of single $G_\mathbb{k}$-orbits $\mathcal{O}_S \subset \mathcal{N}_S$ such that $\mathcal{O}_S \cap \pi^{-1}(0) = \mathsf{F}_1(\mathcal{O}_\mathbb{k})$.

Theorem A claims that our variety $\mathfrak{N}$ behaves well with respect to the specializations. Theorem B claims that we can regard $\mathfrak{N}$ as a model of $\mathcal{N}_\mathbb{k}$ in a certain sense.

To illustrate these, let us describe the orbit correspondence of Theorem B, together with the corresponding Springer correspondences:

Example C (The orbit correspondence for $n = 2$). Let $\mathfrak{t}_\mathbb{C}$ be a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let $R = \{\pm\epsilon_i \pm \epsilon_j\}_{1 \leq i,j \leq 2} \backslash \{0\} \subset \mathfrak{t}^*_\mathbb{C}$ be the set of roots of $\mathfrak{g}_\mathbb{C}$. We choose its positive simple roots as $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = 2\epsilon_2$. Let $\mathbf{x}[\lambda] \in \mathfrak{g}$ and $\mathbf{v}[\lambda] \in \mathbb{V}$ be T-eigenvectors with T-weight λ, respectively. We refer the Springer correspondence of $\mathcal{N}$ by ordinary and that of $\mathfrak{N}$ by exotic. Then, representatives of $G_\mathbb{k}$-orbits of $\mathcal{N}_\mathbb{k}$ and $\mathfrak{N}_\mathbb{k}$ corresponding to each member of $W^\vee$ are:

$W^\vee$	dim.	ordinary (char$\mathbb{k} \neq 2$)	ordinary (char$\mathbb{k} = 2$)	exotic
sign	1	0	0	0
Ssign	1	$\mathbf{x}[2\epsilon_1]$	$\mathbf{x}[2\epsilon_1]$	$\mathbf{v}[\epsilon_1]$
Lsign	1	$\mathbf{x}[\alpha_1]$	$\mathbf{x}[\alpha_1]$	$\mathbf{v}[\alpha_1]$
regular	2	$\mathbf{x}[\alpha_1]$	$\mathbf{x}[\alpha_1] + \mathbf{x}[2\epsilon_1]$	$\mathbf{v}[\alpha_1] + \mathbf{v}[\epsilon_1]$
triv	1	$\mathbf{x}[\alpha_1] + \mathbf{x}[\alpha_2]$	$\mathbf{x}[\alpha_1] + \mathbf{x}[\alpha_2]$	$\mathbf{v}[\alpha_1] + \mathbf{v}[\epsilon_2]$

Theorem B gives an isogeny between the Springer fibers of $\mathcal{N}_\mathbb{k}$ and $\mathfrak{N}_\mathbb{k}$ when char$\mathbb{k} = 2$. This implies that the Springer correspondences associated to $\mathcal{N}_\mathbb{k}$ and $\mathfrak{N}_\mathbb{k}$ must coincide up to scalar multiplication of their basis.

To see this phenomenon more closely, we employ the Joseph model of the Springer representations. Following Joseph [Jos83], we define the orbital variety attached to a $G_{\mathbb{C}}$-orbit $\mathbb{O}_{\mathbb{C}} \subset \mathcal{N}_{\mathbb{C}}$ as an irreducible component of the intersection $\mathbb{O}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}}$. Let us denote the set of orbital varieties attached to $\mathbb{O}_{\mathbb{C}}$ by $\mathrm{Comp}(\mathbb{O}_{\mathbb{C}})$. Similarly, let $\mathcal{O}_{\mathbb{C}} \subset \mathfrak{N}_{\mathbb{C}}$ be a $G_{\mathbb{C}}$-orbit and let $\mathrm{Comp}(\mathcal{O}_{\mathbb{C}})$ be the set of irreducible components of $\mathcal{O}_{\mathbb{C}} \cap \mathbb{V}_{\mathbb{C}}^+$. We also call a member of $\mathrm{Comp}(\mathcal{O}_{\mathbb{C}})$ an orbital variety (attached to $\mathcal{O}_{\mathbb{C}}$).

Joseph found that the leading terms of T-equivariant Hilbert polynomials of $\mathrm{Comp}(\mathbb{O}_{\mathbb{C}})$ yield an irreducible W-module which is contained in the Springer representation attached to $\mathbb{O}_{\mathbb{C}}$. These polynomials are usually called the Joseph polynomials. In representation-theoretic context (originally pursued by Joseph), $\mathrm{Comp}(\mathcal{O}_{\mathbb{C}})$ is precisely the set of Lagrangian subvarieties of $\mathcal{O}_{\mathbb{C}}$ corresponding to highest weight modules of $\mathfrak{g}_{\mathbb{C}}$ and these polynomials build up "asymptotic Goldie ranks" of primitive ideals of $U(\mathfrak{g}_{\mathbb{C}})$ (see eg. [Jos97]).

In view of Hotta [Hot84] (see also Joseph [Jos89] or Chriss-Ginzburg [CG97]), it is straightforward to see that Joseph's construction extends to the case of our exotic Springer correspondence. In particular, we have the notion of Joseph polynomials attached to each orbit of $\mathfrak{N}$.

Theorem D. *Let $\Bbbk = \overline{\mathbb{F}}_2$. Let $\mathbb{O}_{\mathbb{C}}$ be a $G_{\mathbb{C}}$-orbit of $\mathcal{N}_{\mathbb{C}}$. Then, there exists G-stable locally closed subsets $\mathbb{O} \subset \mathcal{N}$ and $\mathcal{O} \subset \mathfrak{N}$ such that*

1. *We have $\mathbb{O}_{\mathbb{C}} = \mathbb{O} \otimes \mathbb{C}$;*

2. *The variety $\mathcal{O} \otimes \Bbbk$ is a single $G_{\Bbbk}$-orbit which corresponds to a unique dense open $G_{\Bbbk}$-orbit of $\mathbb{O} \otimes \Bbbk$;*

3. *The Joseph polynomials of $\mathcal{O}_{\mathbb{C}}$ and that of $\mathbb{O}_{\mathbb{C}}$ are equal up to scalar.*

It may worth to mention that there exists some orbit $\mathcal{O}$ of $\mathfrak{N}$ which does not correspond to an orbit of $\mathcal{N}_{\mathbb{C}}$. In this case, our version of Joseph polynomials realize a Weyl group representation which cannot be realized by the usual Joseph polynomials. To illustrate this phenomenon, we compare Joseph polynomials for $Sp(4)$:

Example E (Joseph polynomials for $n = 2$). Keep the setting of Example C. By the natural W-action on $\mathfrak{t}_{\mathbb{C}}^*$, we have a W-action on $\mathbb{C}[\mathfrak{t}_{\mathbb{C}}] = \mathbb{C}[\epsilon_1, \epsilon_2]$ preserving each degree. Each Joseph polynomial belongs to $\mathbb{C}[\mathfrak{t}_{\mathbb{C}}]$ and hence admits a W-action. The list of Joseph polynomials corresponding to each member of $W^{\vee}$ via Example C is:

$W^{\vee}$	dim.	ordinary (char$\Bbbk \neq 2$)	ordinary (char$\Bbbk = 2$)	exotic
sign	1	$4\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$	$4\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$	$\epsilon_1\epsilon_2(\epsilon_1^2 - \epsilon_2^2)$
Ssign	1	$2(\epsilon_1^2 - \epsilon_2^2)$	$2(\epsilon_1^2 - \epsilon_2^2)$	$(\epsilon_1^2 - \epsilon_2^2)$
Lsign	1	N/A	$4\epsilon_1\epsilon_2$	$\epsilon_1\epsilon_2$
regular	2	$\alpha_1, 2\epsilon_2$	$\alpha_1, 2\epsilon_2$	α_1, ϵ_2
triv	1	1	1	1

Note that the $\mathbb{C}$-span of polynomials in each entry of the above table recovers the original member of $W^{\vee}$.

Since our exotic Springer correspondence shares a similar flavor with the usual Springer correspondence of type A, it is natural to expect a combinatorial description. To state this, we need:

Definition F. Let (μ, ν) be a pair of partitions such that $|\mu| + |\nu| = n$. For a partition λ, we put $\lambda_i^< := \sum_{j<i} \lambda_j$ and $\lambda_i^\leq := \sum_{j \leq i} \lambda_j$. We define

$$\mathsf{D}_i^0(\mu) := \prod_{\mu_i^< < k < l \leq \mu_i^\leq} (\epsilon_k^2 - \epsilon_l^2), \quad \mathsf{D}_i^+(\mu, \nu) := \prod_{\nu_i^< < k < l \leq \nu_i^\leq} (\epsilon_{k+|\mu|}^2 - \epsilon_{l+|\mu|}^2), \text{ and}$$

$$\mathsf{D}(\mu, \nu) := \prod_{i > |\mu|} \epsilon_i \times \prod_{i=1}^\infty \mathsf{D}_i^0(\mu) \mathsf{D}_i^+(\mu, \nu).$$

Theorem G. *For each G-orbit $\mathcal{O}$ of $\mathfrak{N}$, there exists a pair of partitions (μ, ν) and $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$ such that the Joseph polynomial of $\mathfrak{X}$ is a scalar multiplication of $\mathsf{D}(\mu, \nu)$.*

Since (μ, ν) in Theorem G is easily computable, this completes a determination of our exotic Springer correspondence. Taking account into Theorem D, we have determined some special Joseph polynomials which we cannot compute easily from their naive definitions.

The organization of this paper is as follows:

In §1, we fix convention and introduce our variety $\mathfrak{N}$. Then, we describe its set of defining equations in §2. Our system of defining equations is explicit and behaves nice with respect to the restriction to certain linear subvariety. These facts enable us to prove that $\mathfrak{N}$ is normal in §3. This proves the first part of Theorem A. Also, we introduce a parameterization of orbits of $\mathfrak{N}$ over $\mathbb{Z}$ or $\Bbbk$. The §4 contains the main observations of this paper. Namely, we observe:

- the adjoint representation $\mathfrak{g}$ of a symplectic group over characteristic two is not irreducible;

- this reducibility enables one to define a natural deformation of $\mathfrak{g}$, and its subvariety $\mathcal{N}$ in characteristic two;

- the special fiber such that (the deformation of) $\mathfrak{g}$ becomes decomposable is isogenous to $\mathbb{V}$;

- the above three observations are sufficient to construct a "deformation" from $\mathcal{N}$ (general fiber) to $\mathfrak{N}$ (special fiber) in characteristic two.

These observations enable us to prove Theorem B. In §5, we define a particular element of the Weyl group attached to each orbit of $\mathfrak{N}$ which cut-off a particular part of the orbit. In §6, we see that every orbit of $\mathfrak{N}_\Bbbk$ extends to an orbit of $\mathfrak{N}$ in order to prove the second part of Theorem A. The §§7–8 are devoted to the equi-dimensionality of the orbital varieties attached to $\mathfrak{N}$. Its proof is nothing but a minor modification of the Steinberg-Spaltenstein-Joseph theorem, which we present here for the reference purpose. (So I claim no originality here.) These are preparatory steps to the later sections. In §9, we use the results in the previous sections to prove Theorem D. With the help of previous sections and Joseph's theory, the only missing piece boils down to the rigidity of the torus character. In §10, we construct a special orbital variety from an orbit of $\mathfrak{N}$ in order to prove Theorem G. The main difficulty in the couse of its proof is that we cannot expect some orbital variety to be a linear subspace contrary to the type A case. We make a trick coming from the symmetry of Joseph polynomials

to avoid this difficulty.

With the technique developed in this paper, a similar construction applied to $G^{\vee} = SO(2n+1)$ yields an analogue of Theorem D for special representations of the Weyl groups of $Sp(2n)$ and $SO(2n+1)$. However, as the work of Tian Xue [Xue08] suggests, there might be a better formulation of the connection between these two cases.

Finally, one word of caution is in order. We work not over $\mathrm{Spec}\,\mathbb{Z}$ but a neighborhood of $\mathrm{Spec}\,\overline{\mathbb{F}}_2$ in the main body of this paper. The reason is that two is the only bad prime for symplectic groups and the corresponding statements are more or less trivial (or inexistent) with respect to the reduction to the other primes.

1 Preliminaries

1.1 Convention

Consider a ring

$$A := \mathbb{Z}[p^{-1}, \zeta_N; p, N \in \mathbb{Z}_{>0}, (p, 2) = 1, \zeta_N^{2^N - 1} = 1] \subset \overline{\mathbb{Q}}.$$

This is a local ring with a unique maximal ideal (2). Let $\mathbb{K}$ be the quotient field of A and let $\Bbbk$ be the residual field of A. We have $\Bbbk = \overline{\mathbb{F}}_2$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, we define $\lambda_i^< := \sum_{j<i} \lambda_j$ and $\lambda_i^> := \sum_{j>i} \lambda_j$ for each i. We also use the notation $\lambda_i^{\leqq} := \lambda_{i+1}^<$ and $\lambda_i^{\geqq} := \lambda_{i-1}^>$. We put $|\lambda| := (\lambda)_1^{\geqq}$. We denote the dual partition of λ by ${}^t\lambda$.

For a scheme $\mathcal{X}$ over A, we denote its specializations to $\Bbbk$ and $\mathbb{K}$ by $\mathcal{X}_{\Bbbk}$ and $\mathcal{X}_{\mathbb{K}}$, respectively. In addition, assume that $\mathcal{X}$ admits an action of a group scheme G over A. By a G-orbit on $\mathcal{X}$, we refer a flat subfamily $\mathcal{O}$ of $\mathcal{X}$ over A such that $\mathcal{O}_{\mathbb{K}}$ is a single $G_{\mathbb{K}}$-orbit. For a map of commutative rings $A \to D$, we define $\mathcal{X}(D)$ the set of D-valued points of $\mathcal{X}$. For each $e \in G(A)$, we denote by $\mathcal{X}^e$ the e-fixed part of $\mathcal{X}$. We denote by $H_{\bullet}(\mathcal{X}, \mathbb{C})$ the Borel-Moore homology of $\mathcal{X}_{\mathbb{C}}$.

We understand that the intersection $\cap$ of two (sub-)schemes are set-theoretic. (I.e. we consider the reduced part of the scheme-theoretic intersection.) The scheme-theoretic intersection is denoted by $\dot{\cap}$.

For a scheme $\mathcal{Y}$ over $\Bbbk$, we denote its (geometric) Frobenius endomorphism by Fr. Here geometric means that the induced map $\mathsf{Fr}^* : \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$ is $\Bbbk$-linear and (suitable) local coordinates are changed to its 2nd power.

1.2 Notation and Terminology

Let $G = Sp(2n, A)$ be the symplectic group of rank n over A. Let $B \supset T$ be its Borel subgroup and a maximal torus defined over A. Put $N := [B, B]$. Denote the opposite unipotent radical of N (with respect to T) by N^-. Let $W := N_G(T)/T$ be the Weyl group of G. We denote by $X^*(T)$ the weight lattice of T. Let R be the set of roots of (G, T) with its positive part R^+ determined by B. Consider an A-module $V_1 := A^{2n}$, for which G acts by the

multiplication of matrices. Let $V_2 := \wedge^2 V_1$ ($\subset \wedge^2 (V_1)_{\mathbb{C}}$) be the A-module with the natural G-module structure. Let $\mathfrak{g}$ be the Lie algebra of G over A, whose integral structure is $\mathsf{Sym}^2 V_1 = (V_1 \otimes_A V_1)^{\mathfrak{S}_2}$. Let $\mathfrak{b}, \mathfrak{t}, \mathfrak{n}$ be the intersections of Lie algebras corresponding to $B_{\mathbb{C}}, T_{\mathbb{C}}, N_{\mathbb{C}}$ with $\mathfrak{g}$ inside of $\mathfrak{g}_{\mathbb{C}}$, respectively.

Fix a $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_n$ of $X^*(T)$ such that $R^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\}_i \subset X^*(T)$. For each i, we put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i < n)$, $2\epsilon_n$ $(i = n)$. Let s_i be the reflection of W corresponding to α_i. Let $\ell : W \to \mathbb{Z}$ denote the length function on W with respect to $s_1, \ldots, s_n$.

We put $\mathbb{V} := V_1 \oplus V_2$. For a T-weight $\lambda \neq 0$, let $\mathbb{V}[\lambda]$ be the T-eigenpart of $\mathbb{V}$ with its weight λ. Let $\mathbb{V}^+$ be the sum of T-weight spaces of $\mathbb{V}$ with its weights in $\mathbb{Q}_{\geq 0} R^+ - \{0\}$. For a T-subrepresentation $V \subset \mathbb{V}$, we denote by $\Psi(V)$ the set of T-weights λ with $\mathbb{V}[\lambda] \neq \{0\}$.

For each $w \in W$, we denote (one of) its lift by $\dot{w} \in N_G(T)$. For a T-stable subset $\mathcal{S}$ in $\mathbb{V}$ or $\mathfrak{g}$, we define ${}^w \mathcal{S} := \dot{w}\mathcal{S}$.

We denote the flag variety G/B by $\mathcal{B}$.

Let $\mathfrak{N}$ be the G-subscheme of $\mathbb{V}$ defined by the positive degree part $A[\mathbb{V}]_+^G$ of $A[\mathbb{V}]^G$.

Let $\mathcal{N}$ be the space of ad-nilpotent elements of $\mathfrak{g}$.

Theorem 1.1 (Hesselink [Hes79]). *We have $\mathcal{N}(\mathbb{k}) = \bigcup_{g \in G(\mathbb{k})} \mathrm{Ad}(g)\mathfrak{n}(\mathbb{k})$.*

Theorem 1.1 implies that the natural map (the Springer resolution over A)

$$\mu : G \times^B \mathfrak{n} \longrightarrow \mathcal{N}$$

is surjective at the level of points.

For a $G_{\mathbb{k}}$-module V over $\mathbb{k}$, we define its Frobenius twist $V^{[1]}$ as the composition

$$G_{\mathbb{k}} \longrightarrow GL(V) \xrightarrow{\mathsf{Fr}} GL(V) \subset \mathrm{End}_{\mathbb{k}}(V).$$

2 Defining equations of $\mathfrak{N}$

Let $e \in T$ be an element such that $\epsilon_i(e) = c$ (for every $1 \leq i \leq n$), where $c \in A$ is an element with sufficiently high order after taking modulo two. In particular, we assume $Z_G(e) \cong GL(n, A)$, $V_1^e = \{0\}$, and $V_2^e \cong \mathrm{Mat}(n, A)$. Put $G_0 := Z_G(e)$. Consider a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \text{ and } \mathbb{V} = \mathbb{V}_{-2} \oplus \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \qquad (2.1)$$

determined by the eigenvalues of the action of e (indicated as subscript). Here $\mathfrak{g}_{\pm 2}$ and $\mathbb{V}_i$ $(-2 \leq i \leq 2)$ are G_0-modules. We have $\mathfrak{g}_0 \cong \mathbb{V}_0$ as G_0-modules. Let $\mathfrak{n}_0 := \mathrm{Lie} G_0 \cap \mathfrak{n}$, which we may regard as a subspace of $\mathbb{V}_0$. We define G_{-2}, G_2, N_0 to be the unipotent subgroups of G corresponding to $\mathfrak{g}_{-2}, \mathfrak{g}_2$, and $\mathfrak{n}_0$, respectively. We fix an identification $\mathfrak{S}_n = N_{G_0}(T)/T \cong \langle s_i; i < n \rangle \subset W$. We define

$$\mathbf{J} := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

We have $V_2 \cong \mathrm{Alt}(2n, A)$ as $GL(2n, A)$-modules. Hence, it restricts to a G-module isomorphism. Let Pf be the Pfaffian associated to $X = \{x_{ij}\}_{ij} = \{-x_{ji}\}_{ij} \in \mathrm{Alt}(2n)$. It is defined as

$$\mathsf{Pf}(X) = \frac{1}{n!} \sum_{\sigma} \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)}, \qquad (2.2)$$

where σ runs over all permutations of $\mathfrak{S}_{2n}$ such that $\sigma(2m-1) < \sigma(2m)$ for every $1 \le m \le n$. By using Pf, we define polynomials $1 = P_0, P_1, \ldots, P_n$ on V_2 as

$$\sum_{i=0}^{n} t^{n-i} P_i(X) = \mathsf{Pf}(t\mathbf{J} - X) \qquad (X \in V_2 \cong \mathrm{Alt}(2n)).$$

We have $P_i \in A[V_2]^G$ by

$$\mathsf{Pf}(t\mathbf{J} - X) = \det(g)\mathsf{Pf}(t\mathbf{J} - X) = \mathsf{Pf}(tg\mathbf{J}^t g - gX^t g) = \mathsf{Pf}(t\mathbf{J} - gX^t g)$$

for each $g \in G(\mathbb{K})$.

Proposition 2.1. *By means of the G-module isomorphism $V_2 \cong \mathrm{Alt}(2n)$, the variety $(\mathfrak{N} \cap V_2)$ is identified with the common zeros of $P_1, \ldots, P_n$.*

Proof. Under the isomorphism $V_2 \cong \mathrm{Alt}(2n)$, the subspace $\mathbb{V}_0 \subset V_2$ corresponds to

$$\mathrm{Alt}(2n)_0 = \{\{x_{ij}\} \in \mathrm{Alt}(2n); x_{ij} = 0 \text{ for } 1 \le i,j \le n \text{ or } n < i,j \le 2n\}.$$

Substituting them into the definition of Pfaffians, we deduce

$$\mathsf{Pf}(t\mathbf{J} - X) = (-1)^n \det(t 1_n - Y) \quad \text{where } X = \begin{pmatrix} 0 & -Y \\ {}^t Y & 0 \end{pmatrix}.$$

This implies

$$A[V_2]^G \supset A[P_i; i \ge 1] \cong A[\mathbb{V}[0]]^{\mathfrak{S}_n}$$

via the restriction map. By the Dadok-Kac classification [DK85] table 2, we know that

$$\mathbb{C}[V_2]^G \cong \mathbb{C}[\mathbb{V}_0]^{G_0} \cong \mathbb{C}[\mathbb{V}[0]]^{\mathfrak{S}_n}.$$

This implies that $\{P_i\}_{i=1}^n$ generates the ideal $\langle A[V_2]_+^G \rangle$ as desired. $\square$

Corollary 2.2. *We have $\mathfrak{N} \cong A[\mathbb{V}]/(P_i; 1 \le i \le n)$.*

Proof. It is clear from the isomorphism $\mathbb{C}[\mathbb{V}]^G \cong \mathbb{C}[V_2]^G$, which can be read off from the Dadok-Kac classification ([DK85] table 2). $\square$

For the later use, we prove a lemma.

Lemma 2.3. *Let $Y \in \mathrm{Alt}(n)$ and let $Z \in \mathrm{Mat}(n)$. Then, we have*

$$P_i \begin{pmatrix} Y & Z \\ -{}^t Z & 0 \end{pmatrix} = P_i \begin{pmatrix} 0 & Z \\ -{}^t Z & 0 \end{pmatrix}$$

for each i.

Proof. By the pigeon hole principle, if a Pfaffian term

$$\mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)}$$

in (2.2) satisfies $\sigma(2m-1), \sigma(2m) \le n$ for some m, then there exists m' such that $\sigma(2m'-1), \sigma(2m') > n$. Then, this term cannot contribute to $\mathsf{Pf}\begin{pmatrix} Y & Z \\ -{}^t Z & 0 \end{pmatrix}$, which implies the result. $\square$

7

3 Geometric construction of $\mathfrak{N}$

We retain the setting of the previous section.

Let $F := G \times^B \mathbb{V}^+$. Consider the map

$$\nu : F = G \times^B \mathbb{V}^+ \longrightarrow \mathbb{V}.$$

We denote the specialization of ν to $\mathbb{K}$ and $\Bbbk$ by $\nu_{\mathbb{K}}$ and $\nu_{\Bbbk}$, respectively. Since the fiber of ν is naturally isomorphic to a closed subscheme of a flag variety, ν is projective.

Lemma 3.1. *We have* $G(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2) = G(\mathbb{V}[0] \oplus \mathbb{V}^+) = \mathbb{V}$.

Proof. The subgroup $G_0 B \subset G$ is parabolic. It preserves $(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2) \subset \mathbb{V}$. Hence, $G(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)$ is a closed subscheme of $\mathbb{V}$. Thus, it suffices to show the corresponding statement over $\mathbb{C}$. It is well-known that

$$(G_0)_{\mathbb{C}}(\mathbb{V}_0 \cap (\mathbb{V}[0] \oplus \mathbb{V}^+))_{\mathbb{C}} = (\mathbb{V}_0)_{\mathbb{C}}.$$

Therefore, it suffices to show the map

$$\widetilde{\nu} : G \times^B (\mathbb{V}[0] \oplus \mathbb{V}^+) \longrightarrow \mathbb{V}$$

extending ν is surjective after specializing to $\mathbb{C}$. We have $\mathfrak{N}_{\mathbb{C}} = \mathrm{Im}\,\nu_{\mathbb{C}}$ by [K09] 1.2 1). By the proof of Proposition 2.1, we have $\mathbb{V}[0]_{\mathbb{C}} \cap \mathfrak{N}_{\mathbb{C}} = \{0\}$. Therefore, we deduce that

$$\dim \mathrm{Im}\,\widetilde{\nu}_{\mathbb{C}} \geq \dim \mathfrak{N}_{\mathbb{C}} + \dim \mathbb{V}[0]_{\mathbb{C}} = \dim \mathbb{V}_{\mathbb{C}}$$

by $\mathbb{V}[0]_{\mathbb{C}} \subset \mathrm{Im}\,\widetilde{\nu}_{\mathbb{C}}$. Since $\mathrm{Im}\,\widetilde{\nu}_{\mathbb{C}} \subset \mathbb{V}_{\mathbb{C}}$ is a closed subscheme, we conclude the result. $\square$

Corollary 3.2. *We have* $\mathfrak{N}(\Bbbk) = G(\Bbbk)\mathbb{V}^+(\Bbbk) \subset \mathbb{V}(\Bbbk)$. $\square$

Proof. By Lemma 3.1 and the fact that the map $\widetilde{\nu}$ (borrowed from the proof of Lemma 3.1) is flat over A, we deduce that $G(\Bbbk)(\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)(\Bbbk) = \mathbb{V}(\Bbbk)$. Since $\mathfrak{N}(\Bbbk)$ is $G(\Bbbk)$-stable, we have

$$\mathfrak{N}(\Bbbk) = G(\Bbbk)(\mathfrak{N} \cap \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)(\Bbbk).$$

By Lemma 2.3, we have

$$(\mathfrak{N} \cap \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)(\Bbbk) = (\mathfrak{N} \cap \mathbb{V}_0)(\Bbbk) \times (\mathbb{V}_1 \oplus \mathbb{V}_2)(\Bbbk).$$

By the description of the nilpotent cone of $\mathfrak{g}_0(\Bbbk) \cong \mathbb{V}_0(\Bbbk)$, we deduce

$$(\mathfrak{N} \cap \mathbb{V}_0)(\Bbbk) = G_0(\Bbbk)(\mathbb{V}_0 \cap \mathbb{V}^+)(\Bbbk).$$

Taking account into the fact that $\mathbb{V}_1$ and $\mathbb{V}_2$ are G_0-stable, we conclude the result. $\square$

Lemma 3.3. *The map ν is semi-small with respect to the stratification given by G-orbits. The same holds for $\nu_{\Bbbk}$ provided if $G(\Bbbk)\backslash\mathfrak{N}(\Bbbk) < \infty$.*

Proof. This is a straight-forward generalization of the results in [K09] §1.1. $\square$

Remark 3.4. By a result of Borho-MacPherson (c.f. [CG97] §8.9), our exotic Springer correspondence (a bijection between $G_\mathbb{C}$-orbits of $\mathfrak{N}_\mathbb{C}$ and irreducible representations of W) implies that $\nu_\mathbb{C}$ must be strictly semi-small. (Otherwise there must be some $G_\mathbb{C}$-orbit which does not correspond to an irreducible representation of W.)

Proposition 3.5. *The differentials* $dP_1, \ldots, dP_n$ *of the polynomials* $P_1, \ldots, P_n$ *are linearly independent up to codimension two subscheme of* $\mathfrak{N}$.

Proof. By [K09] 1.2 6), a non-dense orbit of $G_\mathbb{K}$ in $\mathfrak{N}_\mathbb{K}$ has codimension at least two. Hence it suffices to check the assertion for an open subset of $\mathfrak{N}_\Bbbk$. By Lemma 3.1, we can replace $\mathfrak{N}_\Bbbk$ with $\mathfrak{N}_\Bbbk \cap (\mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2)_\Bbbk$. By Lemma 2.3 and the proof of Proposition 2.1, it suffices to prove the assertion on the dense open $(G_0)_\Bbbk$-orbit of $(\mathfrak{N}_\Bbbk \cap (\mathbb{V}_0)_\Bbbk)$, which is the regular nilpotent orbit of $GL(n)_\Bbbk$. This is well-known (or is easily checked). $\square$

Corollary 3.6. *The scheme* $\mathfrak{N}$ *is regular in codimension one.*

Proof. The scheme $\mathfrak{N}$ is a complete intersection up to codimension two locus. $\square$

Proposition 3.7. *The scheme* $\mathfrak{N}$ *is Cohen-Macaulay.*

Proof. We have
$$A[\mathbb{V}[0]]^{\mathfrak{S}_n} \cong A[\mathbb{V}]^G = A[P_1, \ldots P_n].$$
As a consequence, we deduce that $A[\mathbb{V}[0]]$ is a free $A[\mathbb{V}[0]]^{\mathfrak{S}_n}$-module by the Pittie-Steinberg theorem. It follows that the multiplication map $A[\mathbb{V}/\mathbb{V}[0]] \otimes_A A[\mathbb{V}]^G \to A[\mathbb{V}]$ equip $A[\mathbb{V}]$ the structure of a free $A[\mathbb{V}/\mathbb{V}[0]] \otimes_A A[\mathbb{V}]^G$-module. We have $A[\mathfrak{N}] \cong A[\mathbb{V}] \otimes_{A[\mathbb{V}]^G} A$. This is a free $A[\mathbb{V}/\mathbb{V}[0]]$-module, which implies that $\mathfrak{N}$ is Cohen-Macaulay. $\square$

Corollary 3.8. *The scheme* $\mathfrak{N}$ *is flat over* A.

Proof. A free $A[\mathbb{V}/\mathbb{V}[0]]$-module is automatically flat over $A[\mathbb{V}/\mathbb{V}[0]]$. $\square$

Theorem 3.9. *The scheme* $\mathfrak{N}$ *is normal.*

Proof. By the Serre criterion and Propositions 3.5 and 3.7, it suffices to show that $\mathfrak{N}$ is integral. The intersection
$$\mathfrak{N} \dot{\cap} (V_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_2)$$
is integral since $\mathfrak{N} \dot{\cap} \mathbb{V}_0$ is so. Let $I_1 = (P_i)$ and $I_2 := (x_{ij} = 0; i, j > n)$ be ideals of $A[\mathbb{V}]$, where $\{x_{ij}\} \in \mathrm{Alt}(2n) \cong V_2$. Then, the ideal $I := I_0 + I_1$ is prime. The set of A-valued points $T(A) \cong (A^\times)^n$ is Zariski dense in T. By the Bruhat decomposition, it follows that $G(A)$ is Zariski dense in G. By Lemma 3.1, this implies that

$$I_1 = I_1 + \bigcap_{g \in G(A)} (g^* I_2) = \bigcap_{g \in G(A)} (I_1 + g^* I_2) = \bigcap_{g \in G(A)} g^* I$$
$$= \bigcap_{g \in G(A)} (g^* I \otimes_A \mathbb{K}) \cap A[\mathbb{V}] = \langle \mathbb{K}[\mathbb{V}]^G_+ \rangle \cap A[\mathbb{V}].$$

We have $(\nu_\mathbb{C})_* \mathcal{O}_{F_\mathbb{C}} \cong \mathbb{C}[\mathfrak{N}]$ by [K09] 1.2 and the Zariski main theorem. In particular, $\mathbb{C}[\mathfrak{N}]$ and its subring $\mathbb{K}[\mathfrak{N}]$ are integral. Therefore, the RHS is an ideal whose quotient does not contain a zero divisor. $\square$

Theorem 3.10. *The image of ν is equal to $\mathfrak{N}$.*

Proof. Since F is smooth over A, the A-algebra

$$B := \Gamma(\mathbb{V}, \nu_* \mathcal{O}_F) = \Gamma(F, \mathcal{O}_F)$$

is torsion-free over A. Hence, B is flat and integral over A.

We have $\mathbb{C} \otimes_A B \cong (\nu_{\mathbb{C}})_* \mathcal{O}_{F_{\mathbb{C}}} \cong \mathbb{C}[\mathfrak{N}]$. By the proof of Theorem 3.9, we have

$$\left\langle A[\mathbb{V}]_+^G \right\rangle = \left\langle \mathbb{C}[\mathbb{V}]_+^{G_{\mathbb{C}}} \right\rangle \cap A[\mathbb{V}].$$

In particular, the natural map $A[\mathbb{V}] \ni f \mapsto f.1 \in B$ factors through $A[\mathfrak{N}]$. Here we have $(\mathrm{Im}\nu)(\mathbb{C}) = \mathfrak{N}(\mathbb{C})$ and $(\mathrm{Im}\nu)(\Bbbk) = \mathfrak{N}(\Bbbk)$ as sets. This implies that B must be normal since $A[\mathfrak{N}]$ is so. Therefore, we conclude $B = A[\mathfrak{N}]$ as desired. $\qquad\qquad\square$

Definition 3.11 (Marked partitions)**.** A marked partition $\boldsymbol{\lambda} = (\lambda, a)$ is a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of n, together with a sequence $a = (a_1, a_2, \ldots)$ of integers such that:

1. $0 \leq a_k \leq \lambda_k$ for each k;

2. $a_k = 0$ if $\lambda_{k+1} = \lambda_k$;

3. $\lambda_p - \lambda_q > a_p - a_q > 0$ if $p < q$ and $a_p \neq 0 \neq a_q$.

Let $X = X_1 \oplus X_2 \in \mathfrak{N}$. Since $V_1^* \cong V_1$, we can regard $X_2 \in \mathrm{End}(V_1)$ via the embedding

$$X_2 \in V_2 \xrightarrow{\cong} \mathrm{Alt}(V_1) \subset (V_1) \boxtimes (V_1) \cong \mathrm{End}_A(V_1). \qquad (3.1)$$

Definition 3.12. Let $\boldsymbol{\lambda} = (\lambda, a)$ be a marked partition of n. An element $X \in \mathfrak{N}$ is said to have $\mathbb{K}$-invariant $\boldsymbol{\lambda}$ if the following conditions hold:

- X_2 is a nilpotent element with its Jordan type $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$;

- There exists a family of vectors $\{\xi(i)\}_{i \geq 0} \in (V_1)_{\mathbb{K}}$ such that:

  1. $\xi(i) \neq 0$ if and only if $a_i \neq 0$;
  2. $X_1 = \sum_j X_2^{\lambda_j - a_j} \xi(j)$;
  3. $X_2^{\lambda_i - 1} \xi(i) \neq 0$ and $X_2^{\lambda_i} \xi(i) = 0$ for each i such that $\xi(i) \neq 0$.

We say that $X \in \mathfrak{N}$ have $\Bbbk$-invariant $\boldsymbol{\lambda}$ if the same conditions hold by replacing every $\mathbb{K}$ by $\Bbbk$. The $\mathbb{K}$-invariant (resp. the $\Bbbk$-invariant) of X is denoted by $\boldsymbol{\lambda}(X_{\mathbb{K}}) = (\lambda(X_{\mathbb{K}}), a(X_{\mathbb{K}}))$ (resp. $\boldsymbol{\lambda}(X_{\Bbbk})$). We define $\mathcal{O}_{\boldsymbol{\lambda}}$ to be the locally closed subscheme of $\mathfrak{N}_{\mathbb{K}}$ whose $\mathbb{K}$-valued points shares the $\mathbb{K}$-invariant $\boldsymbol{\lambda}$.

It is standard that the $\mathbb{K}$-invariants and $\Bbbk$-invariants are invariants under the G-action.

Theorem 3.13 ([K09] Theorem 1.14)**.** *Two points $X, Y \in \mathfrak{N}_{\mathbb{K}}$ are $G_{\mathbb{K}}$-conjugate iff X and Y share the same $\mathbb{K}$-invariant. In particular, $(\mathcal{O}_{\boldsymbol{\lambda}})_{\mathbb{K}}$ is a single $G_{\mathbb{K}}$-orbit.* $\qquad\qquad\square$

4 A geometric family of nilcones

We assume the same setting as in the previous section. The $G_\Bbbk$-module $\mathfrak{g}_\Bbbk$ has a non-trivial B-eigenvector with its weight $\epsilon_1 + \epsilon_2$. This yields the following short exact sequence of $G_\Bbbk$-modules:

$$0 \longrightarrow (V_2)_\Bbbk \longrightarrow \mathfrak{g}_\Bbbk \longrightarrow (V_1)_\Bbbk^{[1]} \longrightarrow 0. \tag{4.1}$$

Thus, we have a $G_\Bbbk$-equivariant flat deformation

$$\pi_V : \mathcal{V} \longrightarrow \mathbb{A}_\Bbbk^1$$

of $\mathfrak{g}_\Bbbk$ such that $\pi_V^{-1}(0) \cong (V_1)_\Bbbk^{[1]} \oplus (V_2)_\Bbbk$ and $\pi_V^{-1}(x) \cong \mathfrak{g}_\Bbbk$ ($x \neq 0$) as $G_\Bbbk$-modules. Let $\mathcal{V}^+ \subset \mathcal{V}$ be its $B_\Bbbk$-equivariant smooth subfamily such that $\mathcal{V}^+ \cap \pi_V^{-1}(0) \cong \left(V_1^+\right)_\Bbbk^{[1]} \oplus (V_2^+)_\Bbbk$ and $\mathcal{V}^+ \cap \pi_V^{-1}(x) \cong \mathfrak{n}_\Bbbk$ ($x \neq 0$).

By (4.1) and the isomorphism $\mathsf{Sym}^2 V_1 \cong \mathfrak{g}$ over A, we have the following $G_\Bbbk$-equivariant map

$$ml : \mathbb{V}_\Bbbk = (V_1)_\Bbbk \oplus (V_2)_\Bbbk \ni (X_1 \oplus X_2) \mapsto \mathsf{Sym}^2 X_1 + X_2 \in \mathfrak{g}_\Bbbk.$$

This map is $G_\Bbbk$-equivariant and finite as a map between affine algebraic varieties. By restriction, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathbb{A}_\Bbbk^1 \times \mathbb{V}_\Bbbk^+ & \xrightarrow{\;\underline{ml}\;} & \mathcal{V}^+ \\
\big\uparrow & & \big\uparrow \\
\{0\} \times \mathbb{V}_\Bbbk^+ & \xrightarrow{\;\mathsf{F}_1\;} & \{0\} \times (V_1^+)_\Bbbk^{[1]} \oplus (V_2^+)_\Bbbk
\end{array} \qquad , \tag{4.2}$$

where the map $\underline{ml}$ is the natural prolongization of the map ml, and the map F_1 is the product of the Frobenius map of the first component of $\mathbb{V}_\Bbbk^+$ and the identity map of the second component of $\mathbb{V}_\Bbbk^+$.

Let $\mathcal{F} := (G_\Bbbk \times \mathbb{A}_\Bbbk^1) \times^{(B_\Bbbk \times \mathbb{A}_\Bbbk^1)} \mathcal{V}^+$. This is a flat family over $\mathbb{A}_\Bbbk^1$. We define $\mathcal{N}_S := (G_\Bbbk \times \mathbb{A}_\Bbbk^1)\mathcal{V}^+ \subset \mathcal{V}$. We define $\pi := \pi_V|_{\mathcal{N}_S}$. The diagram (4.2) gives a $G_\Bbbk$-equivariant commutative diagram

$$\begin{array}{ccccc}
\mathbb{A}_\Bbbk^1 \times F_\Bbbk & \xrightarrow{\;\widetilde{ml}\;} & \mathcal{F} & \longleftarrow & \{0\} \times F_\Bbbk \\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \nu_\Bbbk} \\
\mathbb{A}_\Bbbk^1 \times \mathfrak{N}_\Bbbk & \xrightarrow{\;ml\;} & \mathcal{N}_S & \longleftarrow & \{0\} \times \mathfrak{N}_\Bbbk
\end{array} \qquad . \tag{4.3}$$

Here the vertical arrows are defined as $(G_\Bbbk \times \mathbb{A}_\Bbbk^1)$- or $G_\Bbbk$-translations of (4.2) inside $\mathbb{A}_\Bbbk^1 \times \mathbb{V}_\Bbbk$, $\mathcal{V}$, or $\mathbb{V}_\Bbbk$, respectively. Thanks to the surjectivity of $\underline{ml}$ at (4.2) and Theorem 1.1 and Corollary 3.2, the map ml at (4.3) is surjective at the level of points. Since $\mathcal{F}$ is flat over $\mathbb{A}_\Bbbk^1$, it follows that $\mathcal{N}_S$ is flat over $\mathbb{A}_\Bbbk^1$. Let $\mathsf{m} : \mathfrak{N}_\Bbbk \longrightarrow \mathcal{N}_\Bbbk$ be the map obtained by the specialization of ml to the fiber at the point $\{1\} \in \mathbb{A}_\Bbbk^1$.

Theorem 4.1. *Each $G_\Bbbk$-orbit $\mathcal{O}$ of $\mathfrak{N}_\Bbbk$ extends to a flat family of $G_\Bbbk$-orbits in $\mathcal{N}_S$ with its general fiber isomorphic to $\mathsf{m}(\mathcal{O})$. Moreover, this yields a one-to-one correspondence between $G_\Bbbk$-orbits of $\mathfrak{N}_\Bbbk$ and $\mathcal{N}_\Bbbk$ which preserves the closure relations.*

Proof. The map m is a $G(\Bbbk)$-equivariant isomorphism at the level of $\Bbbk$-valued points. Hence, we have an equi-dimensional one-to-one correspondence between the $G_\Bbbk$-orbits of $\mathfrak{N}_\Bbbk$ and $\mathcal{N}_\Bbbk$.

We have an equi-dimensional family $\mathbb{O}_S := \mathsf{ml}(\mathbb{A}^1_\Bbbk \times \mathcal{O})$ for each $G_\Bbbk$-orbit $\mathcal{O}$. Here the map F_1 is finite. As a result, $\overline{\mathbb{O}_S}$ is an equi-dimensional family such that each fiber contains a unique dense open $G_\Bbbk$-orbit.

The number of $G_\Bbbk$-orbits of $\mathcal{N}_\Bbbk$ is finite by [Hes79]. It is clear that $\overline{\mathbb{O}_S}$ is irreducible. The family $\mathcal{N}_S$ is locally trivial along $\mathbb{A}^1 \backslash \{0\}$. It follows that the open dense $G_\Bbbk$-orbit of $\overline{\mathbb{O}_S}$ is constant along $\mathbb{A}^1 \backslash \{0\}$.

Conversely, each $G_\Bbbk$-orbit $\mathbb{O} \subset \mathcal{N}_\Bbbk$ determines a $G_\Bbbk$-orbit $\mathcal{O}' \subset \mathsf{F}_1(\mathfrak{N})$ via

$$\mathcal{O}' \overset{\text{open}}{\hookrightarrow} \pi^{-1}(0) \cap \overline{\mathbb{O}'_S} \subset \overline{\mathbb{O}'_S}, \tag{4.4}$$

where $\mathbb{O}'_S$ is a family of single $G_\Bbbk$-orbits $\mathbb{O}$ in $\mathcal{N}_S$ along $\mathbb{A}^1 \backslash \{0\}$. Here $\mathcal{O}'$ is uniquely determined since $\overline{\mathbb{O}'_S} = \overline{\mathsf{ml}(\mathbb{A}^1 \times \mathcal{O}'')}$ for some $G_\Bbbk$-orbit $\mathcal{O}''$ of $\mathfrak{N}_\Bbbk$.

Since $\mathcal{O}' = \mathsf{F}_1(\mathcal{O}'')$, this establishes a one-to-one correspondences between $G_\Bbbk$-orbits in $\mathfrak{N}_\Bbbk$, that of $\mathcal{N}_\Bbbk$, and families of single $G_\Bbbk$-orbits of $\mathcal{N}_S$, which is compatible with the bijection induced by m. Now the closure relation of $\mathfrak{N}_\Bbbk$ extends to the closure relation of $\mathbb{O}_S$, which guarantees that the closure relation is preserved by m.

Since $\mathbb{A}^1_\Bbbk$ is one-dimensional, $\overline{\mathbb{O}_S}$ yields the required flat family. $\qquad\square$

In the below, we denote the one-to-one correspondence from the set of $G_\Bbbk$-orbits of $\mathfrak{N}_\Bbbk$ to that of $\mathcal{N}_\Bbbk$ described in Theorem 4.1 by df.

Corollary 4.2. *The numbers of $G_\Bbbk$-orbits in $\mathcal{N}_\Bbbk$ and $\mathfrak{N}_\Bbbk$ are equal to the number of $G_\Bbbk$-orbits in $\mathfrak{N}_\Bbbk$.*

Proof. By Spaltenstein [Spa83] 3.9, we deduce that the number of $G_\Bbbk$-orbits in $\mathcal{N}_\Bbbk$ is equal to the set of bi-partitions of n. Hence, Theorem 3.13 implies the result (see also Theorem 5.1). $\qquad\square$

Corollary 4.3. *Two $G_\Bbbk$-orbits in $\mathfrak{N}_\Bbbk$ are equal if and only if their $\Bbbk$-invariants are equal.* $\qquad\square$

5 Special elements of Weyl groups

Keep the setting of the previous section. In this section, we always work over an algebraically closed field and drop the specific field from the notation.

Theorem 5.1 ([K09] Theorem 1.14 and Proof of Proposition 1.16)**.** *Let $\boldsymbol{\lambda} = (\lambda, a)$ be a marked partition of n. We define a sequence of integers $b = (b_1, b_2, \ldots)$ as*

$$b_i := \begin{cases} a_i & (a_i \neq 0) \\ \max\{\{a_j + \lambda_i - \lambda_j; j < i\} \cup \{a_j; j \geq i\}\} & (a_i = 0) \end{cases}.$$

Then, we define two partitions $\mu^{\boldsymbol{\lambda}}$ and $\nu^{\boldsymbol{\lambda}}$ as

$$\mu_i^{\boldsymbol{\lambda}} = b_i, \nu_i^{\boldsymbol{\lambda}} = \lambda_i - b_i.$$

The pair $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ gives a bi-partition of n. Moreover, this assignment establishes a bijection between the set of marked partitions of n and the set of bi-partitions of n. $\qquad\square$

We refer the bi-partition $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ the associated bi-partition of a marked partition $\boldsymbol{\lambda}$. Let $\mathcal{O}_{\boldsymbol{\lambda}}$ be the G-orbit with its $\mathbb{K}$-invariant (or $\Bbbk$-invariant, depending on the base field) $\boldsymbol{\lambda}$.

Definition 5.2 (Special elements)**.** Let $\boldsymbol{\lambda}$ be a marked partition and let $(\mu, \nu) := (\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ be its associated bi-partition. We define an element $w_{\boldsymbol{\lambda}} \in W$ as

$$
w_{\boldsymbol{\lambda}} \epsilon_i = \begin{cases} \epsilon_{n-m+1} & (i = ({}^t\mu)^{\geq}_{m}) \\ -\epsilon_{|\nu| + ({}^t\mu)^{\leq}_{m} + i - ({}^t\mu)^{\geq}_{m} - m + 1} & (({}^t\mu)^{>}_{m} < i < ({}^t\mu)^{\geq}_{m}) \\ -\epsilon_{({}^t\nu)^{\geq}_{m} + i - ({}^t\nu)^{\leq}_{m} - |\mu|} & (|\mu| + ({}^t\nu)^{<}_{m} < i \leq |\mu| + ({}^t\nu)^{\leq}_{m}) \end{cases},
$$

where m is some natural number. We put $\mathbb{V}^{\boldsymbol{\lambda}} := \mathbb{V}^{+} \cap {}^{w_{\boldsymbol{\lambda}}^{-1}}\mathbb{V}^{+}$. We define a sequence of integers $d^{\boldsymbol{\lambda}} := \{d_i^{\boldsymbol{\lambda}}\}_{i=0}^{\mu_1 + \nu_1}$ as $d_0^{\boldsymbol{\lambda}} := 0$ and

$$
d_1^{\boldsymbol{\lambda}} := ({}^t\mu)^{\geq}_{\mu_1}, \ldots, d_{\mu_1}^{\boldsymbol{\lambda}} := ({}^t\mu)^{\geq}_{1}, d_{\mu_1+1}^{\boldsymbol{\lambda}} := |\mu| + ({}^t\nu)^{\leq}_{1}, \ldots, d_{\mu_1+\nu_1}^{\boldsymbol{\lambda}} := |\mu| + ({}^t\nu)^{\leq}_{\nu_1}.
$$

We may drop the superscript $^{\boldsymbol{\lambda}}$ if the meaning is clear from the context.

Lemma 5.3. *Keep the setting of Definition 5.2.*

1. *We have $\epsilon_i - \epsilon_j \notin \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \geq j$ or one of the following conditions hold for some natural number m:*

   (a) $({}^t\mu)^{>}_{m} < i, j < ({}^t\mu)^{\geq}_{m}$;
   (b) $|\mu| + ({}^t\nu)^{<}_{m} < i, j \leq |\mu| + ({}^t\nu)^{\leq}_{m}$;
   (c) $j = ({}^t\mu)^{\geq}_{m}$, $1 \leq i \leq |\mu|$, and $i \notin \{({}^t\mu)^{\geq}_{l}\}_{l>m}$.

2. *We have $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \neq j$ and $i, j \in \{({}^t\mu)^{\geq}_{m}\}_m$;*

3. *We have $\epsilon_i \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ if and only if $i \in \{({}^t\mu)^{\geq}_{m}\}_m$.*

Proof. Straight-forward. $\qquad\square$

For each $\alpha \in R^{+}$, we denote the corresponding unipotent one-parameter subgroup of G by $U_\alpha \subset N$.

Lemma 5.4. *Keep the setting of Definition 5.2. We have*

$$
\mathbb{V}^{\boldsymbol{\lambda}} \subset \overline{B(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))},
$$

where $\mathbb{V}_0 \subset V_2$ and $\mathbb{V}_1 \subset V_1$ are G_0-stable subspaces defined at (2.1).

Proof. We put $\gamma_m := \epsilon_{({}^t\mu)^{\geq}_{m}}$. By Lemma 5.3, we have $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ only if $i, j \in \{({}^t\mu)^{\geq}_{m}\}_m$. Moreover, $i < j$ and $\epsilon_i + \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$ implies $\epsilon_i - \epsilon_j \in \Psi(\mathbb{V}^{\boldsymbol{\lambda}})$. We put

$$
U_{+} := \prod_{l \leq m} U_{\gamma_l + \gamma_m}, \mathbb{V}_0^{\circ} := \bigoplus_{l > m} \mathbb{V}[\gamma_l - \gamma_m], \text{ and } \mathbb{V}_2^{\circ} := \bigoplus_{l < m} \mathbb{V}[\gamma_l + \gamma_m].
$$

We have $\mathbb{V}_0^{\circ}, \mathbb{V}_2^{\circ} \subset \mathbb{V}^{\boldsymbol{\lambda}}$. By a weight comparison, we deduce

$$
U_{+}\mathbb{V}^{\boldsymbol{\lambda}} = U_{+}((\mathbb{V}_0^{\circ} + \mathbb{V}_2^{\circ}) \cap \mathbb{V}^{\boldsymbol{\lambda}}) + \mathbb{V}^{\boldsymbol{\lambda}}.
$$

Here $U_{+}\mathbb{V}_0^{\circ} \subset \mathbb{V}_0^{\circ} + \mathbb{V}_2^{\circ}$ is a dense open subset. Thus, we conclude

$$
\mathbb{V}^{\boldsymbol{\lambda}} \subset \overline{U_{+}(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))} \subset \overline{B(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))}
$$

as desired. $\qquad\square$

Lemma 5.5. *Keep the setting of Definition 5.2. We define*

$$\mathbb{V}_{01}^{\boldsymbol{\lambda}} := \bigoplus_{i \le d_{\mu_1}} \mathbb{V}[\epsilon_i] \oplus \bigoplus_{l < m} \bigoplus_{\substack{d_l \, < \, i \, \le \, d_{l+1} \\ d_m \, < \, j \, \le \, d_{m+1}}} \mathbb{V}[\epsilon_i - \epsilon_j].$$

Then, we have $\overline{B\mathbb{V}_{01}^{\boldsymbol{\lambda}}} = \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}$.

Proof. We put $\Psi := \Psi(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))$. Since $\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1) \subset \mathbb{V}_{01}^{\boldsymbol{\lambda}}$, it suffices to prove the inclusion

$$\mathbb{V}_{01}^{\boldsymbol{\lambda}} \subset \overline{TN_0(\mathbb{V}^{\boldsymbol{\lambda}} \cap (\mathbb{V}_0 \oplus \mathbb{V}_1))} \subset \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}.$$

Here the second inclusion is obvious. We have

$$\Psi(\mathbb{V}_{01}^{\boldsymbol{\lambda}}) \backslash \Psi = \{\epsilon_i; i \in [1, d_{\mu_1}] \backslash \{(^t\mu)_m^{\ge}\}_m\} \cup \{\epsilon_i - \epsilon_j; i < j, i \notin \{(^t\mu)_m^{\ge}\}_m \ni j\}.$$

We deduce that

$$\left(\prod_{i < |\mu|} U_{\epsilon_i - \epsilon_{|\mu|}} \right) \mathbb{V}[\epsilon_{|\mu|}] \subset \mathbb{V}_{01}^{\boldsymbol{\lambda}} \cap \mathbb{V}_1$$

is a dense open subset by the comparison of weights. We put

$$U_- := \prod_{i < j; j \in \{(^t\mu)_m^{\ge}\}_m} U_{\epsilon_i - \epsilon_j} \subset N_0.$$

It is easy to see that U_- does not depend on the order of the product. By the comparison of weights, we have a dense open subset

$$U_-(\mathbb{V}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0) \subset \mathbb{V}_{01}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0.$$

Applying the T-action, we conclude that the first inclusion is dense. $\qquad\square$

In the below, we denote by $\mathbb{V}_i^{\boldsymbol{\lambda}}$ $(i = 0, 1)$ the spaces $\mathbb{V}_{01}^{\boldsymbol{\lambda}} \cap \mathbb{V}_i$ coming from the statement of Lemma 5.5 for each marked partition $\boldsymbol{\lambda}$.

Proposition 5.6. *Let $\boldsymbol{\lambda}$ be a marked partition. We have* $\overline{G\mathbb{V}^{\boldsymbol{\lambda}}} = \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$.

Proof. By Definition 3.12 and Lemma 5.3, we deduce that $\mathcal{O}_{\boldsymbol{\lambda}} \subset G\mathbb{V}^{\boldsymbol{\lambda}}$. Thus, it suffices to check $\mathbb{V}_{01}^{\boldsymbol{\lambda}} \subset \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$. We define an increasing filtration

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{\mu_1 + \nu_1 - 1} \subsetneq F_{\mu_1 + \nu_1} = V_1^+ := \mathbb{V}^+ \cap V_1$$

as $F_k := \bigoplus_{i \le d_k} \mathbb{V}[\epsilon_i]$. By a weight comparison, each $x \in \mathbb{V}_0^{\boldsymbol{\lambda}}$ preserves the flag $\{F_k\}_k$ when regarded as an element of $\mathrm{End}(V_1)$ as in (3.1). Moreover, the set of elements x in $\mathbb{V}_0^{\boldsymbol{\lambda}}$ which satisfies

$$\dim x^l F_k / F_{k-l-1} = \min\{\dim F_{k-m+1}/F_{k-m}\}_{m=1}^l \text{ for every } l$$

is dense in $\mathbb{V}_0^{\boldsymbol{\lambda}}$. Let $\xi \in F_{\mu_1} \cap V_1^+$ be an element such that there exists $\{\xi(i)\}_i \subset V_1^+$ which satisfies

$$\xi = \sum_{i \ge 1} x^{\lambda_i - b_i} \xi(i), \text{ and } x^{\lambda_i - 1} \xi(i) \ne 0 = x^{\lambda_i} \xi(i) \text{ if } \xi(i) \ne 0.$$

14

Under the above choice of x, this condition is an open condition. We rearrange $\{\xi(i)\}$ according to the following rules: If $b_i = a_j$ for some $j < i$, then we rearrange $\xi(i), \xi(j)$ as $0, x^{\lambda_j - \lambda_i} \xi(i) + \xi(j)$, and let others unchanged. If $b_i = a_j - \lambda_j + \lambda_i$ for $i > j$, then we rearrange $\xi(i), \xi(j)$ with $0, \xi(i) + \xi(j)$, and let others unchanged.

We repeat this procedure for all possible pairs (i, j). We conclude that if we have $\xi(i) \neq 0$ for every i such that $a_i \neq 0$ after applying these procedures, then $\xi \oplus x$ has $\mathbb{K}$-invariant (λ, a). Since the former is an open condition on ξ, we conclude the result. $\qquad\square$

Corollary 5.7. *Under the setting of Proposition 5.6, we have*

$$\dim \mathcal{O}_{\boldsymbol{\lambda}} = \dim \mathcal{O}_{(\lambda, \mathbf{0})} + 2 \, |\mu| \, .$$

Proof. We retain the setting of the proof of Proposition 5.6. Let $\xi \in V_1^+$. Then, we have $\xi \oplus x \in \overline{\mathcal{O}_{\boldsymbol{\lambda}}}$ if and only if $\xi \in \bigoplus_{i \leq |\mu|} V_1[\epsilon_i]$. The Jordan type of x is λ (unchanged) if we regard $x \in \mathrm{End}(V_1)$ as either $x \in \mathrm{End}(\mathbb{V}_1)$ or $x \in \mathrm{End}(\mathbb{V}_{-1})$. Therefore, the fiber of the projection $\mathcal{O}_{\boldsymbol{\lambda}} \to \mathcal{O}_{(\lambda, \mathbf{0})}$ has dimension $2 \dim F_{\mu_1} = 2 \, |\mu|$ as desired. $\qquad\square$

6 Normality of nilcones in characteristic two

We assume the setting of §4.

Theorem 6.1. *For each $G_{\mathbb{k}}$-orbit $\mathcal{O}_{\mathbb{k}}$ of $\mathfrak{N}_{\mathbb{k}}$, there exists a G-orbit $\mathcal{O}$ of $\mathfrak{N}$ such that $\mathcal{O} \otimes \mathbb{k} = \mathcal{O}_{\mathbb{k}}$.*

Proof. Assume that $\mathcal{O}_{\mathbb{k}}$ has $\mathbb{k}$-invariant $\boldsymbol{\lambda}$. Let $\mathcal{O}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit of $\mathfrak{N}_{\mathbb{K}}$ with $\mathbb{K}$-invariant $\boldsymbol{\lambda}$. By Proposition 5.6 and Lemma 5.5, we have $\overline{\mathcal{O}_{\mathbb{K}}} = \overline{G \mathbb{V}_{01}^{\boldsymbol{\lambda}}} \subset \mathbb{V}$ over A. Since $\overline{\mathcal{O}_{\mathbb{K}}}$ is irreducible and dominates $\mathrm{Spec}\, A$, it is flat over A. Therefore, $\overline{\mathcal{O}_{\mathbb{K}}}$ is equi-dimensional over A. We have $B \mathbb{V}_{01}^{\boldsymbol{\lambda}} = G_2 \mathbb{V}_{01}^{\boldsymbol{\lambda}}$. For some (and in fact generic) $X \in \mathbb{V}_{01}^{\boldsymbol{\lambda}}(A)$ such that $X_{\mathbb{K}}$ has $\mathbb{K}$-invariant $\boldsymbol{\lambda}$ and $X_{\mathbb{k}}$ has $\mathbb{k}$-invariant $\boldsymbol{\lambda}$, we have $(\overline{\mathsf{Stab}_{(G_2)_{\mathbb{K}}} X_{\mathbb{K}}})_{\mathbb{k}} = \mathsf{Stab}_{(G_2)_{\mathbb{k}}} X_{\mathbb{k}}$ by identifying defining equations, which are all linear. Applying the G_0-action, we deduce that $B \mathbb{V}_{01}^{\boldsymbol{\lambda}}$ contains a dense open $\mathbb{G}_m$-stable subvariety, which is flat over A. Therefore, we conclude $\overline{B \mathbb{V}_{01}^{\boldsymbol{\lambda}}} \otimes \mathbb{k} = \overline{B_{\mathbb{k}} (\mathbb{V}_{01}^{\boldsymbol{\lambda}})_{\mathbb{k}}}$ via the equality of the leading terms of Hilbert polynomials of $\mathbb{k}[\mathbb{V}_{\mathbb{k}}^+]$-modules. By Proposition 5.6, we have $\overline{\mathcal{O}_{\mathbb{K}}} \otimes \mathbb{k} = \overline{\mathcal{O}_{\mathbb{k}}}$ and the latter is irreducible. Hence, setting

$$\mathcal{O} := \overline{\mathcal{O}_{\mathbb{K}}} - \bigcup_{\mathcal{O}': G_{\mathbb{K}}\text{-orbit}; \overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}}} \overline{\mathcal{O}'_{\mathbb{K}}}$$

yields the result. $\qquad\square$

Corollary 6.2. *The variety $\mathfrak{N}_{\mathbb{k}}$ is normal.*

Proof. Since $\mathfrak{N}$ is Cohen-Macauley and flat over A, it suffices to prove that $\mathfrak{N}_{\mathbb{k}}$ is regular in codimension one. By Theorem 6.1, it follows that every non-dense $G_{\mathbb{k}}$-orbit in $\mathfrak{N}_{\mathbb{k}}$ is codimension at least two. Therefore, we deduce the result. $\qquad\square$

Corollary 6.3. *The map $\nu_{\mathbb{k}}$ is strictly semi-small with respect to the stratification given by $G_{\mathbb{k}}$-orbits.*

Proof. For each G-orbit $\mathcal{O}$ of $\mathfrak{N}$, we choose $X \in \mathcal{O}(A)$. Then, the upper-semicontinuity of fiber dimensions of $\nu^{-1}(X)$ along A implies the strictness of $\nu_{\Bbbk}$ from that of $\nu_{\mathbb{K}}$. $\qquad\square$

Theorem 6.4. *The variety $\mathcal{N}_{\Bbbk}$ is normal.*

Proof. The isogeny $ml : \mathbb{V} \mapsto \mathfrak{g}$ induces an injective map $\Bbbk[\mathfrak{g}_{\Bbbk}] \hookrightarrow \Bbbk[\mathbb{V}_{\Bbbk}]$. Since this map is $G(\Bbbk)$-equivariant, we have an inclusion $\Bbbk[\mathfrak{g}_{\Bbbk}]^G \hookrightarrow \Bbbk[\mathbb{V}_{\Bbbk}]^G$. The variety $\mathcal{N}_{\Bbbk}$ is defined from $\mathcal{N}$ by reduction modulo 2. Hence, its defining equations are coming from $A[\mathfrak{g}_A]^G$, which are given by polynomials of degree $2, 4, \ldots, 2n$ (c.f. [Bou02]). In this proof, we understand that $\epsilon_{-i} = -\epsilon_i$ for every $1 \le i \le n$. We set $\{a_{ij}\}_{ij} \in \mathrm{Alt}(V_1)_{\Bbbk} \cong (V_2)_{\Bbbk}$ as the coordinates with respect to the T-eigenbasis so that a_{ij} has eigenvalue $\epsilon_i + \epsilon_j$. Let $\{v_i\}_i \in (V_1)_{\Bbbk}$ be the T-eigenbasis such that v_i is of weight ϵ_i. Let c_{ij} be the T-eigenbasis of $\mathfrak{g}_{\Bbbk}$ of weight $\epsilon_i + \epsilon_j$. We set $c_i := c_{ii}$ and $\bar{c}_i = c_{-i,-i}$. We have

$$ml^*(c_{ij}) = a_{ij} + v_i v_j \ \ (i \ne j) \ \ \text{or} \ \ v_i^2 \ \ (i = j).$$

Here we have $P_k(a_{ij}^2) \in \Bbbk[\mathfrak{g}_{\Bbbk}] \cap \Bbbk[\mathbb{V}_{\Bbbk}]^G$. By explicit calculation, we have

$$P_k(a_{ij}^2) = \sum_{I \subset [1,n]; \#I=k} \prod_{i \in I} c_i \bar{c}_i + \text{ lower terms with respect to } \{c_i\}.$$

It follows that the differentials of $P_k(a_{ij}^2)$ with respect to $c_1, \ldots, c_n$ defines a collection of linear independent differentials along the generic point, regardless the values of a_{ij}.

By degree counting using the inclusions $\Bbbk[\mathbb{V}_{\Bbbk}^{[1]}] \subset \Bbbk[\mathfrak{g}_{\Bbbk}] \subset \Bbbk[\mathbb{V}_{\Bbbk}]$, these are precisely the defining equations of $\mathcal{N}_{\Bbbk}$ embedded into $\Bbbk[\mathbb{V}_{\Bbbk}]$. The union of non-dense orbit of $\mathcal{N}_{\Bbbk}$ has codimension two (c.f. [Hes79] or Theorem 4.1). Therefore, the defining equations of $\mathcal{N}_{\Bbbk}$ defines a set of linearly independent differentials up to codimension two. In conclusion, the same proof as the normality of $\mathfrak{N}_{\Bbbk}$ implies the result. $\qquad\square$

Remark 6.5. The normality of the nilpotent cone of $\mathfrak{sp}(2n)$ over a field of characteristic $\ne 2$ is well-known. (See eg. Brion-Kumar [BK04] §5.)

7 Exotic orbital varieties: statement

In this section, the term "flat" means that the object is a flat scheme over $\mathrm{Spec} A$.

Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. We denote the set of irreducible components of $\mathcal{O} \cap \mathbb{V}^+$ by $\mathrm{Comp}(\mathcal{O})$. We define $\mathrm{Comp}(\mathbb{O})$ for a G-orbit of $\mathbb{O} \subset \mathcal{N}$ by replacing $\mathbb{V}^+$ with $\mathfrak{n}$.

Similarly, we denote the set of irreducible components of $\mathcal{O}_* \cap \mathbb{V}_*^+$ (or $\mathbb{O}_* \cap \mathfrak{n}_*$) by $\mathrm{Comp}(\mathcal{O}_*)$ or $\mathrm{Comp}(\mathbb{O}_*)$ for $* = \mathbb{C}, \mathbb{K}, \Bbbk$.

An element of $\mathrm{Comp}(\mathcal{O})$ or $\mathrm{Comp}(\mathbb{O})$ is called an orbital variety.

Theorem 7.1. *Let $\mathcal{O}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit. Let $\mathfrak{X}_{\mathbb{K}} \in \mathrm{Comp}(\mathcal{O}_{\mathbb{K}})$. Then, we have*

1. *$\dim \mathfrak{X}_{\mathbb{K}} = \frac{1}{2} \dim \mathcal{O}_{\mathbb{K}}$;*

2. *There exists $w \in W$ such that $\overline{\mathfrak{X}_{\mathbb{K}}} = \overline{B_{\mathbb{K}}(\mathbb{V}_{\mathbb{K}}^+ \cap {}^w \mathbb{V}_{\mathbb{K}}^+)}$.*

Moreover, the same statements hold when we replace $\mathbb{K}$ with $\Bbbk$.

Proof. Postponed to §8. ☐

Remark 7.2. **1)** Theorem 7.1 is an "exotic" analogue of Joseph's version of the Steinberg-Spaltenstein theorem.
2) If the variety $G \times^B \mathbb{V}_{\mathbb{C}}^+$ or $\mathcal{O}_{\mathbb{C}}$ admits a symplectic structure, then Theorem 7.1 follows from Kaledin [Kal06] or [CG97]. However, there exists no G-invariant holomorphic symplectic form on both of them. We do not know whether it exists when we drop the invariance.

Let $Z := F \times_{\mathfrak{N}} F$ and let $\mathsf{p} : Z \to \mathfrak{N}$ be the projection. The following result is a straight-forward generalization of [K09] §1.2 or [Ste74]:

Theorem 7.3 (Steinberg). *The variety Z is flat of relative dimension* $\dim \mathfrak{N}$. *Moreover, it consists of #W irreducible components.* ☐

Lemma 7.4. *Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. Then, $\mathbb{O}_{\Bbbk}$ is a union of a single $G_{\Bbbk}$-orbit $\mathbb{O}'_{\Bbbk}$ of dimension* $\dim \mathbb{O}$ *and $G_{\Bbbk}$-orbits of dimension* $< \dim \mathbb{O}$.

Proof. Consider the natural embedding $\iota : \mathcal{N}_{\mathbb{K}} \subset \mathrm{Mat}(2n)_{\mathbb{K}}$. It is well-known that the induced map $G_{\mathbb{K}} \backslash \mathcal{N}_{\mathbb{K}} \hookrightarrow GL(2n)_{\mathbb{K}} \backslash \mathrm{Mat}(2n)_{\mathbb{K}}$ is injective. (See eg. Tanisaki [Tan85] P152 for this kind of phenomenon.) Hence, $\mathbb{O}_{\Bbbk}$ is a union of $G_{\Bbbk}$-orbits with the same Jordan normal form (in $\mathrm{Mat}(2n, \Bbbk)$.) By Hesselink [Hes79], the maximal dimension of $G_{\Bbbk}$-orbits in $\mathbb{O}_{\Bbbk}$ is attained by a unique orbit as desired. ☐

Definition 7.5. Let $X \in \mathfrak{N}$. Then, we define a subscheme

$$\mathcal{G}_X := \{g \in G; X \in g\mathbb{V}^+\} \subset G.$$

It is clear that $(\mathcal{G}_X)_{\mathbb{K}}$ admits a free left $\mathrm{Stab}_G(X)_{\mathbb{K}}$-action and a free right $B_{\mathbb{K}}$-action. The same statement holds if we replace $\mathbb{K}$ with $\Bbbk$.

Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. For each $X \in \mathcal{O}$, we define

$$\mathcal{E}_X := \mathcal{G}_X/B \subset \mathcal{B}$$

and call it the (exotic) Springer fiber along X. By taking conjugation, we know that $(\mathcal{E}_X)_{\mathbb{K}} \cong (\mathcal{E}_Y)_{\mathbb{K}}$ and $(\mathcal{E}_X)_{\Bbbk} \cong (\mathcal{E}_Y)_{\Bbbk}$ hold if $X, Y \in \mathcal{O}(A)$.

Lemma 7.6. *Keep the setting of Definition 7.5. Let $\mathcal{O}$ be a G-orbit such that $X \in \mathcal{O}(\mathbb{K})$. Let $\{\mathcal{G}_X^i\}_i$ be the set of irreducible components of $(\mathcal{G}_X)_{\mathbb{K}}$. Then, the assignment*

$$\mathrm{Comp}(\mathcal{O}_{\mathbb{K}}) \ni \mathcal{G}_X^i X \mapsto \mathcal{G}_X^i/B_{\mathbb{K}} \subset (\mathcal{E}_X)_{\mathbb{K}}$$

establishes one-to-one correspondences between the sets of irreducible components of $\mathcal{O}_{\mathbb{K}} \cap \mathbb{V}_{\mathbb{K}}^+, (\mathcal{G}_X)_{\mathbb{K}}$, and $(\mathcal{E}_X)_{\mathbb{K}}$. The same statement holds if we replace $\mathbb{K}$ with $\Bbbk$.

Proof. The assignments $\mathcal{G}_X^i \mapsto \mathcal{G}_X^i/\mathrm{Stab}_G(X)_{\mathbb{K}} \cong \mathcal{G}_X^i X \in \mathrm{Comp}(\mathcal{O}_{\mathbb{K}})$ and $\mathcal{G}_X^i \mapsto \mathcal{G}_X^i/B_{\mathbb{K}}$ gives a surjection from the set of irreducible components of $(\mathcal{G}_X)_{\mathbb{K}}$ and the other two sets. Hence, these assignments fail to be bijective only if $\mathrm{Stab}_G(X)_{\mathbb{K}}$ or $B_{\mathbb{K}}$ is not connected. The group $\mathrm{Stab}_G(X)_{\mathbb{K}}$ is connected by [K09] Proposition 4.5. The Borel subgroup $B_{\mathbb{K}} \subset G_{\mathbb{K}}$ is clearly connected. Entirely the same proof works for $\Bbbk$ (including [K09] Proposition 4.5). ☐

17

Let $N(\mathfrak{O})$ be the number of orbital varieties attached to some orbit $\mathfrak{O}$. In the rest of this section, we assume

($\spadesuit$)$_1$ We have $\sum_{\mathcal{O}_{\mathbb{K}} \in G_{\mathbb{K}} \backslash \mathfrak{N}_{\mathbb{K}}} N(\mathcal{O}_{\mathbb{K}})^2 = \#W$.

($\spadesuit$)$_2$ There are at least $N(\mathcal{O}_{\mathbb{K}})$ orbital varieties of $\mathcal{O}_{\mathbb{k}}$ contained in the closures of $\mathrm{Comp}(\mathcal{O}_{\mathbb{K}})$ for every G-orbit of $\mathfrak{N}$.

The statements ($\spadesuit$)$_1$, ($\spadesuit$)$_2$ themselves are proved at Corollary 9.4 and Corollary 9.5, respectively. (The results presented in the rest of this section is used only in the proof of Theorem 9.6.)

Proposition 7.7. *Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. Then, every element of $\mathrm{Comp}(\mathcal{O})$ is flat with relative dimension $\frac{1}{2} \dim \mathcal{O}_{\mathbb{K}}$.*

Proof. Let $\mathfrak{Y} \in \mathrm{Comp}(\mathcal{O})$. If $\mathfrak{Y}$ dominates $\mathrm{Spec}A$, then it is flat since A is one-dimensional. Hence, we assume that $\mathfrak{Y}(\mathbb{K}) = \emptyset$ in order to deduce contradiction. By Theorem 7.1, we have $\mathfrak{Y}(\mathbb{k}) \subset \mathfrak{X}_{\mathbb{k}}(\mathbb{k})$ for some $\mathfrak{X}_{\mathbb{k}} \in \mathrm{Comp}(\mathcal{O}_{\mathbb{k}})$. Since $\mathcal{O}_{\mathbb{k}}$ is an single $G_{\mathbb{k}}$-orbit, we can further assume $\mathfrak{Y} = \mathfrak{X}_{\mathbb{k}}$. Here $\mathfrak{Y}$ does not appear in the specializations of the closure of $\mathrm{Comp}(\mathcal{O}_{\mathbb{K}})$. It follows that $N(\mathcal{O}_{\mathbb{k}}) > N(\mathcal{O}_{\mathbb{K}})$. Hence, we have

$$\sum_{\mathcal{O}_{\mathbb{k}} \in G_{\mathbb{k}} \backslash \mathfrak{N}_{\mathbb{k}}} N(\mathcal{O}_{\mathbb{k}})^2 > \sum_{\mathcal{O}_{\mathbb{K}} \in G_{\mathbb{K}} \backslash \mathfrak{N}_{\mathbb{K}}} N(\mathcal{O}_{\mathbb{K}})^2 = \#W. \tag{7.1}$$

For $X \in \mathcal{O}_{\mathbb{k}}$, we have $\mathsf{p}_{\mathbb{k}}^{-1}(X) \cong \nu_{\mathbb{k}}^{-1}(X) \times \nu_{\mathbb{k}}^{-1}(X)$. We have

$$\dim \mathsf{p}_{\mathbb{k}}^{-1}(\mathcal{O}_{\mathbb{k}}) = 2\dim \nu_{\mathbb{k}}^{-1}(X) + \dim \mathcal{O}_{\mathbb{k}} = \dim \mathfrak{N}_{\mathbb{k}} = \dim Z_{\mathbb{k}}.$$

By Theorem 7.1 1) and Lemma 7.6, $\overline{\mathsf{p}_{\mathbb{k}}^{-1}(\mathcal{O}_{\mathbb{k}})}$ is a union of $N(\mathcal{O}_{\mathbb{k}})^2$ irreducible components of $Z_{\mathbb{k}}$. By (7.1), $Z_{\mathbb{k}}$ has more than $\#W$ irreducible components. This contradicts the existence of $\mathfrak{Y}$ as required. $\square$

Proposition 7.8. *Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$ and let $\mathbb{O}$ be a G-orbit of $\mathcal{N}$.*

1) *Let $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. The variety $\mathfrak{X}_{\mathbb{k}}$ is irreducible.*

2) *Let $\mathfrak{Y} \in \mathrm{Comp}(\mathbb{O})$ be such that $\mathfrak{Y}_{\mathbb{K}} \in \mathrm{Comp}(\mathbb{O}_{\mathbb{K}})$. Then, $\mathfrak{Y}_{\mathbb{k}}$ is irreducible.*

Proof. Let $X \in \mathcal{O}(A)$. For two irreducible components $\mathcal{G}_X^i, \mathcal{G}_X^j \subset \mathcal{G}_X$, we set

$$Z_0 := G(\mathcal{G}_X^i / B \times \mathcal{G}_X^j / B) \subset Z.$$

We have

$$\dim Z_0 = \dim \mathcal{G}_X^i / B + \dim \mathcal{G}_X^j / B + \dim \mathcal{O} = \dim Z. \tag{7.2}$$

Hence, $\overline{Z_0} \subset Z$ is an irreducible component of Z. It follows that $(\mathcal{G}_X^i)_{\mathbb{k}} \cap (\mathcal{G}_X^j)_{\mathbb{k}}$ does not contain an irreducible component of $(\mathcal{G}_X)_{\mathbb{k}}$ unless $\mathcal{G}_X^i = \mathcal{G}_X^j$. By taking quotient by $\mathsf{Stab}_G(X)$, we conclude the first assertion.

For the second assertion, we uniformly change $\mathfrak{N}$ by $\mathcal{N}$ and $\mathbb{V}^+$ by $\mathfrak{n}$. Then, we use the extra assumption to guarantee the flatness of $\mathcal{G}_X^i$. Set C_X to be the component group of $\mathsf{Stab}_{G_{\mathbb{K}}} X$. Then, the problem reduces to show:

($\clubsuit$) $\mathfrak{Y}_{\mathbb{k}} = (\mathsf{Stab}_{G_{\mathbb{k}}} X_{\mathbb{k}}) \backslash (\mathcal{G}_X^+)_{\mathbb{k}}$ is irreducible for each C_X-orbit $\mathcal{G}_X^+$ of $\mathcal{G}_X^i$.

(Notice that C_X is not defined over A, and hence $\mathcal{G}_X^+$ cannot defined directly.) By Lemma 7.4, an irreducible component $\mathfrak{Y}'_\Bbbk \subset \mathfrak{Y}_\Bbbk$ is a union of subvarieties of orbital varieties of some $G_\Bbbk$-orbits of $\mathbb{O}_\Bbbk$. We have $\dim \mathfrak{Y}'_\Bbbk \le \dim \mathfrak{Y}_\Bbbk$ with the equality holds only when $\mathfrak{Y}'_\Bbbk \in \mathrm{Comp}(\mathbb{O}'_\Bbbk)$. By the upper-semicontinuity of fiber dimensions applied to $\mathfrak{Y}$ over A, we have necessarily $\dim \mathfrak{Y}'_\Bbbk = \dim \mathfrak{Y}_\Bbbk$. In particular, we have an irreducible component $\mathcal{G}_X^0 \subset \mathcal{G}_X^+$ corresponding to $\mathrm{Comp}(\mathbb{O}'_\Bbbk)$. Assume that $X_\Bbbk \in \mathbb{O}'_\Bbbk(\Bbbk)$. Then, $G_\Bbbk(\mathcal{G}_X^0 \times \mathcal{G}_X^0)_\Bbbk$ is a union of irreducible components of $Z_\Bbbk$ thanks to the dimension estimate as in (7.2). Therefore, we deduce that $\overline{G_\Bbbk(\mathcal{G}_X^0 \times \mathcal{G}_X^0)_\Bbbk} \subset (\overline{G_\mathbb{K}(\mathcal{G}_X^0 \times \mathcal{G}_X^0)_\mathbb{K}})_\Bbbk \subset Z_\Bbbk$ is an irreducible component, which implies that $(\mathcal{G}_X^0)_\Bbbk$ is irreducible. It follows that an irreducible component of $\mathrm{Comp}(\mathbb{O}'_\Bbbk)$ obtained as

$$(\mathsf{Stab}_{G_\Bbbk} X_\Bbbk)\backslash(\mathcal{G}_X^0)_\Bbbk = \overline{((\mathsf{Stab}_{G_\mathbb{K}} X_\mathbb{K})^\circ\backslash(\mathcal{G}_X^0)_\mathbb{K})}_\Bbbk = \overline{(\mathsf{Stab}_{G_\mathbb{K}} X_\mathbb{K}\backslash(\mathcal{G}_X^+)_\mathbb{K})}_\Bbbk$$

is unique as required. $\qquad\square$

8　Exotic orbital varieties: proof

This section is devoted to the proof of Theorem 7.1.

The proof itself is a modification of the arguments of Steinberg [Ste74], Spaltenstein [Spa77], and Joseph [Jos83]. The only essential diffusion in the proof is contained in the strict semi-smallness of the map ν, which follows from [K09] §1 and §8. Since the literature is little scattered, we provide a proof with its necessary modifications.

In the below, we assume the same settings as in Theorem 7.1, but we drop the subscript $_\mathbb{K}$ or $_\Bbbk$ for the sake of simplicity.

Lemma 8.1. *We have* $\dim \mathfrak{X} \le \frac{1}{2} \dim \mathcal{O}$. *Moreover, there exists* $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$ *which satisfies the equality.*

Proof. Let $X \in \mathfrak{X}$. We have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \nu^{-1}(X) = \dim G/B$$

by the (strict) semi-smallness of ν. Since $\nu^{-1}(X) = \mathcal{G}_X/B$, we have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \mathcal{G}_X - \dim B = \dim G/B.$$

We have $\mathfrak{X} \subset \mathcal{G}_X/\mathrm{Stab}_G(X)$. In particular, we have

$$\frac{1}{2} \dim G\mathfrak{X} + \dim \mathfrak{X} + \dim \mathrm{Stab}_G(X) \le \dim G.$$

Therefore, we have

$$\dim \mathfrak{X} \le \dim G - \dim \mathrm{Stab}_G(X) - \frac{1}{2} \dim G\mathfrak{X} = \frac{1}{2} \dim \mathcal{O}, \qquad (8.1)$$

which proves the first assertion. The second assertion follows by choosing $\mathfrak{X}$ so that $\dim \mathfrak{X} = \dim \mathcal{G}_X/\mathrm{Stab}_G(X)$. $\qquad\square$

Proposition 8.2. *Assume that* $\dim \mathfrak{X} = \frac{1}{2} \dim \mathcal{O}$. *Then there exists* $w \in W$ *such that*

$$\mathfrak{X} \subset \overline{B(\mathbb{V}^+ \cap {}^w\mathbb{V}^+)}$$

is a dense open subset.

Proof. Let $X \in \mathfrak{X}$. We assume that $\mathfrak{X} \cong \mathcal{G}^i X$ for an irreducible component $\mathcal{G}^i$ of $\mathcal{G}_X \subset G$. We put $\mathcal{E}^i_X := \mathcal{G}^i/B$, which is an irreducible component of $\mathcal{E}_X$. By $\dim \mathfrak{X} = \frac{1}{2} \dim \mathcal{O}$, it follows that $\mathcal{E}^i_X$ has the maximal dimension among the irreducible components of $\mathcal{E}_X$. In other words, we have

$$\dim \mathcal{E}^i_X = \frac{1}{2}(\dim \mathfrak{N} - \dim \mathcal{O}). \tag{8.2}$$

Consider the variety

$$\mathcal{S} := \{(g_1 B, g_2 B, v) \in \mathcal{B} \times \mathcal{B} \times \mathbb{V}; v \in g_1\mathbb{V}^+ \cap g_2\mathbb{V}^+ \cap \mathcal{O}\}$$

and its subvarieties

$$\mathcal{S}_w := \{(g_1 B, g_2 B, v) \in \mathcal{S}; g_1^{-1}g_2 \in B\dot{w}B\}$$

for each $w \in W$. It is straight-forward to check $\mathcal{S} = \sqcup_{w \in W} \mathcal{S}_w$ (the arguments in [Ste74] p133 L14-L20 works merely by changing the meaning of the symbols appropriately). By considering the third projection $p_3 : \mathcal{S} \to \mathcal{O}$, we deduce that

$$p_3^{-1}(X) \cong \mathcal{E}_X \times \mathcal{E}_X.$$

Consider the projection $p_{12} : \mathcal{S} \to \mathcal{B} \times \mathcal{B}$ of $\mathcal{S}$ to the first two components. By definition, we have $p_{12}(\mathcal{S}_w) = G([B \times \dot{w}B])$ or $\emptyset$. It follows that

$$\dim \mathcal{S}_w = \dim \mathcal{B} + \ell(w) + \dim(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O})$$
$$\leq \dim \mathcal{B} + \ell(w) + \dim(\mathbb{V}^+ \cap {}^w\mathbb{V}^+) = \dim G - n = \dim \mathcal{S}$$

whenever $\mathcal{S}_w \neq \emptyset$. Define

$$\mathcal{S}^{i,i} := G(\mathcal{E}^i_X \times \mathcal{E}^i_X \times \{X\}) \subset Gp_3^{-1}(X) \subset \mathcal{S}.$$

This is an irreducible component of $\mathcal{S}$. Since $Gp_3^{-1}(X) = \mathcal{S}$ and (8.2), we conclude

$$\dim \mathcal{S}^{i,i} = \dim \mathcal{O} + 2\dim \mathcal{E}^i_X = \dim \mathcal{O} + 2 \times \frac{1}{2}\mathrm{codim}\mathcal{O} = \mathfrak{N} = \dim G - n.$$

There exists $w \in W$ such that $\mathcal{S}_w \cap \mathcal{S}^{i,i} \subset \mathcal{S}^{i,i}$ is a dense open subset. By dimension counting, we deduce $\mathcal{S}^{i,i} = \overline{\mathcal{S}_w}$. Now we have

$$\mathcal{S}_w = \{(g_1 B, g_2 B, v); g_1^{-1}g_2 \in B\dot{w}B, v \in g_1(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O})\}. \tag{8.3}$$

Since $\dim \mathcal{S} = \dim \mathcal{S}^{i,i} = \dim \mathcal{S}_w$, we deduce that

$$\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O} \subset \mathbb{V}^+ \cap {}^w\mathbb{V}^+$$

is dense.

Consider the image $\mathcal{G}_w$ of $\mathcal{S}_w$ under the first and third projection $p_{13} : \mathcal{S} \to \mathcal{B} \times \mathbb{V}$.

Its second projection $q_3 : \mathcal{G}_w \to \mathcal{O}$ satisfies $q_3 \circ p_{13} = p_3$. In the RHS of (8.3), g_2 plays no rôle for the restriction on v. Therefore, we deduce $q_3^{-1}(X) \subset \mathcal{E}_X^i$ (dense open subset). By construction, we have

$$q_3^{-1}(X) = \{(gB, X); X \in g(\mathbb{V}^+ \cap {}^w\mathbb{V}^+ \cap \mathcal{O})\}.$$

As a consequence, we deduce

$$\mathfrak{X} = \mathcal{G}^i X \subset \overline{\{g^{-1}X; (gB, X) \in q_3^{-1}(X)\}} = \overline{B(\mathbb{V}^+ \cap {}^w\mathbb{V}^+)}.$$

Since the second inclusion is dense by construction, we conclude the result. $\square$

Lemma 8.1 and Proposition 8.2 claim that the both assertions of Theorem 7.1 hold for at least one $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. To derive Theorem 7.1 for general irreducible components, we need some preparation:

For each $1 \le i \le n$, we put $\mathbb{V}(i) := U(\mathfrak{b})^{s_i}\mathbb{V}^+$. We define $\mathbb{V}_i^+ := \mathbb{V}^+/(\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ and $\mathbb{V}_i := \mathbb{V}(i)/(\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$.

For each $1 \le i \le n$, we put $P_i := B\dot{s}_iB \cup B$. It is a parabolic subgroup of G.

The derived group of the Levi part of P_i is isomorphic to $SL(2)$. Its action on $\mathbb{V}_i$ is equivalent to either $\mathfrak{sl}(2)$ (adjoint representation, $1 \le i < n$) or $\mathbb{K}^2$ (vector representation, $i = n$).

Since the both of $\mathbb{V}(i)$ and $(\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ are P_i-stable, it follows that $\mathbb{V}_i$ admits a natural P_i-action. Let $\pi_i : \mathbb{V}(i) \longrightarrow \mathbb{V}_i$. The map π_i is P_i-equivariant. We define $\mathfrak{X}_i := \pi_i(\mathfrak{X})$.

Lemma 8.3. *Let $1 \le i \le n$. Assume that the both assertions of Theorem 7.1 hold for $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. Then, $P_i\mathfrak{X} \cap \mathbb{V}^+$ is a union of elements of $\mathrm{Comp}(\mathcal{O})$ which satisfy the both assertions of Theorem 7.1.*

Proof. By Proposition 8.2, it suffices to verify Theorem 7.1 1).

By construction, $\mathfrak{X}_i \subset \mathbb{V}_i^+$ is a B-stable subset. We have $\dim \mathbb{V}_i^+ = 1$. Hence, we have $\mathfrak{X}_i = \{0\}$ or $\overline{\mathfrak{X}_i} = \mathbb{V}_i^+$. If $\mathfrak{X}_i = \{0\}$, then $\mathfrak{X}$ is P_i-stable. Thus, the assertion trivially holds.

Therefore, we concentrate ourselves to the case $\overline{\mathfrak{X}_i} = \mathbb{V}_i^+$ in the below. Let $\mathfrak{X}' \in \mathrm{Comp}(\mathcal{O})$ such that $\mathfrak{X}' \cap P_i\mathfrak{X} \not\subset \mathfrak{X}$. If $\mathfrak{X}'$ does not exist, then we have $P_i\mathfrak{X} \cap \mathbb{V}^+ = \mathfrak{X}$. Hence, the assertion trivially holds. Thus, we assume the existence of $\mathfrak{X}'$.

Let $\mathfrak{D} := \mathfrak{X} \cap \pi_i^{-1}(\{0\}) \subset \mathfrak{X}$. This is a purely codimension one subscheme of $\mathfrak{X}$. Since $\pi_i^{-1}(\{0\}) = (\mathbb{V}^+ \cap {}^{s_i}\mathbb{V}^+)$ is P_i-stable, it follows that

$$P_i\mathfrak{D} \subset P_i\mathfrak{X} \cap \pi_i^{-1}(\{0\}) \subset \mathbb{V}^+.$$

Let $0 \ne X \in \mathbb{V}_i^+$. By an explicit $SL(2)$-computation, we have $gX \in \mathbb{V}_i^+$ ($g \in P_i$) if and only if $g \in P_i \cap B$. This implies

$$P_i(\mathfrak{X} - \mathfrak{D}) \cap \mathbb{V}^+ = \mathfrak{X} - \mathfrak{D}.$$

Hence, we have $\mathfrak{X}' \cap P_i\mathfrak{D} \ne \emptyset$. Let $\mathfrak{D}_0 \subset \mathfrak{D}$ be an irreducible component such that $\mathfrak{X}' \cap P_i\mathfrak{D}_0 \not\subset \mathfrak{D}$. We have necessarily $P_i\mathfrak{D}_0 \ne \mathfrak{D}_0$. This implies

$$\dim P_i\mathfrak{D}_0 = \dim \mathfrak{D}_0 + 1 = \dim \mathfrak{X}.$$

As a consequence, $\overline{P_i\mathfrak{D}_0}$ contains a (unique) element of $\mathrm{Comp}(\mathcal{O})$ which is different from $\mathfrak{X}$. Letting $\mathfrak{X}'$ and $\mathfrak{D}_0$ vary arbitrary, we conclude the result. $\square$

In order to complete the proof of Theorem 7.1, it suffices to prove that a successive application of Lemma 8.3 eventually exhausts the whole of $\mathrm{Comp}(\mathcal{O})$. This is guaranteed by the following:

Proposition 8.4. *Let $\mathfrak{X}, \mathfrak{X}' \in \mathrm{Comp}(\mathcal{O})$. Assume that $\mathfrak{X}'$ satisfies the both assertions of Theorem 7.1. Then, there exists a sequence of integers $i_1, i_2, \ldots, i_m \in [1, n]$ and a sequence $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m \in \mathrm{Comp}(\mathcal{O})$ such that*

$$\mathfrak{X}' = \mathfrak{X}_1, \ \ \mathfrak{X} = \mathfrak{X}_m, \ and \ \mathfrak{X}_{k-1} \subset P_{i_k} \mathfrak{X}_k$$

hold for every $2 \le k \le m$.

Proof. Let $i_1, \ldots i_m \in [1, n]$ be a sequence of integers such that

$$\mathfrak{X}' \subset P_{i_1} P_{i_2} \cdots P_{i_m} \overline{\mathfrak{X}}. \tag{8.4}$$

We assume that $(\star)$: $\mathfrak{X}' \not\subset P_{i'_1} P_{i'_2} \cdots P_{i'_{m'}} \mathfrak{X}$ does not holds for any sequence $i'_1, \ldots, i'_{m'}$ if $m' < m$. This implies that $s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression. We prove that there exists a sequence $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m \in \mathrm{Comp}(\mathcal{O})$ which satisfies the required condition. By (8.4) and $(\star)$, we deduce

$$\mathfrak{Z} := B\dot{s}_{i_1} \mathfrak{X}' \cap B\dot{s}_{i_2} B \cdots B\dot{s}_{i_m} \overline{\mathfrak{X}} \neq \emptyset.$$

Claim 8.5. *We have $\dim \mathfrak{Z} = \dim \mathfrak{X}'$.*

Proof. By $(\star)$, we have $P_{i_1} \mathfrak{X}' \neq \mathfrak{X}'$. Hence, we have

$$P_{i_1} \mathfrak{X}' = \dim \mathfrak{X}' + 1.$$

Since $\mathfrak{Z}$ is an open subset of a codimension one subscheme of $\overline{P_{i_1} \mathfrak{X}'}$, we deduce the result. $\square$

We return to the proof of Proposition 8.4.

We have $B\dot{s}_{i_1} \mathfrak{X}' \subset \mathbb{V}(i_1)$. We put $w = s_{i_2} \cdots s_{i_m}$. This is a reduced expression. Since $\ell(w) < \ell(s_{i_1} w)$, we have $\mathbf{v}[-\alpha_{i_1}^\vee] \notin \dot{w}\mathbb{V}^+$, where $\alpha_i^\vee = \alpha_i$ $(i \neq n)$ or ϵ_n $(i = n)$. It follows that

$$(\mathbb{K}^\times \mathbf{v}[-\alpha_{i_1}^\vee] + \mathbb{V}[0] + \mathbb{V}^+) \cap B\dot{w}\mathbb{V}^+ = \emptyset$$

by a weight comparison. (Here we need to replace $\mathbb{K}$ with $\Bbbk$ when the subscripts are $\Bbbk$.) Hence, we have $\mathbb{V}(i_1) \cap B\dot{w}\mathbb{V}^+ \subset \mathbb{V}^+$. Taking account into $(\star)$, this implies that

$$\emptyset \neq B\dot{s}_{i_1} \mathfrak{X}' \cap B\dot{w}\overline{\mathfrak{X}} \subset \mathbb{V}^+.$$

In particular, there exists an irreducible component $\mathfrak{X}'' \subset P_{i_1} \mathfrak{X}' \cap \mathbb{V}^+$ such that

$$\mathfrak{X}'' \cap B\dot{s}_{i_2} B \cdots B\dot{s}_{i_m} \overline{\mathfrak{X}} \neq \emptyset, \mathfrak{X}' \subset P_{i_1} \mathfrak{X}'',$$

and the intersection at the most LHS is a maximal dimensional irreducible component of $\mathfrak{Z}$. By Lemma 8.3, it suffices to prove the assertion for $\mathfrak{X}, \mathfrak{X}'' \in \mathrm{Comp}(\mathcal{O})$ with

$$\mathfrak{X}'' \subset \overline{B\dot{s}_{i_2} B \cdots B\dot{s}_{i_m} \mathfrak{X}} = P_{i_2} P_{i_3} \cdots P_{i_m} \overline{\mathfrak{X}}.$$

Since the assertion for $m = 1$ is proved in Lemma 8.3, the downward induction on m yields the result. $\square$

9 Comparison of Springer correspondences

We work under the same setting as in §1.2.

Let $\mathcal{X}$ be a T-equivariant scheme over A. Let $K^{T_*}(\mathcal{X}_*)_{\mathbb{Q}}$ be the $\mathbb{Q}$-coefficient Grothendieck group of T_*-equivariant coherent sheaves on $\mathcal{X}_*$ which are flat over the base ($* = A, \mathbb{K}, \Bbbk$). Let $R(T)_{\mathbb{Q}}$ be the representation ring of T with coefficient $\mathbb{Q}$. For a T-module V, we define $\mathrm{ch}_T V$ to be the class $[V] \in R(T)_{\mathbb{Q}}$. Consider a map

$$p : K^T(\mathfrak{n})_{\mathbb{Q}} \longrightarrow R(T)_{\mathbb{Q}}$$

which sends a T-equivariant closed subset $C \subset \mathfrak{n}$ to the ratio

$$\mathrm{ch}_T \Gamma(\mathfrak{n}, \mathcal{O}_C) / \mathrm{ch}_T \Gamma(\mathfrak{n}, \mathcal{O}_\mathfrak{n}) \in R(T)_{\mathbb{Q}}.$$

Replacing T and $\mathfrak{n}$ with T_* and $\mathfrak{n}_*$ (where $* = A, \mathbb{K}, \Bbbk$), we define the corresponding maps p_*. Similarly, consider a map

$$q : K^T(\mathbb{V}^+)_{\mathbb{Q}} \longrightarrow R(T)_{\mathbb{Q}}$$

which sends a T-equivariant closed subset $C \subset \mathbb{V}^+$ to the ratio

$$\mathrm{ch}_T \Gamma(\mathbb{V}^+, \mathcal{O}_C) / \mathrm{ch}_T \Gamma(\mathbb{V}^+, \mathcal{O}_{\mathbb{V}^+}) \in R(T)_{\mathbb{Q}}.$$

Replacing T and $\mathbb{V}^+$ with T_* and $\mathbb{V}_*^+$ (where $* = \mathbb{K}, \Bbbk$), we define the corresponding maps q_*.

Let $\mathsf{fx} : R(T) \to \mathbb{C}[[\mathsf{t}]]$ be the map given by the formal expansion of a function on T along 1. For $f \in R(T)$, we denote the lowest non-zero homogeneous term of $\mathsf{fx}(f)$ by $\mathsf{lt}(f)$. By definition, $\mathsf{lt}(f)$ is a homogeneous polynomial on t.

Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. For each $\mathfrak{Y}_* \in \mathrm{Comp}(\mathbb{O}_*)$ ($* = A, \mathbb{K}, \Bbbk$), we define the Joseph polynomial attached to $\mathfrak{Y}_*$ as $\mathsf{lt}\left(p_*(\overline{\mathfrak{Y}}_*)\right)$. Let $\mathcal{O}$ be a G-orbit in $\mathfrak{N}$. For each $\mathfrak{X}_* \in \mathrm{Comp}(\mathcal{O}_*)$ ($* = \mathbb{K}, \Bbbk$), we define the Joseph polynomial attached to $\mathfrak{X}_*$ as $\mathsf{J}(\mathfrak{X}_*) := \mathsf{lt}\left(q_*(\overline{\mathfrak{X}}_*)\right)$.

We denote the set of $\mathbb{Q}$-multiples of Joseph polynomials attached to orbital varieties of $\mathcal{O}_*$ or $\mathbb{O}_*$ ($* = A, \mathbb{K}, \Bbbk$) by $\mathrm{Jos}(\mathcal{O}_*)$ or $\mathrm{Jos}(\mathbb{O}_*)$, respectively.

Proposition 9.1. *Let $\mathcal{X} \subset \mathbb{V}^+$ and $\mathcal{Y} \subset \mathfrak{n}$ be T-equivariant flat subfamilies over A. Then, we have*

$$q_{\mathbb{K}}(\mathcal{X}_{\mathbb{K}}) = q_{\Bbbk}(\mathcal{X}_{\Bbbk}) \text{ and } p_{\mathbb{K}}(\mathcal{Y}_{\mathbb{K}}) = p_{\Bbbk}(\mathcal{Y}_{\Bbbk}).$$

Proof. Each character of tori is defined over A. In particular, specialization (to $\mathbb{K}$ or $\Bbbk$) of a T-equivariant flat A-module of rank one preserves the character. Hence, the assumption implies that the coordinate rings $\mathbb{K}[\mathcal{X}_{\mathbb{K}}]$ and $\Bbbk[\mathcal{X}_{\Bbbk}]$ share the same character. Hence, we conclude the result for $\mathcal{X}$. The case $\mathcal{Y}$ is entirely the same. □

Proposition 9.2. *Let $\mathcal{C}$ be a $T_{\Bbbk}$-stable flat subfamily of $\mathcal{V}^+$ over $\mathbb{A}_{\Bbbk}^1$ whose fibers are irreducible schemes. Let $\mathcal{C}_t := C \dot{\cap} \pi^{-1}(t)$. Then, we have*

$$\mathsf{lt}(p_{\Bbbk}([\mathcal{C}_1])) \in \mathbb{Z}_{\geq 1} \mathsf{lt}(q_{\Bbbk}([\mathsf{F}_1^{-1}(\mathcal{C}_0)])).$$

Proof. For the sake of simplicity, we drop the subscripts $_\Bbbk$ during this proof. A T-character does not admit a non-trivial deformation. We express $\mathcal{O}_{\mathcal{C}_0}$ by a T-equivariant free resolution

$$0 \to \mathcal{F}_N \to \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{O}_{\mathcal{C}_0} \to 0$$

such that $N < \infty$ and each $\mathcal{F}_i$ is isomorphic to a product of T-module and $\mathcal{O}_{\mathsf{F}_1(\mathbb{V}^+)}$ (c.f. Chriss-Ginzburg [CG97] §5.1 or Sumihiro [Sum74]). Utilizing the Frobenius splitting of V_1, we deduce that $\mathcal{O}_{\mathbb{V}^+}$ is a free $\mathcal{O}_{\mathsf{F}_1(\mathbb{V}^+)}$-module of rank 2^n. It follows that $q([\mathsf{F}_1^*(\mathcal{C}_0)]) = p([\mathcal{C}_1])$. Now the sheaf $\mathsf{F}_1^*(\mathcal{O}_{\mathcal{C}_0})$ is a vector bundle of rank m along an open subset of $\mathsf{F}_1^{-1}(\mathcal{C}_0)$ for some $m > 0$. Taking the lowest term of q neglects the effects from subsets of $\mathsf{F}_1^{-1}(\mathcal{C}_0)$ which has codimension ≥ 1. Therefore, we conclude that

$$m\,\mathsf{lt}q([\mathsf{F}_1^{-1}(\mathcal{C}_0)]) = \mathsf{lt}q([\mathsf{F}_1^*(\mathcal{C}_0)]) = \mathsf{lt}p([\mathcal{C}_1])$$

as desired. $\square$

Theorem 9.3 (Joseph [Jos83, Jos89]). *Let $\mathbb{O}$ be a G-orbit of $\mathcal{N}$ and let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$. The $\mathbb{C}$-span of $\mathrm{Jos}(\mathbb{O}_\mathbb{K})$ or $\mathrm{Jos}(\mathcal{O}_\mathbb{K})$ form an irreducible W-module with a basis $\mathrm{Jos}(\mathbb{O}_\mathbb{K})$ or $\mathrm{Jos}(\mathcal{O}_\mathbb{K})$.*

Proof. The proof for the case $\mathbb{O}_\mathbb{K}$ is the original case and is treated in [CG97] 6.5.13 and 7.4.1. The case $\mathcal{O}_\mathbb{K}$ follows from the same construction as in [CG97] 6.5 and 7.4 if we replace $\mathfrak{n}_\mathbb{K}$ with $\mathbb{V}_\mathbb{K}^+$, $\mathcal{N}_\mathbb{K}$ with $\mathfrak{N}_\mathbb{K}$, and $\mathbb{O}_\mathbb{K}$ with $\mathcal{O}_\mathbb{K}$ uniformly. $\square$

Corollary 9.4 (($\spadesuit$)$_1$ in §7). *We have $\sum_{\mathcal{O}_\mathbb{K} \in G_\mathbb{K} \backslash \mathfrak{N}_\mathbb{K}} (\mathrm{Comp}(\mathcal{O}_\mathbb{K}))^2 = \#W$.*

Proof. Since our exotic Springer correspondence is a bijection between the orbits of $\mathfrak{N}$ and $W^\vee$, Theorem 9.3 and the Wedderburn theorem yields the result. $\square$

Corollary 9.5 (($\spadesuit$)$_2$ in §7). *For a G-orbit $\mathcal{O}$ of $\mathfrak{N}$, there are at least $\#\mathrm{Comp}(\mathcal{O}_\mathbb{K})$ orbital varieties of $\mathcal{O}_\Bbbk$ contained in the closure of that of $\mathcal{O}_\mathbb{K}$.*

Proof. Straight-forward consequence of Proposition 9.1 and Theorem 9.3. $\square$

Theorem 9.6. *Let $\mathbb{O}$ be a G-orbit in $\mathcal{N}$. Let $\mathcal{O}$ be a G-orbit in $\mathfrak{N}$ such that $\mathsf{df}(\mathcal{O}_\Bbbk) \subset \mathbb{O}_\Bbbk$ is a open dense subset. Then, we have*

$$\mathrm{Jos}(\mathbb{O}_\mathbb{K}) = \mathrm{Jos}(\mathcal{O}_\mathbb{K}).$$

Proof. Since the construction of Joseph polynomials factors through the closures of orbital varieties, we may refer an orbital variety closure as an orbital variety during this proof (for the sake of simplicity). We prove the following identities:

$$\mathrm{Jos}(\mathbb{O}_\mathbb{K}) = \mathrm{Jos}(\mathbb{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}_\Bbbk) = \mathrm{Jos}(\mathcal{O}_\mathbb{K}). \tag{9.1}$$

(Proof of $\mathrm{Jos}(\mathbb{O}_\mathbb{K}) \subset \mathrm{Jos}(\mathbb{O}_\Bbbk)$) Let $\mathfrak{Y} \in \mathrm{Comp}(\mathbb{O})$ be such that $\mathfrak{Y}_\mathbb{K} \in \mathrm{Jos}(\mathbb{O}_\mathbb{K})$. The variety $\mathfrak{Y}_\Bbbk$ is irreducible by Proposition 7.8 2). Since $\mathfrak{Y}$ dominates A, we deduce that $\mathfrak{Y}$ is a flat over A. Therefore, $\mathrm{Jos}(\mathbb{O}_\mathbb{K}) \subset \mathrm{Jos}(\mathbb{O}_\Bbbk)$ follows from Proposition 9.1 as desired.

(Proof of $\mathrm{Jos}(\mathbb{O}_{\Bbbk}) = \mathrm{Jos}(\mathcal{O}_{\Bbbk})$) Let $\mathfrak{X}_{\Bbbk} \in \mathrm{Comp}(\mathcal{O}_{\Bbbk})$. Consider a family $\mathsf{ml}(\overline{\mathfrak{X}}_{\Bbbk})$. This is a $B_{\Bbbk}$-stable equidimensional subfamily of $\mathcal{V}^{+}$. By the comparison of dimensions, it is a flat family of orbital varieties over $\mathbb{A}_{\Bbbk}^{1}$. Hence the equality $\mathrm{Jos}(\mathbb{O}_{\Bbbk}) = \mathrm{Jos}(\mathcal{O}_{\Bbbk})$ follows from Proposition 9.2.

(Proof of $\mathrm{Jos}(\mathcal{O}_{\mathbb{K}}) = \mathrm{Jos}(\mathcal{O}_{\Bbbk})$) Let $\mathfrak{X} \in \mathrm{Comp}(\mathcal{O})$. The variety $\mathfrak{X}_{\Bbbk}$ is irreducible by Proposition 7.8 1). Since $\mathfrak{X}$ is flat over A, the equality $\mathrm{Jos}(\mathcal{O}_{\mathbb{K}}) = \mathrm{Jos}(\mathcal{O}_{\Bbbk})$ follows from Proposition 9.1.

Thanks to Theorem 9.3, we deduce (9.1), which implies the result. $\qquad\square$

Let $\mathbb{O}_{\mathbb{K}}$ be a $G_{\mathbb{K}}$-orbit of $\mathcal{N}_{\mathbb{K}}$. Let $Y \in \mathbb{O}_{\mathbb{K}}$. We define

$$\mathcal{B}_Y := \{g \in G_{\mathbb{K}}; Y \in \mathrm{Ad}(g)\mathfrak{n}_{\mathbb{K}}\}/B_{\mathbb{K}} \subset \mathcal{B}_{\mathbb{K}}$$

and call it the Springer fiber along Y.

Corollary 9.7. *Let $\mathbb{O}$ be a G-orbit of $\mathcal{N}$ with its codimension $2d$. Let $\mathcal{O}$ be a G-orbit of $\mathfrak{N}$ such that $\mathsf{df}(\mathcal{O}_{\Bbbk}) \subset \mathbb{O}_{\Bbbk}$ is a dense open subset. Let $X \in \mathcal{O}_{\mathbb{K}}$ and let $Y \in \mathbb{O}_{\mathbb{K}}$. Let $C_Y := \mathrm{Stab}_{G_{\mathbb{K}}}(Y)/\mathrm{Stab}_{G_{\mathbb{K}}}(Y)^{\circ}$. Then, we have a W-equivariant isomorphism*

$$H_{2d}(\mathcal{B}_Y, \mathbb{C})^{C_Y} \cong H_{2d}(\mathcal{E}_X, \mathbb{C}),$$

compatible with their embeddings into $H_{2d}(\mathcal{B}, \mathbb{C})$. Moreover, the bases given by irreducible components of $\mathcal{B}_Y$ and $\mathcal{E}_X$ coincide up to scalar multiplication.

Proof. This is a direct consequence of Theorem 9.6 and [CG97] 6.5.13. Here the counter-part of [CG97] 6.5.13 for $\mathfrak{N}$ is obtained by merely by replacing the meaning of the symbols as $\mathcal{N}$ by $\mathfrak{N}$, $\mathbb{O}$ by $\mathcal{O}$, and $\mathcal{B}_Y$ by $\mathcal{E}_X$. $\qquad\square$

10 An explicit description of the correspondence

Keep the setting of the previous section, but fix the base to be $\mathbb{K}$ (or rather its scalar extension to $\mathbb{C}$). Here we use the notation $\mathbb{V}^{\boldsymbol{\lambda}}, \mathbb{V}_{01}^{\boldsymbol{\lambda}}, \ldots$ defined in §5.

We start from a dimension formula which seems to go back to Kraft-Procesi [KP82] §8.1. Here we present a slightly modified form which is suitable for applications.

Theorem 10.1 (Kraft-Procesi [KP82]). *Let λ be a partition of n and let $\mathbf{0} = (0, 0, \ldots)$ be a sequence of zeros. We put $\boldsymbol{\lambda} := (\lambda, \mathbf{0})$. Then, we have*

$$\dim \mathcal{O}_{(\lambda, \mathbf{0})} = 2 \dim(\mathcal{O}_{(\lambda, \mathbf{0})} \cap \mathbb{V}_0) = 4 \sum_{i < j} (d_i^{\boldsymbol{\lambda}} - d_{i-1}^{\boldsymbol{\lambda}})(d_j^{\boldsymbol{\lambda}} - d_{j-1}^{\boldsymbol{\lambda}}),$$

where $\{d_i^{\boldsymbol{\lambda}}\}_{i \geq 1}$ is a sequence obtained from $\boldsymbol{\lambda}$ as in Definition 5.2.

Lemma 10.2. *For each marked partition $\boldsymbol{\lambda}$, there exists $\mathfrak{X}_{\boldsymbol{\lambda}} \in \mathrm{Comp}(\mathcal{O}_{\boldsymbol{\lambda}})$ such that*

$$\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} = \overline{B\mathbb{V}^{\boldsymbol{\lambda}}}.$$

Proof. We set $d_i^{\nabla} := d_i^{\boldsymbol{\lambda}} - d_{i-1}^{\boldsymbol{\lambda}}$. By Theorem 7.1 and Corollary 5.7, it suffices to prove $\dim B\mathbb{V}^{\boldsymbol{\lambda}} \geq |\mu| + 2\sum_{i<j} d_i^{\nabla} d_j^{\nabla}$. The subspace $\mathbb{V}_1^{\boldsymbol{\lambda}}$ is B-stable and has dimension $|\mu|$. Since $B\mathbb{V}^{\boldsymbol{\lambda}} = B\mathbb{V}_{01}^{\boldsymbol{\lambda}}$, we have only to prove $\dim B\mathbb{V}_0^{\boldsymbol{\lambda}} \geq 2\sum_{i<j} d_i^{\nabla} d_j^{\nabla}$. Here $\mathbb{V}_0^{\boldsymbol{\lambda}}$ is N_0-stable and $\dim \mathbb{V}_0^{\boldsymbol{\lambda}} = \sum_{i<j} d_i^{\nabla} d_j^{\nabla}$. Thus, it suffices

to prove that $\dim G_2 x \geq \sum_{i<j} d_i^\nabla d_j^\nabla$ for a generic element $x \in \mathbb{V}_0^{\boldsymbol{\lambda}}$. Since the dimension of the G_2-stabilizer is an upper semi-continuous function along $\mathbb{V}_0^{\boldsymbol{\lambda}}$, it is enough to show $\dim G_2 x \geq \sum_{i<j} d_i^\nabla d_j^\nabla$ for some $x \in \mathbb{V}_0^{\boldsymbol{\lambda}}$. For $x \in \mathbb{V}_0$, we have

$$\dim \mathfrak{g}_2 x = \dim \mathfrak{g}_{-2} x.$$

Theorem 10.1 implies that

$$\dim \mathcal{O}_{(\lambda, \mathbf{0})} = \dim \mathfrak{g} x = 2 \dim \mathfrak{g}_0 x = 4 \sum_{i<j} d_i^\nabla d_j^\nabla.$$

In particular, we have $\dim \mathfrak{g}_2 x = \sum_{i<j} d_i^\nabla d_j^\nabla$ as desired. $\qquad\square$

Lemma 10.3. *Let $\boldsymbol{\lambda} = (\lambda, a)$ be a marked partition. Then, we have an equality*

$$\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} = \mathbb{V}_1^{\boldsymbol{\lambda}} \oplus \overline{\mathfrak{X}}$$

for some orbital variety $\mathfrak{X}$ of $\mathcal{O}_{(\lambda, \mathbf{0})}$.

Proof. The number d_i^∇ from the proof of Lemma 10.2 is some parts of ${}^t(\mu^{\boldsymbol{\lambda}})$ if $i \leq \mu_1^{\boldsymbol{\lambda}}$ and count some parts of ${}^t(\nu^{\boldsymbol{\lambda}})$ if $i > \mu_1^{\boldsymbol{\lambda}}$. We define $\{d_i'\}$ to be the parts of the partition ${}^t\lambda$. Since $\lambda_j = \mu_j^{\boldsymbol{\lambda}} + \nu_j^{\boldsymbol{\lambda}}$ for all j, we deduce that the two sequences $\{d_i^\nabla\}_i$ and $\{d_i'\}_i$ coincides up to changing their ordering.

Here $\{d_i'\}_i$ is a decreasing sequence. It follows that the dense open part of $\mathbb{V}_0^{\boldsymbol{\lambda}}$ has the same Jordan normal form as that of $\mathbb{V}_0^{(\lambda, \mathbf{0})}$. Here, $\mathbb{V}_1^{\boldsymbol{\lambda}}$ is B-stable. Therefore, Lemma 10.2 and Corollary 5.7 implies that

$$\dim \overline{\mathfrak{X}} = \dim \overline{\mathfrak{X}_{\boldsymbol{\lambda}}} - d_{\mu_1^{\boldsymbol{\lambda}}} = \frac{1}{2} \dim \mathcal{O}_{\boldsymbol{\lambda}} - |\mu^{\boldsymbol{\lambda}}| = \frac{1}{2} \dim \mathcal{O}_{(\lambda, \mathbf{0})}.$$

By Theorem 7.1, the closure $\overline{B \mathbb{V}_0^{\boldsymbol{\lambda}}}$ must be the closure of an orbital variety of $\mathcal{O}_{(\lambda, \mathbf{0})}$ as desired. $\qquad\square$

Definition 10.4 (Special vectors). Let $\boldsymbol{\lambda}$ be a marked partition of n and let $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ be its associated bi-partition (c.f. Theorem 5.1 and the line below it). We define

$$\mathsf{D}_i^0(\boldsymbol{\lambda}) := \prod_{d_i < k < l \leq d_{i+1}} (\epsilon_k^2 - \epsilon_l^2), \text{ and } \mathsf{D}_i^+(\boldsymbol{\lambda}) := \mathsf{D}_i^0(\boldsymbol{\lambda}) \prod_{d_i < k \leq d_{i+1}} \epsilon_k.$$

Using this, we define

$$\mathsf{D}(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}) := \prod_{i=0}^{\mu_1 - 1} \mathsf{D}_i^0(\boldsymbol{\lambda}) \times \prod_{i=\mu_1}^{\mu_1 + \nu_1 - 1} \mathsf{D}_i^+(\boldsymbol{\lambda}).$$

For an arbitrary bi-partition (μ, ν) of n, we define the Macdonald representation attached to (μ, ν) as

$$L(\mu, \nu) := \mathbb{C}[W] \mathsf{D}(\mu, \nu) \subset \mathbb{C}[\mathfrak{t}].$$

We remark that $L(\mu, \nu)$ is well-defined due to Theorem 5.1. Let $\mathbb{C}[\mathfrak{t}]_m$ denote the degree m-part of the polynomial ring $\mathbb{C}[\mathfrak{t}]$. For a subset $I \subset \mathbb{C}[\mathfrak{t}]$, we put $I_m := I \cap \mathbb{C}[\mathfrak{t}]_m$.

Theorem 10.5 (Macdonald c.f. Lusztig-Spaltenstein [LS79])**.** *For each bi-partition (μ, ν) of n, the Macdonald representation $L(\mu, \nu)$ is an irreducible representation of W. Moreover, we have*

$$\operatorname{Hom}_W(L(\mu, \nu), \mathbb{C}[\mathfrak{t}]_m) = \begin{cases} 0 & (m < \deg \mathsf{D}(\mu, \nu)) \\ 1 & (m = \deg \mathsf{D}(\mu, \nu)) \end{cases}.$$

We define

$$W_l := \langle s_i s_{i+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_i; 1 \le i \le n \rangle \subset W.$$

We have $W_l \cong (\mathbb{Z}/2\mathbb{Z})^n$. Moreover, we have a short exact sequence of groups

$$\{1\} \longrightarrow W_l \longrightarrow W \longrightarrow \mathfrak{S}_n \longrightarrow \{1\}.$$

Corollary 10.6. *Let λ be a partition of n. For each integer m, we have*

1. $\operatorname{Hom}_W(L(\lambda, \emptyset), \mathbb{C}[\mathfrak{t}]_m) = \operatorname{Hom}_{\mathfrak{S}_n}(L(\lambda, \emptyset), \mathbb{C}[\epsilon_1^2, \ldots, \epsilon_n^2]_m)$;

2. $\operatorname{Hom}_W(L(\emptyset, \lambda), \mathbb{C}[\mathfrak{t}]_m) = \operatorname{Hom}_{\mathfrak{S}_n}(L(\emptyset, \lambda), \left(\epsilon_1 \epsilon_2 \cdots \epsilon_n \mathbb{C}[\epsilon_1^2, \ldots, \epsilon_n^2]\right)_m)$.

Proof. An irreducible W-module for which W_l acts trivially yields a $\mathfrak{S}_n$-module. The invariant ring $\mathfrak{C}[\mathfrak{t}]^{W_l}$ is generated by $\epsilon_1^2, \ldots, \epsilon_n^2$. This implies the first assertion. The whole ring $\mathfrak{C}[\mathfrak{t}]$ is a free $\mathfrak{C}[\mathfrak{t}]^{W_l}$-module of rank 2^n. The sign representation of W_l also defines a rank one $\mathfrak{C}[\mathfrak{t}]^{W_l}$-submodule $\mathfrak{C}[\mathfrak{t}]^{W_l}(\epsilon_1 \epsilon_2 \cdots \epsilon_n) \subset \mathfrak{C}[\mathfrak{t}]$ via taking the isotypical component. Hence, we deduce the second assertion. $\square$

Theorem 10.7. *Let $\boldsymbol{\lambda}$ and $(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ be a marked partition of n and its associated bi-partition. Let $\mathfrak{X}_{\boldsymbol{\lambda}}$ be the orbital variety of $\mathcal{O}_{\boldsymbol{\lambda}}$ defined in Lemma 10.2 (c.f. Definition 5.1). We have*

$$\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in \mathbb{Q}\mathsf{D}(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}).$$

In particular, the $\mathbb{Q}$-span of $\operatorname{Jos}(\mathcal{O}_{\boldsymbol{\lambda}})$ is equal to $L(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$ as W-submodules of $\mathbb{Q}[\mathfrak{t}]$.

Proof of Theorem 10.7. We define

$$q_0(\boldsymbol{\lambda}) := \prod_{i > |\mu|} \epsilon_i \times \prod_{i \ge 0} \prod_{d_i < j < k \le d_{i+1}} (\epsilon_j - \epsilon_k).$$

The map $\mathtt{pr} : \mathbb{V}^+ \longrightarrow \mathbb{V}^+/(\mathbb{V}_1 \oplus \mathbb{V}_2)$ is a B-equivariant fibration. Hence, it induces the associated map $\mathfrak{X}_{\boldsymbol{\lambda}} \longrightarrow \mathtt{pr}(\mathfrak{X}_{\boldsymbol{\lambda}})$, which is generically a flat fibration. By Lemma 5.5, we have $\overline{\mathtt{pr}(\mathfrak{X}_{\boldsymbol{\lambda}})} = \mathbb{V}^{\boldsymbol{\lambda}} \cap \mathbb{V}_0$. Moreover, each $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}}$ is written as a product of $\bigoplus_{1 \le i \le |\mu|} V_1[\epsilon_i]$ and $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} \cap V_2$. It follows that

$$q_0(\boldsymbol{\lambda}) \text{ divides } \mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}). \tag{10.1}$$

By a dimension counting, we deduce that

$$d_Y := \dim \mathcal{E}_Y = \frac{1}{2} \operatorname{codim}_{\mathfrak{N}} GY = \deg \mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) = \deg \mathsf{D}(\mu, \nu) \tag{10.2}$$

for each $Y \in \mathcal{O}_{\boldsymbol{\lambda}}$.

Applying the argument of [CG97] 6.5.3 and 7.4.1, we know that

$$H_{2d_Y}(\mathcal{E}_Y, \mathbb{C}) \hookrightarrow H_\bullet(\mathcal{B}, \mathbb{C}) \hookrightarrow \mathbb{C}[\mathfrak{t}]$$

is some Macdonald representation. (The second inclusion is realized by the harmonic polynomials determined by T-equivariant fundamental classes.) By [K09] Theorem 8.3 and [CG97] §8.9, this establishes a one-to-one correspondence between the set of Macdonald representations and the set of G-orbits of $\mathfrak{N}$.

By Lemma 10.3, we have $\overline{\mathfrak{X}_{\boldsymbol{\lambda}}} \cap V_2 = \overline{\mathfrak{X}_{\widetilde{\boldsymbol{\lambda}}}}$ for some $\mathfrak{X}_{\widetilde{\boldsymbol{\lambda}}} \in \mathrm{Comp}(\mathcal{O}_{(\lambda, \mathbf{0})})$. Hence, we have an equality

$$(\prod_{i \leq |\mu|} \epsilon_i) \mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) = \mathsf{J}(\mathfrak{X}_{\widetilde{\boldsymbol{\lambda}}}). \tag{10.3}$$

For $f \in \mathbb{C}[\epsilon_1^2, \ldots, \epsilon_n^2]$, it is standard to see

$(\epsilon_i - \epsilon_j)$ (resp. ϵ_i) divides f if and only if $(\epsilon_i + \epsilon_j)$ (resp. ϵ_i) divides f. (10.4)

By Corollary 10.6, if $\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in L(\gamma, \emptyset)$ or $\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in L(\emptyset, \gamma)$ for a partition γ, then we have

$$\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in \mathbb{C}D(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}) \tag{10.5}$$

by a degree counting.

We prove the condition (10.5) by induction on the usual partition λ of n constituting $\boldsymbol{\lambda} = (\lambda, a)$.

Claim 10.8. *If $\lambda = (1, 1, \ldots, 1)$, then (10.5) holds.*

Proof. We have either $a = \{0\}$ or $\{1, 0, \cdots\}$. If $a = \{0\}$, then the weight configuration of $\mathbb{V}^+$ is the same as the positive roots of $SO(2n+1)$. It follows that $\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) = D(\emptyset, \lambda)$, which implies $\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in L(\emptyset, \lambda)$. If $a = \{1, 0, \cdots\}$, then we have $\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) = D(\lambda, \emptyset) \in L(\lambda, \emptyset)$. This proves (10.5) in all cases. $\square$

Claim 10.9. *Assume that (10.5) holds for $\boldsymbol{\lambda} = (\lambda, \mathbf{0})$. Then, we have (10.5) for all marked partition of shape $(\lambda, *)$.*

Proof. We have $L(\mu^{(\lambda, \mathbf{0})}, \nu^{(\lambda, \mathbf{0})}) \subset \mathbb{C}[\mathfrak{t}]^{W_l}(\epsilon_1 \cdots \epsilon_n)$. Hence, for every marked partition of shape (λ, a'), we have $\mathsf{J}(\mathfrak{X}_{\widetilde{(\lambda, a')}}) \in \mathbb{C}[\mathfrak{t}]^{W_l}(\epsilon_1 \cdots \epsilon_n)$. Hence, we deduce (10.5) for (λ, a') from (10.1), (10.3), and (10.4). $\square$

We return to the proof of Theorem 10.7

Assume that we know (10.5) for all $\boldsymbol{\lambda} = (\lambda, a)$ when $\deg D(\mu^{(\lambda, \mathbf{0})}, \nu^{(\lambda, \mathbf{0})}) > M$. We prove the case $\deg D(\mu^{(\lambda, \mathbf{0})}, \nu^{(\lambda, \mathbf{0})}) = M$. We a priori knows that $\{L(\mu, \nu)\}_{(\mu, \nu)}$ gives a complete collection of $W^\vee$. Therefore, the induction hypothesis implies that each $D(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}}) \in \mathbb{C}[\mathfrak{t}]_M$ which we have not yet proved (10.5) admits a factorization by $(\epsilon_1 \ldots \epsilon_n)$ and a W_l-invariant polynomial. Hence, we deduce (10.5) for $\boldsymbol{\lambda} = (\lambda, \mathbf{0})$ such that $\deg D(\mu^{(\lambda, \mathbf{0})}, \nu^{(\lambda, \mathbf{0})}) = M$ from (10.1) and (10.4). Now Claim 10.9 proceeds the induction and we obtain (10.5) for every marked partition.

Since an equivariant Hilbert polynomial has rational coefficients, we deduce

$$\mathsf{J}(\mathfrak{X}_{\boldsymbol{\lambda}}) \in \mathbb{Q}D(\mu^{\boldsymbol{\lambda}}, \nu^{\boldsymbol{\lambda}})$$

as desired. $\square$

References

[Bou02]   N. Bourbaki, Lie groups and Lie algebras, Chapters 4–6, Elements of Mathematics, Springer-Verlag Berlin 2002 xii+300pp ISBN 0-8176-4191-2

[BK04]   M. Brion, and S. Kumar, Frobenius splitting methods in Geometry and Representation theory, PM **234** Birkhäuser, 2004. viii+250 pp. ISBN 0-8176-4191-2

[CG97]   N. Chriss, and V. Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp. ISBN 0-8176-3792-3

[DK85]   J. Dadok, and V. Kac, Polar representations. J. Algebra 92 (1985), no. 2, 504–524.

[Gin97]   V. Ginzburg, Geometric methods in representation theory of Hecke algebras and quantum groups. Notes by Vladimir Baranovsky. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 127–183, Kluwer Acad. Publ., Dordrecht, 1998.

[Hot84]   R. Hotta, On Joseph's construction of Weyl group representations, Tôhoku Math. J. (2) 36 (1984), no. 1, 49–74.

[Hes79]   W. H. Hesselink, Nilpotency in Classical Groups over a Field of Characteristic 2, Math. Zeit. 166, 165–181, (1979)

[Jos83]   A. Joseph, On a variety associated to highest weight modules, J. Algebra 88 238–278, (1984).

[Jos89]   A. Joseph, On the characteristic polynomials of orbital varieties, Ann. Sci. École Norm. Sup. (4) 22 no.4 569–603, (1989).

[Jos97]   A. Joseph, Orbital varieties, Goldie rank polynomials, and unitary highest weight modules, in: Algebraic and analytic methods in representation theory, Perspectives in Math. **17** 53–99, Academic Press, 1997.

[Kal06]   D. Kaledin, Symplectic singularities from the Poisson point of view, J. reine angew. Math. 600, 135–156 (2006).

[K09]   S. Kato, An exotic Deligne-Langlands correspondence for symplectic groups, Duke Math. **148** no. 2, 305–371 (2009)

[KL87]   D. Kazhdan and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987), no. 1, 153–215.

[KP82]   H. Kraft, and C. Procesi, On the geometry of conjugary clases in classical groups, Comment. Math. Helv. 57 (1982), 539–602.

[LS79]   G. Lusztig, and N. Spaltenstein, Induced unipotent classes, J. London Math. Soc., **19**, (1979), 41–52.

[Lus84]   G. Lusztig, Intersection cohomology complexes on a reductive group, Invent. Math. 75 (1984), 205–272.

[Lus88]   G. Lusztig, Cuspidal local systems and graded Hecke algebras. I, Inst. Hautes Études Sci. Publ. Math., **67**, (1988), 145–202.

[Oht86]   T. Ohta, The singularities of the closures of nilpotent orbits in certain symmetric pairs, Tôhoku Math. J., 38 (1986), 441-468..

[Spa77]   N. Spaltenstein, On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology. 16, 203–204 (1977).

[Spa83]   N. Spaltenstein, Nilpotent Classes and Sheets of Lie algebras in Bad characteristic, Math. Zeit. 181 31–48, (1983).

[Spr76]   T. A. Springer, Trigonometric sums, Green functions of finite groups and representation theory of Weyl groups, Invent. Math. 36 (1976) 173–207.

[Ste74]   R. Steinberg, Conjugacy classes in algebraic groups, LMN **366**, vi+159pp, Springer 1974.

[Sum74]   H. Sumihiro, Equivariant completion. J. Math. Kyoto Univ. 14 (1974), 1–28.

[Tan85]   T. Tanisaki, Holonomic systems on a flag variety associated to Harish-Chandra modules and representation of a Weyl group, In: Algebraic groups and related topics, ASPM **6**, 1985 139–154.

[Xue08]   T. Xue, Nilpotent orbits in classical Lie algebras over $\mathbf{F}_{2^n}$ and the Springer correspondence, Proceedings of National Academy of Science, online edition Jan.17, 2008, arXiv:0709.4515.