What is the higher-dimensional infinitesimal groupoid of a manifold?

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Abstract

The construction (by Kapranov) of the space of infinitesimal paths on a manifold is extended to include higher dimensional infinitesimal objects, encoding contractions of infinitesimal loops. This full infinitesimal groupoid is shown to have the algebra of polyvector fields as its non-linear cohomology.

What is the infinitesimal version of the fundamental groupoid of a manifold $\mathcal{M}$? The standard answer is that it is the Lie-Rinehart algebra (also called Lie algebroid) of vector fields on $\mathcal{M}$. However, this answer is not precise.

Let α, β be two vector fields on $\mathcal{M}$, and assume for simplicity that they commute, i.e. $[\alpha, \beta] = 0$. The left hand side of this equation represents a loop, while the right hand side stands for the constant path. By equating the two sides we actually contract a loop.

In [Ka07] another Lie-Rinehart algebra was constructed for each $\mathcal{M}$. Starting with the space $\mathfrak{X}^1$ of vector fields, one builds the free Lie-Rinehart algebra $\mathcal{R}(\mathfrak{X}^1)$, generated by $\mathfrak{X}^1$. The action of $\mathcal{R}(\mathfrak{X}^1)$ on functions on $\mathcal{M}$ is generated by the action of $\mathfrak{X}^1$, but otherwise the Lie bracket is free.

The Lie bracket being free means that one doesn't contract non-degenerate loops (we still have $[\![\alpha, \alpha]\!] = [\![\alpha, 0]\!] = 0$), and therefore one can call $\mathcal{R}(\mathfrak{X}^1)$ **the space of infinitesimal paths**. This is a module over the algebra $\mathfrak{X}^0$ of functions on $\mathcal{M}$, and it defines a vector bundle on $\mathcal{M}$, that we will denote by $\mathcal{R}T\mathcal{M}$. Unless $dim(\mathcal{M}) \leq 1$, $\mathcal{R}T\mathcal{M}$ is obviously infinite dimensional.

While $\mathfrak{X}^1$ is obtained by contracting all loops, $\mathcal{R}(\mathfrak{X}^1)$ is built by avoiding contractions. In this paper we add higher dimensional components to $\mathcal{R}(\mathfrak{X}^1)$, that represent higher dimensional submanifolds, needed to parametrize contractions. We obtain **the full infinitesimal groupoid** $\mathbb{X}^*$, graded by the dimension of submanifolds.

The algebraic structure on $\mathbb{X}^*$ is considerably more complicated than that of a Lie-Rinehart algebra. In particular there are the **homotopy maps**

$$\mathbb{X}^k \to \mathbb{X}^{k-1}, \tag{1}$$

that represent contractions of loops. For example: $\mathbb{X}^1 = \mathcal{R}(\mathfrak{X}^1)$, but $\mathbb{X}^1$ modulo the image of $\mathbb{X}^2 \to \mathbb{X}^1$ is just $\mathfrak{X}^1$.

We also consider the "non-linear cohomology" $\mathcal{H}(\mathbb{X}^*)$ of $\mathbb{X}^*$ (for $k > 1$ the set $\mathbb{X}^k$ is not additive), i.e. homotopy classes of elements of $\mathbb{X}^*$, that themselves do not define non-trivial equivalence relations. We obtain that $\mathcal{H}(\mathbb{X}^*)$ is the algebra of polyvector fields $\sum_{k=1}^{-\infty} \wedge^{-k+1} \mathfrak{X}^1$. It is important to note that this algebra is not Gerstenhaber, since the Lie bracket has degree 0, while the wedge product is of degree -1.[1]

The idea of construction of $\mathbb{X}^*$ is as follows. A loop like $[\alpha, \beta]$ is contracted by a 2-morphism, i.e. it happens inside a jet of a submanifold of dimension 2. While α, β are given as 1-jets, it is not enough to take the 1-jet of the surface, indeed, if α, β are given as morphisms

$$\alpha : \mathfrak{X}^0 \to \mathfrak{X}^0 \otimes \mathbb{R}[\epsilon_1]/(\epsilon_1^2), \quad \beta : \mathfrak{X}^0 \to \mathfrak{X}^0 \otimes \mathbb{R}[\epsilon_2]/(\epsilon_2^2),$$

their bracket requires $\epsilon_1\epsilon_2$, and hence we need a morphism

$$\mathfrak{X}^0 \to \mathfrak{X}^0 \otimes \mathbb{R}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2).$$

In other words we need to consider sections of the second tangent bundle $T^2\mathcal{M}$. If $\nu \in \mathfrak{X}^2$ is such a section, it has α, β as its two projections to $\mathfrak{X}^1$, and it defines a homotopy relation in $\mathbb{X}^1$ by equating $[\alpha, \beta] = [\![\alpha, \beta]\!]$, where $[\![-, -]\!]$ is the free bracket on $\mathbb{X}^1 = \mathcal{R}(\mathfrak{X}^1)$.

To understand how ν provides the identification $[\alpha, \beta] = [\![\alpha, \beta]\!]$ one should note the two ways to view ν as a one-parameter family of vector fields on $\mathcal{M}$. On one hand ν is tangent to α, on the other it is tangent to β. Choosing one of the ways is equivalent to choosing an order on the pair $\{\alpha, \beta\}$, i.e. choosing an orientation on ν.

The symmetric group $\mathbb{S}_2$ acts on the set of these choices, and if $\sigma \in \mathbb{S}_2$ is the non-trivial element, its action is well known to produce $[\alpha, \beta]$, indeed,

[1] We use the cohomological notation, i.e. differentials raise degrees.

let $\alpha_*(\beta), \beta_*(\alpha)$ be the sections of $T^2\mathcal{M}$, obtained by using functoriality of T. Then

$$\alpha_*(\beta) - \beta_*(\alpha) = [\alpha, \beta], \tag{2}$$

where on the left hand side we use the *strong difference* of points in $T^2\mathcal{M}$ ([KL84], [MR91]), and on the right hand side we use identification of points in $T\mathcal{M}$ with tangents to the fibers of $T\mathcal{M} \to \mathcal{M}$.

Having two brackets $[-,-]$ and $[\![-,-]\!]$ on $\mathcal{R}T\mathcal{M}$ we have two actions of $\mathbb{S}_2$ on $T(\mathcal{R}T\mathcal{M})$, and taking their strong difference we obtain $[\alpha,\beta] - [\![\alpha,\beta]\!]$.

There are many sections of $T^2\mathcal{M}$, that have α, β as their projections to $\mathfrak{X}^1$, and therefore there are many different ways to contract the loop $[\alpha,\beta]$. To equate the different ways we need to consider jets of submanifolds of dimension 3, again these jets should be 3-jets of a particular kind, i.e. maps to $\mathbb{R}[\epsilon_1, \epsilon_2, \epsilon_3]/(\epsilon_i^2)$. So we need sections of $T^3\mathcal{M}$.

Since the combinatorics of $\{T^k\mathcal{M}\}$ is not globular but cubical, a section $\mu : \mathcal{M} \to T^3\mathcal{M}$ defines several homotopy relations on $\mathbb{X}^2$. There are 3 pairs of generators in $\mathbb{R}[\epsilon_1, \epsilon_2, \epsilon_3]/(\epsilon_i^2)$, and hence there are three homotopy maps $\mathfrak{X}^3 \to \mathbb{X}^2$. For an arbitrary k, $\nu \in \mathfrak{X}^k$ defines $\frac{k!}{2(k-2)!}$ homotopies.

To continue this construction further we have to work with iterations of $\mathcal{R}T$, rather than with iterations of the usual tangent bundle, i.e. instead of $T^2\mathcal{M}$ we should take sections in $(\mathcal{R}T)^2\mathcal{M}$.

One can construct $(\mathcal{R}T)^2\mathcal{M}$, but it is too big. It contains infinitesimal loops, that are completely inside the fibers of $\mathcal{R}T\mathcal{M} \to \mathcal{M}$. Tangents to these fibers represent infinitesimal automorphisms of tangents to $\mathcal{M}$, and, as far as $\mathcal{M}$ is concerned, infinitesimal loops in these fibers should be contracted.

This leads us to the construction of **relatively free Lie-Rinehart algebras**. We formulate this in general terms: let $\pi : \mathcal{N}_1 \to \mathcal{N}_2$ be a smooth map, that locally (on $\mathcal{N}_1$) is a trivial bundle (not necessarily linear). We define $\mathcal{R}(\mathfrak{X}^1(\mathcal{N}_1), \pi)$ to be the space of infinitesimal paths, obtained from $\mathcal{R}(\mathfrak{X}^1(\mathcal{N}_1))$ by contracting all "vertical loops" with respect to π.

If $\pi : \mathcal{N}_1 \to pt$ is the unique map to a point, we obtain the usual space of vector fields, if $\pi : \mathcal{N}_1 \to \mathcal{N}_1$ is the identity map, we obtain the space $\mathcal{R}(\mathfrak{X}^1(\mathcal{N}_1))$ of all infinitesimal paths from [Ka07].

Applying this construction to $\mathcal{M}$, and iterating, we obtain a sequence $\{\mathbb{T}^k\mathcal{M}\}$, that, just like $\{T^k\mathcal{M}\}$, is a semi-simplicial diagram of linear bundles. We define $\mathbb{X}^k$ as the set of sections $\mathcal{M} \to \mathbb{T}^k\mathcal{M}$.

For $k > 1$, the set $\mathbb{X}^k$ is not additive, but it has k different additions *over* $\mathbb{X}^{k-1}$. Also there is a cup product and a composition product, encoding infinitesimal automorphisms and Lie derivatives respectively. Taking non-linear cohomology as described above, and factoring out jets of degenerate submanifolds we obtain the algebra of polyvector fields.

Here is the structure of the paper. In section 1 we recall the construction and some algebraic properties of the usual iterated tangent bundles, and describe the additional structure one has on the sets of sections.

In section 2 we give the construction of relatively free Lie-Rinehart algebras and discuss their functorial properties.

In section 3 we construct the full infinitesimal groupoid $\mathbb{X}^*$, discuss some of the algebraic operations defined on this groupoid, including actions of symmetric groups, and compute its cohomology $\mathcal{H}(\mathbb{X}^*)$.

Everything in this paper is formulated for smooth real manifolds. All the statements and proofs are also valid, if one uses complex analytic manifolds instead.

Contents

1 k-vectors and k-vector fields

In this section we recall the basic properties of points and sections of iterated tangent bundles, in particular we describe decompositions of sections into sets of vector fields, subject to action by symmetric groups. These decompositions, and the action will be central to our treatment of full infinitesimal groupoids in section 3.

Let $\mathcal{M}$ be a smooth manifold of dimension n. Let $T^k\mathcal{M}$, $k \geq 0$, be its k-th iterated tangent bundle, i.e. $T^k\mathcal{M}$ is the tangent bundle on $T^{k-1}\mathcal{M}$, and $T^0\mathcal{M} = \mathcal{M}$.

It is well known (e.g. [Wh82], [Be08]) that for each $k \geq 1$ there are k vector-bundle projections $\{\pi_{k,i} : T^k\mathcal{M} \to T^{k-1}\mathcal{M}\}_{0 \leq i \leq k-1}$, and $\{\pi_{k,i}\}_{k \geq 1}$ satisfy the usual equations of simplicial boundaries.

From the (semi-)simplicial properties of $\{\pi_{k,i}\}_{k \geq 1}$ and the equality

$$\pi_1 \circ \pi_{2,0} = \pi_1 \circ \pi_{2,1},$$

it follows easily, that for any $k \geq 1$ all possible projections $T^k\mathcal{M} \to \mathcal{M}$ are equal, and we will denote by $\mathfrak{X}^k$ the set of smooth sections $\mathcal{M} \to T^k\mathcal{M}$. We will call such sections **k-vector fields**, and their values at points of $\mathcal{M}$ **k-vectors**.

There are different ways to interpret points in $T^k\mathcal{M}$, and we will use the notion of F-equivalence, introduced in [Wh82], since it is very well suited for treatment of k-morphisms, defined as jets of k-dimensional sub-manifolds in $\mathcal{M}$.

Let $\nu, \nu' : (\mathbb{R}^k, 0) \to (\mathcal{M}, p)$ be two k-jets, ν **is F-equivalent to** ν' if for any function f on $\mathcal{M}$ around p, and any $1 \leq m \leq k$, we have

$$\frac{\partial^m}{\partial_{i_1} \dots \partial_{i_m}}(f \circ \nu)|_0 = \frac{\partial^m}{\partial_{i_1} \dots \partial_{i_m}}(f \circ \nu')|_0, \tag{3}$$

if i_j's are pairwise different. Obviously this equivalence relation depends on the choice of coordinate system on $\mathbb{R}^k$.

Proposition 1 *([Wh82], [Be08]) Let p be a point on $\mathcal{M}$. There is a bijective correspondence between k-vectors and F-equivalence classes of k-jets of maps $(\mathbb{R}^k, 0) \to (\mathcal{M}, p)$.*

This correspondence is natural in $\mathcal{M}$.

The semi-simplicial structure on $\{T^k\mathcal{M}\}_{k \geq 1}$ corresponds to the diagram of coordinate subspaces of $\mathbb{R}^n$. More precisely, let $\{x_i\}_{0 \leq i < k}$ be the natural coordinate system on $\mathbb{R}^k$, then each $\nu : (\mathbb{R}^k, 0) \to (\mathcal{M}, p)$ has k faces:

$$\nu_i : (\mathbb{R}^{k-1}, 0) \to (\mathcal{M}, p), \qquad 0 \leq i \leq k-1, \tag{4}$$

with ν_i being the restriction of ν to the linear subspace of $\mathbb{R}^k$, given by vanishing of x_i.

If we choose a simplicial model for the n-groupoid of $\mathcal{M}$, i.e. if we consider k-simplices in $\mathcal{M}$ (submanifolds with corners) as k-morphisms, it is clear, that we can realize ν as the jet of a k-morphism between jets of $k-1$-morphisms $\{\nu_0, \dots, \nu_{k-1}\}$.

Note that k-morphisms are represented by k-jets, while $k-1$-morphisms are represented by $k-1$-jets. This is not really an inconsistency, since for a $k-1$-dimensional submanifold of $\mathcal{M}$, the F-equivalence class of its k-jet is completely determined by its $k-1$-jet.

To describe reparametrizations of k-vectors and k-fields, i.e. suitable changes of coordinates on $\mathbb{R}^k$, we use the dual language of morphisms between algebras of functions. Consider a sequence of Weil algebras

$$\mathcal{W}_k := \mathbb{R}[\epsilon_0, \dots, \epsilon_{k-1}]/\{\epsilon_i^2\}_{0 \leq i \leq k-1}, \quad k \geq 1.$$

For any point $p \in \mathcal{M}$, let $\mathfrak{X}^0_p$ be the stalk at p of the sheaf of functions on $\mathcal{M}$. Then points in $T^k\mathcal{M}$ over p correspond to morphisms of $\mathbb{R}$-algebras

$$\mathfrak{X}^0_p \to \mathcal{W}_k,$$

that factor the evaluation map $\mathfrak{X}^0_p \to \mathbb{R}$. Automorphisms of W_k provide reparametrizations of k-vectors, and lead to polynomial groups ([Be08]).

Among all the automorphisms we will be particularly interested in the action of the symmetric group $\mathbb{S}_k$, permuting generators of $\mathcal{W}_k$. These permutations induce an action of $\mathbb{S}_k$ on $T^k\mathcal{M}$, and this action is important to us, since choosing the order on the generators of $\mathcal{W}_k$ allows us to decompose any section $\mathcal{M} \to T^k\mathcal{M}$ into a set of sections $\mathcal{M} \to T\mathcal{M}$.

This decomposition will allow us to introduce several important operations on sections of $T^k\mathcal{M}$, and these operations are well defined since they are $\mathbb{S}_k$-invariant.

It is well known (see e.g. [Be08]) that any k-vector field $\nu \in \mathfrak{X}^k$ can be decomposed into a set $\{\alpha_\phi\}$ of 1-vector fields, indexed by *non-emtpy* subsets $\phi \subseteq \{0, \dots, k-1\}$.

Since there are different possible ways to decompose, we discuss this here in detail, and we start with an example of $\beta \in \mathfrak{X}^2$.

The two projections $T^2\mathcal{M} \rightrightarrows T\mathcal{M}$ map β to two 1-vector fields α_0, α_1. The image of $\alpha_0 : \mathcal{M} \to T\mathcal{M}$ is transversal to the fibers of $T\mathcal{M}$ over $\mathcal{M}$, and hence at the points of the image we have a decomposition of the tangent spaces into vertical and horizontal parts. Applying this decomposition to β we obtain

$$\beta \mapsto \{\alpha_0, \alpha_1, \alpha_{01}\}, \quad \alpha_0, \alpha_1, \alpha_{01} \in \mathfrak{X}^1, \tag{5}$$

where α_{01} is tangent to the fibers of $T\mathcal{M} \to \mathcal{M}$.

This decomposition, however, depends on the choice of α_0 as the first projection. More precisely, there is the canonical action of the symmetric group $\mathbb{S}_2$ on $T^2\mathcal{M}$, and of course we can permute α_0 and α_1 in (5). If $\sigma \in \mathbb{S}_2$ is the non-trivial element, we have

$$\sigma(\beta) \mapsto \{\alpha_1, \alpha_0, \alpha_{01} + [\alpha_0, \alpha_1]\}. \tag{6}$$

Appearance of $[\alpha_0, \alpha_1]$ in (6) is due to the following. Every function f on $\mathcal{M}$ defines two functions on $T\mathcal{M}$: one by composition with the projection $\pi_1 : T\mathcal{M} \to \mathcal{M}$, and the other is df. Reflecting this fact we can combine the two lifts into one **total lift** $f \circ \pi_1 + \epsilon df$ on $T\mathcal{M}$, where $\epsilon^2 = 0$.

Being a collection of tangents to $T\mathcal{M}$, β acts on the total lift $f\circ\pi_1+\epsilon df$, and using the decomposition into horizontal/vertical parts, provided by α_0, we can write this action as follows:

$$f\circ\pi_1+\epsilon df\mapsto\alpha_1(f)\circ\pi_1+\epsilon(\alpha_{01}(f)+\alpha_1\alpha_0(f))\circ\pi_1. \tag{7}$$

In fact, β is determined by its first projection to $T\mathcal{M}$, i.e. α_0, and its action (7) on the total lifts of functions from $\mathcal{M}$. On the ϵ-part this action is that of a second order differential operator, and given some choices, as for example the order on the pair α_0, α_1, we can extract the first order part: α_{01}. The opposite choice produces a different extraction: $\alpha_{01}+[\alpha_0,\alpha_1]$.

In general, a section $\nu:\mathcal{M}\to T^k\mathcal{M}$ is uniquely determined by 2^k-1 sections $\{\alpha_\phi:\mathcal{M}\to T\mathcal{M}\}$, if we fix an order on ϕ's. We use the lexicographical one.

Then a set $\{\alpha_\phi\}$ of 1-vector fields uniquely determines a k-vector field: for each ϕ of size m, the corresponding differential operator of order m is

$$\sum_{\phi_i>\ldots>\phi_1}\alpha_{\phi_i}\circ\ldots\circ\alpha_{\phi_1}, \tag{8}$$

where the sum is taken over all decompositions $\phi=\bigcup\limits_{1\le j\le i}\phi_j$ into pairwise disjoint subsets. The action of the symmetric group $\mathbb{S}_k$ on $T^k\mathcal{M}$ is expressed as follows: let $0\le i<k-1$, and let $\sigma_{i,i+1}\in\mathbb{S}_k$ be the swapping of i and $i+1$. Then $\sigma_{i,i+1}(\nu_k)$ is given by $\{\alpha'_\phi\}$, where $\alpha'_\phi=\alpha_\psi$, if $\sigma_{i,i+1}(\phi)=\psi$, $\phi\neq\psi$, and for the rest of $\phi\subseteq\{0,\ldots,k-1\}$

$$\alpha'_\phi=\alpha_\phi+\sum_{\phi'<\phi''}[\alpha_{\phi'},\alpha_{\phi''}], \tag{9}$$

where the sum is taken over all decompositions $\phi=\phi'\cup\phi''$, s.t. $\phi'\cap\phi''=\emptyset$, and $\sigma_{i,i+1}(\phi')>\sigma_{i,i+1}(\phi'')$.

There are operations on k-vector fields that are performed pointwise, i.e. they can be defined also for k-vectors, and there are operations that require sections. Now we describe some of the both types of these operations in terms of k-vector fields.

We have already mentioned that for each $k\ge1$ there are k vector bundle structures on $T^k\mathcal{M}$ over $T^{k-1}\mathcal{M}$. Obviously these k additions on $T^k\mathcal{M}$ translate to k additions on $\mathfrak{X}^k$: two sections $\{\mu_\phi\}$, $\{\nu_\phi\}$ can be added if there is a $k-1$-dimensional face $\psi\subset\{0,\ldots,k-1\}$, i.e. ψ has exactly $k-1$ elements, s.t.

$$\mu_\phi=\nu_\phi,\quad\forall\phi\subseteq\psi. \tag{10}$$

Then $\mu +_\psi \nu$ is given as follows: if $\phi \subseteq \psi$, then $(\mu +_\psi \nu)_\phi = \mu_\phi = \nu_\phi$, if $\phi \nsubseteq \psi$ then $(\mu +_\psi \nu)_\phi = \mu_\phi + \nu_\phi$.

An operation, closely related to the additions, is the **strong difference** between two 2-vectors. It is not defined for every couple of $\beta_1, \beta_2 \in T^2\mathcal{M}$, but only for those that have same projections to $T\mathcal{M}$, i.e.

$$\pi_{2,0}(\beta_1) = \pi_{2,0}(\beta_2), \quad \pi_{2,1}(\beta_1) = \pi_{2,1}(\beta_2). \tag{11}$$

In this case it is easy to see that the difference between β_1, β_2 as vectors on $\pi_{2,0}(\beta_1) \in T\mathcal{M}$ is a vector tangent to the fiber of $\pi_{2,0}(\beta_1)$ over $\mathcal{M}$. This vector can be obtained by vertical lift of some $\alpha \in T\mathcal{M}$, which is by definition the strong difference of β_1, β_2

$$\alpha := \beta_1 - \beta_2. \tag{12}$$

Clearly, one can take strong difference of a pair of k-vectors for any $k \geq 2$, considered as 2-vectors on $T^{k-2}\mathcal{M}$. This operation is defined also for k-fields, and in terms of decompositions into 1-fields it is represented as follows.

Let $\mu = \{\alpha_\phi\}$, $\nu = \{\beta_\phi\}$ be two k-fields. We can define $\mu - \nu$ only if all $k-1$-faces of μ, ν coincide, i.e. for any $\phi \subset \{0, \ldots, k-1\}$ of of size $\leq k-1$, we have $\alpha_\phi = \beta_\phi$. Then $\mu - \nu = \{\gamma_\psi\}$ is the $k-1$-field defined as follows: for any $\psi \subseteq \{0, \ldots, k-3\}$

$$\gamma_\psi = \alpha_\psi = \beta_\psi, \quad \gamma_{\psi \cup \{k-2\}} = \alpha_{\psi \cup \{k-2,k-1\}} - \beta_{\psi \cup \{k-2,k-1\}}. \tag{13}$$

There is another important pointwise operation: **the cup product**, however, it is only partially defined. Consider the following Weil algebras

$$\mathcal{V}_k := \mathbb{R}[\varepsilon_1, \ldots, \varepsilon_k]/(\varepsilon_i \varepsilon_j).$$

For $k > 1$ $\mathcal{V}_k \neq \mathcal{W}_k$, and there are several $\mathbb{R}$-algebra morphisms $\mathcal{V}_k \to \mathcal{W}_k$.

Now let p be a point in $\mathcal{M}$, and let $x : \mathfrak{X}^0_p \to \mathcal{W}_k$, $y : \mathfrak{X}^0_p \to \mathcal{W}_m$ be a k-vector and an m-vector at p. Suppose that y factors through some $g : \mathcal{V}_m \to \mathcal{W}_m$. Then we can define a $k+m$-vector $x \cup y$, using the following $\mathbb{R}$-algebra morphism

$$\mathcal{W}_k \prod_{\mathbb{R}} \mathcal{V}_m \to \mathcal{W}_{k+m}, \quad \epsilon_i \mapsto \epsilon_i, \quad \varepsilon_j \mapsto h \circ g(\varepsilon_j)\epsilon_0 \ldots \epsilon_{k-1}, \tag{14}$$

where $h : \mathcal{W}_m \to \mathcal{W}_{k+m}$ maps ϵ_i to ϵ_{k+i}.

For example: if $m = k = 1$, then $x \cup y$ is the evaluation at x of the well known "vertical lift" of y to $T\mathcal{M}$. Similarly there are vertical lifts of

1-vectors to $T^k\mathcal{M}$ for $k>1$. If $m>1$, this operation is not everywhere defined anymore, but only for those $y:\mathfrak{X}^0_p \to \mathcal{W}_m$, that factor through $\mathcal{V}_m$.

Notice that the cup product is invariant with respect to the action of symmetric groups, i.e. if $\sigma_k \in \mathbb{S}_k$, $\sigma_m \in \mathbb{S}_m$, then

$$(\sigma_k x) \cup (\sigma_m y) = (\sigma_k \times \sigma_m)(x \cup y). \tag{15}$$

In terms of sequences of 1-vector fields cup product is represented as follows: let $\{\alpha_\phi\} = \mu \in \mathfrak{X}^k$ and $\{\beta_\psi\} = \nu \in \mathfrak{X}^m$, and suppose their cup product $\{\gamma_\chi\} = \mu \cup \nu \in \mathfrak{X}^{k+m}$ is everywhere defined (i.e. $\nu : \mathfrak{X}^0 \to \mathcal{W}_m \otimes_{\mathbb{R}} \mathfrak{X}^0$ factors through $\mathcal{V}_m \otimes_{\mathbb{R}} \mathfrak{X}^0$), then

$$\gamma_\phi = \alpha_\phi, \quad \gamma_{\phi\cup\psi} = \beta_\psi, \tag{16}$$

and the rest of components are 0. Here we consider $\phi \subseteq \{0,\dots,k-1\}$ and $\psi \subseteq \{0,\dots,m-1\}$ as subsets of $\{0,\dots,k+m-1\}$ given by the (lexicographical) order preserving bijection

$$\{0,\dots,k-1\} \coprod \{0,\dots,m-1\} \xrightarrow{\simeq} \{0,\dots,k+m-1\}. \tag{17}$$

An important operation, that is not defined pointwise, is **the composition product** of vector fields. Let $\mu \in \mathfrak{X}^k$, $\nu \in \mathfrak{X}^m$. Since μ is a map $\mathcal{M} \to T^k\mathcal{M}$, applying the tangent functor m times we obtain a map $\mu_* : T^m\mathcal{M} \to T^{m+k}\mathcal{M}$. Evaluating at $\nu : \mathcal{M} \to T^m\mathcal{M}$ we get a section $\mu \times \nu : \mathcal{M} \to T^{m+k}\mathcal{M}$.

In terms of sequences of 1-vector fields this operation is written as follows. Let $\{\alpha_\phi\}$, $\{\beta_\psi\}$, $\{\gamma_\chi\}$ be $\mu, \nu, \mu \times \nu$ respectively. Then

$$\gamma_\phi = \alpha_\phi, \quad \gamma_\psi = \beta_\psi, \tag{18}$$

and the rest of components are 0. Here again we consider ϕ, ψ as subsets of $\{1,\dots,k+m\}$ using (17).

Notice that also composition product is invariant with respect to the action of symmetric groups, i.e. for any $\sigma_k \in \mathbb{S}_k$, $\sigma_m \in \mathbb{S}_m$ we have

$$(\sigma_k \mu) \times (\sigma_m \nu) = (\sigma_k \times \sigma_m)(\mu \times \nu). \tag{19}$$

2 Relatively free Lie-Rinehart algebras

In this section we recall (from [Ka07]) the construction of the bundle of infinitesimal paths on a manifold, and relativize it, i.e. construct bundles of infinitesimal paths, where we contract some of the loops. This relative version has nice functorial properties, and we use them in the next section to build the iterated bundles of infinitesimal paths on a manifold.

In [Ka07] the space of infinitesimal paths on $\mathcal{M}$ is defined as the free Lie-Rinehart algebra, generated by $\mathfrak{X}^1$. Here are the details of this free construction in the general case.

Let A be an $\mathbb{R}$-algebra, and let M be an A-module **with an anchor**, i.e. there is an A-map $M \to Der_{\mathbb{R}}(A)$. Let $\mathcal{L}(M)$ be the free Lie algebra over $\mathbb{R}$, generated by M. By its construction $\mathcal{L}(M)$ is a graded space, with $\mathcal{L}(M)_d$ being generated by Lie monomials of length d. There is an obvious extension of the anchor $\mathcal{L}(M) \to Der_{\mathbb{R}}(A)$.

The free Lie-Rinehart algebra $\mathcal{R}(M)$, generated by M, is a filtered A-module, inductively defined as follows:

- $\mathcal{R}(M)_{\leq 1} = M$,
- for $n > 1$ $\mathcal{R}(M)_{\leq d}$ is $\bigoplus_{1 \leq i \leq d} \mathcal{L}(M)_i$ modulo the following relations

$$[x, fy] - [fx, y] = x(f)y + y(f)x, \quad [x, q] = 0, \tag{20}$$

  where $x, y \in \mathcal{L}(M)$, $f \in A$, q is in the kernel of

$$\bigoplus_{1 \leq i \leq d-1} \mathcal{L}(M)_i \to \mathcal{R}(M)_{\leq d-1},$$

  and for any $x \in \mathcal{L}(M)$ we write $x(f)$ for the action of x on f through the anchor.

It is easy to see that the kernel of $\mathcal{L}(M) \to \mathcal{R}(M)$ is a Lie ideal, and hence $\mathcal{R}(M)$ inherits a Lie structure. Also it is easy to check that the anchor $\mathcal{L}(M) \to Der_{\mathbb{R}}(A)$ vanishes on the kernel of $\mathcal{L}(M) \to \mathcal{R}(M)$, and hence there is a well defined action of $\mathcal{R}(M)$ on A. Finally, the action of A on $\mathcal{R}(M)$ is given by

$$f[x, y] = [fx, y] + y(f)x = [x, fy] - x(f)y. \tag{21}$$

Applying the above construction to the set $\mathfrak{X}^1$ of vector fields on $\mathcal{M}$, we get a new Lie-Rinehart algebra $\mathcal{R}(\mathfrak{X}^1)$, which is the space of infinitesimal paths, without contractions of surfaces, except for the degenerate ones (i.e. we do have $[\alpha, \alpha] = 0$).

The sheaf $\mathcal{R}(\mathfrak{X}^1)$ is locally free, and hence we can form the linear bundle $\mathcal{R}T\mathcal{M}$, having $\mathcal{R}(\mathfrak{X}^1)$ as the set of sections. We would like to iterate this construction, i.e. we would like to consider spaces of infinitesimal paths on spaces of infinitesimal paths on $\mathcal{M}$, and so on. First we need to establish some of the functorial properties of $\mathcal{R}$.

Proposition 2 *Let $\mathcal{M}$, $\mathcal{N}$ be two manifolds, and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map, that locally (on $\mathcal{M}$) is an embedding. Then F extends to a map of pairs*

$$\begin{array}{ccc} \mathcal{R}T\mathcal{M} & \xrightarrow{\mathcal{R}F} & \mathcal{R}T\mathcal{N} \\ \downarrow & & \downarrow \\ \mathcal{M} & \xrightarrow[F]{} & \mathcal{N} \end{array} \tag{22}$$

and this extension is functorial in F.

Proof: Since construction of $\mathcal{R}T$ can be done locally ([Ka07]), we can (choosing local coordinates) assume that $\mathcal{M} = \mathbb{R}^m$, $\mathcal{N} = \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$, and $F : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$ is inclusion of a coordinate subspace.

Using the natural flat structure on $\mathbb{R}^n$ we have a map

$$F_* : \mathfrak{X}^1(\mathbb{R}^m) \to \mathfrak{X}^1(\mathbb{R}^n), \tag{23}$$

that is a morphism of Lie algebras. Consequently there is an induced morphism

$$\mathcal{L}(F_*) : \mathcal{L}(\mathfrak{X}^1(\mathbb{R}^m)) \to \mathcal{L}(\mathfrak{X}^1(\mathbb{R}^n)).$$

There is also a projection $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$, and hence we have an inclusion of algebras of functions

$$\mathfrak{X}^0(\mathbb{R}^m) \to \mathfrak{X}^0(\mathbb{R}^n). \tag{24}$$

Using this inclusion it is easy to see that $\mathcal{L}(F_*)$ maps kernel of $\mathcal{L}(\mathfrak{X}^1(\mathbb{R}^m)) \to \mathcal{R}(\mathfrak{X}^1(\mathbb{R}^m))$ to the kernel of $\mathcal{L}(\mathfrak{X}^1(\mathbb{R}^m)) \to \mathcal{R}(\mathfrak{X}^1(\mathbb{R}^m))$, and hence we have an $\mathbb{R}$-linear map

$$\mathcal{R}(F_*) : \mathcal{R}(\mathfrak{X}^1(\mathbb{R}^m)) \to \mathcal{R}(\mathfrak{X}^1(\mathbb{R}^n)).$$

This map is also $\mathfrak{X}^0(\mathbb{R}^m)$-linear, where we see $\mathcal{R}(\mathfrak{X}^1(\mathbb{R}^n))$ as an $\mathfrak{X}^0(\mathbb{R}^m)$-module through (24).

Now we compose $\mathcal{R}(F_*)$ with the projection

$$\mathcal{R}(\mathfrak{X}^1(\mathbb{R}^n)) \to \mathcal{R}(\mathfrak{X}^1(\mathbb{R}^n)) \otimes_{\mathfrak{X}^0(\mathbb{R}^n)} \mathfrak{X}^0(\mathbb{R}^m),$$

and obtain a morphism of bundles

$$\mathcal{R}F : \mathcal{RTM} \to F^*(\mathcal{RTN}). \tag{25}$$

We claim that $\mathcal{R}F$ is independent of the choice of local coordinates on $\mathcal{N}$. Indeed, a different choice produces different maps in (23), (24), but they become the same, when restricted to the image of $\mathcal{M}$ in $\mathcal{N}$, and it is easy to check that this implies the resulting $\mathcal{R}F$'s are equal.

Having two smooth maps $\mathcal{M} \to \mathcal{N} \to \mathcal{N}'$, s.t. each one is a local embedding, it is clear that locally we can present them as inclusions of coordinate subspaces $\mathbb{R}^m \to \mathbb{R}^{m+k} \to \mathbb{R}^{m+k+k'}$, and the corresponding choice of local coordinates implies functoriality. ■

Now we would like to iterate the $\mathcal{RT}$ construction. In some of our arguments we assume that the manifolds in question are finite dimensional, while $\mathcal{RTM}$ is infinite dimensional, if dimension of $\mathcal{M}$ is greater than 1. However, since $\mathcal{RTM}$ is filtered, and each $\mathcal{RTM}_{\leq d}$ is finite dimensional, we can treat $\mathcal{RTM}$ as if it is finite dimensional itself, as long as everything that we do happens in some $\mathcal{RTM}_{\leq d}$ for d large enough.

We are interested in particular in tangent vectors to $\mathcal{RTM}$, and we will always assume that each vector is tangent to some $\mathcal{RTM}_{\leq d}$. Therefore, as long as there are only finitely many tangents involved, there is a finite dimensional manifold $\mathcal{RTM}_{\leq d}$, where these tangents live.

Let $\mathfrak{X}^1(\mathcal{RTM})$ be the space of vector fields on $\mathcal{RTM}$, s.t. for every field α there is $d < \infty$, s.t. α is tangent to $\mathcal{RTM}_{\leq d}$. Applying $\mathcal{R}$ for each d, and using functoriality of $\mathcal{R}$ with respect to local embeddings, we obtain a bundle $(\mathcal{RT})^2\mathcal{M}$ of infinitesimal paths on $\mathcal{RTM}$.

Iterating this procedure further, we get a sequence of (filtered infinite dimensional) manifolds $\{(\mathcal{RT})^k\mathcal{M}\}_{k\geq 1}$. However, this sequence is not the right substitute for the (semi-simplicial) sequence $\{T^k\mathcal{M}\}_{k\geq 1}$ of iterated tangent bundles.

For $k \geq 2$ $(\mathcal{RT})^k\mathcal{M}$ is too big. For example, there are loops in $(\mathcal{RT})^2\mathcal{M}$ that are built of vector fields, tangent to the fibers of $\mathcal{RTM} \to \mathcal{M}$. These fibers are linear spaces and have a natural flat connection. As far as paths and surfaces on $\mathcal{M}$ are concerned, we are interested only in flat vertical fields on $\mathcal{RTM}$, and the corresponding loops have unique flat fillings.

All this forces us to introduces a relative version of the free Lie-Rinehart algebra construction. Instead of anchored modules we have the following.

Definition 1 *Let A be a commutative $\mathbb{R}$-algebra, and let $(\mathfrak{g} \xrightarrow{a} Der_{\mathbb{R}}(A))$ be a Lie-Rinehart algebra.* **An anchored $(\mathfrak{g}, A)$-module** *is an A-module M,* *together with A-maps*

$$\mathfrak{g} \xrightarrow{\iota} M \xrightarrow{b} Der_{\mathbb{R}}(A),$$

s.t. ι *is injective,* $a = b\iota$*, and having a Lie module structure* $\mathfrak{g} \otimes_{\mathbb{R}} M \to M$*, s.t.*

$$\gamma(fm) = \gamma(f)m + f\gamma(m), \quad (f\gamma)(m) = f\gamma(m) - m(f)\iota(\gamma), \tag{26}$$

$$b(\gamma(m)) = [a(\gamma), b(m)]. \tag{27}$$

An example of an anchored $(\mathfrak{g}, A)$-module is given by a morphism of Lie-Rinehart algebras $\mathfrak{g} \to \mathfrak{h}$ over A, where we take $\mathfrak{h}$ to be M.

If we fix $\mathfrak{g}$ and A we have a forgetful functor from the category of Lie-Rinehart algebras under $\mathfrak{g}$ to the category of anchored $(\mathfrak{g}, A)$-modules. This functor has a left adjoint, that we now describe.

Let M be an anchored $(\mathfrak{g}, A)$-module. Recall that $\mathcal{L}(M)$ is the free Lie algebra over $\mathbb{R}$, generated by M. The action of $\mathfrak{g}$ on M extends to an action on $\mathcal{L}(M)$, by requiring that $\gamma([x, y]) = [\gamma(x), y] + [x, \gamma(y)]$.

We inductively define $\mathcal{R}(M, \mathfrak{g})$ as follows:

- $\mathcal{R}(M, \mathfrak{g})_{\leq 1} = M$,
- for $d > 1$ $\mathcal{R}(M, \mathfrak{g})_{\leq d}$ is $\bigoplus\limits_{1 \leq i \leq d} \mathcal{L}(M)_i$ modulo the following relations

$$[x, fy] - [fx, y] = x(f)y + y(f)x, \quad [x, q] = 0, \tag{28}$$

$$[\iota(\gamma), x] = \gamma(x), \tag{29}$$

  where $x, y \in \mathcal{L}(M)$, $f \in A$, $\gamma \in \mathfrak{g}$, q is in the kernel of

$$\bigoplus_{1 \leq i \leq n-1} \mathcal{L}(M)_i \to \mathcal{R}(M)_{\leq n-1},$$

  and for any $x \in \mathcal{L}(M)$ we write $x(f)$ for the action of x on f through the anchor.

Here we consider $\mathcal{L}(M)$ as an $\mathbb{R}$-space, and divide it by the subspace, generated by (28), (29). From the construction it is clear that $\mathcal{R}(M, \mathfrak{g})$ is a Lie algebra over $\mathbb{R}$, inheriting the Lie structure from $\mathcal{L}(M)$. We claim that in addition $\mathcal{R}(M, \mathfrak{g})$ is an A-module, and the projection $\mathcal{L}(M) \to \mathcal{R}(M, \mathfrak{g})$ factors through $\mathcal{R}(M)$.

Notice that the action of $\mathfrak{g}$ on $\mathcal{L}(M)$ is compatible with (28), i.e. the kernel of $\mathcal{L}(M) \to \mathcal{R}(M)$ is stable under the action of $\mathfrak{g}$. Therefore this action extends to $\mathcal{R}(M)$, and hence (29) are well defined on $\mathcal{R}(M)$. This implies that $\mathcal{L}(M) \to \mathcal{R}(M, \mathfrak{g})$ factors through $\mathcal{R}(M)$.

There is an A-module structure on $\mathcal{R}(M)$, given by $f[x, y] = [fx, y] + y(f)x = [x, fy] - x(f)y$. We claim that the kernel of $\mathcal{R}(M) \to \mathcal{R}(M, \mathfrak{g})$ is an A-submodule. Indeed, we have

$$f([\iota(\gamma), x] - \gamma(x)) = [\iota(\gamma), fx] - \gamma(fx), \tag{30}$$

for any $x \in \mathcal{L}(M)$, $f \in A$, and $\gamma \in \mathfrak{g}$. Consequently $\mathcal{R}(M, \mathfrak{g})$ is an A-module.

Finally we note that $\mathcal{R}(M, \mathfrak{g})$ inherits an action on A from $\mathcal{R}(M)$. This is rather obvious, since elements of the kernel of $\mathcal{R}(M) \to \mathcal{R}(M, \mathfrak{g})$ act trivially on A (this is a consequence of (27)).

For us the main applications of the relatively free Lie-Rinehart algebra construction are for **locally trivial bundles**, i.e. smooth maps $F : \mathcal{M} \to \mathcal{N}$, s.t. locally (on $\mathcal{M}$) F is a trivial bundle. Let $\mathfrak{X}^1(F)$ be the vector fields tangent to the fibers of F, then we have the relatively free Lie-Rinehart algebra $\mathcal{R}(\mathfrak{X}^1(\mathcal{M}), \mathfrak{X}^1(F))$ over $\mathfrak{X}^0(\mathcal{M})$, that defines the bundle $\mathcal{R}T_{\mathcal{N}}\mathcal{M}$ of **F-horizontal infinitesimal paths** on $\mathcal{M}$.

If we choose F to be the unique map $\mathcal{M} \to pt$, we have $\mathcal{R}T_{pt}\mathcal{M} = T\mathcal{M}$, the usual tangent bundle. If we choose F to be the identity map $\mathcal{M} = \mathcal{M}$, we have $\mathcal{R}T_{\mathcal{M}}\mathcal{M} = \mathcal{R}T\mathcal{M}$, the space of infinitesimal paths from [Ka07].

A nice property of $\mathcal{R}T_{\mathcal{N}}\mathcal{M}$, that will be used in the next section, is that it is functorial in both arguments, as the following proposition shows.

Proposition 3 *Let $F_1 : \mathcal{M}_1 \to \mathcal{N}_1$, $F_2 : \mathcal{M}_2 \to \mathcal{N}_2$ be two locally (on domains) trivial bundles. Suppose we are given a commutative diagram of smooth maps*

$$\begin{array}{ccc} \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 \\ \downarrow & & \downarrow \\ \mathcal{N}_1 & \longrightarrow & \mathcal{N}_2 \end{array} \tag{31}$$

s.t. also the horizontal arrows are locally trivial bundles. Then we have a smooth map

$$\mathcal{R}T_{\mathcal{N}_1}\mathcal{M}_1 \to \mathcal{R}T_{\mathcal{N}_2}\mathcal{M}_2, \tag{32}$$

and (32) is functorial in (31).

Proof: First we prove functoriality in the second variable, i.e. consider the diagram

$$\begin{array}{ccc} & \mathcal{M}_1 & \\ {}^{F_1}\swarrow & & \searrow^{F_2} \\ \mathcal{N}_1 & \longrightarrow & \mathcal{N}_2 \end{array} \tag{33}$$

It is clear that $\mathfrak{X}^1(F_1) < \mathfrak{X}^1(F_2) < \mathfrak{X}^1(\mathcal{M}_1)$ as Lie algebras, and therefore it is easy to check that there is a canonical surjective morphism of Lie-Rinehart algebras

$$\mathcal{R}(\mathfrak{X}^1(\mathcal{M}_1), \mathfrak{X}^1(F_1)) \to \mathcal{R}(\mathfrak{X}^1(\mathcal{M}_1), \mathfrak{X}^1(F_2)),$$

and hence a morphism of bundles $\mathcal{R}T_{\mathcal{N}_1}\mathcal{M}_1 \to \mathcal{R}T_{\mathcal{N}_2}\mathcal{M}_1$, that is obviously functorial in (33).

Now we prove functoriality in the first variable. Choosing local coordinates we can represent F_1 as the projection $\mathbb{R}^{k+n} = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, and then every element of $\mathcal{R}(\mathfrak{X}^1(\mathcal{M}_1), \mathfrak{X}^1(F_1))$ can be written as follows:

$$\sum_{i=1}^{\infty} f_{j_1,\ldots,j_i}[\partial_{j_1}, [\ldots [\partial_{j_{i-1}}, \partial_{j_i}] \ldots], \tag{34}$$

where the sum is finite, and $i > 1$ implies that $f_{j_1,\ldots,j_i} = 0$ if at least one of ∂_j's is tangent to the fibers of F_1.

Consider the following diagram

$$\begin{array}{ccc} \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 \\ {}_{F_1}\searrow & & \swarrow_{F_2} \\ & \mathcal{N}_2 & \end{array} \tag{35}$$

Since all maps are locally trivial bundles, we can choose local coordinates in a compatible way, i.e. locally (35) becomes

$$\begin{array}{ccc} \mathbb{R}^{n+m_1+m_2} & \longrightarrow & \mathbb{R}^{n+m_2} \\ {}_{F_1}\searrow & & \swarrow_{F_2} \\ & \mathbb{R}^n & \end{array} \tag{36}$$

and it is clear how to define the map $\mathcal{R}T_{\mathcal{N}_2}\mathcal{M}_1 \to \mathcal{R}T_{\mathcal{N}_2}\mathcal{M}_2$ locally. Functoriality and independence of the choice of coordinates are easy to check. ■

3 The full infinitesimal groupoid

In this section we use the relatively free Lie-Rinehart construction of the previous section to define the sequence $\{\mathbb{T}^k\mathcal{M}\}_{k\geq 0}$ of iterated bundles of spaces of infinitesimal paths on a manifold $\mathcal{M}$. In particular we get a semi-simplicial structure on $\{\mathbb{T}^k\mathcal{M}\}_{k\geq 1}$.

For $k \geq 1$ we define $\mathbb{X}^k$ to be the set of sections $\mathcal{M} \to \mathbb{T}^k\mathcal{M}$, and $\mathbb{X}^0 := \mathfrak{X}^0(\mathcal{M})$. We show how every $\nu \in \mathbb{X}^k$, $k \geq 1$, can be decomposed into a sequence $\{\alpha_\phi\}$, with $\alpha_\phi \in \mathbb{X}^1$, and ϕ running over non-empty subsets of $\{0, \ldots, k-1\}$. This allows us to define a rich algebraic structure on $\mathbb{X}^* := \{\mathbb{X}^k\}_{k\geq 0}$, and we call it *the full infinitesimal groupoid* of $\mathcal{M}$.

Finally we show that the non-linear cohomology of $\mathbb{X}^*$ is the algebra of polyvector fields on $\mathcal{M}$.

Definition 2 *Let $\mathcal{M}$ be a manifold. We define a sequence of (in general non-linear) locally trivial bundles $\{\pi_k : \mathbb{T}^k\mathcal{M} \to \mathcal{M}\}_{k\geq 0}$ as follows: $\mathbb{T}^0\mathcal{M} := \mathcal{M}$, if we have defined $\pi_k : \mathbb{T}^k\mathcal{M} \to \mathcal{M}$, then*

$$\mathbb{T}^{k+1}\mathcal{M} := \mathcal{R}T_{\mathcal{M}}(\mathbb{T}^k\mathcal{M}), \tag{37}$$

and the projection $\pi_{k+1} : \mathbb{T}^{k+1}\mathcal{M} \to \mathcal{M}$ is obvious.

Notice that, just like $\{T^k\mathcal{M}\}_{k\geq 1}$, the sequence $\{\mathbb{T}^k\mathcal{M}\}_{k\geq 1}$ has a semi-simplicial structure, i.e. for each $k \geq 1$ there are k projections

$$\pi_{k,i} : \mathbb{T}^k\mathcal{M} \to \mathbb{T}^{k-1}\mathcal{M}, \quad 0 \leq i \leq k-1.$$

When $i = 0$ $\pi_{k,0}$ is the projection of the bundle $\mathcal{R}T_{\mathcal{M}}(\mathbb{T}^{k-1}\mathcal{M})$ on $\mathbb{T}^{k-1}\mathcal{M}$, when $i > 0$ $\pi_{k,i}$ is obtained from $\pi_{k-i,0}$ by functoriality of $\mathcal{R}T$.

Moreover, each $\pi_{k,i}$ is a linear bundle. This is a general fact: consider a morphism of locally trivial bundles

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{F} & \mathcal{M}_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & \mathcal{B} & \end{array}$$

s.t. F is a linear bundle. By functoriality of $\mathcal{R}T$ we have the diagram

$$\begin{array}{ccc} \mathcal{R}T_{\mathcal{B}}\mathcal{M}_1 & \xrightarrow{\mathcal{R}T_{\mathcal{B}}(F)} & \mathcal{R}T_{\mathcal{B}}\mathcal{N} \\ & {\scriptstyle \mathcal{R}T_{\mathcal{B}}(\pi_1)}\searrow \quad \swarrow{\scriptstyle \mathcal{R}T_{\mathcal{B}}(\pi_2)} & \\ & \mathcal{R}T_{\mathcal{B}}\mathcal{B} & \end{array} \tag{38}$$

It is easy to see that fibers of $\mathcal{R}T_{\mathcal{B}}(\pi_1)$ are the tangent bundles to the fibers of π_1, and similarly for π_2, and therefore fibers of $\mathcal{R}T_{\mathcal{B}}(F)$ are the same as fibers of $T(F)$. It is well known (e.g. [MK05]) that the latter is a linear bundle, when F is.

Let $\mathbb{X}^0 := \mathfrak{X}^0(\mathcal{M})$, and let $\mathbb{X}^k$ be the set of sections of $\mathbb{T}^k\mathcal{M} \to \mathcal{M}$. Just like with $\mathfrak{X}^k$, using functoriality of $\mathcal{R}T$ with respect to local embeddings (proposition 2), we obtain a decomposition of each $\nu \in \mathbb{X}^k$ into a set $\{\alpha_\phi\}$, where each $\alpha \in \mathbb{X}^1$, and ϕ runs over all non-empty subsets of $\{1,\dots,k\}$.

As with $\mathfrak{X}^k$, for $\nu \in \mathbb{X}^k$ the decomposition into $\{\alpha_\phi\}$ depends on the order on the set of projections of ν on $\mathbb{X}^1$. Also here we have an action of the symmetric group $\mathbb{S}_k$ given as in (9), but with the Lie bracket substituted by the free bracket $[\![-,-]\!]$ on $\mathbb{X}^1 = \mathcal{R}T(\mathfrak{X}^1)$.

It is clear how to extend additions, strong differences, the cup product, and the composition product from $\{\mathfrak{X}^k\}$ to $\{\mathbb{X}^k\}$, and we will freely use the notation of section 1. We will call $\mathbb{X}^* := \{\mathbb{X}^k\}$ together with these (and other) operations **the full infinitesimal groupoid** of $\mathcal{M}$.

In addition to the operations listed above, $\mathbb{X}^*$ has **homotopy operations**, defined as maps $\{\mathbb{X}^k \overset{h^k_{ij}}{\to} \mathbb{X}^{k-1}\}_{0\le i<j\le k-1}$ for each $k \geq 2$.

To define the homotopy operations we notice that $\mathbb{X}^1$ has actually two Lie structures. They come from two Lie structures on $\mathcal{L}(\mathfrak{X}^1)$. The first one is the free bracket $[\![-,-]\!]$ given by $\mathcal{L}$, and the other is the Lie bracket $[-,-]$ on $\mathfrak{X}^1$, extended to $\mathcal{L}(\mathfrak{X}^1)$ by the requirement that

$$[x,[\![y,z]\!]] = [\![[x,y],z]\!] + [\![y,[x,z]]\!].$$

It is easy to check that the kernel of $\mathcal{L}(\mathfrak{X}^1) \to \mathcal{R}(\mathfrak{X}^1)$ is a Lie ideal also for $[-,-]$, and hence $\mathbb{X}^1$ inherits $[-,-]$.

Now, having an additional bracket on $\mathbb{X}^1$, we have an additional action of $\mathbb{S}_k$ on $\mathbb{X}^k$, written in terms of $\{\alpha_\phi\}$, $\alpha_\phi \in \mathbb{X}^1$. That is, we apply the same formula (9), but instead of the free bracket we use $[-,-]$.

For $k \geq 2$ let $\nu \in \mathbb{X}^k$, and let $\sigma_{ij} \in \mathbb{S}_k$ be the swapping of i and j. Let $\mu \in \mathbb{X}^{k-2}$ be the projection of ν on the ϕ-face, where $\phi = \{0,\dots,\widehat{i},\dots,\widehat{j},\dots,k-1\}$ (if $k=2$ μ is just $\mathcal{M}$ itself). Clearly μ is also projection of $\sigma_{ij}(\nu), \sigma'_{ij}(\nu)$, where σ_{ij} acts using $[\![-,-]\!]$, and σ'_{ij} acts using $[-,-]$.

As 2-vector fields on the image of μ in $\mathbb{T}^{k-2}\mathcal{M}$, $\sigma_{ij}(\nu), \sigma'_{ij}(\nu)$ have the same boundaries, and we can take their strong difference. It is an element of $\mathbb{X}^{k-1}$, and we define $h^k_{ij}(\nu)$ to be this element. A straightforward but long computation shows that modulo homotopies $h^{<k}_{**}$, h^k_{ij} is well defined with respect to the actions of symmetric groups, i.e. for any $\sigma \in \mathbb{S}_k$ there is $\tau \in \mathbb{S}_{k-1}$ s.t.

$$h^k_{\sigma(i)\sigma(j)}(\sigma\nu) \sim \tau(h^k_{ij}(\nu)).$$

The equivalence relation on $\mathbb{X}^{k-1}$, defined by $h^k_{ij}(\nu)$ is the following: for $\xi, \xi' \in \mathbb{X}^{k-1}$ $\xi \sim \xi'$ if $\xi -_\phi \xi' = h^k_{ij}(\nu)$. Consequently, it is natural to say that $\nu \in \mathbb{X}^k$ defines **trivial homotopies**, if $\sigma_{ij}(\nu) = \sigma'_{ij}(\nu)$ for any $0 \leq i < j \leq k-1$, i.e. if $h^k_{ij}(\nu)$ consists of trivial vectors on the ϕ-face of ν.

What do we get if we take the subset of $\mathbb{X}^*$, consisting of elements, that define trivial homotopies, and divide it by the homotopy equivalence? We denote the result by $\mathcal{H}(\mathbb{X}^*)$ and call it **the cohomology of $\mathbb{X}^*$.**

First of all it is clear that any $\nu \in \mathbb{X}^k$ is equivalent to some $\nu' \in \mathfrak{X}^k$, and no two $\nu', \nu'' \in \mathfrak{X}^k$ are equivalent.

Secondly, a $\{\alpha_\phi\} = \nu \in \mathfrak{X}^k$ defines trivial homotopies if and only if for any $\phi, \psi \subseteq \{0, \dots, k-1\}$, s.t. $\phi \cap \psi = \emptyset$ we have that either $\alpha_\phi = \alpha_\psi$ or at least one of α_ϕ, α_ψ is 0. This is just the condition for $[\![\alpha_\phi, \alpha_\psi]\!] = 0$.

There are quite many such ν's, but not that many if we discard degenerate k-submanifolds and divide by reparametrizations. Note, that taking the canonical zero section, we can consider any $\alpha \in \mathfrak{X}^1$ as a section of $T^2\mathcal{M}$, i.e. we take $\alpha \cup 0$, and this is obviously a degenerate 2-vector field. Also $\alpha \cup \alpha$ is degenerate, since it is the jet of a 1-dimensional submanifold. Together we have

$$\alpha \sim \alpha \cup 0 \sim \alpha \cup \alpha \sim 0 \cup \alpha,$$

In addition, for $k \geq 2$ we have reparametrizations of k-vectors, i.e. maps $\mathcal{W}_k \to \mathcal{W}_k$, that have 1 as their Jacobian.

Discarding degenerate fields implies that any $\nu \in \mathfrak{X}^*$, that defines trivial homotopies, is equivalent to some $\nu' = \{\alpha'_\phi\}$, s.t. $\alpha'_\phi = 0$, unless $\phi = \{0, \dots, i\}$ for some $i \leq k-1$. Dividing by reparametrizations means that ν' is linear over $\mathfrak{X}^0$ in each one of α'_ϕ's, and discarding degenerate fields again we get that ν' is an element of some alternating power of $\mathfrak{X}^1$ over $\mathfrak{X}^0$, i.e. $\mathcal{H}(\mathbb{X}^*)$ is the set of decomposable elements of $\wedge^*\mathfrak{X}^1$. Taking the $\mathfrak{X}^0$-module generated by $\mathcal{H}(\mathbb{X}^*)$ we get all of $\wedge^*\mathfrak{X}^1$.

Finally we discuss algebraic operations on $\mathcal{H}(\mathbb{X}^*)$. On $\mathbb{X}^*$ we have some additions, cup product, and the composition product. Additions translate to the addition on $\wedge^*\mathfrak{X}^1$, and the cup product becomes the wedge product.

With the composition product it is not as simple. Recall how one defines the Lie derivative of a vector field α along another vector field β: one takes the composition product $\alpha \times \beta \in \mathfrak{X}^2$, uses trivialization of $T\mathcal{M}$ over the integral curves of β to find the projection of $\alpha \times \beta$ on the fibers of $T\mathcal{M}$, and then uses the linear structure on these fibers to identify tangents to fibers with their points. The resulting section of $T\mathcal{M}$ is $[\beta, \alpha]$.

In terms of higher categories this is what is called **a thin structure**, i.e. $\alpha \times \beta$ is not the composition, but one of its faces is. Since $\mathcal{H}(\mathbb{X}^*)$ is not additive, but multi-linear, we need not one face but many, and the resulting operation is the Schouten bracket.[2]

We would like to stress that $\wedge^* \mathfrak{X}^1$, obtained as above from $\mathcal{H}(\mathbb{X}^*)$, is not a Gerstenhaber algebra. Elements of $\wedge^k \mathfrak{X}^1$ represent k-morphisms, and hence sit in degree $-k+1$, if we use cohomological notation. Therefore, while the bracket is of degree 0, the cup product is of degree -1.

References

[Be08] W.Bertram. *Differential geometry over general base fields and rings.* Mem. Amer. Math. Soc. 192, no. 900, x+202 pp. (2008).

[Ka07] M.Kapranov. *Free Lie algebroids and the space of paths.* Sel. math., New ser. 13, pp. 277-319 (2007).

[KL84] A.Kock, R.Lavendhomme. *Strong infinitesimal linearity, with applications to strong difference and affine connections.* Cahiers de Topologie et Geom. Diff., 25, pp. 311-324 (1984).

[MK05] K.C.H.Mackenzie. *General theory of Lie groupoids and Lie algebroids.* London Mathematical Society, Lecture Note Series 213 (2005), 539 pages.

[MR91] I.Moerdijk, G.E.Reyes. *Models for smooth infinitesimal analysis.* Springer (1991), 399+X pages.

[Wh82] J.E.White. *The method of iterated tangents with applications in local Riemannian geometry.* Pitman, Monographs and Studies in Mathematics 13 (1982), 272 pages.

[2]Composition product is also a thin structure on all of $\mathbb{X}^*$, but its algebra of faces is beyond the scope of this paper.