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Interpolating coefficient systems and p-ordinary cohomology of arithmetic groups

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For Fritz Grunewald on the occasion of his 60th birthday

Abstract. We present an approach to Hida's theory of p-adically interpolating the ordinary cohomology of arithmetic groups and we discuss some application of this approach to special values of L-functions.

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Contents

1. Introduction

For the basic notations we refer to [Ha-Coh], Chap. 3, Sect. 1 and Sect. 2, 2.1. We consider a reductive group $G/\mathbb{Q}$, an open compact subgroup $K_f \subset G(\mathbb{A}_f)$, a subgroup $K_\infty \subset G(\mathbb{R})$, and put

$$S^G_{K_f} = G(\mathbb{Q}) \setminus \big(G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f\big).$$

We choose a highest weight λ and use the resulting highest weight module $\mathcal{M}_\lambda$ to construct a sheaf $\widetilde{\mathcal{M}}_\lambda$ on the space $S^G_{K_f}$, we are interested in the sheaf cohomology groups $H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_\lambda)$.

We pick a prime p. We assume in addition that we have chosen a lattice $\mathcal{M}_\mathbb{Z}$ which is K_f stable (see loc. cit. 2, 2.1). Then we can consider $H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}})$, actually we study the cohomology groups $H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p)$. The choice of these lattices $\widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}}$ or $\widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p$ has to be done with some care.

We recall the construction of Hecke operators from [Ha-Coh], Chap. 2, 2.2. We construct certain specific operators $T(t_{p^k}, u_{t_{p^k}})$ acting on our cohomology groups $H^\bullet(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p)$ (see 2.2 below).

These operators allow us to define a quotient of the cohomology, on which these operators act as isomorphisms. This quotient is called the ordinary cohomology and denoted by $H^\bullet_{\mathrm{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p)$. This definition goes back to Hida.

In this note we want to investigate how these ordinary cohomology groups vary if we vary the highest weight p-adically. Let us assume that $G/\mathbb{Q}$ is split semi-simple, let $\{\alpha, \beta, \dots\}$ be the simple roots and $\{\gamma_\alpha, \gamma_\beta, \dots\}$ be the corresponding dominant fundamental weights. Let $\lambda_0 = \sum \nu_\alpha \gamma_\alpha$, be a dominant weight, where we assume $0 \le \nu_\alpha < p-1$. We consider the weights

$$\Lambda_{\lambda_0} = \{\lambda \mid \lambda = \lambda_0 + (p-1)\sum z_\alpha \gamma_\alpha, z_\alpha \in \mathbb{N}\}.$$

We want to interpolate the cohomology groups $H^\bullet_{\mathrm{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p)$ p-adically, in a certain sense we want to replace the integers z_α by p-adic numbers. To achieve this goal we construct interpolating sheaves $\widetilde{\mathcal{P}}_{\tilde{\chi}}$ on our locally symmetric space. These sheaves depend on a character $\chi\colon T(\mathbb{Z}_p) \to \mathbb{Z}_p^\times$ which is written as

$$\chi = (\chi^{[1]}_{\lambda_0}, \sum \theta_\alpha \gamma_\alpha).$$

Such a character has two components where $\chi^{[1]}_{\lambda_0}$ is a character on $T(\mathbb{F}_p) \to \mathbb{Z}_p^\times$ and the second component is a character on $T^{(1)}(\mathbb{Z}_p) = \ker(T(\mathbb{Z}_p) \to T(\mathbb{F}_p))$. To any such a character χ we define some kind of an induced module $\mathcal{P}_\chi$. Any λ yields a χ_λ and we have a natural inclusion $\widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p \hookrightarrow \widetilde{\mathcal{P}}_{\tilde{\chi}_\lambda}$.

We also consider finite quotients $\widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}/p^m\mathbb{Z}$ and $\widetilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}$. It will turn out that $\widetilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}$ only depends on the numbers z_α mod p^{m-1}. Hence we get a bunch of cohomology groups. These cohomology groups are finitely generated (or even finite) $\mathbb{Z}_p$-modules. On these cohomology groups we construct Hecke operators $T(t_{p^k}, u_{t_{p^k}})$. We will see that we have a greater degree of freedom in the choice of the second component $u_{t_{p^k}}$. (See 3.5 and 3.6.)

Let $H^\bullet$ be any such a cohomology group. A Hecke operators $T(t_{p^k}, u_{t_{p^k}})$ decomposes cohomology groups $H^\bullet$ into a direct sum $H^\bullet_{\text{nilpt}} \oplus H^\bullet_{\text{ord}}$ such that $T(t_{p^k}, u_{t_{p^k}})$ acts (topologically) nilpotently on the first summand and as an isomorphism on the second one. Two ordinary summands which are defined with respect to different choices of $T(t_{p^k}, u_{t_{p^k}})$ are canonically isomorphic.

Our first theorem (Thm. 2.1) asserts that under some assumptions on λ_0 the morphisms $\widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}/p^m\mathbb{Z} \to \widetilde{\mathcal{P}}_{\tilde{\chi}_\lambda^{[m]}}$ induce isomorphisms of Hecke modules

$$H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{P}}_{\tilde{\chi}_\lambda^{[m]}}).$$

This is a first approximation to interpolating $H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p)$; at least it implies that for $\lambda, \lambda' \in \Lambda_{\lambda_0}$ which satisfy $z_\alpha \equiv z'_\alpha \bmod p^{m-1}$ we get a canonical isomorphism $H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda_{\lambda'},\mathbb{Z}} \otimes \mathbb{Z}/p^m\mathbb{Z})$, i.e., we get congruences between cohomology groups if the highest weight of the coefficient system satisfy congruences.

The second theorem (Thm. 4.4 and 4.6) proved in this note says that the p-torsion of the ordinary cohomology groups $H^\bullet_{\text{ord}}(S^G_{K_f}, \mathcal{M}_\lambda \otimes \mathbb{Z}_p)$ is bounded independently of $\lambda \in \Lambda_{\lambda_0}$ as long as λ stays away from certain sets $Y_i^\bullet$ of irregular p-adic weights. These theorems use the first theorem but they are not a direct consequence. In a first version of this note I stated a much stronger result which turned out to be false, Hida showed a counter-example to me (see Section 4).

In a subsequent paper with J. Mahnkopf we will show that this theorem implies that under certain assumptions the ordinary cohomology groups $H^\bullet_{\text{ord}}(S^G_{K_f}, \widetilde{\mathcal{P}}_{\tilde{\chi}})$ form a nice analytic p-adic family in the variable $(\chi^{[1]}_{\lambda_0}, \sum \theta_\alpha \gamma_\alpha)$, the first entry is fixed, the variables are the θ_α. A precise version of this result will be formulated in Section 4.2, Theorem 4.7.

As long as everything happens on the level of sheaf cohomology, there is no mentioning of automorphic forms so far. In the paper with J. Mahnkopf we implement methods from analysis and representation theory. Using the trace formula, some vanishing results and results on Eisenstein cohomology we can find some λ_0 such that Theorem 4.7 applies to all $\lambda \in \Lambda_{\lambda_0}$.

The results explained so far are perhaps not really new. There is certainly an overlap with the results proved by Hida, Ash–Stevens, Urban, Tilouine, Mauger, Emerton and others. But I claim that the approach to the subject is new, very simple, direct and transparent. In any case the results as they are stated here are just in right form for the applications I have in mind and which are discussed at the end of this paper.

The boundedness theorem provides a tool to make further progress in the questions which are discussed in my paper "A congruence …" in [1-2-3]. In my paper on rank-one Eisenstein cohomology [Ha-Bom2] I discuss rationality of certain ratios of special values of L-functions. The theorems we hope to prove in the forthcoming paper with J. Mahnkopf provide a method to prove that in a certain sense these ratios of special values of L-functions are p-adic analytic functions.

These applications are discussed in Sections 4.3 and 4.4, and they are my main motivation for writing this paper.

For the discussion of the Hecke operators and also for some basic notions and notations used in this note I refer to my book project "Cohomology of arithmetic groups" ([Ha-Coh], Chap. 2–6), which exists in preliminary form on my homepage:

www.math.uni-bonn.de/people/harder/Manuscripts/buch/

Part of this paper was prepared when I was visiting the Institute for Advanced Study in Princeton in the fall term 2006. The idea that the (wrong) boundedness theorem may be true and may be interesting came to me when I walked down the Strudlhofstiege in Vienna during my stay at the workshop on Automorphic Forms in 2006 at the Erwin Schrödinger Institute. I thank both institutions for their support.

2. The case Gl_2

2.1. The coefficient systems. Let $\mathbb{C}_p$ the completion of the algebraic closure $\bar{\mathbb{Q}}_p$, let $O_{\mathbb{C}_p}$ be its ring of integers. We assume for a moment that $p > 2$. We have the canonical homomorphism $r\colon \mathbb{Z}_p^\times \to \mathbb{F}_p^\times$ the kernel is the group $\mathbb{Z}_p^{(1)}$ of 1-units and the Teichmüller character provides a section $\omega\colon \mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$. It is defined by the requirements that $\omega(x)$ is a $(p-1)$-th root of unity and $\omega(x) \equiv x \bmod p$. Hence we have $x/\omega(x) = 1 + l(x)p \in \mathbb{Z}_p^{(1)}$. Any $x \in \mathbb{Z}_p^\times$ can be written uniquely as $\omega(x)(1 + l(x)p)$ with $l(x) \in \mathbb{Z}_p$.

We will forget the homomorphism r and we will write $\omega(x)$ for $\omega(r(x))$. Then $\mathbb{Z}_p^\times$ is a direct product $\mathbb{F}_p^\times \times \mathbb{Z}_p^{(1)}$. We consider a pair (i,θ) where $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $\theta \in O_{\mathbb{C}_p}$. We denote such a pair by $\chi = (i,\theta)$. Any such χ defines a character

$$\chi\colon \mathbb{Z}_p^\times \to O_{\mathbb{C}_p}^\times, \quad \chi\colon x = \omega(x)(1 + l(x)p) \mapsto \omega(x)^i (1 + l(x)p)^\theta.$$

The assumption $p > 2$ guaranties the convergence of the series

$$(1 + l(x)p)^\theta = 1 + \theta l(x)p + \binom{\theta}{2} l(x)^2 p^2 + \cdots.$$

If $p = 2$ then we replace $\mathbb{F}_2$ by $(\mathbb{Z}/4\mathbb{Z})^\times$ and ω by the character $\omega_2\colon \mathbb{Z}_2^\times \to \{\pm 1\}$, which satisfies $\omega_2(x) \equiv x \bmod 4$. Again we can write $x = \omega_2(x)(1 + l(x)4)$ and then we can proceed as before. In the following we always pretend that $p > 2$, our arguments are – mutatis mutandis – also correct for $p = 2$.

We are not forced to assume that $\theta \in O_{\mathbb{C}_p}$. We have the homomorphism $\mathrm{ord}_p\colon \mathbb{C}_p^\times \to \mathbb{Q}$ which has value zero on $O_{\mathbb{C}_p}^\times$ the units and satisfies $\mathrm{ord}_p(p) = 1$. Then we can define the open disc $\mathcal{D}(-\frac{1}{p-1}) = \{\theta | \mathrm{ord}_p(\theta) > -\frac{1}{p-1}\}$ and it is well known that the series defining $(1+xp)^\theta$ converges for $\theta \in \mathcal{D}(-\frac{1}{p-1})$ and it defines a homomorphism $\mathbb{Z}_p^{(1)} \to \mathbb{C}_p^\times$.

For any integer m we get

$$\chi^{[m]}\colon (\mathbb{Z}/p^m\mathbb{Z})^\times \to (O_{\mathbb{C}_p}/p^m O_{\mathbb{C}_p})^\times.$$

By Gl_2 we mean the semi-simple group scheme $\mathrm{Gl}_2/\operatorname{Spec}(\mathbb{Z})$, for any commutative ring R with identity the group of R-valued points is $\mathrm{Gl}_2(R)$. Let $A(\mathrm{Gl}_2)$ be the $\mathbb{Z}$-algebra of regular functions on Gl_2.

Let $B = B_+$ be the Borel subgroup of upper triangular matrices. Its quotient by its unipotent radical U_+ of strictly upper triangular matrices is equal to the torus of diagonal matrices

$$T = \{t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)\}.$$

We define the character $\tilde{\chi}^{[m]}\colon B_+(\mathbb{Z}/(p^m)) \to (O_{\mathbb{C}_p}/p^m O_{\mathbb{C}_p})^\times$, which is given by

$$\tilde{\chi}^{[m]}\colon \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \chi^{[m]}(t_1).$$

We can consider the induced $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$-module

$$\begin{aligned} I_{\tilde{\chi}^{[m]}} &:= \mathrm{Ind}_{B(\mathbb{Z}/p^m\mathbb{Z})}^{G(\mathbb{Z}/p^m\mathbb{Z})} \tilde{\chi}^{[m]} \\ &= \Big\{ f : G(\mathbb{Z}/p^m\mathbb{Z}) \to O_{\mathbb{C}_p}/(p^m)) \mid f\Big(\left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\Big) \\ &\qquad\qquad = \tilde{\chi}^{[m]}(\left(\begin{smallmatrix} t_1 & * \\ 0 & t_2 \end{smallmatrix}\right)) f\Big(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\Big)\Big\}. \end{aligned}$$

Of course as usual the group acts on this module by translations from the right, i.e., $R_x(f)(g) = f(gx)$. Sometimes it is convenient to consider the submodule of those functions which take their values in a subring of $R \subset O_{\mathbb{C}_p}$: This ring must receive the values of χ, i.e., we must require that $\theta \in R$. Then we put $R_m = R/(p^m)$. Hence we get that $I_{\tilde{\chi}^{[m]}}$ is a free R_m-module of rank $\#\mathbb{P}^1(\mathbb{Z}/(p^m))$.

We formulate a Lipschitz condition which defines a submodule

$$\mathcal{P}_{\tilde{\chi}^{[m]}} = \{f \in I_{\tilde{\chi}^{[m]}} | \text{for all } 0 < \nu < m, g_\nu \equiv 1 \bmod p^\nu, f(gg_\nu) \equiv f(g) \bmod p^\nu\}.$$

For this system of submodules we have $G(\mathbb{Z}/(p^m))$ equivariant homomorphisms

$$r_{\chi^{[m]}}\colon \mathcal{P}_{\tilde{\chi}^{[m]}} \longrightarrow \mathcal{P}_{\tilde{\chi}^{[m-1]}}.$$

We choose an open compact subgroup $K_f = \prod_\ell K_\ell$ and assume that $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$. Let $S_{K_f}^G$ the associated modular curve (see [Ha-Coh], Chap. 3, 1.2), it has an adelic description as

$$S_{K_f}^G = \mathrm{Gl}_2(\mathbb{Q})\backslash \mathrm{Gl}_2(\mathbb{R})/\mathrm{SO}(2) \times G(\mathbb{A}_f)/K_f.$$

The $\mathcal{P}_{\tilde{\chi}^{[m]}}$ are $\mathrm{Gl}_2(\widehat{\mathbb{Z}}) = K_f$-modules and hence we get sheaves $\widetilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}$ on $S^G_{K_f}$. These sheaves can be considered as sheaves for the analytic topology on $S^G_{K_f}$. But in our case $S^G_{K_f}$ is the set of complex points of a scheme $\mathcal{S}^G_{K_f}/\mathbb{Q}$ (or a number field) and then we may also interpret $\tilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}$ as sheaves for the etale topology on $\mathcal{S}^G_{K_f}$.

We get a projective system of sheaves on $S^G_{K_f}$, and we define

$$H^1(S^G_{K_f}, \widetilde{\mathcal{P}}_{\tilde{\chi}}) := \varprojlim_m H^1(S^G_{K_f}, \widetilde{\mathcal{P}}_{\tilde{\chi}^{[m]}}).$$

For any integer n we can define the special character

$$\chi_n = (n \mod (p-1), n).$$

For integers $n \geq 0$ we can relate the sheaves $\widetilde{\mathcal{P}}_{\tilde{\chi}_{-n}}$ to sheaves which are obtained from rational representations.

Let us denote by $\mathcal{M}_n$ the module of homogenous polynomials in two variables X, Y with $\mathbb{Z}$-coefficients, we define an action of $\mathrm{Gl}_2(\mathbb{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX+cY, bX+dY) \det\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)^{-n}.$$

The resulting $\mathrm{Gl}_2(\mathbb{Z})$-module will be denoted by $\mathcal{M}_n[-n]$.

Remark. It is important that we view $\mathcal{M}_n[-n]$ as a module for the group scheme $\mathrm{Gl}_2/\mathbb{Z}$, this means that for any commutative ring R with identity we have an action of $\mathrm{Gl}_2(R)$ on $\mathcal{M}_n \otimes R$. We notice that for any n we have a family of modules $\mathcal{M}_n[r]$ where we replace the determinant factor by $\det(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^r$. These modules become the same, if we restrict them to the derived group $\mathrm{Sl}_2/\mathbb{Z}$. This restriction is in a sense the "essential part" of the representation. Two such representations differ by their restriction to the centre where they are given by the central character $z \mapsto z^{n+2r}$. This central character is less relevant and the choice of $r = -n$ is natural (see [Ha-MM]).

We review briefly the standard construction of Gl_2-modules. On our torus we have the two rational characters $\gamma\colon t \mapsto t_1$, $\det\colon t \mapsto t_1 t_2$. To any character $\gamma\colon T \to \mathcal{G}_m$ we define the line bundle $\mathcal{L}_\gamma$ on $B\backslash \mathrm{Gl}_2$, its global sections are

$$H^0(B\backslash \mathrm{Gl}_2, \mathcal{L}_\gamma) = \{f \in A(\mathrm{Gl}_2) \mid f(bg) = \gamma(b) f(g)\}.$$

This is a Gl_2-module, the group scheme acts by translations from the right. Now it is well known and easily verified that we have a canonical isomorphism of Gl_2-modules

$$\mathcal{M}_n = \mathcal{M}_n[-n] \xrightarrow{\sim} H^0(B\backslash \mathrm{Gl}_2, \mathcal{L}_{-n\gamma_1}).$$

If we restrict the character $-n\gamma_1$ to the $\mathbb{Z}/p^m\mathbb{Z}$-valued points then we obviously get

$$-n\gamma_1 | T(\mathbb{Z}/p^m\mathbb{Z}) = \chi^{[m]}_{-n}$$

and hence restriction of the sections in $H^0((B\backslash \mathrm{Gl}_2, \mathcal{L}_{-n\gamma_1})$ to $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$ yields a $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$-module homomorphism $H^0((B\backslash \mathrm{Gl}_2, \mathcal{L}_{-n\gamma_1}) \otimes \mathbb{Z}/p^m\mathbb{Z} \to I_{\chi_n^{[m]}}$. This morphism clearly factors through $\mathcal{P}_{\tilde{\chi}^{[m]}}$.

Hence we get a $\mathrm{Gl}_2(\mathbb{Z}/p^m)$ invariant homomorphism

$$j_m\colon \mathcal{M}_n/p^m\mathcal{M}_n \to \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}},$$

which sends a polynomial $P \in \mathcal{M}_n$ to the function

$$f_P(\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right)) = P(x,y)\det\left(\left(\begin{smallmatrix} u & v \\ x & y \end{smallmatrix}\right)\right)^{-n}$$

on $\mathrm{Gl}_2(\mathbb{Z}/p^m)$. It will neither injective nor surjective in general. This homomorphism induces a homomorphism

$$H^\bullet(S^G_{K_f}, \mathcal{M}_n/p^m\mathcal{M}_n) \longrightarrow H^\bullet(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}}).$$

In this case we may take $R_m = \mathbb{Z}/p^m\mathbb{Z}$.

2.2. The Hecke operators. We want to construct Hecke operators $T(g, u_g)$ which act on these cohomology groups as endomorphisms.

We recall the construction of Hecke operators acting on the cohomology with coefficients as it is outlined in [Ha-Coh], Chap. II, in the section on Hecke operators. It has to be translated into the adelic language, but this is a minor point.

Let Γ be any arithmetic congruence group. Any Γ-module M gives us a coefficient system $\tilde{M}$ and we study the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{M})$. To get Hecke operators acting on these cohomology groups we need two data:

a) an element $g \in G(\mathbb{Q})$.

The group $\Gamma(g) = g\Gamma g^{-1} \cap \Gamma$ has finite index in Γ. We define a new $\Gamma(g)$-module $M^{(g)}$, which is equal to M as an abelian group, but the action of $\gamma \in \Gamma(g)$ is given by

$$(g, m) \mapsto (g^{-1}\gamma g)m.$$

The second datum is

b) a $\Gamma(g)$-homomorphism $u_g\colon M^{(g)} \to M$.

Then such a pair (g, u_g) induces an endomorphism in the cohomology (see [Ha-Coh], Chap. III, 2)

$$T^\bullet(g, u_g)\colon H^\bullet(\Gamma\backslash X, \tilde{M}) \to H^\bullet(\Gamma\backslash X, \tilde{M}).$$

It also induces an endomorphism on the cohomology with compact supports and the cohomology of the boundary of the Borel–Serre compactification. Of course it is compatible with the fundamental long exact sequence.

In our case we will take for g the element $t_{p^k} = \begin{pmatrix} p^k & 0 \\ 0 & 1 \end{pmatrix}$ and we have to look for the possible choices for a morphism $u_g = u_{t_{p^k}}$.

If for instance our M is one of the modules $\mathcal{M}_n$ (this is a $\mathbb{Z}$-module), then we have essentially only one good choice for $u_{t_{p^k}}$ this is the so called canonical or classical choice in Chap. 2 loc. cit. We briefly recall how it looks in this case. The element $u_{t_{p^k}} \in \mathrm{Gl}_2(\mathbb{Q})$ induces a linear map $\mathcal{M}_n \otimes \mathbb{Q} \to \mathcal{M}_n \otimes \mathbb{Q}$. This map sends $X^i Y^{n-i} \to p^{ik} p^{-nk} X^i Y^{n-i} = p^{k(i-n)} X^i Y^{n-i}$. Up to a scalar this is the unique homomorphism from $(\mathcal{M}_n \otimes \mathbb{Q})^{(t_{p^k})} \to \mathcal{M}_n \otimes \mathbb{Q}$. But it does not send the integral lattice to the integral lattice. We have to multiply it by p^{nk} to get a homomorphism

$$u_{t_{p^k}}^{\text{class}} \colon \mathcal{M}_n^{(t_{p^k})} \to \mathcal{M}_n.$$

Under this homomorphism Y^n is mapped to itself, all other monomials get multiplied by a strictly positive power of p.

But if we pass to $\mathcal{M}_n \otimes \mathbb{Z}/(p^m)$, or if we consider one the modules $\mathcal{P}_{\tilde{\chi}[m]}$, $I_{\tilde{\chi}[m]}$, then we will find many more such $u_{t_{p^k}}$. We always can consider the reduction of $u_{t_{p^k}}^{\text{class}}$ to $\mathcal{M}_n/p^m\mathcal{M}_n$, and then we also call this reduction $u_{t_{p^k}}^{\text{class}}$.

We will not consider all of them. For any χ and any pair of integers $k, m > 0$ we want to construct a set

$$\mathcal{H}_\chi^{[m]} = T(\{t_{p^k}, u_{t_{p^k}}))\}$$

which means to give a collection of $u_{t_{p^k}}$. In general we assume that $k \geq m$, unless we discuss the classical operator, in this case this assumption does not make sense.

We assume that $n > 0$ and formulate certain requirements, that should be fulfilled by this families of operators.

(i) We want to have diagrams

$$\begin{array}{ccc} \mathcal{P}_{\tilde{\chi}[m]}^{(t_{p^k})} & \xrightarrow{u_{t_{p^k}}} & \mathcal{P}_{\tilde{\chi}[m]} \\ \downarrow & & \downarrow \\ I_{\tilde{\chi}[m]}^{(t_{p^k})} & \xrightarrow{u_{t_{p^k}}} & I_{\tilde{\chi}[m]}, \end{array}$$

i.e., they are defined on the I and restrict to the $\mathcal{P}$.

(ii) We want them to form a projective system if we restrict them to the sheaves $\mathcal{P}_{\tilde{\chi}[m]}$. This means that given $k \geq m+1$ and an homomorphism

$$u_{t_{p^k}} : \mathcal{P}_{\tilde{\chi}[m+1]}^{(t_{p^{m+1}})} \to \mathcal{P}_{\tilde{\chi}[m+1]}$$

we require that it "pushes down" to an $u_{t_{p^k}}$, i.e., we get diagrams

$$\begin{tikzcd} \mathcal{P}^{(t_{p^k})}_{\tilde{\chi}^{[m+1]}} \arrow[r, "u_{t_{p^k}}"] \arrow[d] & \mathcal{P}_{\tilde{\chi}^{[m+1]}} \arrow[d] \\ \mathcal{P}^{(t_{p^k})}_{\tilde{\chi}^{[m]}} \arrow[r, "u_{t_{p^k}}"] & \mathcal{P}_{\tilde{\chi}^{[m]}}. \end{tikzcd}$$

This means that we can define $u_{t_{p^k}}$ which induce morphisms on all levels $m \leq k$. We get a projective system of Hecke operators $r_{m+1} : \mathcal{H}_\chi^{[m+1]} \to \mathcal{H}_\chi^{[m]}$, we do not require that the r_{m+1} are surjective. The set $\mathcal{H}_\chi^{[m]}$ depends only on $\chi^{[m]}$ as the notation indicates.

(iii) We want to construct a principal operator

$$u^{\text{princ}} = (\dots, T(t_{p^{k+1}}, u^{\text{princ}}_{t_{p^{k+1}}}), T(t_{p^k}, u^{\text{princ}}_{t_{p^k}}), \dots)$$

in the projective system of $\mathcal{H}_\chi^{[m]}$, which has the following properties:

a) $u^{\text{princ}}_{t_{p^k}}(I_{\tilde{\chi}^{[m]}}) \subset pI_{\tilde{\chi}^{[m]}} + \mathcal{P}_{\tilde{\chi}^{[m]}}$.
If we know in addition that $\chi^{[m]} = \chi_\lambda^{[m]}$ with some highest weight λ then we even have

$$u^{\text{princ}}_{t_{p^k}}(I_{\tilde{\chi}^{[m]}}) \subset pI_{\tilde{\chi}^{[m]}} + j_m(\mathcal{M}_n \otimes \mathbb{Z}/(p^m)).$$

b) For any $u_{t_{p^k}} \in \mathcal{H}_\chi^{[m]}$ we find an integer $b \in \mathbb{Z}_p$ such that

$$(u_{t_{p^k}} - bu^{\text{princ}}_{t_{p^k}})(I_{\tilde{\chi}^{[m]}}) \subset pI_{\tilde{\chi}^{[m]}}. \tag{Princ}$$

(iv) And finally we want: If $\chi = \chi_{-n}$ as above then the classical Hecke operator on $\mathcal{M}_n \otimes \mathbb{Z}/(p^m)$ extends to an operator in $\mathcal{H}^{[m]}_{\chi_{-n}}$.

Our first and in some sense main result will be the existence of such a system of Hecke operators. We will construct such a system in Section 3. Actually for the case Gl_2 this discussion will be much to detailed, we will see that we really have quite a lot of Hecke operators, many more than we need. After that we will discuss the more general case of reductive group schemes.

2.3. First consequences. The existence of such a system of sheaves together with the Hecke operators acting on their cohomology has interesting consequences. Let $\mathcal{F}$ be any of the above sheaves.

We introduce some notation. We write $H^\bullet_{*,*}$ for the various variants of cohomology. We explain the options for the first $*$. It indicates whether we take the cohomology with compact supports, then $* = c$, whether we take the cohomology without supports, then $* = \natural$. We may also take the cohomology of the boundary. Then we write

$$H^\bullet_*(\partial S^G_{K_f}, \mathcal{F}) = H^\bullet_{\partial,*}(S^G_{K_f}, \mathcal{F}).$$

These cohomology groups are related by the exact sequence

$$H^{\bullet-1}_{\partial,*}(S^G_{K_f}, \mathcal{F}) \to H^\bullet_{c,*}(S^G_{K_f}, \mathcal{F}) \to H^\bullet_{\natural,*}(S^G_{K_f}, \mathcal{F}) \to H^\bullet_{\partial,*}(S^G_{K_f}, \mathcal{F}) \to .$$

Finally we may take $* = !$; this is the "inner cohomology" and it is the image of the cohomology with compact supports in the cohomology without supports.

Now we explain the options for the second $*$. It may be blank or ord, we explain what we mean by that.

It follows from (iiia) that all Hecke operators in $\mathcal{H}^{[m]}_\chi$ act nilpotently on $H^\bullet_{*,*}(S^G_{K_f}, I_{\tilde{\chi}[m]}/\mathcal{P}_{\tilde{\chi}[m]})$. A first consequence of the properties of the system of Hecke operators is the following assertion:

Let us denote by $\mathcal{J}_{\chi,m}$ the set of operators $u_{t_{p^k}}$ for which the number b in (Princ) is zero modulo p. Then it is clear that any composition of operators in $\mathcal{H}^{[m]}_\chi$ which contains more than m factors from $\mathcal{J}_{\chi,m}$ annihilates the cohomology $H^\bullet_{*,}(S^G_{K_f}, I_{\tilde{\chi}[m]}/\mathcal{P}_{\tilde{\chi}[m]})$ for any choice of the first $*$.

Using the long exact sequence we get the same consequence for the cohomology of the sub sheaves

$$H^\bullet_{*,*}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}[m]}),$$

but now we may need $2m$ factors.

Hence we see: If X is any of the above cohomology groups. We know that it is finitely generated over $\mathbb{Z}_p$. Let us assume for the moment that it is torsion, i.e., finite. Then we can take the principal Hecke operator $T^{\text{princ}}_{p^k} = T(t_{p^k}, u^{\text{princ}}_{t_{p^k}})$ and consider the image $(T^{\text{princ}}_{p^k})^N(X)$ for a high power N. This becomes stationary and will be called $X^{\text{princ}}_{\text{ord}}$. The operator $T^{\text{princ}}_{p^k}$ induces an isomorphism on $X^{\text{princ}}_{\text{ord}}$. We denote by $X^{\text{princ}}_{\text{nilpt}}$ the maximal submodule of X on which $u^{\text{princ}}_{t_{p^m}}$ acts nilpotently. We get a decomposition

$$X = X^{\text{princ}}_{\text{nilpt}} \oplus X^{\text{princ}}_{\text{ord}}.$$

It is also clear that we can replace $T^{\text{princ}}_{p^k}$ by any other operator $T(t_{p^k}, u_{t_{p^k}})$ where $u_{t_{p^k}} = u^{\text{princ}}_{t_{p^k}} + v_m$ with $v_m \in \mathcal{J}_{\chi[m]}$. Then $u_{t_{p^k}}$ will define a second decomposition

$$X = X^{(u_{t_{p^k}})}_{\text{nilpt}} \oplus X^{(u_{t_{p^k}})}_{\text{ord}}.$$

In both cases we have an identification $X_{\text{ord}} \xrightarrow{\sim} X/X_{\text{nilpt}}$. We claim that $X^{(u_{t_{p^k}})}_{\text{nilpt}} \cap X^{\text{princ}}_{\text{ord}} = (0)$.

To see this let us pick an element $x \in X_{\text{nilpt}}^{(u_{t_{p^k}})} \cap X_{\text{ord}}^{\text{princ}}$. Then we get

$$T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})x = T(t_{p^k}, u_{t_{p^k}})x + v_m x.$$

We repeatedly apply $T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})$ to this equation and get

$$T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})^N x = T(t_{p^k}, u_{t_{p^k}})^N x + \tilde{v}_m(x),$$

where $\tilde{v}_m$ is an endomorphism of X which is a sum of products of endomorphisms and where each such product contains a factor v_m. This implies the following: If $x \in p^r X$ then $\tilde{v}_m(x) \in p^{r+1}X$. If now N is large enough the first summand on the right-hand side becomes zero. Then we can conclude that $T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})^N x \in p^{r+1}X$. But since $X_{\text{ord}}^{(u_{t_{p^k}})}$ is a direct summand it follows that $T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})^N x \in p^{r+1}X_{\text{ord}}^{\text{princ}}$, and hence we can conclude that $x \in p^m X_{\text{ord}}^{\text{princ}}$ for any $m > 0$ and therefore $T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})^N x = 0$. But this implies $x = 0$ because $T(t_{p^k}, u_{t_{p^k}}^{\text{princ}})$ induces an isomorphism on $X_{\text{ord}}^{\text{princ}}$.

Clearly we have a similar conclusion if X is only finitely generated, we simply have to replace nilpotent by topologically nilpotent. We still call the summand on which the given operator $T(t_{p^k}, u_{t_{p^k}})$ acts topologically nilpotently X_{nilp}.

From this it follows that we get a canonical identification

$$X_{\text{ord}}^{\text{princ}} \xrightarrow{\sim} X/X_{\text{nilpt}}^{(u_{t_{p^k}})} \xrightarrow{\sim} X_{\text{ord}}^{(u_{t_{p^k}})}$$

and this allows us to speak of the module X_{ord}. It is the collection of modules $\{X_{\text{ord}}^{(u_{t_{p^k}})}\}_{u_{t_{p^k}}}$, which are identified to each other.

It is clear that a homomorphism $X \to X_1$ which is compatible with the action of the Hecke algebra sends $X_{\text{ord}} \to X_{1,\text{ord}}$ and from an exact sequence of Hecke modules

$$X' \to X' \to X''$$

we get an exact sequence

$$X'_{\text{ord}} \to X'_{\text{ord}} \to X''_{\text{ord}}.$$

We consider the cohomology groups $H^\bullet_*(S^G_{K_f}, \mathcal{M}_n/p^m\mathcal{M}_n)$. We use the Hecke operator $T(t_{p^k}, u_{t_{p^k}}^{\text{class}})$ to define $H^\bullet_{*,\text{ord}}(S^G_{K_f}, \mathcal{M}_n/p^m\mathcal{M}_n)$. Hence the second $*$ in the subscript $*, *$ may be ord if we consider the ordinary cohomology, it is a blank if we consider all the cohomology, in principle it could also be a nilpt.

We could ask ourselves whether we can have a mild form of a commutation relation:

(v) For any $v \in \mathcal{J}_{\chi,m}$ we can find an endomorphism $w \in \mathrm{End}(X)$ such that

$$T(t_{p^k}, u_{t_{p^k}}^{\text{princ}}) * v = (pw)T(t_{p^k}, u_{t_{p^k}}^{\text{princ}}).$$

If our system of Hecke operators has this property then it is rather easy to see that the submodule X_{nilpt} does not depend on the choice of $u_{t_{p^k}}$ provided that the element b in (Princ) is a unit.

Now we consider the group $\mathrm{Gl}_2/\mathbb{Q}$ and formulate the following result, which is more or less an obvious consequence of the existence of the family of Hecke operators having the property above. We formulate it in the special case $\mathrm{Gl}_2/\mathbb{Q}$, but later on we give a more general statement.

Theorem 2.1. *Assume that $n > 0$. We get a sequence of isomorphisms*

$$H^{\bullet}_{*,\text{ord}}(S^G_{K_f}, \mathcal{M}_n/p^m\mathcal{M}_n) \xrightarrow{\sim} H^{\bullet}_{*,\text{ord}}(S^G_{K_f}, \mathcal{P}_{\chi_n^{[m]}}) \xrightarrow{\sim} H^{\bullet}_{*,\text{ord}}(S^G_{K_f}, I_{\chi_n^{[m]}}).$$

This holds for any choice of the first $*$.

This is a control theorem in the sense of Hida. Theorems of this kind have been proved by Hida himself, by Ash and Stevens and by Mauger. We will formulate a general result for arbitrary reductive groups later (Thm. 4.1)

The proof is simple. Our previous considerations imply that the second homomorphism is an isomorphism. If we investigate the morphism $\mathcal{M}_n/p^m\mathcal{M}_n \to I_{\tilde{\chi}_n^{[m]}}$ a little bit more closely then we see easily that the kernel has no ordinary cohomology, i.e., $\mathcal{H}_{\chi_n^{[m]}}$ acts nilpotently on the cohomology of the kernel of this homomorphism (for a detailed argument we refer to the discussion of the general case in 3.6). Our requirement (iiib) implies $u_{t_{p^k}}(I_{\tilde{\chi}_n^{[m]}}) \subset pI_{\tilde{\chi}_n^{[m]}} + j_n(\mathcal{M}_n/p^m\mathcal{M}_n)$ and then it is also clear that $\mathcal{H}_{\chi_n^{[m]}}$ acts nilpotently on the cohomology of the cokernel.

Therefore the claim is proved: All the homomorphisms above are isomorphism. The proof depends on the existence and properties of the $\widetilde{\mathcal{P}}_\chi$ and the existence of the Hecke operators, we will see why we need $n > 0$.

I want to call these sheaves $\widetilde{\mathcal{P}}_\chi$ *interpolating* sheaves, they have the property that for two such characters $\chi = (i, \theta), \psi = (i_1, \theta_1)$ we have $\mathcal{P}_{\tilde{\chi}_{-n}^{[m]}} = \mathcal{P}_{\tilde{\psi}_{-n}^{[m]}}$ if $i = i_1$ and $\theta \equiv \theta_1 \bmod p^{m-1}$. It seems impossible to find such congruences for the $\mathcal{M}_n$. But still we know that

$$H^{\bullet}_{*,\text{ord}}(S^G_{K_f}, \mathcal{M}_n/p^m\mathcal{M}_n) \xrightarrow{\sim} H^{\bullet}_{*,\text{ord}}(S^G_{K_f}, \mathcal{M}_{n_1}/p^m\mathcal{M}_{n_1}))$$

if $n, n_1 > 0$ and $n \equiv n_1 \bmod (p-1)p^{m-1}$.

3. The construction of Hecke operators on the interpolating coefficient systems

3.1. The case $m = 1$. We assume that $0 \leq n < p - 1$ and consider the $\mathrm{Gl}_2(\mathbb{F}_p)$ homomorphism

$$j_n : \mathcal{M}_n/p\mathcal{M}_n \to I_{\tilde{\chi}_{-n}^{[1]}},$$

under our assumption on n this is always an inclusion. Of course we can define this homomorphism j_n for any integer $n \geq 0$ and it is important to notice that the right-hand side depends only on $n \bmod (p-1)$.

We want to compute the spaces of Hecke operators for these modules and investigate the spaces of Hecke operators which are compatible with j_n. In the beginning of 3.3 we show that for any $\mathrm{Gl}_2(\mathbb{F}_p)$-module M we have to compute $\mathrm{Hom}_{T(\mathbb{F}_p)}(M_{U_-(\mathbb{F}_p)}, M^{U_+(\mathbb{F}_p)})$ (see 3.3 (Hom)). Hence we have to compute the spaces of $U_+(\mathbb{F}_p)$ coinvariants and $U_-(\mathbb{F}_p)$ invariants for our modules and then we have to determine the spaces of $T(\mathbb{F}_p)$-invariant homomorphism between the coinvariants and the invariants. We use the Bruhat decomposition. Let w the nontrivial element in the Weyl group, let $u \in U_+(\mathbb{F}_p)$. We define functions Ψ_0 (resp. $\Psi_{w,u}$) $\in I_{\tilde{\chi}_{-n}^{[1]}}$ by the condition, that they are supported on $B_+(\mathbb{F}_p)$ (resp. $B_+(\mathbb{F}_p)wu$, see 3.3) and assume the value one at e resp. w.

Let $\bar{\Psi}_e$ (resp. $\bar{\Psi}_w$) be the images of Ψ_0 (resp. $\Psi_{w,u}$) in the space of coinvariants, the element $\bar{\Psi}_{w,u}$ does not depend on u. The space of $U_-(\mathbb{F}_p)$ invariants is generated by the functions $\Phi_e(g) = \Phi_e(bu_-) = \tilde{\chi}^{[1]}(b)$ and $\Phi_w(g) = \Phi_e(bwu_-) = \Phi_e(bw) = \tilde{\chi}^{[1]}(b)$. Then an easy computation shows (here we use $n \leq p-1$)

$$\begin{array}{ccc}
(\mathcal{M}_n/p\mathcal{M}_n)_{U_+(\mathbb{F}_p)} = \mathbb{F}_p Y^n & \longrightarrow & (I_{\tilde{\chi}_n^{[1]}})_{U_+(\mathbb{F}_p)} = \mathbb{F}_p\bar{\Psi}_e \oplus \mathbb{F}_p\bar{\Psi}_w \\
\Big\downarrow {\scriptstyle u_{t_{pk}}^{\mathrm{class}}} & & \Big\downarrow \\
(\mathcal{M}_n/p\mathcal{M}_n)^{U_-(\mathbb{F}_p)} = \mathbb{F}_p Y^n & \longrightarrow & (I_{\tilde{\chi}_n^{[1]}})^{U_-(\mathbb{F}_p)} = \mathbb{F}_p\Phi_e \oplus \mathbb{F}_p\Phi_w .
\end{array}$$

It is easy to see that the $T(\mathbb{F}_p)$ rational points of our standard torus acts by the characters $t \mapsto t_1^{-n}$ on $\bar{\Psi}_e$, Φ_e and $t \mapsto t_2^{-n}$ on $\bar{\Psi}_w$, Φ_w. Hence we see that for $0 < n < p-1$ the space of possible $u_{t_{pk}}$ is of dimension two: We have the operator – which will be called the *principal* operator later –

$$u_{t_{pk}}^{\mathrm{princ}} : \bar{\Psi}_e \mapsto \Phi_e, \; \bar{\Psi}_w \mapsto 0$$

and a second one which just does the opposite.

We observe that for all n the polynomial Y^n goes to $\bar{\Psi}_e$ in the module of coinvariants. If $n > 0$ the polynomial Y^n maps to Φ_e in the module of invariants and hence we see that under the assumption $n > 0$ the principal operator is an extension of the classical operator. Especially we have that the classical Hecke operator induces the zero map on $H^\bullet(S_{K_f}^G, I_{\tilde{\chi}_n^{[1]}}/(\mathcal{M}_n/p\mathcal{M}_n))$.

Hence we see that for $n > 0$ we get isomorphisms

$$j_n^\bullet : H^\bullet_{*,\mathrm{ord}}((S_{K_f}^G, \mathcal{M}_n/p\mathcal{M}_n) \xrightarrow{\sim} H^\bullet_{*,\mathrm{ord}}(S_{K_f}^G, I_{\tilde{\chi}_n^{[1]}}).$$

But we also see that this is not the case if $n = 0$. In this case Y^n is mapped to $\Phi_e + \Phi_w$ in the space of invariants. So we see that (iii) is not valid. But since the torus action becomes trivial, we see that the space of possible $u_{t_{pk}}$ has dimension 4 and we have several extensions of the classical operator.

We also can replace n by $n_1 = n + k(p-1)$, where $k > 0$ is an integer. This does not change the module $I_{\tilde{\chi}_n^{[1]}}$. It is easy to see that then the homomorphism $\mathcal{M}_{n_1}/p\mathcal{M}_{n_1} \to I_{\tilde{\chi}_{-n}^{[1]}}$ is surjective, if $k > 0$ and $0 < n < p-1$. But then the homomorphism j_{n_1} has a kernel and clearly the classical Hecke operator induces the trivial map on the cohomology of the kernel. Hence we get for any $n > 0$ as above and any $n_1 = n + k(p-1)$ that the natural maps

$$H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_{n_1}/p\mathcal{M}_{n_1}) \to H^\bullet_{\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\chi_n^{[1]}}) \to H^\bullet_{\mathrm{ord}}(S^G_{K_f}, I_{\chi_n^{[1]}})$$

are isomorphisms. This is our Theorem 2.1 for $m = 1$, the exceptional case is $n = 0$.

If $n = 0$ the situation is a little bit different. Let us look at the more general case that $n \equiv 0 \bmod (p-1)$. In this case we have

$$I_{\chi_n^{[1]}} = I_{\chi_0^{[1]}} = \mathrm{Ind}_{B(\mathbb{F}_p)}^{G(\mathbb{F}_p)} \mathbb{F}_p,$$

and this module decomposes into the one-dimensional subspace of constant functions and a p-dimensional space of functions which are orthogonal to the constant functions. It is clear that the constant functions are simply the given by the image of j_0 and the complement is given as the image of j_{p-1}. Then j_{p-1} induces an isomorphism on ordinary cohomology, but j_0 does not if we define "ordinary" with respect to the principal operator.

If we take for $n = p^m(p-1)$ with $m > 0$, then j_n becomes surjective. Then we have that in the p-adic topology $n = p^m(p-1) \to 0$, but we do not have $H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_n/p^k\mathcal{M}_n) \to H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_0/p^k\mathcal{M}_0)$.

3.2. More general group schemes. These considerations generalize. Let us consider a semi-simple (or reductive) group over $G/\mathbb{Q}$. Let us assume that $G/\mathbb{Q}$ is quasi-split. We find a minimal extension $E/\mathbb{Q}$ over which it splits. Let p be a prime, we assume that this extension is unramified at p. We choose a prime $\mathfrak{p}$ above p, let $E_\mathfrak{p}/\mathbb{Q}_p$ the local extension. Then we can extend $G/\mathbb{Q}$ to a flat group scheme of finite type $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$, which is reductive over an open subset V_p containing p. This open subset may shrink during the following considerations. We can choose a maximal torus $\mathcal{T}/V_p \subset \mathcal{G} \times_{\operatorname{Spec}(\mathbb{Z})} V_p = \mathcal{G}_{V_p}$ which is contained in a Borel subgroup $\mathcal{B}/V_p$. This means that we can find a Weyl chamber $\mathcal{C} \subset X^*(\mathcal{T})$ which is invariant under the action of the Galois group $\mathrm{Gal}(E_\mathfrak{p}/\mathbb{Q}_p)$. We have a unique element w_0 in the Weyl group, which sends this chamber to the opposite chamber $-\mathcal{C}$. Let $\mathcal{U}_+$ be the unipotent radical of $\mathcal{B}$, let $\mathcal{U}_- = \mathcal{U}_+^{w_0}$, it is the unipotent radical of the opposite Borel $\mathcal{B}_- \supset \mathcal{T}$.

Let us assume that the derived group $\mathcal{G}^{(1)}$ is simply connected. Let $\mathcal{T}^{(1)} \subset \mathcal{T}$ be the torus $\mathcal{G}^{(1)} \cap \mathcal{T}$. Let π be the set of simple positive roots, the Galois group $\mathrm{Gal}(E_\mathfrak{p}/\mathbb{Q}_p)$ acts by permutations on π. The dominant fundamental weight corresponding to $\alpha \in \pi$ is denoted by γ_α. These dominant weights $\gamma_\alpha, \gamma_\beta, \ldots$ are in $X^*(\mathcal{T}^{(1)})$. If $\lambda \in X^*(\mathcal{T})$, then its restriction to $\mathcal{T}^{(1)}$ is a linear combination $\sum n_\alpha \gamma_\alpha$, if all $n_\alpha \geq 0$ (or $\sum n_\alpha \gamma_\alpha \in \mathcal{C}$), then λ is a highest weight. The homomorphism $X^*(\mathcal{T}) \to X^*(\mathcal{T}^{(1)})$ is surjective, hence any $\sum n_\alpha \gamma_\alpha$ extends to a λ, which is also considered as a highest weight.

For any highest weight λ we have the highest weight module $\mathcal{M}_\lambda$. Here a few words of explanation seem to be in order. Let $\mathcal{O}_E$ be its ring of integers of E, let $V'_p \subset \operatorname{Spec}(\mathcal{O}_E)$ be the inverse image of V_p. Then we can extend the Borel subgroup $\mathcal{B}/\operatorname{Spec}(\mathbb{Z}_p)$ to a Borel subgroup $\widetilde{\mathcal{B}}/V'_p$, the weight $w_0(\lambda)$ defines a line bundle $\mathcal{L}_{w_0(\lambda)}$ on the flag variety of Borel subgroups $(\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V_p}$ (see [De-Gr], Exp. XXII, 5.8). This line bundle is ample if and only if $n_\alpha > 0$ for all α. It has non trivial global sections if and only if $n_\alpha \geq 0$ for all α. The space of global sections $H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ is a finitely generated projective $\mathcal{O}(V'_p)$-module on which the group scheme $\mathcal{G}_{V_p}$ acts. By definition we get an action of $\mathcal{G}(\mathbb{Z}_p)$ on this module, it is easy to see that we can extend $H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ to an $\mathcal{O}_E$-module $\mathcal{M}_\lambda$ on which we have an action of $\mathcal{G}(\mathcal{O}_E)$. This extension is not unique, but we are only interested in what happens at p, so this does not matter.

The module $H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)})$ has two specific elements. Let $\mathcal{A}$ be the ring of regular functions of $\mathcal{G}_{V_p}$, then we have by definition

$$H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V'_p}, \mathcal{L}_{w_0(\lambda)}) = \{f \in \mathcal{A} \mid f(bg) = w_0(\lambda)(b)f(g)\},$$

where b, g are elements in $\mathcal{B}(\mathcal{O}(V'_p)), \mathcal{G}(\mathcal{O}(V'_p))$.

The subsets $\mathcal{B}\cdot\mathcal{U}_-$ (resp. $\mathcal{B}w_0\mathcal{U}_+$) $\subset \mathcal{G}_{V'_p}$ are open and Zariski dense. The complement of these subsets is a divisor D_- (resp. D_+) whose irreducible components correspond to the simple roots. We can write (see proof of Satz 1.3.1 in [Ha-vK])

$$D_- = \sum_{\alpha\in\pi} Y_\alpha^-, \quad D_+ = \sum_{\alpha\in\pi} Y_\alpha^+.$$

Now it is well known that the two functions, which are defined on $\mathcal{B}\cdot\mathcal{U}_-$ and $\mathcal{B}w_0\mathcal{U}_+$ respectively, namely

$$e_{w_0(\lambda)}(bu_-) = w_0(\lambda)(b), \quad e_\lambda(bw_0u_+) = w_0(\lambda)(b)$$

extend to regular functions on $\mathcal{G}_{V'_p}$. So they yield elements in $H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V'_p}, w_0(\mathcal{L}_\lambda))$ (at this point we need that the coefficients $n_\alpha \geq 0$).

More precisely we know that the divisor of zeroes of these two sections are given by

$$\operatorname{Div}(e_{w_0(\lambda)}) = \sum_\alpha n_\alpha Y_\alpha^-, \quad \operatorname{Div}(e_\lambda) = \sum_\alpha n_\alpha Y_\alpha^+$$

and hence we see that $e_{w_0(\lambda)}$ vanishes on the complement of $\mathcal{B}\cdot\mathcal{U}_-$ if all the $n_\alpha > 0$, i.e., our weight is regular.

Moreover it is clear that the two sections are eigensections for the action of the torus: For an element $t \in \mathcal{T}(R)$ we have

$$te_{w_0(\lambda)} = w_0(\lambda)(t)e_{w_0(\lambda)}, \quad te_\lambda = \lambda(t)e_\lambda.$$

The vector $e_{w_0(\lambda)}$ is invariant under the action of $\mathcal{U}_-$ hence it is a highest weight vector with respect to the Borel subgroup $\mathcal{B}_- = \mathcal{B}^{w_0}$. It is a lowest weight vector for $\mathcal{B}$. An analogous statement holds for e_λ.

We will also consider the reduction $\bmod \mathfrak{p}$, i.e., the module $H^0((\widetilde{\mathcal{B}}\backslash\mathcal{G})_{V_p'} \times \mathrm{Spec}(\mathcal{O}_E/\mathfrak{p}), \mathcal{L}_{w_0(\lambda)})$. This is a module for the group $\mathcal{G}(\mathbb{F}_p)$. This module is irreducible if all coefficients satisfy $0 \le n_\alpha \le p-1$ and is equal to $\mathcal{M}_\lambda \otimes \mathcal{O}_E/\mathfrak{p}$ (see [Ja], Chap. 3).

Now we consider the group $\mathcal{T}(\mathbb{Z}_p)$, let $\mathcal{T}^{(1)}$ be its intersection with $\mathcal{G}^{(1)}$. It sits in an exact sequence (see notations in the introduction to Chap. 3 of [Ha-Coh]).

$$1 \to \mathcal{T}^{(1)}(\mathbb{Z}_p) \to \mathcal{T}(\mathbb{Z}_p) \to C'(\mathbb{Z}_p) \to 1.$$

We consider continuous characters $\chi\colon \mathcal{T}(\mathbb{Z}_p) \to \mathcal{O}_{\mathbb{C}_p}^\times$, basically we are only interested in their restriction to $\mathcal{T}^{(1)}(\mathbb{Z}_p)$. We want to construct the interpolating modules. The group $\mathcal{T}^{(1)}(\mathbb{Z}_p) \subset \mathcal{T}^{(1)}(\mathcal{O}_{E_\mathfrak{p}})$ and it is the subgroup of Galois invariant elements. The group

$$\mathcal{T}^{(1)}(\mathcal{O}_{E_\mathfrak{p}}) = \prod_{\alpha\in\pi} \mathcal{O}_{E_\mathfrak{p}}^\times = \{(\dots, x_\alpha, \dots)_{\alpha\in\pi} | x_\alpha \in \mathcal{O}_{E_\mathfrak{p}}^\times\}$$

the Galois group acts by $\sigma(\dots, x_\alpha, \dots) = (\dots, x_{\sigma(\alpha)}^\sigma \dots)$. Hence $\mathcal{T}(\mathbb{Z}_p)$ is the subgroup of elements which satisfy $\sigma(x_\alpha) = x_{\sigma(\alpha)}$. This tells us that the torus $\mathcal{T}^{(1)}$ is a product over induced tori, the factors in this product correspond to the orbits of the Galois group on π. If we denote such an orbit by $\bar{\alpha}$ and if we choose representatives $\alpha \in \bar{\alpha}$ then this defines a subfield $E_\alpha \subset E_\mathfrak{p}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_\alpha)$ is the stabilizer of α. Since $E_\mathfrak{p}$ is unramified, we know that $E_\alpha/\mathbb{Q}_p$ is cyclic of order r_α, where r_α is the length of the orbit. The Galois group $\mathrm{Gal}(E_\alpha/\mathbb{Q}_p)$ is cyclic and generated by the Frobenius element σ. Then the factor corresponding to $\bar{\alpha}$ is denoted by $\mathcal{T}_{\bar{\alpha}}$ and we have $\mathcal{T}_{\bar{\alpha}} = R_{\mathcal{O}_{E_\alpha}/\mathbb{Z}_p}(\mathcal{G}_m)$.

Then

$$\mathcal{T}_{\bar{\alpha}}(\mathbb{Z}_p) = \mathcal{O}_{E_\alpha}^\times \subset \prod_{\sigma^i : i=0}^{r_\alpha - 1} \mathcal{O}_{E_\mathfrak{p}}^\times$$

where the embedding is given by

$$x \mapsto (x, \sigma(x), \dots, \sigma^{r_\alpha-1}(x)).$$

To get our interpolating modules we consider characters on $\mathcal{T}^{(1)}(\mathcal{O}_{E_\mathfrak{p}})$ and restrict them to $\mathcal{T}^{(1)}(\mathbb{Z}_p)$. We still have the Teichmüller character $\omega\colon (\mathcal{O}_{E_\mathfrak{p}}/\mathfrak{p})^\times \hookrightarrow \mathcal{O}_{E_\mathfrak{p}}^\times$. We choose an embedding $\mathcal{O}_{E_\mathfrak{p}} \to \mathcal{O}_{\mathbb{C}_p}$ and then we put as before

$$\chi((\dots, x_\alpha, \dots)) = \prod \omega(x_\alpha)^{\nu_\alpha} (\tfrac{x_\alpha}{\omega(x_\alpha)})^{z_\alpha}$$

where $z_\alpha \in \mathcal{O}_{\mathbb{C}_p}$ and $\nu_\alpha \in \mathbb{Z}$.

We restrict χ to $\mathcal{T}(\mathbb{Z}_p)$, more precisely we look at the restriction to the components. Clearly we have $\omega(\sigma(x_\alpha)) = \omega(x_\alpha)^p$ and hence the factor in front is

$$\prod_{\bar{\alpha}} \prod_{i=0}^{r_\alpha - 1} \omega(x_\alpha)^{\nu_{\sigma^i(\alpha)} p^\nu} = \prod_{\bar{\alpha}} \omega(x_\alpha)^{\sum_i \nu_{\sigma^i(\alpha)} p^\nu} = \prod_{\bar{\alpha}} \omega(x_\alpha)^{\nu_{\bar{\alpha}}}.$$

Hence we see that the $\bar{\alpha}$ component of the factor in front only depends on

$$\nu_{\bar{\alpha}} = \sum_i \nu_{\sigma^i(\alpha)} p^\nu \mod (p^{r_\alpha} - 1).$$

Now we can define the induced modules $\mathcal{P}_{\chi^{[m]}} \subset I_{\chi^{[m]}}$ as before. At this moment we assume $m = 1$, then the z_α do not play a role and we have $\mathcal{P}_{\chi^{[1]}} = I_{\chi^{[1]}}$.

We observe that by definition λ is a rational character on the torus $\mathcal{T} \times E$. Any such character defines a homomorphism $\chi_\lambda \colon \mathcal{T}(\mathbb{Z}_p) \to \mathcal{O}_{E_\mathfrak{p}}^\times \subset \mathcal{O}_{\mathbb{C}_p}^\times$. If we consider the reduction $\mod p$ then we get a homomorphism $\chi_\lambda^{[1]} \colon \mathcal{T}(\mathbb{F}_p) \to (\mathcal{O}_{E_\mathfrak{p}}/(p))^\times \subset (\mathcal{O}_{\mathbb{C}_p}/(p))^\times$. We have $\mathcal{O}_{E_\mathfrak{p}}/(p) = \mathbb{F}_{p^r}$. We want to write this homomorphism in the form above, we can forget the z_α and

$$\chi_\lambda^{[1]}((\dots, x_\alpha, \dots)) = \prod \omega(x_\alpha)^{\nu_\alpha}.$$

Now we have to analyze the relation between the coefficients n_i in λ and the ν_α. It suffices to investigate what happens on the factors $\mathcal{T}_\alpha$. We pick a simple root α and we consider its orbit $\alpha, \sigma(\alpha), \dots, \sigma^{r-1}(\alpha)$.

Since $E_\mathfrak{p}$ is the splitting field of the entire torus it can happen, that our root α is fixed under the action of the Galois group. Then we have

$$\mathcal{T}_\alpha(\mathbb{F}_p) = \mathcal{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times \subset \mathcal{G}_m(\mathbb{F}_{p^r}) = \mathbb{F}_{p^r}^\times.$$

The component of λ corresponding to this root is a rational character $x \mapsto x^{n_\alpha}$; if we restrict this to $\mathcal{G}_m(\mathbb{F}_p)$ it depends only on $n_\alpha \mod (p-1)$. On the other side ν_α is an integer $\mod (p^r - 1)$, but if we restrict this to $\mathcal{G}_m(\mathbb{F}_p)$ this restriction only depends on $\nu_\alpha \mod (p-1)$.

Now we consider the other extreme case namely the length of the orbit is r. Then we have

$$\mathcal{T}_\alpha(\mathbb{F}_p) = \mathbb{F}_{p^r}^\times \subset \prod_\sigma \mathbb{F}_{p^r}^\times$$

and the embedding is given by

$$x \mapsto (x, x^p, \dots, x^{p^{r-1}}).$$

The $\bar{\alpha}$ component $\lambda_{\bar{\alpha}} = \sum_\nu n_{\sigma^\nu(\alpha)} \gamma_{\sigma^\nu(\alpha)}$ induces on $\mathcal{T}_\alpha(\mathbb{F}_p) = \mathbb{F}_{p^r}^\times$ the homomorphism

$$x \mapsto x^{\sum_\nu n_{\sigma^\nu(\alpha)} p^\nu} = \prod (x^{p^\nu})^{n_{\sigma^\nu(\alpha)}},$$

and this implies that for any $\nu_{\bar{\alpha}} \in \mathbb{Z}/(p^r-1)$ we can find coefficients $0 \leq n_{\sigma^\nu(\alpha)} \leq p-1$ such that

$$\sum_\nu n_{\sigma^\nu(\alpha)} p^\nu = \nu_{\bar{\alpha}} \mod (p^r-1).$$

Hence we see that for any χ we can find a $\lambda = \sum_\alpha n_\alpha \gamma_\alpha, 0 \leq n_\alpha < p-1$ such that $w_0(\lambda)|\mathcal{T}(\mathbb{F}_p) = \chi^{[1]}$, and then we get a homomorphism

$$j_\lambda \colon \mathcal{M}_\lambda \otimes \mathcal{O}_{E_\mathfrak{p}}/(\mathfrak{p})) = \mathcal{M}_\lambda/\mathfrak{p}\mathcal{M}_\lambda \to I_{\chi^{[1]}}.$$

We can define Hecke operators $T(t_{p^k}, u_{t_{p^k}})$: We choose an element $t_{p^k} \in T(\mathbb{Q}_p)$ such that for all positive simple roots $|\alpha(t_{p^k})|_\mathfrak{p} < 1$. ($t_{p^k}$ is our $g \in G(\mathbb{Q})$, the parameter k is not an integer anymore; we should think of it as an array of positive integers giving us the orders of the $\mathfrak{p}$-adic valuations of the $\alpha(t_{p^k})$.) Then it is again clear that the possible $u_{t_{p^k}}$ are given by elements in

$$\begin{gathered}\mathrm{Hom}_{\mathcal{T}(\mathbb{F}_p)}\left((\mathcal{M}_\lambda/\mathfrak{p}\mathcal{M}_\lambda)_{\mathcal{U}_+(\mathbb{F}_p)}, (\mathcal{M}_\lambda/\mathfrak{p}\mathcal{M}_\lambda)^{\mathcal{U}_-(\mathbb{F}_p)}\right),\\ \mathrm{Hom}_{\mathcal{T}(\mathbb{F}_p)}\left((I_{\chi^{[1]}})_{\mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)}\right).\end{gathered}$$

The $\mathcal{T}(\mathbb{F}_p)$-modules $I_{\chi^{[1]},\mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)}$ are easy to compute, if we use the Bruhat decomposition. We have the action of $\mathcal{U}_+(\mathbb{F}_p), \mathcal{U}_-(\mathbb{F}_p)$ on $\mathcal{B}(\mathbb{F}_p)\backslash\mathcal{G}(\mathbb{F}_p)$ we write

$$\begin{aligned}\mathcal{G}(\mathbb{F}_p) &= \bigcup_w \mathcal{B}(\mathbb{F}_p) w \mathcal{U}_+(\mathbb{F}_p) = \mathcal{B}(\mathbb{F}_p) w_0 \mathcal{U}_+(\mathbb{F}_p) \cup \cdots \cup \mathcal{B}(\mathbb{F}_p),\\ \mathcal{G}(\mathbb{F}_p) &= \bigcup_w \mathcal{B}(\mathbb{F}_p) w \mathcal{U}_-(\mathbb{F}_p) = \mathcal{B}(\mathbb{F}_p) \mathcal{U}_-(\mathbb{F}_p) \cup \cdots \cup \mathcal{B}(\mathbb{F}_p) w_0.\end{aligned}$$

Clearly the module $I_{\chi^{[1]}}$ decomposes into direct sums under the action of the two unipotent radicals $\mathcal{U}_+(\mathbb{F}_p), \mathcal{U}_-(\mathbb{F}_p)$ according to the decompositions. Then we have the function Φ_e supported on the smallest orbit $\mathcal{B}(\mathbb{F}_p)$, its image $\overline{\Phi}_e$ in $I_{\chi^{[1]},\mathcal{U}_+(\mathbb{F}_p)}$ generates a copy of $\mathbb{F}_p = \mathbb{F}_p\overline{\Phi}_e$. We have the function $\Psi_e \in I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)}$ which has support in $\mathcal{B}(\mathbb{F}_p)\mathcal{U}_-(\mathbb{F}_p)$, i.e., it is given by $\Psi_e(bu_-) = \lambda(b)$. If we send $\overline{\Phi}_e$ to Ψ_e and all other summands to zero then this gives us an element in $\mathrm{Hom}_{\mathcal{T}(\mathbb{F}_p)}((I_{\chi^{[1]}})_{\mathcal{U}_+(\mathbb{F}_p)}, I_{\chi^{[1]}}^{\mathcal{U}_-(\mathbb{F}_p)})$ and this is our principal operator $u_{t_{p^k}}^{\mathrm{princ}}$.

If now all $n_\alpha > 0$ (this is the regularity condition), then we have seen that the function Ψ_e can be interpreted as the restriction of the $\mathcal{U}_-$ highest weight $e_{w_0(\lambda)}$ vector to $G(\mathbb{F}_p)$. It is clear that the classical operator $u_{t_{p^k}}^{\mathrm{class}}$ sends $e_{w_0(\lambda)}$ to $e_{w_0(\lambda)}$ and the subspace $\mathbb{F}_p e_{w_0(\lambda)}$ is the image of $u_{t_{p^k}}^{\mathrm{class}}$. Hence we see that the classical and the principal Hecke operator coincide in this case, and under the above regularity condition we get an isomorphism

$$j_\lambda^\bullet \colon H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda/\mathfrak{p}\mathcal{M}_\lambda) \to H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, I_{\chi^{[1]}}).$$

It is easily checked that the Hecke operators induce the zero map on the kernel and cokernel of j_λ. (See also 3.6.)

3.3. The case $m > 1$ for the group Gl_2. For this special group we give a very detailed discussion (may be too detailed but perhaps useful) of the possible choices of operators $u_{t_{p^k}}$ for our various coefficient systems.

The point is that these modules are $\mathbb{Z}/(p^m)$-modules and the action of Γ factors through the quotient $\mathrm{Gl}_2(\mathbb{Z}/(p^m)$. Let us assume that now M is any $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$-module. We assume $k \geq m$ and consider

$$\Gamma(t_{p^k}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b \equiv 0 \mod p^m\right\}$$

and its image in $\mathrm{Gl}_2(\mathbb{Z}/p^m\mathbb{Z})$ is the Borel subgroup

$$B_-(\mathbb{Z}/p^m\mathbb{Z}) = \left\{\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \mid a, d \in (\mathbb{Z}/p^m\mathbb{Z})^\times,\, c \in \mathbb{Z}/p^m\mathbb{Z}\right\}.$$

The group $\Gamma(t_{p^k})$ acts in two different ways on M we have the modules $M^{(t_{p^k})}$ and M. The action on M is the action induced by the inclusion $\Gamma(t_{p^k}) \subset \Gamma$. The module $M^{(t_{p^k})}$ is as an abelian group equal to M but the action of $\Gamma(t_{p^k})$ is the one where we include $\Gamma(t_{p^k})$ via the conjugation $\gamma \to t_{p^k}^{-1}\gamma t_{p^k}$ into Γ. To make it clear: An element $u_{t_{p^k}} \in \mathrm{Hom}_{\Gamma(t_{p^k})}(M^{(t_{p^k})}, M)$ is a homomorphism $u_{t_{p^k}} : M \to M$ which satisfies

$$u_{t_{p^k}}(t_{p^k}^{-1}\gamma t_{p^k} f) = \gamma u_{t_{p^k}}(f)$$

or in terms of matrices

$$u_{t_{p^k}}\left(\left(\begin{smallmatrix} a & b \\ p^k c & d \end{smallmatrix}\right) f\right) = \left(\left(\begin{smallmatrix} a & p^k b \\ c & d \end{smallmatrix}\right) u_{t_{p^k}} f\right).$$

Suppose that $U_-(\mathbb{Z}/p^m\mathbb{Z}), U_+(\mathbb{Z}/p^m\mathbb{Z})$ are the two unipotent radicals of $B_-(\mathbb{Z}/p^m\mathbb{Z}), B_+(\mathbb{Z}/p^m\mathbb{Z})$. Then the module $M_{U_+(\mathbb{Z}/p^m\mathbb{Z})}$ of coinvariants and $M^{U_-(\mathbb{Z}/p^m\mathbb{Z})}$ of invariants become $T(\mathbb{Z}/p^m\mathbb{Z})$-modules and it is clear that

$$\mathrm{Hom}_{\Gamma(t_{p^k})}(M^{(t_{p^k})}, M) \xrightarrow{\sim} \mathrm{Hom}_{T(\mathbb{Z}/p^m\mathbb{Z})}(M_{U_+(\mathbb{Z}/p^m\mathbb{Z})}, M^{U_-(\mathbb{Z}/p^m\mathbb{Z})}). \quad \text{(Hom)}$$

We have to understand the $T(\mathbb{Z}/(p^m))$-modules $I_{\chi[m],U_+(\mathbb{Z}/(p^m))}$ and $I_{\chi[m]}^{U_-(\mathbb{Z}/(p^m))}$. To do this we have to investigate the action of $U_\pm(\mathbb{Z}/(p^m))$ on

$$B_+(\mathbb{Z}/(p^m))\backslash \mathrm{Gl}_2(\mathbb{Z}/(p^m)) = \mathbb{P}^1(\mathbb{Z}/(p^m)).$$

We write the elements of $\mathbb{P}^1(\mathbb{Z}/(p^m))$ in the form (a, b) and the group acts by multiplication from the right.

We consider the action of $U_+(\mathbb{Z}/(p^m))$. We see that $x_0 = (0, 1)$ is the fixed points for $B_+(\mathbb{Z}/(p^m))$. Let w be the non trivial element in the Weyl group , i.e., $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and

$$(0, 1) w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (1, u).$$

This gives us the "big cell"

$$B_+(\mathbb{Z}/(p^m)) \cdot wU_+(\mathbb{Z}/(p^m)) \subset \mathrm{Gl}_2(\mathbb{Z}/(p^m)).$$

The remaining points are of the form

$$(v,1) \quad \text{where } v \equiv 0 \mod p.$$

The group $U_+(\mathbb{Z}/(p^m))$ acts by

$$(v,1)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (v, 1+uv) = (v(1+uv)^{-1}, 1),$$

and hence we see that two elements v, v' are in the same orbit for the action of $U_+(\mathbb{Z}/(p^m))$ if and only if

$$\mathrm{ord}(v) = \mathrm{ord}(v') \quad \text{and} \quad v \equiv v' \mod p^{2\,\mathrm{ord}(v)}.$$

The stabilizer of $(v,1)$ in $U_+(\mathbb{Z}/(p^m))$ is the congruence subgroup

$$U_+^{m-2\,\mathrm{ord}(v))}(\mathbb{Z}/(p^m)) = \left\{\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \mid u \equiv 0 \bmod p^{m-2\,\mathrm{ord}(v)}\right\},$$

which becomes the full group after $2\,\mathrm{ord}(v) \geq m$. We put

$$l(v) = \begin{cases} m - 2\,\mathrm{ord}(v) & \text{if } 2\,\mathrm{ord}(v) \leq m, \\ 0 & \text{else.} \end{cases}$$

then $p^{l(v)}$ is also the length of the orbit. We denote the orbits of $U_+(\mathbb{Z}/(p^m))$ on the set of v's by $\bar{v}$.

For any of the orbits we choose a representative $(1,0)$, $(v,1)$ and write it

$$\begin{aligned} (1,u) &= (0,1) \cdot w \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = (0,1) \cdot \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}, \\ (v,1) &= (0,1) \cdot \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}. \end{aligned}$$

Then we consider the double cosets $\mathcal{X}_w = B(\mathbb{Z}/p^m\mathbb{Z})wU_+(\mathbb{Z}/p^m\mathbb{Z})$, the intermediate cosets $\mathcal{X}_{\bar{v}} = B(\mathbb{Z}/p^m\mathbb{Z})\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)U_+(\mathbb{Z}/p^m\mathbb{Z})$, where $\bar{v}$ runs over the orbits with $\mathrm{ord}(\bar{v}) = 1, \ldots, m-1$ and the again special orbit $\mathcal{X}_0 = B(\mathbb{Z}/p^m\mathbb{Z}) = B(\mathbb{Z}/p^m\mathbb{Z})U_+(\mathbb{Z}/p^m\mathbb{Z})$.

We group the orbits according to the number $\nu = \mathrm{ord}(\bar{v})$ and write the decomposition into double cosets in decreasing order

$$\mathrm{Gl}_2(\mathbb{Z}/(p^m)) = \mathcal{X}_w \cup \bigcup_{\bar{v}:\nu=\mathrm{ord}(\bar{v})=1} \mathcal{X}_{\bar{v}} \cup \cdots \cup \bigcup_{\bar{v}:\nu=i} \mathcal{X}_{\bar{v}} \cup \cdots \cup \bigcup_{\bar{v}:\nu=m-1} \mathcal{X}_{\bar{v}} \cup B(\mathbb{Z}/p^m\mathbb{Z}).$$

In this decomposition the number ν goes up from zero to m, the number $p^{l(\nu)}$ start with p^m and drops to p^{m-2}, p^{m-4} until we reach the middle and then it becomes constant equal to one. Another number $\mu = \min(\nu, m-\nu)$ goes up from zero to $\left[\frac{m}{2}\right]$ and after that drops again in steps by one to zero.

We choose a χ and a ring $\mathbb{Z}_p \subset R \subset O_{\mathbb{C}_p}$ which receives the values of χ. We define $I_{\tilde{\chi}}$ as the induced module with values in R. We get a decomposition of $I_{\tilde{\chi}^{[m]}}$ into $U_+(\mathbb{Z}/(p^m))$-submodules

$$I_{\tilde{\chi}^{[m]}} = I^{(w)}_{\tilde{\chi}^{[m]}} \oplus \bigoplus_{\nu=1}^{\nu=m-1} \bigoplus_{\bar{v}:\mathrm{ord}(\bar{v})=i} I^{(\bar{v})}_{\tilde{\chi}^{[m]}} \oplus I^{(0)}_{\tilde{\chi}^{[m]}},$$

the submodules consist of functions which are supported on the orbits.

Now we define the functions: For $u \in U_+(\mathbb{Z}/(p^m)$ we put

$$\Psi_{w,u}(g) = \begin{cases} \chi^{[m]}(b) & \text{if } g = bwu, \\ 0 & \text{else,} \end{cases} \tag{1}$$

$$\Psi_v(g) = \begin{cases} \Psi_v(b \cdot \left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)) = \chi^{[m]}(b) & \text{if } g \in B_+(\mathbb{Z}/(p^m))\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right), \\ 0 & \text{else,} \end{cases} \tag{2}$$

which form a basis of $I_{\tilde{\chi}^{[m]}}$. These functions are essentially like δ-functions. If we want to be completely consistent, we should denote these functions by $\Psi^{[m]}_{w,u}, \ldots$, but as long as we work on a fixed level we suppress the superscript.

It is clear that the image of the elements $\Psi_{w,u}$ in $I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}$ is independent of u. Let us call it $\bar{\Psi}_w$. Two elements $\Psi_v, \Psi_{v'}$ have the same image in $I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}$ if and only if v, v' are conjugate under the action of $U_+(\mathbb{Z}/(p^m)$. This means that each orbit $\bar{v}$ of v's contributes by a cyclic R_m-module and hence we get a direct sum decomposition

$$I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))} = R_m \bar{\Psi}_w \oplus \bigoplus_{\bar{v}} R_m \bar{\Psi}_{\bar{v}},$$

where $\bar{\Psi}_{\bar{v}}$ is the image of any of the Ψ_v, $v \in \bar{v}$. The summands $R_m \bar{\Psi}_{\bar{v}}$ are not necessarily free R_m-modules. But by definition they are cyclic and hence we have to determine their annihilators.

To understand these annihilators we have to take into account that these elements $(v, 1)$ still have stabilizers in $U_+(\mathbb{Z}/(p^m))$, we described them further above. If we denote such a stabilizer by $U_+^{(v)}$, then it is clear that $U_+^{(v)}$ acts upon the free R_m-modules $R_m \Psi_v$ and

$$(R_m \Psi_v)_{U_+(\mathbb{Z}/(p^m))} \simeq R_m \bar{\Psi}_{\bar{v}}.$$

For $x_w = (0, 1)$ the stabilizer is trivial and we get

$$R_m \Psi_w = R_m \bar{\Psi}_w.$$

Now we saw that the stabilizer becomes bigger and bigger if $\mathrm{ord}(v)$ goes from 1 to m and once $2\,\mathrm{ord}(v) \geq m$ we have $U_+^{(v)} = U_+(\mathbb{Z}/(p^m))$.

We have to find out how these stabilizers act upon

$$R_m \Psi_v.$$

This is easy: Since $uv^2 \equiv 0 \bmod p^m$ we have

$$\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}(1+uv)^{-1} & u\\ 0 & (1+uv)\end{pmatrix}\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix}.$$

Hence we see that

$$\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\Psi_v = \chi(1+uv)\Psi_v,$$

and the annihilator of $\overline{\Psi}_v$ in R_m is the ideal generated by the elements

$$\chi(1+uv) - 1 = (1+uv)^\theta - 1 = \theta uv + \cdots .$$

If we take into account that u satisfies $u \equiv 0 \bmod p^{m-2\,\mathrm{ord}(v)}$, then this ideal is

$$(\theta uv) = (\theta p^{m-\mathrm{ord}(v)})$$

as long as we have $\mathrm{ord}(v) < \frac{m}{2}$. So the ideal becomes bigger and bigger as long as $\mathrm{ord}(v) < \frac{m}{2}$. But after that the stabilizer becomes $U_+(\mathbb{Z}/(p^m))$ and the ideal will be

$$\left(\theta p^{\mathrm{ord}(v)}\right) \quad \text{for } \mathrm{ord}(v) \geq \frac{m}{2}$$

and eventually for $v \equiv 0 \bmod p^m$ it becomes trivial. We defined

$$\mu = \min(\nu, m-\nu).$$

Then we see that this ideal is $(\theta p^{m-\mu})$.

We have

$$R_m\Psi_0 = R_m\overline{\Psi}_0.$$

Now we observe that the $R_m\overline{\Psi}_{\bar{v}}$ do not depend on the choice of a $v \in \bar{v}$ and

$$(R_m\Psi_v)_{U_+^{(v)}} \simeq R_m/(\theta p^{m-\mu})\Psi_{\bar{v}} = R_m\overline{\Psi}_{\bar{v}} \subset I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))}.$$

This means that we have a direct sum decomposition

$$I_{\tilde{\chi}^{[m]}, U_+(\mathbb{Z}/(p^m))} = R_m\overline{\Psi}_w \oplus \Big(\bigoplus_{\nu=1}^{\nu=m-1} \bigoplus_{\bar{v}:\mathrm{ord}(\bar{v})=\nu} R_m\overline{\Psi}_{\bar{v}}\Big) \oplus R_m\overline{\Psi}_0.$$

We recall that we have to understand the module of coinvariants as a $T(\mathbb{Z}/(p^m))$-module. The summands are $T(\mathbb{Z}/(p^m))$-modules and we have to investigate the action of the torus $T(\mathbb{Z}/(p^m))$ on these modules.

The torus leaves the two outer terms invariant, it acts on $R_m\overline{\Psi}_w$ by

$$(\tilde{\chi}^{[m]})^w : t = \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \chi^{[m]}(t_1),$$

which is the conjugate by the Weyl group of the character $\tilde{\chi}^{[m]}$. On $R_m\Psi_0$ it acts by $\chi^{[m]}$.

The individual summands in the middle are not invariant. The torus acts on the set of orbits for $U_+(\mathbb{Z}/(p^m))$ and the orbits under the torus action are given by the numbers $\nu = \mathrm{ord}(v)$ which vary from 1 to $m-1$. We group the summands according to the order $\nu = \mathrm{ord}(v)$ and consider the summand

$$\bigoplus_{\bar{v}:\mathrm{ord}(\bar{v})=i} R_m\overline{\Psi}_{\bar{v}},$$

where $\bar{v}$ runs over the orbits of $U_+(\mathbb{Z}/(p^m))$. This sum is invariant under the torus $T(\mathbb{Z}/(p^m))$. The stabilizer of an orbit $\mathcal{X}_{\bar{v}}$ is the torus

$$T^{(\mu)}(\mathbb{Z}/(p^m)) = \left\{\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \mid t_1/t_2 \equiv 1 \mod p^\mu\right\}.$$

An easy calculation shows that the $T^{(\mu)}(\mathbb{Z}/(p^m))$ fixes the module $R_m\overline{\Psi}_{\bar{v}}$. The restriction of our character $\chi^{[m]}$ to $T^{(\mu)}(\mathbb{Z}/(p^m))$ induces a character

$$\tilde{\chi}^{[m,\mu]} : T^{(\mu)}(\mathbb{Z}/(p^m)) \to (R_m/(\theta p^{m-\mu}))^\times,$$

by this character $T^{(\mu)}(\mathbb{Z}/(p^m))$ acts on $R_m\overline{\Psi}_{\bar{v}}$. If we choose a representative $\bar{v}$ with $\mathrm{ord}(\bar{v}) = \nu$, we find for the term in the middle

$$\bigoplus_{0<\nu<m} \mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m\overline{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]}.$$

A completely analogous computation gives us the modules of invariants $I_{\tilde{\chi}^{[m]}}^{U_-(\mathbb{Z}/(p^m))}$.

We start with the list of double cosets

$$\begin{aligned} \mathcal{Y}_w &= B_+(\mathbb{Z}/p^m\mathbb{Z})wU_-(\mathbb{Z}/p^m\mathbb{Z}) = B_+(\mathbb{Z}/p^m\mathbb{Z})w, \ldots \\ \ldots, \mathcal{Y}_{\bar{v}} &= B_+(\mathbb{Z}/p^m\mathbb{Z}) \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix} U_-(\mathbb{Z}/p^m\mathbb{Z}), \ldots \\ \ldots, \mathcal{Y}_0 &= B_+(\mathbb{Z}/p^m\mathbb{Z})U_-(\mathbb{Z}/p^m\mathbb{Z}), \end{aligned}$$

where in the middle the v runs over the elements in $p\mathbb{Z}$ mod $p^m\mathbb{Z}$ and the $\bar{v}$ are the equivalence classes with respect to the equivalence relation above. But this time we order them in descending order of $\mathrm{ord}(\bar{v})$.

The two extremal terms are again easy. We have the $U_-(\mathbb{Z}/(p^m))$ invariant function Φ_- which is supported on the big cell

$$\Phi_-(bu) = \tilde{\chi}(b),$$

and the function Φ_0 which is supported on $B(\mathbb{Z}/(p^m))w$ (the smallest cell)

$$\Phi_0(bw) = \tilde{\chi}(b),$$

and we know that $T(\mathbb{Z}/(p^m))$ acts by $\tilde{\chi}^{[m]}$ on $R_m\Phi_-$ and by $\tilde{\chi}^{[m],w}$ on $R_m\Phi_0$.

We investigate the terms in the middle. Again we denote the orbit of v under $U_-(\mathbb{Z}/(p^m))$ by $\bar{v}$. For any such an orbit we can take a representative

$$B_+(\mathbb{Z}/(p^m)) \cdot \begin{pmatrix} 0 & -1 \\ 1 & v \end{pmatrix}$$

and define a function Φ_v with support on $B_+(\mathbb{Z}/p^m\mathbb{Z})\left(\begin{smallmatrix} 0 & -1 \\ 1 & v \end{smallmatrix}\right)$ by the rule

$$\Phi_v\big(b \cdot \left(\begin{smallmatrix} 0 & -1 \\ 1 & v \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)\big) = \tilde{\chi}^{[m]}(b).$$

The function Φ_v is not invariant under the stabilizer $U_-^{(v)}(\mathbb{Z}/(p^m))$ because it picks up a factor $\chi(1-uv)^{-1}$ if we translate it by $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right) \in U_-^{(v)}(\mathbb{Z}/(p^m))$. An easy computation gives the formula

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \Phi_v = \chi(1-uv)^{-1}\Phi_{\frac{v}{(1-uv)}}.$$

If $u \in U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$ then $\frac{v}{(1-uv)} = v$. We saw that the expressions $1-\chi(1-uv)^{-1}$ with $u \in U^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$ generate the ideal $(\theta p^{m-\mu})$: The annihilator of this ideal is a principal ideal $(\beta(\mu)) \subset R_m$. Clearly $\beta(\mu)$ divides p^μ. If θ is a unit then we see that it is given by $(\beta(\mu)) = (p^\mu)$. The element $\beta(\mu)\Phi_v \in R_m\Phi_v$ is a generator for the $U_-^{(v)}(\mathbb{Z}/(p^m))$-invariants in the space of the functions supported on $B_+(\mathbb{Z}/p^m\mathbb{Z})\left(\begin{smallmatrix} 0 & -1 \\ 1 & v \end{smallmatrix}\right)U_-^{(v)}(\mathbb{Z}/p^m\mathbb{Z})$.

We define

$$\tilde{\Phi}_v = \beta(\mu) \sum_{u \in \bar{U}_-(\mathbb{Z}/(p^m))/U_-^{(v)}(\mathbb{Z}/(p^m))} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \Phi_v = \sum_{v' \in \bar{v}} \chi(1-uv)^{-1}\Phi_{\frac{v}{(1-uv)}};$$

here we parametrized the elements of the orbit $\bar{v}$ by $v' = v/(1-uv)$.

Then we find that $R_m\beta(\mu)\tilde{\Phi}_v$ is the space of $U_-(\mathbb{Z}/(p^m))$-invariant functions with support on the orbit $\bar{v}$ containing v, this is actually a free $R_m/(\theta p^{m-\mu})$-module. An easy calculation shows that $T^{(\mu)}(\mathbb{Z}/(p^m)$ acts again by $\tilde{\chi}^{[m,\mu]}$ on this summand. We can summarize: The coinvariants give

$$R_m\overline{\Psi}_w \oplus \bigoplus_{\nu=1}^{m-1} \mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\theta p^{m-\mu})\overline{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\overline{\Psi}_0, \qquad \text{(coinv)}$$

and the invariants (we arrange them in the opposite order) give

$$R_m\Phi_0 \oplus \bigoplus_{\nu=m-1}^{1} \mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\theta p^{m-\mu})(\beta(\mu)\tilde{\Phi}_v) \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\Phi_-, \text{ (inv)}$$

where v runs over a set of representatives of elements of given order $\mathrm{ord}(v) = \nu$ (we simply will take $v = p^{\nu}$). Note that the extremal terms are also induced, but in this case $\mu = 0$, so the induction step is trivial.

Now it is easy to see how to construct Hecke operators on our coefficient systems. First of all we have the operator

$$u_{t_{p^k}}^{(\mathrm{princ})} \colon \Psi_0 \mapsto \Phi_-$$

which sends all the other summands to zero. This is the *principal* Hecke operator. Since the function $\Phi_- \in \mathcal{P}_{\tilde{\chi}[m]}$ we see that it induces the zero operator on the quotient $I_{\tilde{\chi}[m]}/\mathcal{P}_{\tilde{\chi}[m]}$. I claim that the system of principal operators satisfies (ii), hence it yields an element in the projective limit. To see this we observe that we have for any $f \in \mathcal{P}_{\tilde{\chi}[m+1]}^{(t_{p^{m+1}})}$ the formula $u_{t_{p^{m+1}}}^{(\mathrm{princ})}(f) = f(e_{m+1})\Phi_-^{(m+1)}$, where e_{m+1} is the identity in $\mathrm{Gl}_2(\mathbb{Z}/(p^{m+1}))$ and where $\Phi_-^{(m+1)}$ is our function Φ_-, but on the next higher level. We have to check that for an element $x \in \mathrm{Gl}_2(\mathbb{Z}/(p^{m+1}))$ congruent to e_{m+1} modulo p^m we have

$$u_{t_{p^{m+1}}}^{(\mathrm{princ})}(R_x(f)) \equiv u_{t_{p^{m+1}}}^{(\mathrm{princ})}(f) \mod p^m.$$

This is clear from the definition of $\mathcal{P}_{\tilde{\chi}[m+1]}$.

We can enlarge the supply of Hecke operators by adding linear combinations of correction terms $u_{t_{p^m}}^{(i,\nu')}$ which are homomorphisms

$$\begin{array}{c}
\mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\theta p^{m-\mu})\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]} \\
\Big\downarrow u_{t_{p^m}}^{(i,\nu')} \\
\mathrm{Ind}_{T^{(\mu')}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(\theta p^{m-\mu'})\beta(\mu')\tilde{\Phi}_{v'} \otimes \tilde{\chi}^{[m,\mu']}
\end{array}$$

where $\mathrm{ord}(v') > 0$ if $\nu = 0$ and where we may require that $\mathrm{ord}(v) + \mathrm{ord}(v') \le m$. We consider maps of the form $u_{t_{p^k}} = u_{t_{p^k}}^{(\mathrm{princ})} + \sum u_{t_{p^k}}^{(i,\nu')}$, where the $u_{t_{p^k}}^{(i,\nu')}$ are as above.

By Frobenius reciprocity this means that our $u_{t_{p^k}}^{(i,\nu')}$ are elements in

$$\begin{aligned}
\mathrm{Hom}_{T^{(\mu')}(\mathbb{Z}/(p^m))}(\mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))}&(R_m/(\theta p^{m-\mu})\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]}), \\
&R_m/(\theta p^{m-\mu'})\beta(\mu')\tilde{\Phi}_{v'} \otimes \tilde{\chi}^{[m,\mu']}).
\end{aligned}$$

We assume $\mu \le \mu'$. Then $T^{(\mu')}(\mathbb{Z}/(p^m)) \subset T^{(\mu)}(\mathbb{Z}/(p^m))$ and as a $T^{(\mu)}(\mathbb{Z}/(p^m))$-module we get a direct sum over $\xi \in T(\mathbb{Z}/(p^m))/T^{(\mu)}(\mathbb{Z}/(p^m))$,

$$\mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))}(R_m/(\theta p^{m-\mu})\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]} = \bigoplus_{\xi}(R_m/(\theta p^{m-\mu})\tilde{\Psi}_{\xi\bar{v}}, \otimes \tilde{\chi}^{[m,\mu]}$$

and hence we see that our module of homomorphisms is given by

$$\bigoplus_{\xi} \mathrm{Hom}_{T^{(\mu')}(\mathbb{Z}/(p^m)))}(R_m/(\theta p^{m-\mu})\bar{\Psi}_{\xi\bar{v}}, R_m/(\theta p^{m-\mu})\beta(\mu')\tilde{\Phi}_{v'})$$
$$= \bigoplus_{\xi \in T(\mathbb{Z}/(p^m))/T^{(\mu)}(\mathbb{Z}/(p^m))} R_m/(\theta p^{m-\mu}).$$

We have a large supply of Hecke operators for the $I_{\tilde{\chi}^{[m]}}$. Not all of them are good because we also want that they send $\mathcal{P}^{(t_{p^k})}_{\tilde{\chi}^{[m]}}$ to $\mathcal{P}_{\tilde{\chi}^{[m]}}$.

3.4. Discuss the further requirements formulated at the beginning of this section. The first step is to modify the situation slightly. We want to get rid of the dependence of θ and this means that we pass to the quotient $R_m/(p^{m-\mu})$ on the left-hand side and on the right-hand side we replace the factor $\beta(\mu')$ by $p^{\mu'}$ and hence we get a submodule. We only look at correction terms that go from the quotient on the left to the submodule on the right. Hence we see that the quotients on the left get smaller and smaller if we go from right to left until we reach the middle. The analogous assertion holds for the submodules on the right. We still go one step further. After we pass the middle we continue with the drop rate, i.e., in the decomposition of the module of coinvariants we replace $p^{m-\mu}$ by p^{ν}, and in the decomposition of the module of invariants we replace $\beta(\mu)$ by p^{ν}. Then we get a quotient of the coinvariants and a submodule of the invariants. This means that we go to a still smaller quotient on the left and a still smaller submodule on the right.

Then we look for operators between the $T(\mathbb{Z}/(p^m))$-modules

$$I_{\tilde{\chi}^{[m]},U_+(\mathbb{Z}/(p^m)),\mathrm{small}} = R_m/(p^0)\bar{\Psi}_w \oplus \bigoplus_{\nu=1}^{m-1} \mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(p^i)\bar{\Psi}_{\bar{v}} \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\bar{\Psi}_0,$$

which is a quotient of the coinvariants and

$$I^{U_+(\mathbb{Z}/(p^m))}_{\tilde{\chi}^{[m]},\mathrm{small}} = R_m/(p^0)p^m\Phi_0 \oplus \bigoplus_{\nu=m-1}^{1} \mathrm{Ind}_{T^{(\mu)}(\mathbb{Z}/(p^m))}^{T(\mathbb{Z}/(p^m))} R_m/(p^{m-i})(p^{\nu}\tilde{\Phi}_v) \otimes \tilde{\chi}^{[m,\mu]} \oplus R_m\Phi_-.$$

Now it is easily verified that the elements in $I^{U_+(\mathbb{Z}/(p^m))}_{\tilde{\chi}^{[m]},\mathrm{small}}$ are actually in $\mathcal{P}_{\tilde{\chi}^{[m]}}$.

It is clear that $T(\mathbb{Z}/(p^m))$ invariant homomorphisms from $I_{\tilde{\chi}^{[m]},U_+(\mathbb{Z}/(p^m)),\mathrm{small}}$ to $I^{U_+(\mathbb{Z}/(p^m))}_{\tilde{\chi}^{[m]},\mathrm{small}}$ satisfy (i). The property (iiia, iiib) has been verified above. This system of Hecke operators clearly satisfies (ii). So we are left to show that the classical Hecke operator extends.

3.5. The extension of the classical Hecke operator to an element in $\mathcal{H}_\chi^{[m]}$. Now we consider the case $\chi = \chi_{-n}$ and the morphism

$$\mathcal{M}_n/p^m\mathcal{M}_n \to \mathcal{P}_{\tilde{\chi}_{-n}^{[m]}} \subset I_{\tilde{\chi}_{-n}^{[m]}}.$$

We have the classical Hecke operator on the cohomology of the sheaf $\widetilde{\mathcal{M}}_n$ which is given by the map $u_{t_{pk}}^{\mathrm{class}}\colon X^iY^{n-i} \mapsto p^{ki}X^iY^{n-i}$. If $k \geq m$ then $u_{t_{pk}}^{\mathrm{class}}$ maps Y^n to Y^n and all other monomials go to zero.

The monomial Y^n has as image in the module of coinvariants

$$Y^n \mapsto \bar{f}_{Y^n} = \Big(\sum_{u\in U_+(\mathbb{Z}/(p^m))} u^n\Big)\Psi_w + \sum_{\bar{v}} p^{l(v)}\overline{\Psi}_{\bar{v}} + \Psi_0.$$

Under the homomorphism given by the projection to the last component $I_{\tilde{\chi}_{-n}^{[m]}} \to R_m\Phi_0$ we find $\bar{f}_{Y^n} \mapsto \Psi_0$. Any $T(\mathbb{Z}/p^m\mathbb{Z})$-invariant homomorphism from $R_m\Phi_0$ – viewed as quotient of $I_{\tilde{\chi}_{-n}^{[m]}}$ – to the submodule of small invariants gives us a Hecke operator. Such a homomorphism is given by

$$\Psi_0 \mapsto \sum_{v\in p\mathbb{Z}/(p^m)} v^n \cdot \Phi_v + \Phi_-,$$

where the right-hand side is the image of the polynomial Y^n in $\mathcal{P}_{\tilde{\chi}_{-n}^{[m]}}^{U_-(\mathbb{Z}/p^m)}$. This gives us the desired extension of $u_{t_{pk}}^{\mathrm{class}}$ to a homomorphism $u_{t_{pk}}\colon I_{\tilde{\chi}_{-n}^{[m]}}^{(t_{pk})} \to I_{\tilde{\chi}_{-n}^{[m]}}$.

Since we assumed that $n > 0$ – this is the regularity condition –, we see that (iiib) is satisfied. In the next section we will see that this argument works for general reductive group schemes.

3.6. The case of a general reductive group scheme. At this point it turns out, that our previous considerations are much to detailed. What we actually need is that our system $\mathcal{H}_\chi^{[m]}$ contains the principal operator and in the case that $\chi = \chi_\lambda$ we only want to extend the classical operator. We will see that we have the same extension procedure to extend the classical Hecke operator in general. The regularity condition guarantees that we have the essential properties (iiia), (iiib) for this operator.

We give a few comments. I recall the situation in 3.2. We put $\mathcal{O}(V_p')$ and consider the R-module $\mathcal{M}_\lambda = H^0(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{w_0(\lambda)})$. It provides us a well-defined representation of the group scheme $\mathcal{G}/\operatorname{Spec}(R)$. We can restrict this representation to the torus $\mathcal{T}$, this representation is semi-simple, i.e., we have a decomposition into weight spaces

$$\mathcal{M}_\lambda = \bigoplus_\mu \mathcal{M}_{\lambda,\mu} = \mathcal{M}_{\lambda,\lambda} \oplus \cdots \oplus \mathcal{M}_{\lambda,w_0(\lambda)},$$

where μ runs over a finite set of weights of the form $\mu = \lambda - \sum_\alpha m_\alpha\alpha$ with $m_\alpha \geq 0$. These weight spaces are free $\mathcal{O}(V_p')$-modules. A weight vector $e_\mu \in \mathcal{M}_{\lambda,\mu}$ is also a regular function on $\mathcal{G}$. We observe that such a weight function e_μ vanishes at the identity element $e \in \mathcal{G}(\mathcal{O}(V_p')$ provided $\mu \neq w_0(\lambda)$.

We define the character $\chi_\lambda : \mathcal{T}(\mathbb{Z}_p) \to \mathcal{O}^\times_{E_p}$ by $\chi_\lambda(x) = w_0(\lambda)(x)$. Then we have the family of $\mathcal{G}(\mathbb{Z}_p)$-homomorphisms

$$\mathcal{M}_\lambda / p^m \mathcal{M}_\lambda \xrightarrow{j_m} \mathcal{P}_{\tilde{\chi}[m]} \to I_{\tilde{\chi}_\lambda^{[m]}}.$$

We consider the $\mathcal{B}(R)$- homomorphism $\psi_\lambda : I_{\tilde{\chi}_\lambda^{[m]}} \to R$ which is given by evaluation at the identity element. This linear map sends $e_{w_0(\lambda)}$ to 1, our observation above implies that the kernel $\ker(j_m)$ is contained in $\oplus_{\mu : \mu \neq w_0(\lambda)} \mathcal{M}_{\lambda,\mu}$.

Now we need a closer look at our classical Hecke operators. Recall that these operators $T(t_{p^k}, u^{\text{class}}_{t_{p^k}})$ induce endomorphisms on the cohomology groups $H^\bullet_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ and $H^\bullet_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}/p^m)$ and they are of course compatible with the reduction morphism. They only depend on the first variable t_{p^k}. The identity map $\mathcal{M}_\lambda^{(t_{p^k})} \to \mathcal{M}_\lambda$ is compatible with the action of the torus $\mathcal{T}/\mathrm{Spec}(R)$, especially it is clear that the identity $\mathcal{M}_\mathbb{Q}^{(t_{p^k})} \to \mathcal{M}_\mathbb{Q}$ commutes with the action of $\mathcal{T}(\mathbb{Q})$. The element $u^{\text{class}}_{t_{p^k}} : \mathcal{M}_\lambda^{(t_{p^k})} \to \mathcal{M}_\lambda$ is now given by applying t_{p^k} and then multiplying the result by a scalar $n(t_{p^k})$ such that $u^{\text{class}}_{t_{p^k}} = n(t_{p^k}) t_{p^k}$ maps $\mathcal{M}_\lambda^{(t_{p^k})}$ to $\mathcal{M}_\lambda$. This is achieved if $n(t_{p^k}) t_{p^k} e_{w_0(\lambda)} = e_{w_0(\lambda)}$. If now out t_{p^k} is sufficiently deep in the chamber, i.e., it satisfies $|\alpha(t_{p^k})|_p < 1$ for all simple positive roots α then it becomes clear that for all weight vectors $e_\mu \in \mathcal{M}_\mu$ we have

$$u^{\text{class}}_{t_{p^k}}(e_\mu) = \begin{cases} e_\mu & \text{if } \mu = w_0(\lambda), \\ p^{n(\mu,k))} e_\mu & \text{with } n(\mu,k) > 0 \text{ else.} \end{cases}$$

If we choose k very deep inside the chamber, then we will get $n(\mu, k) \geq m$ and hence we see that $u^{\text{class}}_{t_{p^k}}$ annihilates all weight vectors e_μ with $\mu \neq w_0(\lambda)$.

We come to a small intermission: This has the consequence that for two choices $t_{p^k}, t_{p^{k'}}$ which both are sufficiently deep inside the chamber we have on any of our cohomology groups

$$T(t_{p^k}, u^{\text{class}}_{t_{p^k}}) = T(t_{p^{k'}}, u^{\text{class}}_{t_{p^{k'}}}) + pV,$$

where V is an endomorphism of the cohomology.

Since this relation also holds on the cohomology groups $H^\bullet_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}/p^m)$ we see again that the ordinary cohomology groups $H^\bullet_{*,\text{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}/p^m)$ can be defined with respect to any Hecke operator $T(t_{p^k}, u^{\text{class}}_{t_{p^k}})$ provided we choose t_{p^k} deep inside the chamber.

The algebra of endomorphisms generated by the classical operators $T(t_{p^k}, u^{\text{class}}_{t_{p^k}})$ is commutative (this will be proved in [Ha-Coh], Chap. 2, but is not yet written) we see easily that the submodule $H^\bullet_{*,\mathrm{nilpt}}(S^G_{K_f}.\widetilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}/p^m)$ is not depending on the parameter k in the operator $T(t_{p^k}, u^{\mathrm{class}}_{t_{p^k}})$ which we choose to define it. The ends the intermission.

We resume and see that $u^{\mathrm{class}}_{t_{p^k}}$ annihilates $\bigoplus_{\mu \neq w_0(\lambda)} \mathcal{M}_{\lambda,\mu}$ and hence the kernel of j_m if k is very deep inside the chamber. Hence we see that it also acts trivially on the cohomology of this kernel. (This point was left open in the discussion of Theorem 2.1.) On the other hand the principal operator

$$u^{\mathrm{princ}}_{t_{p^k}} : (I_{\tilde{\chi}^{[m]}_\lambda})_{\mathcal{U}_+(\mathbb{Z}/p^m\mathbb{Z})} \to (I_{\tilde{\chi}^{[m]}_\lambda})^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$$

is the composition of ψ_λ and the homomorphism $R \to I_{\tilde{\chi}^{[m]}_\lambda}{}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$ which sends 1 to the function Φ_- , which is supported on the big cell $\mathcal{B}(\mathbb{Z}/p^m\mathbb{Z})\mathcal{U}_-(\mathbb{Z}_p/p^m\mathbb{Z})$. Hence we see that the principal operator sends $e_{w_0(\lambda)}$ to Φ_- whereas the classical operator sends $e_{w_0(\lambda)}$ to itself, here we consider $e_{w_0(\lambda)}$ as an element in $I_{\tilde{\chi}^{[m]}_\lambda}{}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}$ or $\mathcal{P}^{\mathcal{U}_-(\mathbb{Z}/p^m\mathbb{Z})}_{\tilde{\chi}^{[m]}_\lambda}$. Then the regularity condition asserts that $e_{w_0(\lambda)}$ vanishes $\bmod p$ on the cells which are different from the big cell and hence we have shown that

$$(u^{\mathrm{princ}}_{t_{p^k}} - u^{\mathrm{class}}_{t_{p^k}})(I_{\tilde{\chi}^{[m]}_\lambda}) \subset p(I_{\tilde{\chi}^{[m]}_\lambda}),$$

which establishes the validity in the requirements (i) to (iv) made in Section 2. Everything is ready to prove the generalization of Theorem 2.1

4. Further consequences

4.1. A boundedness result for ordinary torsion. We return to 3.6. For simplicity we assume that our group scheme $\mathcal{G}/\mathbb{Z}$ is a split Chevalley scheme. To any dominant weight λ we can attach the character $\chi_{w_0(\lambda)}$, and we get the $\mathcal{G}(\mathbb{Z}_p)$ invariant homomorphisms

$$\mathcal{M}_\lambda/p^m\mathcal{M}_\lambda \to \mathcal{P}_{\tilde{\chi}^{[m]}_\lambda} \to I_{\tilde{\chi}^{[m]}_\lambda}.$$

Now we can take over the arguments in the proof of Theorem 2.1 verbatim and get

Theorem 4.1. *If $\lambda = \sum_\alpha n_\alpha \gamma_\alpha$ is regular, i.e., $n_\alpha > 0$ for all α, then we get isomorphisms*

$$j^\bullet_\lambda : H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) \to H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, I_{\chi^{[m]}}).$$

In the following we only allow $* = \natural, c, \partial$ for the first $*$. The space of characters χ is $\Omega \times (X^*(T) \otimes \mathcal{O}_{\mathbb{C}_p})$, where Ω is the finite set of $\eta = \chi^{[1]}_{\lambda_0}$, where $\lambda_0 = \sum \nu_\alpha \gamma_a$

and ν_α such that $0 \le \nu_\alpha < p-1$. These characters $\eta = \chi_{\lambda_0}^{[1]}$ should be viewed as characters $\eta\colon \mathcal{T}(\mathbb{F}_p) \to \mathbb{Z}_p^\times$.

This space contains the subspace $\Omega \times (X^*(T) \otimes \mathbb{Z}_p)$. It is easy to see that the characters χ_λ are dense in $\Omega \times (X^*(T) \otimes \mathbb{Z}_p)$. We define a distance between two characters $\chi_1, \chi_2 \in \Omega \times (X^*(T) \otimes \mathbb{Z}_p)$ by the rule

$$d(\chi_1, \chi_2) \le \frac{1}{p^m} \quad \text{if and only if} \quad \chi_1^{[m]} = \chi_2^{[m]}.$$

We want to state and prove a theorem which is the consequence of the existence of the interpolating family. At various occasions and especially in my talk in Luminy I stated a much stronger theorem and several people raised doubts about the correctness of this stronger theorem. For instance E. Urban expressed some skepticism after my talk at the Graduate center of CUNY and eventually H. Hida even produced a counterexample. Now the new theorem is weaker and much less precise. But at the same time it raises some questions which seem to me interesting.

For the formulation of the theorem I need some preparation. For an arbitrary degree ν and a fixed λ we study the growth of the functions

$$m \mapsto \# H^\nu_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda / p^m \mathcal{M}_\lambda).$$

We write the exact sequence (*the level p^{m+1}-exact sequence*)

$$0 \to p\mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda \to \mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda \to \mathcal{M}_\lambda / p\mathcal{M}_\lambda \to 0$$

and the resulting long exact sequence for cohomology

$$\begin{aligned} \cdots \to H^{\nu-1}_{*,*}((S^G_{K_f}, \mathcal{M}_\lambda / p\mathcal{M}_\lambda) &\xrightarrow{\delta_\lambda^{\nu-1}} H^\nu_{*,*}(S^G_{K_f}, p\mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda) \\ \to H^\nu_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda) &\to H^\nu_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda / p\mathcal{M}_\lambda) \xrightarrow{\delta_\lambda^\nu} \cdots . \end{aligned}$$

The two extremal modules are finite dimensional vector spaces over $\mathbb{F}_p$ and clearly the dimensions of the kernels of $\delta_\lambda^{\nu-1}$, δ_λ^ν can only drop if m goes up. Hence we find minimal values $m^{\nu-1}_{*,*}(\lambda)$, $m^\nu_{*,*}(\lambda)$ such that these kernels of $\delta_\lambda^{\nu-1}$, δ_λ^ν become stationary (minimal) if $m \ge m^{\nu-1}_{*,*}(\lambda), m^\nu_{*,*}(\lambda)$. Let us now put $c^\nu_{*,*}(\lambda) = \dim H^\nu_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda / p\mathcal{M}_\lambda)$ and let $\kappa^\nu_{*,*}(\lambda)$ be the minimal value of the dimension of $\ker(\delta^\nu_\lambda)$. Then given any λ the sequence above yields the exact sequence

$$\begin{aligned} 0 \to H^{\nu-1}_{*,*}((S^G_{K_f}, \mathcal{M}_\lambda / p\mathcal{M}_\lambda) / \ker(\delta_\lambda^{\nu-1}) &\to H^\nu_{*,*}(S^G_{K_f}, p\mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda) \\ \to H^\nu_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda) &\to \ker(\delta^\nu_\lambda) \to 0, \end{aligned}$$

and since $p\mathcal{M}_\lambda / p^{m+1}\mathcal{M}_\lambda \xrightarrow{\sim} \mathcal{M}_\lambda / p^m \mathcal{M}_\lambda$ the following holds:

Under the assumption $m \geq m^{\nu-1}_{*,*}(\lambda), m^{\nu}_{*,*}(\lambda)$ *we get*

$$\frac{\# H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda/p^{m+1}\mathcal{M}_\lambda)}{\# H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda/p^{m}\mathcal{M}_\lambda)} = p^{-c^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu}_{*,*}(\lambda)}. \tag{A}$$

For any m we may also consider the exact sequence

$$\begin{aligned} 0 \to H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m &\to H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) \\ &\to H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^m] \to 0, \end{aligned}$$

where $[p^m]$ in the term below denotes the p^m-torsion of the module. For a given λ the order the torsion is stationary for $m \geq e^{\nu+1}_{*,*}(\lambda)$ = the exponent of this torsion group. For any m,

$$\# H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) = \#(H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m) \cdot \# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^m].$$

We write this for $m \geq m^{\nu-1}_{*,*}(\lambda), m^{\nu}_{*,*}(\lambda)$ and for $m+1$, and take the ratio:

$$\begin{aligned} &\frac{\#(H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^{m+1})}{\#(H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^{m})} \frac{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m+1}]}{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m}]} \\ &\quad = p^{-c^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu}_{*,*}(\lambda)}. \end{aligned} \tag{B}$$

We know that $H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}_p$ is finitely generated and write it as a direct sum

$$H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda) = \mathbb{Z}_p^{b^{\nu}_{*,*}(\lambda)} \oplus H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}}.$$

This tells us that for $m \geq m^{\nu-1}_{*,*}(\lambda), m^{\nu}_{*,*}(\lambda)$ we have the equality

$$\begin{aligned} &\frac{\#(H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^{m+1})}{\#(H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^{m})} \frac{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m+1}]}{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m}]} \\ &\quad = p^{-b^{\nu}_{*,*}(\lambda)-c^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu-1}_{*,*}(\lambda)+\kappa^{\nu}_{*,*}(\lambda)}. \end{aligned}$$

For any given λ and $m \gg 0$ the term on the left is equal to 1, this implies

$$b^{\nu}_{*,*}(\lambda) = -c^{\nu-1}_{*,*}(\lambda) + \kappa^{\nu-1}_{*,*}(\lambda) + \kappa^{\nu}_{*,*}(\lambda). \tag{C}$$

We go back to the previous equality and observe that the two factors on the left-hand side are ≥ 1. Since their product is one, we conclude

For $m \geq m^{\nu-1}_{*,*}(\lambda), m^{\nu}_{*,*}(\lambda)$ *we have the equality*

$$\frac{\# H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^{m+1})}{\# H^{\nu}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^{m})} = \frac{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m+1}]}{\# H^{\nu+1}_{*,*}(S^G_{K_f}, \mathcal{M}_\lambda)[p^{m}]} = 1, \tag{D}$$

and moreover we have that $e^{\nu+1}_{*,*}(\lambda) \leq \max(m^{\nu-1}_{*,*}(\lambda), m^{\nu}_{*,*}(\lambda))$.

Now we consider the functions $\lambda \to m^{\nu-1}_{*,\mathrm{ord}}(\lambda), \lambda \to m^{\nu}_{*,\mathrm{ord}}(\lambda)$. In my earlier version of this note I stated that these functions are continuous, provided I stay away from a very small set of irregular elements which was easy to describe. But this assertion is not necessarily true, we have to stay away from a larger and much more complicated set of "irregular" elements. (Continuous means of course locally constant because $\mathbb{N}$ has the discrete topology.)

What is clear is that the functions $m^{\nu}_{*,\mathrm{ord}}(\lambda)$ are lower semi-continuous and the $\kappa^{\nu}_{*,\mathrm{ord}}(\lambda)$ are upper semi-continuous. More precisely we have the following

Lemma 4.2. *If we pick a weight λ_0, a degree ν and an $m \geq m^{\nu}_{*,\mathrm{ord}}(\lambda_0)$. Then for any λ which satisfies $\lambda \equiv \lambda_0 \bmod p^m$ we have the inequalities*

$$m^{\nu}_{*,\mathrm{ord}}(\lambda) \geq m^{\nu}_{*,\mathrm{ord}}(\lambda_0), \quad \kappa^{\nu}_{*,\mathrm{ord}}(\lambda) \leq \kappa^{\nu}_{*,\mathrm{ord}}(\lambda_0).$$

A strict inequality in one line implies strict inequality in the other.

Proof. This is clear. We consider the two-level p^{m+1} exact sequences above for λ_0 and λ. We have $\chi^{[m+1]}_{\lambda} = \chi^{[m+1]}_{\lambda_0}$. Then $j^{\bullet,[m+1]}$, provides isomorphisms which yield $\ker(\delta^{\nu}_{\lambda}) = \ker(\delta^{\nu}_{\lambda_0})$. We conclude that $m^{\nu}_{*,\mathrm{ord}}(\lambda) \geq m^{\nu}_{*,\mathrm{ord}}(\lambda_0)$: If we pass to higher level sequences then the dimension of $\ker(\delta^{\nu}_{\lambda})$ may drop, but not the dimension of $\ker(\delta^{\nu}_{\lambda_0})$. $\square$

I claim that we can extend these functions to lower continuous functions $\{\eta\} \times (X^*(T) \otimes \mathcal{O}_{\mathbb{C}_p})$. To see that this is the case, we consider the exact sequence

$$0 \to \mathcal{P}_{\chi^{[m,\prime]}} \to \mathcal{P}_{\tilde{\chi}^{[m+1]}} \to \mathcal{P}_{\chi^{[1]}} \to 0,$$

where by definition $\mathcal{P}_{\chi^{[m,\prime]}}$ is the kernel of the restriction. It is equal to the intersection $\mathcal{P}_{\tilde{\chi}^{[m+1]}} \cap pI_{\tilde{\chi}^{[m+1]}}$. We have a canonical homomorphism $r : \mathcal{P}_{\chi^{[m,\prime]}} \to I_{\tilde{\chi}^{[m]}}$, which is given by $f = pg \mapsto g \bmod p^m$. It is clear that $\mathcal{P}_{\tilde{\chi}^{[m]}} \subset r(\mathcal{P}_{\chi^{[m,\prime]}}) \subset I_{\tilde{\chi}^{[m]}}$ and hence we get isomorphisms

$$H^{\bullet}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}^{[m]}}) \xrightarrow{\sim} H^{\bullet}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\chi^{[m,\prime]}}) \xrightarrow{\sim} H^{\bullet}_{*,\mathrm{ord}}(S^G_{K_f}, I_{\tilde{\chi}^{[m]}}).$$

This gives us again a long exact sequence (the level p^{m+1} long exact sequence)

$$\cdots \to H^{\nu-1}_{*,\mathrm{ord}}((S^G_{K_f}, \mathcal{P}_{\chi^{[1]}}) \xrightarrow{\delta^{\nu-1}_{\lambda}} H^{\nu}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}^{[m]}}) \to H^{\nu}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}^{[m+1]}})$$
$$\to H^{\nu}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\chi^{[1]}}) \xrightarrow{\delta^{\nu}_{\lambda}} \cdots .$$

This allows us to extend the definition of $m^{\nu-1}_{*,\mathrm{ord}}(\chi), m^{\nu}_{*,\mathrm{ord}}(\chi)$ to all characters χ. The above lemma extends to these characters.

Especially we can consider the extension to the compact space $\{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p)$. For any index ν and any integer $k > 0$ we define the open subsets

$$\{\eta\} \times D^{\nu}_{*,\eta,k} = \{\chi \in \{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p) \mid m^{\nu}_{*,\mathrm{ord}}(\chi) \geq k\}.$$

Clearly we have

$$\bigcap_k D^{\nu}_{*,\eta,k} = \emptyset.$$

But the sets $D^{\nu}_{*,\eta,k}$ are not necessarily closed, we consider their closure $\bar{D}^{\nu}_{*,\eta,k}$ and put $Y^{\nu}_{*i,k} = \bar{D}^{\nu}_{*,\eta,k} \setminus D^{\nu}_{*,\eta,k}$. Then we get

$$\bigcap_k \bar{D}^{\nu}_{*,\eta,k} = \bigcap_k Y^{\nu}_{*,\eta,k} = Y^{\nu}_{*,\eta},$$

where the $Y^{\nu}_{*,\eta}$ are compact.

We can characterize these sets in a different way

Lemma 4.3. *The set $Y^{\nu}_{*,\eta}$ consists of those points χ_1 which satisfy any of the two conditions:*

(i) *The function $\chi \mapsto m^{\nu}_{*,\mathrm{ord}}(\chi)$ is discontinuous in χ_1.*

(ii) *The function $\chi \mapsto m^{\nu}_{*,\mathrm{ord}}(\chi)$ is unbounded in any neighborhood of χ_1.*

Proof. The assertion (ii) is immediate from the definition. To see (i) we have to show that in any point $\chi_0 \notin Y^{\nu}_{*,\eta}$ the function $\chi \mapsto m^{\nu}_{*,\mathrm{ord}}(\chi)$ is continuous. We find an integer k such that $\chi_0 \notin \bar{D}^{\nu}_{*i,k}$. It follows from Lemma 4.2 that we can choose a small box $B(\chi_0, \varepsilon) = \{\chi \mid d(\chi, \chi_0) \leq \varepsilon\}$ such that

$$B(\chi_0, \varepsilon) \cap \bar{D}^{\nu}_{*i,k} = \emptyset$$

such that for all χ in this box we have $m^{\nu}_{*,\mathrm{ord}}(\chi) \geq m^{\nu}_{*,\mathrm{ord}}(\chi_0)$. For any point χ in this box for which $m^{\nu}_{*,\mathrm{ord}}(\chi) > m^{\nu}_{*,\mathrm{ord}}(\chi_0)$ we can choose another such box (a second type box) such that on this box $m^{\nu}_{*,\mathrm{ord}}$ does not drop. The values $m^{\nu}_{*,\mathrm{ord}}(\chi)$ are $\leq k$ on $B(\chi_0, \varepsilon)$. Lemma 4.2 tells us that these second type of boxes can be taken such that their volume is bounded from below by a fixed constant $c > 0$. But since any two of these second type boxes are either disjoint or one of them is contained in the other, we can conclude that there are only finitely many disjoint boxes of the second type. These second type boxes are open compact, hence their complement is open in $B(\chi_0, \varepsilon)$ and contains χ_0. □

Of course this has consequences for the functions $\lambda \to \kappa^{\nu}_{*,\mathrm{ord}}(\lambda)$. If we are in a point $y \in Y^{\nu}_{*,\eta}$ then this means that we can find λ arbitrarily close to y such that $m^{\nu}_{*,\mathrm{ord}}(\lambda) > m^{\nu}_{*,\mathrm{ord}}(y)$. But this clearly implies that $\kappa^{\nu}_{*,\mathrm{ord}}(\lambda) < \kappa^{\nu}_{*,\mathrm{ord}}(y)$. Hence we can conclude that the functions $\lambda \to \kappa^{\nu}_{*,\mathrm{ord}}(\lambda)$ are upper semicontinuous, their values only can drop in a neighborhood of a point.

We can define a distance $d(\chi, Y^{\nu}_{*,\eta})$ from a point $\chi \in \{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p)$ to $Y^{\nu}_{*,\eta}$: This distance is simple the smallest distance from χ to a point $y \in Y^{\nu}_{*,\eta}$. Now for any $\varepsilon > 0$ we define the tubular neighborhood

$$N_{\varepsilon}(Y^{\nu}_{*,\eta}) = \{\chi \in \{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p) | d(\chi, Y^{\nu}_{*,\eta}) \leq \varepsilon\}.$$

This is an open compact subset in $\{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p)$.

Theorem 4.4. *For all indices ν we choose an open compact neighborhood $N_\varepsilon(Y^\nu_{*,\eta})$. If λ varies over all dominant weights λ in the complement X_ε of $\eta \times \bigcup_\nu N_\varepsilon(Y^\nu_{*,\eta})$, then the order of the p-torsion of the ordinary cohomology groups $H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda \otimes \mathbb{Z}_p)$ is bounded.*

Proof. This is now almost obvious. We can find an integer m_0 which is larger than $m^{\nu-1}_{*,\mathrm{ord}}(\lambda), m^\nu_{*,\mathrm{ord}}(\lambda)$ for all λ in the compact open set X_ε. Then we get from (D) that $\# H^\nu_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^m)$ does not depend on m provided $m \geq m_0$. Hence for $m \geq m_0$ we conclude that

$$\#(H^\nu_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^m) = \# H^\nu_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}}$$

for all $\lambda \in X_\varepsilon$. Now $\#(H^\nu_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \otimes \mathbb{Z}/p^m)$ divides $\#(H^\nu_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda \otimes \mathbb{Z}/p^{m_0}))$, and since this group only depends on λ mod p^{m_0}, the proof is finished. $\square$

Now we encounter the fundamental problem to understand the sets $Y^\nu_{*,\eta}$. Of course they may depend on the prime p, we have some informations concerning these sets which will be discussed below. All these informations do not depend on p.

We recall the relation (C). The first term on the right-hand is p-adically continuous, the next two terms are upper continuous, i.e., in a suitably small neighborhood of a point they only can drop. Hence we see that $\lambda \mapsto b^\nu_{*,\mathrm{ord}}(\lambda)$ is also upper semi-continuous. Moreover it is clear:

Lemma 4.5. *The function $\lambda \mapsto b^\nu_{*,\mathrm{ord}}(\lambda)$ is continuous in a point λ_0 if and only if the two functions $\lambda \mapsto \kappa^{\nu-1}_{*,\mathrm{ord}}(\lambda), \lambda \mapsto \kappa^\nu_{*,\mathrm{ord}}(\lambda)$ are continuous in that point.*

Hence we know in the above theorem that the Betti numbers $b^\nu_{*,\mathrm{ord}}(\lambda)$ are constant on X_ε. We may turn this around and then we get a simpler version of Theorem 4.4.

Theorem 4.6. *Let us fix a degree q. Let $X \subset \{\eta\} \times X^*(T) \otimes \mathbb{Z}_p$ an open subset such that $\lambda \to b^q_{*,\mathrm{ord}}(\lambda)$ is constant on X. Then the function*

$$\lambda \mapsto \#(H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}}) \cdot \#(H^{q+1}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}})$$

is constant on X.

Especially if there exist a $\lambda_0 \in X$ such that

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_{\lambda_0})_{\mathrm{tors}} = H^{q+1}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_{\lambda_0})_{\mathrm{tors}} = 0,$$

then all these torsion groups vanish for all $\lambda \in X$.

The proof is obvious, just look at the standard exact sequence for $m \gg 0$.

We can draw some further conclusions. Let us consider an open subset $U_i \subset \{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p)$ which does not meet any of the $Y^{\nu}_{*,\eta}$. From our last formula it follows that $\lambda \to b^q_{*,\mathrm{ord}}(\lambda)$ is locally constant, more precisely it depends only on $\lambda \bmod p^{m_1}$, provided $m_1 \geq m^{\nu-1}_{*,\mathrm{ord}}(\lambda), m^{\nu}_{*,\mathrm{ord}}(\lambda)$. We consider a $\chi \in U_i$ and we approximate χ by a λ . Then we look at our exact sequence

$$0 \longrightarrow H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m \longrightarrow$$

$$\begin{array}{ccccc} \longrightarrow H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda) & \longrightarrow & H^{q+1}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)[p^m] & \longrightarrow & 0 \\ \downarrow & & & & \\ H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}[m]}). & & & & \end{array}$$

The vertical arrow is an isomorphism, the modules in the diagram stay the same if we modify λ into a better approximation of χ.

Hence we see that $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}[m]})$ has a submodule S_m which is the image of the module on the left and which is isomorphic to $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda) \otimes \mathbb{Z}/p^m$ and where the cokernel by this submodule is $H^{q+1}_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)[p^m]$.

If we now take the projective limit then we see that for the terms on the right the projective limit is zero. This yields for the projective limit

$$\varprojlim S_m = \varprojlim H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\tilde{\chi}[m]}) = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_\chi).$$

This gives us for $\chi \in \{\nu\} \times (X^*(T) \otimes \mathbb{Z}_p)$ the structure

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_\chi) \xrightarrow{\sim} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda)_{\mathrm{tors}} \oplus \mathbb{Z}_p^{b^q_{*,\mathrm{ord}}(\chi)},$$

where $b^q_{*,\mathrm{ord}}(\chi) = b^q_{*,\mathrm{ord}}(\lambda)$ and λ is approximating χ well enough.

4.2. p-adic families. We want to present some further consequences of our theorems. These results will be discussed and proved in detail in a subsequent paper with J. Mahnkopf.

As always we fix our prime p. For simplicity we consider the case of a split semi-simple group scheme $G/\mathrm{Spec}(\mathbb{Z})$. Look at our various coefficient systems $\mathcal{M}_\lambda \otimes \mathbb{Z}_{(p)}, \mathcal{M}_\lambda/p^m\mathcal{M}_\lambda, \mathcal{P}_{\chi[m]}$. So far we only considered Hecke operators at the prime p these are the operators $T(t_{p^k}, u_{t_{p^k}})$. They allow the definition of ordinary cohomology groups. We still have the other Hecke operators which are obtained from elements $g \in G(\mathbb{Z}[\frac{1}{q}])$ where q is a prime different from p (or an integer prime to p).

For these g we do not have any problem to choose a second component u_g because all our modules are $G(\mathbb{Z}_{(p)})$-modules.

Hence we get a Hecke algebra

$$\mathcal{H} = \mathcal{H}_p \otimes \prod_{q \neq p} \mathcal{H}_q = \mathcal{H}_p \times \mathcal{H}^{(p)},$$

where $\mathcal{H}_p$ is generated by the $T(t_{p^k}, u_{t_{p^k}})$. If our module is $\mathcal{M}_\lambda \otimes \mathbb{Z}_{(p)}$ then we only allow $T(t_{p^k}, u_{t_{p^k}}^{\text{class}})$. We assume (for simplicity) that $K_f = G(\hat{\mathbb{Z}})$, which has the effect that the Hecke algebra $\mathcal{H}^{(p)}$ is commutative.

We consider the cohomology $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. We localize at our given prime p, i.e., we replace $\mathcal{M}_\lambda$ by $\mathcal{M}_\lambda \otimes \mathbb{Z}_{(p)}$ – this module will still be called $\mathcal{M}_\lambda$ –, but we do not take the completion. At this point is does not make sense to speak of ordinary cohomology.

We may decompose the inner cohomology into isotypical components

$$H^q_{!,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = \bigoplus_{\Pi_f} H^q_{!,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\Pi_f).$$

If we take other values of $* = \natural, \partial, c$ we do not have such a decomposition, but we may consider a Jordan–Hölder series for the cohomology groups $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$. The irreducible subquotients X are irreducible modules for the Hecke algebra. If $I(X)$ is the annihilator of X in the Hecke algebra, then $\mathcal{H} \otimes \mathbb{Q}/(I(X) \otimes \mathbb{Q})$ is a field $\mathbb{Q}(X)$, which only depends on the isomorphism type of X.

Now we can find a minimal finite normal extension $F/\mathbb{Q}$ such that all the $\mathbb{Q}(X)$ admit an embedding into F. Let $\mathcal{O}_F$ be its ring of integers localized at p.

We consider the cohomology $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)$. Then the above decomposition (resp. the Jordan–Hölder series) refines and now the summands (resp. subquotients) become absolutely irreducible. This means that

$$H^q_{!,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) = \bigoplus_{\pi_f} H^q_{!,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\pi_f),$$

were the π_f are absolutely irreducible. In general we have a Jordan–Hölder filtration with absolutely irreducible subquotients. Since we assumed that the Hecke algebra is commutative, such a subquotient X is a one-dimensional vector space over F and the isomorphism type as Hecke module is simply a homomorphisms $\pi_f(X) = \pi_f : \mathcal{H} \to \mathcal{O}_F$.

The Jordan–Hölder filtration induces such a filtration on $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_F)/\text{tors}$. We choose a prime ideal $\mathfrak{p} \supset (p)$, we localize at this prime (we do not take the completion), divide by the torsion and consider $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})/\text{tors}$. We say that a π_f is ordinary at $\mathfrak{p}$ ($\mathfrak{p}$-ordinary) if the image of $T(t_{p^k}, u_{p^k})$ is a unit in $\mathcal{O}_{F,(\mathfrak{p})}$.

Then we can regroup our Jordan–Hölder series and collect the ordinary π_f in one group to get a decomposition into

$$H^q_{*,*}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})/\mathrm{tors} \\ = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})/\mathrm{tors} \oplus H^q_{*,\mathrm{nilpt}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})/\mathrm{tors},$$

where by definition the first summand has a Jordan–Hölder series containing the $\mathfrak{p}$-ordinary π_f and the second summand the others. We can lift this decomposition and obtain

$$H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}) = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}) \oplus H^q_{*,\mathrm{nilpt}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}).$$

We consider the inclusion $\mathbb{Z}_p \hookrightarrow \mathcal{O}_{F,\mathfrak{p}}$ (here we took the completion) and clearly

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_p) \otimes \mathcal{O}_{F,\mathfrak{p}} \xrightarrow{\sim} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}) \otimes \mathcal{O}_{F,\mathfrak{p}}.$$

Let us choose an highest weight $\lambda_0 = \sum_\alpha \nu_\alpha \gamma_\alpha$ where $0 < \nu_\alpha < p-1$. We also consider the character $\chi^{[1]}_{\lambda_0}$. We may also view $\chi^{[1]}_{\lambda_0}\colon T(\mathbb{F}_p) \to \mathbb{F}_p^\times$ as a character $\chi_0\colon T(\mathbb{F}_p) \to \mathbb{Z}_p^\times$. To formulate our goal we write two postulates on λ_0 which may or may not be true for a given λ_0:

$$\text{The ordinary cohomology groups } H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_0}) \text{ are torsion-free} \qquad (\lambda_0\text{-tf})$$

and

$$\begin{aligned}&\textit{For a given pair of consecutive degrees } q,\ q+1 \textit{ and for}\\ &\textit{all values } * = \natural,\ !,\ \partial,\ c \textit{ we have that } b^q_{*,\mathrm{ord}}(\lambda) \textit{ is constant}\\ &\textit{for } \lambda \in \Lambda_{\chi_0} = \{\lambda = \lambda_0 + (p-1)\textstyle\sum n_\alpha \gamma_\alpha\}.\end{aligned} \qquad (\lambda_0\text{-Bc})$$

Theorem 4.6 shows that all $H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$, $\lambda \in \Lambda_{\lambda_0}$, are torsion-free if ($\lambda_0$-tf) is true.

We choose a $\lambda \in \Lambda_{\lambda_0}$ and consider the Hecke module $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{F}_p) = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_{\chi^{[1]}_0})$. At this point I want to restrict the action of the Hecke algebra to the action of $\mathcal{H}^{(p)}$. The Hecke operators in $\mathcal{H}_p$ only serve the purpose to define the ordinary cohomology and they depend on the choice of $u_{t_{p^k}}$.

Let $\mathcal{I} \subset \mathcal{H}^{(p)} \otimes \mathbb{F}_p$ is the annihilator of $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{F}_p)$, then the quotient by this ideal $\mathcal{A}^{(p)}_{\chi_0} = \mathcal{H}^{(p)} \otimes \mathbb{F}_p/\mathcal{I}$ is the direct sum of local algebras. It defines a zero dimensional scheme $V_{\chi_0} = \mathrm{Spec}(\mathcal{A}^{(p)}_{\chi_0})/\mathrm{Spec}(\mathbb{F}_p)$ We find a (smallest) finite extension $\mathbb{F}_q \supset \mathbb{F}_p$ such that the algebra $\mathcal{A}^{(p)}_{\chi_0} \otimes \mathbb{F}_q$ decomposes into a direct sum of absolutely local algebras, i.e., all its (reduced) irreducible components are absolutely irreducible. We write

$$\mathcal{A}^{(p)}_{\chi_0} \otimes \mathbb{F}_q = \bigoplus_{\phi \in \Phi} \mathcal{A}^{(p)}_{\chi_0} \otimes \mathbb{F}_q e_\phi,$$

here the e_ϕ are a system of orthogonal idempotents and Φ is the set of these idempotents. It is also the set of geometric points of $\mathcal{A}^{(p)}_{\chi_0} \otimes \mathbb{F}_q$. For our given $\lambda \in \Lambda_{\lambda_0}$ we choose a field F as above and a prime $\mathfrak{p} \supset (p)$ as above. We look exact sequence in cohomology obtained from

$$0 \to \mathcal{M}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})} \to \mathcal{M}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})} \to \mathcal{M}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})} \otimes \mathcal{O}_{F,\mathfrak{p}}/\mathfrak{p} \to 0$$

and see that our assumptions imply

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}/\mathfrak{p}) = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_{\chi_0}) \otimes \mathcal{O}_{F,\mathfrak{p}}/\mathfrak{p}. \qquad \text{(red)}$$

Let $\{0\} \subset X_1 \subset X_2 \subset \cdots \subset \cdots \subset X_r = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})$ be a filtration obtained from intersecting with a Jordan–Hölder series of $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)$. Then the subquotients X_i/X_{i-1} provide homomorphisms $\pi_f^{(p)}(i)\colon \mathcal{H}^{(p)} \to \mathcal{O}_{F,(\mathfrak{p})}$. Let Σ_{ord} be the set of isomorphism classes occurring in the list $\{\pi_f^{(p)}(i)\}_i$. It is called the $\mathfrak{p}$ ordinary spectrum of $H^q_{*,}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)$.

On the other hand we have the decomposition

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}/\mathfrak{p}) = \bigoplus_{\phi\in\Phi} e_\phi H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}/\mathfrak{p}).$$

We consider the reduction $\bmod \mathfrak{p}$ of the filtration

$$\{0\} \subset \bar{X}_1 \subset \bar{X}_2 \cdots \subset \cdots \subset \bar{X}_r = H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_{\chi_0}) \otimes \mathcal{O}_{F,\mathfrak{p}}/\mathfrak{p}.$$

It is clear that $\bar{X}_1$ maps into a summand $e_\phi H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}/\mathfrak{p})$. Dividing by $\bar{X}_1$ on both sides we can apply this reasoning to $\bar{X}_2/\bar{X}_1$ and by induction we get map from $r_\mathfrak{p}\colon \Sigma_{\mathrm{ord}} \to \Phi$. It follows from (red) that this map is surjective. The set $r_\mathfrak{p}^{-1}(\phi)$ of isomorphism classes $\pi_f^{(p)}$ which map to a given e_ϕ form a set of isomorphism classes of Hecke modules, which are congruent to each other $\bmod \mathfrak{p}$. In other words we get a decomposition

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})}) = \bigoplus_{\phi\in\Phi} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})(\phi),$$

where

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\phi) = \bigoplus_{\pi_f^{(p)} \in r_\mathfrak{p}^{-1}(\phi)} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f^{(p)}).$$

We know that the rank of $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,(\mathfrak{p})})(\phi)$ is equal to the rank of $e_\phi H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_{\chi_0}) \otimes \mathcal{O}_F/\mathfrak{p}$.

Our assumption implies that, for any character $\chi = (\eta, \theta)$ with $\eta = \chi^{[1]} = \chi^{[1]}_{\lambda_0}$ and $\theta \in X^*(T) \otimes \mathbb{Z}_p$, the cohomology $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi)$ is a free $\mathbb{Z}_p$-module of rank $b^q_{*,\mathrm{ord}}(\lambda_0)$. If $\mathcal{O}_\mathfrak{p}/\mathbb{Z}_p$ is an unramified extension with residue field our $\mathbb{F}_q$ above then

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi \otimes \mathcal{O}_\mathfrak{p}) = \bigoplus_{\phi \in \Phi} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi \otimes \mathcal{O}_\mathfrak{p})(\phi).$$

On our cohomology groups we always have an action of the group $\pi_0(G(\mathbb{R}))$ of connected components of $G(\mathbb{R})$ (See [Ha-Coh], Chap. 3, 1.1 and 2.8). This action commutes with the action of the Hecke algebra and since this group is an elementary 2-group each isotypic component is split into a direct sum according to characters $\epsilon \colon \pi_0(G(\mathbb{R})) \to \{\pm 1\}$.

We say that an element $\phi \in \Phi$ has minimal multiplicity if $\mathcal{A}^{(p)}_{\chi_0} \otimes \mathbb{F}_q e_\phi = \mathbb{F}_q$ and if for any character $\epsilon \colon \pi_0(G(\mathbb{R})) \to \{\pm 1\}$ the eigenspace $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi \otimes \mathcal{O}_\mathfrak{p})(\phi)(\epsilon)$ has dimension ≤ 1. We say that ϵ is admissible for ϕ if this dimension is equal to 1. This condition implies that the congruence class $r_\mathfrak{p}(\phi)$ consist of one element and if this is so, then we need some input from the theory of automorphic forms to verify this condition (multiplicity one theorems).

If ϕ is of minimal multiplicity then it is clear that for all $\chi \in \{\eta\} \times X^*(T) \otimes \mathbb{Z}_p$ and any admissible choice of $\epsilon \colon \pi_0(G(\mathbb{R})) \to \{\pm 1\}$ we have a unique $\pi_f(\chi, \phi, \epsilon)$ which occurs in $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi \otimes \mathcal{O}_\mathfrak{p})$ with multiplicity one and maps to ϕ. If $\chi = \chi_\lambda, \lambda \in \Lambda_{\chi_0}$ then we denote this Hecke module by $\pi(\lambda, \phi, \epsilon)$. The minimal normal extension $F_1/\mathbb{Q}$ over which $\pi_f(\lambda, \phi, \epsilon)$ is defined, has a unique prime $\mathfrak{p} \supset (p)$ which is unramified and has residue field $\mathbb{F}_q$ such that $\pi_f(\lambda, \phi, \epsilon)$ is an isotypical summand of $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{M}_\lambda \otimes \mathcal{O}_{(\mathfrak{p})})$. Of course this summand is simply a homomorphism $\pi_f(\lambda, \phi, \epsilon) \colon \mathcal{H}^{(p)} \to \mathcal{O}_{(\mathfrak{p})}$.

In the paper with J. Mahnkopf we want to show that we get a family

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{P}}_\chi \otimes \mathcal{O}_\mathfrak{p})(\pi_f(\chi, \phi, \epsilon)),$$

which depends "analytically" on the parameter $\chi \in \bar{\Lambda}_{\chi_0} = \{\eta\} \times (X^*(T) \otimes \mathbb{Z}_p)$; this is what is called a Hida family.

We have to make it more precise what we mean by "depends analytically" on χ. To do this we introduce a ring of power series in the variables $\ldots, \Theta_\alpha, \ldots$ where α runs through the set of simple roots, $\underline{i} = (\ldots, i_\alpha, \ldots)$ will be multi-indices:

$$\mathbb{I} = \{\textstyle\sum_{\underline{i}=0}^\infty a_{\underline{i}} \prod_\alpha \Theta_\alpha^{i_\alpha} \mid ag_{\underline{i}} \in \mathbb{Z}_p,\ \mathrm{ord}_p(a_{\underline{i}}) \geq \sum i_\alpha - \sum \mathrm{ord}_p(i_\alpha!)\}.$$

This is a subring of the power series ring $\mathbb{Z}_p[[\ldots, \Theta_\alpha, \ldots]]$. Let $\mathbb{I}_m$ be the image of Λ in the power series ring $\mathbb{Z}/p^m[[\underline{\Theta}]]$, where $\underline{\Theta} = (\ldots, \Theta_\alpha, \ldots)$.

We consider maps $f \colon G(\mathbb{Z}_p) \to \mathbb{I}$ which satisfy the standard Lipschitz condition and

$$f(bg) = \prod (\omega(\gamma_\alpha(t))^{\nu_\alpha} (\frac{\gamma_\alpha(t)}{\omega(\gamma_\alpha(t)})^{\Theta_\alpha} f(g),$$

where we observe that $\frac{\gamma_\alpha(t)}{\omega(\gamma_\alpha(t))} = 1 + l_\alpha(t)p$ and if $x = l_\alpha(t)$ then

$$(1+xp)^{\Theta_\alpha} = 1 + px\Theta_\alpha + p^2x^2\binom{\Theta_\alpha}{2} + \cdots = 1 + a_1\Theta_\alpha + a_2\Theta_\alpha^2 \cdots \in \mathbb{I}.$$

If we now interpret X as the capital Greek letter chi $= \chi$ and put $\mathrm{X} = (\eta, \Theta)$ then we just defined the induced module $\widetilde{\mathcal{P}}_{\mathrm{X}}$ and obviously we can write it as a projective limit

$$\mathcal{P}_{\mathrm{X}} = \varprojlim \mathcal{P}_{\mathrm{X}[m]}.$$

We can define the cohomology groups $H^q_{*,*}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}}) = \varprojlim H^q_{*,*}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}[m]})$ and Hecke operators $T(t_{p^k}, u_{t_{p^k}})$ on $H^q_{*,*}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}[m]})$, and therefore we can define the ordinary cohomology groups

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}}) = \varprojlim H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}[m]}).$$

On these cohomology groups we have an action of the Hecke algebra $\mathcal{H}^{(p)}$. In the paper with J. Mahnkopf we hope to prove

Theorem 4.7. *Under the above assumptions* (λ_0-tf) *and* (λ_0-Bc) *the cohomology* $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}})$ *is a free* $\mathbb{I}$*-module of rank* $b^q_{*,\mathrm{ord}}(\lambda_0)$. *Any* $\underline{\theta} = (\ldots, \theta_\alpha, \ldots)$, $\theta_\alpha \in \mathbb{Z}_p$, *yields an ideal* $(\underline{\Theta} - \underline{\theta}) = (\ldots, \Theta_\alpha - \theta_\alpha, \ldots)$ *and for any* $\chi = (\eta, \theta)$ *we get an isomorphism*

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}})/(\underline{\Theta} - \underline{\theta}) \xrightarrow{\sim} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\chi}).$$

If we drop the assumption about non existence of torsion and replace it by an assumption about constancy of Betti numbers on $\widetilde{\Lambda}_{\chi_0}$ (or in a neighborhood of λ_0) then we can formulate slightly weaker assertions.

If we have a $\phi \in \Phi$ which has minimal multiplicity and if ϵ is admissible, then we can find a projective (free?) $\mathbb{I} \otimes \mathcal{O}_{\mathfrak{p}}$-submodule $H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}})(\tilde{\pi}_f(\phi, \epsilon)) \subset H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}}) \otimes \mathcal{O}_{\mathfrak{p}}$ of rank 1 such that for any $\chi = (\eta, \theta)$ we have

$$H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\mathrm{X}})\pi_f(\chi, \phi, \epsilon)/(\Theta - \theta) \xrightarrow{\sim} H^q_{*,\mathrm{ord}}(S^G_{K_f}, \mathcal{P}_{\chi})(\chi, \pi_f, \epsilon).$$

Of course $\tilde{\pi}_f : \mathcal{H}^{(p)} \to \mathcal{O}_{\mathfrak{p}}$, the ring $\mathcal{O}_{\mathfrak{p}} = W(\mathbb{F}_q)$ is the Witt ring over $\mathbb{F}_q$ or in other words the unramified extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_q$.

4.3. Some input from analysis and representation theory. In some cases the theory of automorphic forms combined with our knowledge on the cohomology of unitary $(\mathfrak{g}, K_\infty)$-modules and on Eisenstein cohomology gives us some control over the ordinary Betti numbers. This allows us to apply the last assertion in the subsection above to obtain some information on the subsets $Y^\bullet_{*,\eta}$.

A) The simplest case is the case of the group $\mathrm{Gl}_2/\mathbb{Q}$ and our modules $\mathcal{M}_n$. Then it is clear that $n \mapsto b^0_{\natural,\mathrm{ord}}(n)$ is not continuous in the point $n = 0$ and continuous in all other points. We also know that $b^2_{\natural,\mathrm{ord}}(n) = 0$ for all n. This implies that $n \mapsto \kappa^0_{\natural,\mathrm{ord}}(n)$ is discontinuous at $n = 0$ and this is the only discontinuity. Hence we know that in this special case $Y^0_{\natural,0} = \{(0,0)\}$ and $Y^\nu_{\natural,i} = \emptyset$ for $\nu \neq 0$ or $i \neq 0$.

We have other cases in which we have some control over the sets, for instance those cases in which we know the vanishing of some Betti numbers in the $* =!$ cohomology. In this case all the cohomology is given by Eisenstein cohomology and we only need to control the ordinary Eisenstein cohomology.

B) A first example is given by the group $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$ were $F/\mathbb{Q}$ is an imaginary quadratic extension. Let $\mathcal{O}_F$ be the ring of integers of F. For simplicity we enlarge it by inverting the discriminant D_F. Let us call the resulting ring $\mathcal{O}$. Then $\mathcal{O}/\mathbb{Z}[1/D_F]$ is an unramified extension and

$$\mathcal{G}/\mathbb{Z}[1/D_F] = R_{\mathcal{O}/\mathbb{Z}[1/D_F]}(\mathrm{Gl}_2/\mathcal{O}).$$

The extension $\mathcal{G} \times_{\mathbb{Z}[1/D_F]} \mathcal{O} = \mathrm{Gl}_2 \times_{\mathcal{O}} \mathrm{Gl}_2/\mathcal{O}$ here the factors are labeled by the two isomorphisms $\mathcal{O} \to \mathcal{O}$. For any pair n, m of positive integers we can define the $\mathcal{O}$-module $\mathcal{M}_{n,m} = (\mathcal{M}_n \otimes \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{M}_m \otimes \mathcal{O})$. On this module we have an action of $\mathcal{G} \times_{\mathbb{Z}[1/D_F]} \mathcal{O}$ where $g = (g_1, g_2) \in \mathcal{G} \times_{\mathbb{Z}[1/D_F]} (\mathcal{O})$ acts on $x_1 \otimes x_2 \in (\mathcal{M}_n \otimes \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{M}_m \otimes \mathcal{O})$ by $g_1 x_1 \otimes g_2 x_2$. (Again we ignore the possible variation of the central character; see Remark on p. 398.)

Now we pick our prime $p > 2$, we also assume that p does not divide D_F. We choose a $K_f \subset G(\mathbb{A}_f)$ such that $K_p = \mathcal{G}(\mathbb{Z}_p)$ and we consider the cohomology

$$H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{n,m} \otimes \mathbb{Z}_p).$$

Here we know that $H^\bullet_{!,}((S^G_{K_f}, \tilde{\mathcal{M}}_{n,m} \otimes F) = 0$ if we have $n \neq m$ (See [Ha-Coh], 3.3) We conclude that in this case all the cohomology is given by Eisenstein cohomology and hence it is not difficult to see that the ordinary Betti numbers $b^\nu_{*,\mathrm{ord}}(\lambda)$ are constant in the domain where $n \neq m$ (The central character is inessential).

Therefore we see that the sets $Y^\nu_{*,\eta}$ are contained in the set of λ where $n = m$. But on this diagonal we will have some jumps in the Betti numbers, i.e., we may have $b^\nu_{!,\mathrm{ord}}(\lambda_0) \neq 0$ and then we have a discontinuity in λ_0 and unboundedness of ordinary cohomology in a neighborhood of λ_0. This is actually the counter example that was communicated to me by H. Hida and it actually it goes back to an argument in Richard Taylor's thesis.

At this point we encounter another interesting question. We can restrict our attention to the modules $\mathcal{M}_{n,n}$ or to say it differently we fix $(i,i) \equiv (n,n) \bmod p$ and then we take the closure $\Delta(i)$ of these weights in $\Omega \times X^*(T) \otimes \mathbb{Z}_p$. Now we can ask the same question again:

What is the locus of discontinuity of the functions $m^{\nu}_{*,\mathrm{ord}}(\lambda)$ when we restrict them to $\Delta(i)$, or, what amounts to the same, are there points where a Betti number $b^{\mu}_{!,\mathrm{ord}}(\lambda)$ jumps?

If we restrict the representations to $\mathcal{G}^{(1)} = R_{\mathcal{O}/\mathbb{Z}[1/D_F]}(\mathrm{Sl}_2/\mathcal{O})$ then the representations $\mathcal{M}_{n,n}$ are exactly those whose conjugate is isomorphic to their dual. (See [Ha-Coh], Chap. 3, Sec. 3.3)

C) We get an analogous situation if we consider the group $\mathrm{Sl}_3/\mathbb{Z}$. In this case our dominant weights are of the form $\lambda = n_\alpha\gamma_\alpha + n_\beta\gamma_\beta$. Again we know that the inner Betti numbers vanish if $n_\alpha \neq n_\beta$ and then the computation of the Eisenstein cohomology should yield that the sets $Y^{\bullet}_{*,\mathrm{ord}}$ are contained in the closure of the diagonal $n_\alpha = n_\beta$. Therefore we can ask what happens if we restrict to the modules $\mathcal{M}_{n(\gamma_\alpha+\gamma_\beta)}$, i.e., those which are self dual. Again we may ask whether the Betti numbers $b^{\nu}_{!,\mathrm{ord}}(\lambda)$ are locally constant.

D) We may also consider the case of a group $G/\mathbb{Q}$ where $G^{(1)} \times \mathbb{R}$ has an anisotropic maximal torus. This means that $G^{(1)}(\mathbb{R})$ has discrete series representations. In this case we know that we can find a union finite number of proper linear subspaces $\bigcup_\mu H_\mu = Z \subset X^*(T) \otimes \mathbb{Q}$ such that for any $\lambda \notin Z$ we have $H^{\nu}_!(S^G_{K_f}, \mathcal{M}_\lambda) = 0$ unless $\nu = \dim S^G_{K_f}/2$. Combining this with some results on Eisenstein cohomology (see [Li-Schw]) this is good enough to prove that the functions $\kappa^{\bullet}_{*,\mathrm{ord}}$ are continuous in all points $\lambda \notin Z$. This implies that all the $Y^{\nu}_{*,\eta} \subset \bar{Z} \cap \{\eta\} \otimes (X^*(T) \otimes \mathbb{Z}_p)$.

We will encounter cases where already the first component $\eta \in \Omega$ tells us that $\bar{Z} \cap \{\eta\} \otimes (X^*(T) \otimes \mathbb{Z}_p = \emptyset$ and therefore we can conclude $Y^{\nu}_{*,\eta} = \emptyset$.

4.4. What is the arithmetic meaning of the "second term" in the constant term of "cohomological" Eisenstein series? We refer to [Ha-Coh], Chap. 6, and [Ha-Bom2]. We consider rank one Eisenstein classes, i.e., we induce from "cuspidal" classes $\omega \in H^{\bullet}_{\mathrm{cusp}}(\partial_P S^G_{K_f}, \mathcal{M}_\lambda)$ on a maximal parabolic subgroup. These "cuspidal" classes lie in some isotypical piece $\chi_\infty \times \sigma_f$ (see [Ha-Bom2], 1.2.3). We construct an Eisenstein class $\mathrm{Eis}(\omega)$, this is a class which is represented by a closed differential form obtained from an infinite summation. To get the restriction of this class to the boundary we have to compute its constant term. This constant term has two summands: The first summand is basically our original class and the "second" term involves the L-function of $\chi_\infty \times \sigma_f$ (see loc.cit. 1.3).

It may happen that the second term is "zero in cohomology" and if this is the case it seems to be totally uninteresting. But this impression is wrong as we will see later.

At first we discuss the case where this second term gives a non zero contribution to cohomology. In this case the second term enters into the description of the global cohomology in the cohomology of the boundary. From this we get rationality results for special values of L-functions (see [Ha-Bom2]). But our theorems above also allow us to draw some more arithmetical consequences. We explain this in an example and refer to [Ha-Bom2].

The group is $G = \mathrm{Sl}_3/\mathbb{Q}$ we have the two maximal parabolic subgroups $P/\mathbb{Q}$, $Q/\mathbb{Q}$. Since we want to say something about integral cohomology, we fix a level. For simplicity we assume that K_f is the standard maximal compact subgroup. Let $\mathcal{M} = \mathcal{M}_\lambda$ be a $\mathbb{Z}$-module of highest weight $\lambda = n_\alpha\gamma_\alpha + n_\beta\gamma_\beta$. Let $d = n_\alpha + n_\beta + 1$.

Since the Manin–Drinfeld principle is available for $\mathrm{Gl}_2/\mathbb{Q}$ we get a rational decomposition (see [Ha-Coh], Chap. 3, 5.5.2)

$$H^2(\partial S^G_{K_f}, \mathcal{M}_\mathbb{Q}) = H^2_!(\partial S^G_{K_f}, \mathcal{M}_\mathbb{Q}) \oplus H^2_{\mathrm{Eis}}(\partial S^G_{K_f}, \mathcal{M}_\mathbb{Q}).$$

Now we invert certain primes, this are the primes which allow congruences between !-cohomology and Eisenstein cohomology on Gl_2, i.e., primes dividing certain values $\zeta(-1-k)$. We get a ring $R = \mathbb{Z}[\dots, 1/l, \dots]$ and a decomposition

$$H^2(\partial S^G_{K_f}, \mathcal{M}_R) = H^2_!(\partial S^G_{K_f}, \mathcal{M}_R) \bigoplus H^2_{\mathrm{Eis}}(\partial S^G_{K_f}, \mathcal{M}_R).$$

Since the boundary has two strata which correspond to the two maximal parabolic subgroups, we get

$$H^2_!(\partial S^G_{K_f}, \mathcal{M}_R) = H^2_!(\partial_P S^G_{K_f}, \mathcal{M}_R) \bigoplus H^2_!(\partial_Q S^G_{K_f}, \mathcal{M}_R).$$

We can find a (minimal) normal extension $\mathbb{Q} \subset F \subset \mathbb{C}$ such that we can decompose $H^2_!(\partial S^G_{K_f}, \mathcal{M}_F)$ into absolutely irreducible modules and if we extend $\mathcal{O}_F$ to a larger ring R_1 by inverting some more congruence primes we get (see [Ha-Bom2], [Ha-Coh] Chap. 3, 1.2.3)

$$\begin{aligned} H^2_!(\partial S^G_{K_f}, \mathcal{M}_{R_1}) &= \bigoplus_{\sigma_f} H^2_!(\partial_P S^G_{K_f}, \mathcal{M}_{R_1})(\sigma_f^P) \oplus H^2_!(\partial_Q S^G_{K_f}, \mathcal{M}_{R_1})(\sigma_f^Q) \\ &= \bigoplus_{\sigma_f} H^1_!(S^{M_\beta}_{K_f^\alpha}, \mathcal{M}(s_\beta\cdot\lambda)_{R_1})(\sigma_f) \\ &\qquad\qquad \oplus H^1_!(S^{M_\beta}_{K_f^\beta}, \mathcal{M}(s_\alpha\cdot\lambda)_{R_1})(\sigma_f|\gamma_P|^{-3}). \end{aligned}$$

The Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on these cohomology groups via its action on R_1 and permutes the summands. We denote an individual summand by $H^2_!(\partial S^G_{K_f}, \mathcal{M}_{R_1})[\sigma_f]$. Notice that $\mathcal{M}(s_\alpha\cdot\lambda)$ sits in degree one. The summands

$$H^1_!(S^{M_\alpha}_{K_f^\alpha}, \mathcal{M}(s_\beta\cdot\lambda)_{R_1})(\sigma_f), H^1_!(S^{M_\beta}_{K_f^\beta}, \mathcal{M}(s_\alpha\cdot\lambda)_{R_1})(\sigma_f|\gamma_P|^{-3})$$

are projective R_1 modules of rank 1.

The Eisenstein intertwining operator is a map

$$\mathrm{Eis}\colon H^1_!(S^{M_\alpha}_{K_f^\alpha}, \mathcal{M}(s_\beta\cdot\lambda))_\mathbb{C}(\sigma_f) \to H^2(S^G_{K_f}, \mathcal{M}_\mathbb{C}),$$

and if we compose the Eisenstein intertwining operator with the restriction to the boundary we get

$$r \circ \mathrm{Eis}\colon H^1(S^{\alpha}_{K_f^{M_\alpha}}, \mathcal{M}(s_\beta \cdot \lambda)_{\mathbb{C}})(\sigma_f)$$
$$\to H^1(S^{M_\alpha}_{K_f^{\alpha}}, \mathcal{M}(s_\beta \cdot \lambda)_{\mathbb{C}})(\sigma_f) \bigoplus H^1(S^{M_\beta}_{K_f^{\beta}}, M(s_\alpha \cdot \lambda)_{\mathbb{C}})(\sigma_f|\gamma_P|^{-3}).$$

The target of r is an isotypical submodule under the Hecke algebra, the summands are one-dimensional $\mathbb{C}$-vector spaces. These vector spaces are defined over F and the image of the composition $r \circ \mathrm{Eis}$ is given by (see [Ha-Bom2])

$$[\omega] \otimes \psi_f + \frac{(-1)^{\frac{d}{2}+1}}{\Omega(\sigma_f)^{(-1)^{n_\beta}}} \frac{\Lambda(f, n_\beta + 1)}{\Lambda(f, n_\beta + 2)} T^{\mathrm{loc}}_{\mathrm{fin}}([\omega] \otimes \psi_f).$$

Here $f = f(\sigma_f)$ is the normalized Hecke eigenform attached to σ_f and $\Lambda(f, s)$ is the motivic L-function attached to f, i.e., it is the Hecke L-function attached to f completed by the Γ-factor at infinity. (See [Ha-Bom2], 2.1.4.)

In [Ha-Bom2] we explained that $[\omega] \otimes \psi_f \mapsto T^{\mathrm{loc}}_{\mathrm{fin}}([\omega] \otimes \psi_f)$ is an isomorphism between the two one-dimensional $\mathbb{C}$-vector spaces

$$H^1(S^{\alpha}_{K_f^{M_\alpha}}, \mathcal{M}(s_\beta \cdot \lambda))_{\mathbb{C}})(\sigma_f) \xrightarrow{\sim} H^1(S^{M_\beta}_{K_f^{\beta}}, M(s_\alpha \cdot \lambda))_{\mathbb{C}})(\sigma_f|\gamma_P|^{-3}),$$

which is the base extension of an isomorphism

$$T^{\mathrm{loc}}_{\mathrm{fin}}\colon H^1_!(S^{M_\alpha}_{K_f^{\alpha}}, \mathcal{M}(s_\beta \cdot \lambda)_{\mathcal{O}_F[\frac{1}{N}]})(\sigma_f) \xrightarrow{\sim} H^1_!(S^{M_\alpha}_{K_f^{\beta}}, \mathcal{M}(s_\alpha \cdot \lambda)_{\mathcal{O}_F[\frac{1}{N}]})(\sigma_f|\gamma_P|^{-3}).$$

This isomorphism was used to define the period $\Omega(\sigma_f)$, which is unique up to an element in $\mathcal{O}_F[\frac{1}{N}]^\times$.

The image

$$\mathrm{Im}(H^2(S^G_{K_f}, \mathcal{M}_{\mathcal{O}_F[\frac{1}{N}]} \to H^2(\partial S^G_{K_f}, \mathcal{M}_{R_1})) = H^2_{\mathrm{global}}(S^G_{K_f}, \mathcal{M}_{\mathcal{O}_F[\frac{1}{N}]})$$

intersected with the isotypical submodule

$$H^2_!(\partial S^G_{K_f}, \mathcal{M}_{R_1})[\sigma_f] \cap H^2_{\mathrm{global}}(S^G_{K_f}, \mathcal{M}_{\mathcal{O}_F[\frac{1}{N}]}) = H^2_{\mathrm{global}}(S^G_{K_f}, \mathcal{M}_{\mathcal{O}_F[\frac{1}{N}]})(\sigma_f)$$

is a projective rank one $\mathcal{O}_F[\frac{1}{N}]$-submodule in the one-dimensional F-vector space

$$\psi_f + \frac{(-1)^{\frac{d}{2}+1}}{\Omega(\sigma_f)^{(-1)^{n_\beta}}} \frac{\Lambda(f, n_\beta + 1)}{\Lambda(f, n_\beta + 2)} T^{\mathrm{loc}}_{\mathrm{fin}}(\psi_f).$$

We assume for simplicity that the number $d = n_\alpha + n_\beta + 1 = 10, 14, 16, 18, 20, 24$, therefore the space of modular cusp forms describing the boundary cohomology is of dimension 1. Then σ_f is defined over $\mathbb{Q}$ and $H^2_!(\partial S^G_{K_f}, \mathcal{M}_{R_1})(\sigma_f) =$

$H^2_!(\partial S^G_{K_f}, \mathcal{M}_{R_1})$ and $R_1 = \mathbb{Z}[\frac{1}{N}]$, where N is the numerator of $\zeta(-1-d)$. We choose an ordinary prime p for σ_f, not dividing N. We know that the set Φ consists of one element. We can interpolate p-adically our weight $\lambda = n_\alpha\gamma_\alpha + n_\beta\gamma_\beta$, i.e., we consider weights

$$\tilde{\lambda} = (n_\alpha + (p-1)z_\alpha)\gamma_\alpha + (n_\beta + (p-1)z_\beta)\gamma_\beta, \quad z_\alpha, z_\beta \in \mathbb{N}.$$

As usual we denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at p, and we consider the cohomology groups

$$H^1_!(S^{M_\alpha}_{K^\alpha_f}, \mathcal{M}(s_\beta \cdot \tilde{\lambda})_{\mathbb{Z}_{(p)}}), H^1_!(S^{M_\beta}_{K^\alpha_f}, \mathcal{M}(s_\beta \cdot \tilde{\lambda})_{\mathbb{Z}_{(p)}})$$

as modules over the Hecke algebra. We know that we have a minimal finite normal extension $F(\lambda) = F/\mathbb{Q}$ such that these cohomology groups decompose into absolutely irreducible $\mathcal{H}^{(p)}$-modules. For any $\mathcal{O}_F \supset \mathfrak{p} \supset (p)$ the $\mathcal{O}_{F,\mathfrak{p}}$-modules $H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\alpha_f}, \widetilde{\mathcal{M}}(s_\beta \cdot \tilde{\lambda}) \otimes \mathcal{O}_{F,\mathfrak{p}})$, $H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\beta_f}, \widetilde{\mathcal{M}}(s_\alpha \cdot \tilde{\lambda}) \otimes \mathcal{O}_{F,\mathfrak{p}})$ are free of rank one, and $\mathcal{H}^{(p)}$ acts by the interpolating homomorphisms $\sigma_f(s_\alpha \cdot \tilde{\lambda}, \phi, \epsilon)$, $\sigma_f(s_\beta \cdot \tilde{\lambda}, \phi, \epsilon)$ on these modules. The rings $\mathcal{O}_{F,\mathfrak{p}}$ admit an embedding into $\mathbb{Z}_p$. Therefore we can say

$$\begin{aligned} H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\beta_f}, \widetilde{\mathcal{M}}(s_\alpha \cdot \lambda) \otimes \mathbb{Z}_p) &= H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\beta_f}, \widetilde{\mathcal{M}}(s_\alpha \cdot \lambda) \otimes \mathbb{Z}_p)(\sigma_f(s_\alpha \cdot \tilde{\lambda}, \phi, \epsilon)), \\ H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\beta_f}, \widetilde{\mathcal{M}}(s_\alpha \cdot \tilde{\lambda}) \otimes \mathbb{Z}_p) &= H^1_{\mathrm{ord},!}(S^{M_\alpha}_{K^\beta_f}, \widetilde{\mathcal{M}}(s_\alpha \cdot \tilde{\lambda}) \otimes \mathbb{Z}_p)(\sigma_f(s_\beta \cdot \tilde{\lambda}, \phi, \epsilon)) \end{aligned}$$

are free $\mathbb{Z}_p$-modules of rank one. Because we have only one option for ϕ, ϵ we introduce an abbreviation and call the modules on the right-hand side $\mathbb{Z}_p(\sigma_f(s_\alpha \cdot \tilde{\lambda}))$, $\mathbb{Z}_p(\sigma_f(s_\beta \cdot \tilde{\lambda}))$, respectively. They only differ by a twist and provide a modular cusp form $f(\tilde{\lambda})$. (See [Ha-Bom2], 2.1.4.)

We construct a system of isomorphisms

$$\phi(\sigma_f, \tilde{\lambda}) \colon \mathbb{Z}_p(\sigma_f(s_\alpha \cdot \tilde{\lambda})) \xrightarrow{\sim} \mathbb{Z}_p(\sigma_f(s_\beta \cdot \tilde{\lambda})),$$

we want that this system is compatible with the congruences or even better depends analytically on $\tilde{\lambda}$. This defines a system of periods $\Omega(\sigma_f, \tilde{\lambda})$ and an identification $\mathrm{Hom}(\mathbb{Z}_p(\sigma_f(s_\alpha \cdot \tilde{\lambda})), \mathbb{Z}_p(\sigma_f(s_\beta \cdot \tilde{\lambda})) = \mathbb{Z}_p$. The choice of this system of periods is delicate.

The cuspidal part of the cohomology of the boundary is given by

$$H^2_{!,\mathrm{ord}}(\partial S^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda}}) = \mathbb{Z}_p(\sigma_f(s_\alpha \cdot \tilde{\lambda})) \oplus \mathbb{Z}_p(\sigma_f(s_\beta \cdot \tilde{\lambda})).$$

Hence we can view $H^2_{\mathrm{global,\,ord}}(S^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathbb{Q}_p)$ as a point in $\mathbb{P}^1(\mathbb{Q}_p)$, it is the point $(1, c(\tilde{\lambda}))$ where

$$c_{\mathrm{global}}(\tilde{\lambda}) = \frac{(-1)^{\frac{\tilde{d}}{2}+1}}{\Omega(\sigma_f, \tilde{\lambda})^{(-1)^{\tilde{n}_\beta}}} \frac{\Lambda(f(\tilde{\lambda}), \tilde{n}_\beta + 1)}{\Lambda(f(\tilde{\lambda}), \tilde{n}_\beta + 2)}$$

it defines a line $H_{\mathrm{global}}(\tilde{\sigma}_f) \subset \mathbb{Z}_p^2$ and we saw that

$$H_{\mathrm{global}}(\tilde{\sigma}_f)/H^2_{\mathrm{global}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}_p})(\tilde{\sigma}_f) \hookrightarrow H^3_{c,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}_p})_{\mathrm{tors}}.$$

Since in our case we always have $\tilde{n}_\alpha \neq \tilde{n}_\beta$ it follows that $H^3_!(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}_p}) = 0$, and simple arguments using Eisenstein cohomology show that the dimensions of $H^q_{c,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}_p})$, $H^q_{\natural,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}_p})$ for $q = 2, 3$ are constant. Hence we can apply our Theorem 4.6 and find that the torsion of $H^q_{c,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}_p})$ is bounded.

Looking at the standard long exact sequence which relates the $* = c$, $* = \natural$, $* = \partial$ cohomology we easily can derive congruences between the numbers $c(\tilde{\lambda})$. These congruences will be of the following form:

If $\tilde{\lambda} = \lambda_0 + p^m(p-1)(z_\alpha\gamma_\alpha + z_\beta\gamma_\beta)$ *then*

$$c_{\mathrm{global}}(\tilde{\lambda}) \equiv c_{\mathrm{global}}(\lambda_0) \mod p^{m'}$$

where we assume that $c_{\mathrm{global}}(\lambda_0)$ *is integral and where* $p^{m-m'}$ *is the exponent of the torsion.*

In our paper with J. Mahnkopf we hope to prove stronger results. Let us start from a fixed highest weight of $\lambda_0 = \nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta, 0 \leq \nu_\alpha, \nu_\beta < p-1$. We assume that we do not have any ordinary torsion in our cohomology groups $H^\bullet_{*,\mathrm{ord}}(S^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_0} \otimes \mathcal{O}_{F,(\mathfrak{p})})$ for any value of the first $*$. Then we see of course that the above congruences become more precise. But in the forthcoming paper we hope to prove that the $c_{\mathrm{global}}(\tilde{\lambda})$ are p-adic analytic; more precisely we have a Taylor expansion

$$\begin{aligned} c_{\mathrm{global}}(\tilde{\lambda}) &= c_{\mathrm{global}}((\nu_\alpha + (p-1)z_\alpha)\gamma_\alpha + (\nu_\beta + (p-1)z_\beta)\gamma_\beta) \\ &= c_{\mathrm{global}}(\nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta) + c'_\alpha(\lambda_0)(pz_\alpha) + c'_\beta(\lambda_0)(pz_\beta) + \cdots \\ &\qquad \cdots + c_{r,s}(\lambda_0)p^{r+s}z_\alpha^r z_\beta^s + \cdots, \end{aligned}$$

where the coefficients $c_{r,s} \in \mathbb{Z}_p$ satisfy the usual condition of moderate growth of the form $\mathrm{ord}_p(c_{r,s}r!s!) \geq 0$.

Since the proofs are not yet written I made some numerical experiments. We assume that σ_f is given by the modular form Δ of weight 12, which means that $\nu_\alpha + \nu_\beta = 9$. In this case the primes 11 and 13 are ordinary. The prime 11 is not so good because $11 - 1 = 10$ and our coefficient system is $\mathcal{M}_{10}$. So we pick $p = 13$ and compute a list of eigenvalues for the first three modular forms in the Hida family, namely f_{24} (f_{36} and f_{48}, respectively) of weight 24 (36 and 48, respectively) (see 4.3 above).

Then we can compute the values

$$c_{\mathrm{global}}(\tilde{\lambda}) = c_{\mathrm{global}}(\nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta + (p-1)z_\alpha\gamma_\alpha + (p-1)z_\beta)\gamma_\beta)$$

where $z_\alpha, z_\beta \in \mathbb{N}$ and for ν_α, ν_β satisfying $\nu_\alpha + \nu_\beta + (p-1)(z_\alpha + z_\beta) + 1 = d$.

We computed $c_{\text{global}}(\tilde{\lambda})$ for all these values of $\tilde{\lambda}$ and listed them according to indices $j = 1, 2, 3 \ldots$ such that

$$d - j = \nu_\beta + 12 z_\beta = \tilde{n}_\beta.$$

(This means the j-th member of the list is $\frac{(-1)^{\frac{\tilde{d}}{2}+1}}{\Omega(\tilde{\sigma}_f)^{(-1)^{\tilde{n}_\beta}}} \frac{\Lambda(f,\tilde{n}_\beta+1)}{\Lambda(f,\tilde{n}_\beta+2)}$. Recall also that our values are in a finite extension of $\mathbb{Q}$ and we have selected a split prime *13* over 13).

Of course at this point it is not clear what it means to compute these values, because we have to specify the periods $\Omega(f)$ for $f = \Delta$, f_{24}, f_{36}, f_{48}. Here we do something that might be considered problematic. We simply choose the periods by the rule that the first $c_{\text{global}}(\tilde{\lambda})$ in the list becomes 1, i.e.,

$$\frac{(-1)^{\frac{\tilde{d}}{2}+1}}{\Omega(\sigma_f, \tilde{\lambda})^{(-1)^d}} \frac{\Lambda(f(\tilde{\lambda}), d)}{\Lambda(f(\tilde{\lambda}), d+1)} = 1.$$

Then we got the following lists of values $\bmod 13^2$ (for $d = 10, 22, 34, 46$, respectively):

{1, 36, 14, 113, 70, 99, 3, 157, 108, 1},
{1, 140, 66, 35, 122, 34, 29, 105, 17, 87, 35, 29, 68, 10, 66, 35, 5, 151, 29, 105, 134, 1},
{1, 75, 118, 126, 5, 138, 55, 53, 17, 139, 126, 3, 159, 114, 118, 126, 57, 86, 55, 53, 43, 152, 113, 55, 107, 10, 118, 126, 109, 34, 55, 53, 160, 1},
{1, 10, 1, 48, 57, 73, 81, 1, 17, 22, 48, 146, 81, 49, 1, 48, 109, 21, 81, 1, 43, 35, 35, 29, 29, 114, 1, 48, 161, 138, 81, 1, 69, 48, 22, 81, 146, 10, 1, 48, 44, 86, 81, 1, 17, 1}.

The reader will easily check that for $j = 1, \ldots, 9$ the j-th entry in the first list is congruent the j-th entry in the second, third and fourth list, but this congruence does not hold for $j = 10$. Furthermore we see that all entries in the second list are congruent to the corresponding entry in the third list, again with the exception of the last entry. These exceptions have a simple explanation. If we start from the highest weights $9\gamma_\alpha$ or $9\gamma_\beta$ then these weights are not regular, they have to be exempted from our considerations. This should raise some doubts whether we chose the right period.

Finally we notice that for $j = 1, 2, \ldots, 8$ our numerical data are consistent with some "analyticity" or may be better with some differentiability, namely we have

$$\begin{aligned} &c_{\text{global}}(\lambda_0 + (p-1)z_\alpha\gamma_\alpha + (p-1)z_\beta\gamma_\beta) \\ &\quad = c_{\text{global}}(\lambda_0) + c'_\alpha(\lambda_0) p z_\alpha + c'_\beta(\lambda_0) p z_\beta \mod p^2. \end{aligned}$$

But again we see that for $j = 9$ the congruence does not hold and the only way out is that for this value of j we must have some ordinary torsion in $H^3_{c,\text{ord}}(S^G_{K_f}, \tilde{M}_{\lambda_0})$ where $\lambda_0 = \gamma_\alpha + 8\gamma_\beta$.

4.5. Denominators of Eisenstein classes. We come back to the question raised in the headline of 4.4. We consider the case where the second term is zero in cohomology. In this case this second term seems to be uninteresting. But this is not the case, we have a lot of experimental evidence that it influences the denominator of the Eisenstein class.

In this section our p is called ℓ. At various occasions we discuss the conjectural relationship between denominators of Eisenstein classes and special values of L-functions. These special values enter in the second term of the constant term of Eisenstein series. (See [Ha-MM], 3.1.4., 3.2.7, [1-2-3], [Ha-Bom2], [Ha-Coh], Chap. 3, 5.5). The guiding principle is that the divisibility of such an special value by a prime ℓ (or a power of ℓ) should imply the divisibility of the denominator of an Eisenstein cohomology class by ℓ (or a power of ℓ.) To be a little bit more precise we recall that the second term in the constant term essentially of the form

$$c(\sigma_f)T_\infty^{\mathrm{loc}}(\omega_\infty)T_{\mathrm{fin}}^{\mathrm{loc}}(\psi_f),$$

where the scalar factor $c(\sigma_f)$ can be expressed in terms of special values of L-functions attached to σ_f, the object $\omega_\infty \otimes \psi_f$ is essentially a differential form representing a cohomology class.

Let us look at the two examples $G = \mathrm{Sl}_3/\mathbb{Z}$ and $G = \mathrm{Sp}_2/\mathbb{Z}$ simultaneously. In both cases the Hecke module σ_f corresponds to a holomorphic modular form f of some weight k and the scalar factor in front is essentially

$$\Omega(\sigma_f)^{(-1)^m}\frac{\Lambda(f,m-1)}{\Lambda(f,m)}, \quad \Omega(\sigma_f)^{(-1)^m}\frac{\Lambda(f,m-1)}{\Lambda(f,m)}\frac{\zeta'(-a)}{\zeta(-1-a)},$$

respectively. Here m, a a certain positive integers, the number a is even and therefore $\zeta(-1-a) \in \mathbb{Q}$.

We discussed the first case already. In the second case the cohomology class of $T_\infty^{\mathrm{loc}}(\omega_\infty)T_{\mathrm{fin}}^{\mathrm{loc}}(\psi_f)$ will be trivial, it is essentially the operator $T_\infty^{\mathrm{loc}}(\omega_\infty)$ which is responsible for that. (Actually this is good, otherwise we could prove $\zeta'(-a) \in \mathbb{Q}$ and this is false as we all believe.) But the first factor in front – the ratio of the two L values – is an algebraic number, let us assume it is even rational.

I believe that high powers of ℓ in the denominator of this rational number create high powers of ℓ in the denominator of a certain Eisenstein class. These denominators in turn should create high congruences $\bmod\, \ell$ between eigenvalues of Hecke operators on the Eisenstein class and an the eigenvalues on an eigenclass in the inner cohomology.

At this point a side remark seems to be in order. If we look for primes dividing the denominator we look for primes dividing $\frac{1}{\Omega(f)_{\epsilon(m)}}\Lambda(f,m)$, where $\Omega(f)_\pm$ is a suitably normalized period (see [1-2-3]). Once we found such a prime ℓ dividing this number we hope that it does not divide the numerator $\frac{1}{\Omega(f)_{\epsilon(m-1)}}\Lambda(f,m-1)$. In all our examples this turns out to be the case. But of course it is the ratio that matters.

This last conclusion – namely that denominators create congruences – has to be commented. So far I avoided to discuss a problem which may built up an obstacle against this last conclusion.

To illustrate the problem we discuss the specific example in our contribution in [1-2-3]. We consider the group $G/\mathbb{Z} = \mathrm{Sp}_2/\mathbb{Z}$ and the coefficient system $\mathcal{M}_{4,7}$ with highest weight $\lambda = 4\gamma_\beta + 7\gamma_\alpha$ (here β is the short root). In our special situation we assume that $R = \mathbb{Z}_{(41)}$, then it is an easy exercise to see that $H^3(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R) = R$. We write a sequence

$$0 \to H^3_!(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \to H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \\ \to H^3(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R) \to 0,$$

and we assume exactness of this sequence. More or less by definition the exactness of this sequence follows if we know that

$$H^4_{c,\mathrm{ord}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) = 0. \qquad \text{(no torsion)}$$

This is very likely to be the case, but we have very few tools to investigate the torsion, except that we compute the cohomology explicitly.

In [1-2-3] we give a heuristic argument why we should expect a factor 41 in the denominator, provided the above assumption (no torsion) is fulfilled. This heuristic argument is based on the hope that their are no "exotic" mixed Tate motives (our mixed Tate motives constructed in our case are not "exotic"). What we mean by "exotic" mixed Tate motives is discussed in the note MixMot-Intro.pdf in www.math.uni-bonn.de/people/harder/Manuscripts/Eisenstein/.

But if the assumption (no torsion) is not fulfilled then we can not exclude the possibility that the image of

$$H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{4,7} \otimes R) \to H^3(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R) = R.$$

is not surjective. If $\omega \in H^3(\partial\overline{(\Gamma\backslash\mathbb{H}_2)}, \tilde{\mathcal{M}}_{4,7} \otimes R)$ is a generator then its image under the boundary map can be a non zero torsion element. Such an element will be called an Eisenstein torsion class. If we now consider the Eisenstein class $\mathrm{Eis}(\omega)$, then this is a rational cohomology class and obviously it has a factor 41 in its denominator because already its restriction to the boundary has this factor in its denominator. But then this information is not interesting anymore, for instance it does not imply the existence of congruences.

But there is a way out of this difficulty. In our discussion in [1-2-3] we had a second assumption, namely that the prime ℓ dividing the special value of the L-function should also be "large". I think that this assumption can be replaced by the assumption that the isotypical component σ_f (For the notation see [Ha-Bom2]) in $H^\bullet(S^M_{K^M_f}, \mathcal{M}(w \cdot \lambda))$ should be ordinary at ℓ. Under this assumption it seems to be possible to use Theorem 4.6 to avoid the assumption (no torsion) and to prove something which is weaker in a certain sense but covers a more general situation.

We give a short outline of this strategy. We refer to the description of the cohomology of the boundary strata in [Ha-Bom2]. We define the ℓ ordinary cohomology

$$H^\bullet_{\mathrm{ord}}(\partial_P S^G_{K_f}, \mathcal{M}_\lambda) = H^\bullet_{\mathrm{ord}}(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \mathcal{M}_\lambda).$$

Now it is rather clear that we have an "ordinary" version of the van Est theorem. We can define an ordinary sub-sheaf $R^\bullet_{\mathrm{ord}}\pi_{P*}(\tilde{\mathcal{M}}) \subset R^\bullet\pi_{P*}(\tilde{\mathcal{M}})$, which is given by

$$R^\bullet_{\mathrm{ord}}\pi_{P*}(\tilde{\mathcal{M}}) = \bigoplus_{w\in W^P} \tilde{\mathcal{M}}(w\cdot\lambda)_{\mathrm{ord}}.$$

We have a degenerating spectral sequence

$$H^p_{\mathrm{ord}}(S^M_{K^M_f(\xi_f)}, R^\bullet_{\mathrm{ord}}\pi_{P*}(\tilde{\mathcal{M}})) \Rightarrow H^{p+q}_{\mathrm{ord}}(P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f), \mathcal{M}_{\lambda,R_F}).$$

We return to our example, we still have $\ell = 41$. We interpolate our weight $\tilde{\lambda} \in \Lambda_{\lambda_0} = (4+z_\beta(\ell-1))\gamma_\beta + (7+z_\alpha(\ell-1))\gamma_\alpha$, here z_α, z_β are positive integers. Since all these weights $\tilde{\lambda}$ are regular we know that the inner cohomology $H^\bullet(S, \mathcal{M}_{\tilde{\lambda}})$ is concentrated in degree three (see [Li-Schw]) and the Eisenstein cohomology is equal to the boundary cohomology in degrees $\bullet \geq 3$. In the subsequent paper with Mahnkopf we will show that the Betti numbers $\tilde{\lambda} \mapsto b^q_{*,\mathrm{ord}}(\tilde{\lambda})$ are constant if $\tilde{\lambda}$ varies. Then it is also clear that we have the sequence

$$\begin{aligned} 0 \to H^3_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathcal{O}_{F,\mathfrak{p}}) &\to H^3_{\natural,\mathrm{ord}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathcal{O}_{F,\mathfrak{p}}) \\ &\xrightarrow{r_\partial} H^3_{\mathrm{ord}}(\partial(\overline{\Gamma\backslash\mathbb{H}_2}), \tilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathcal{O}_{F,\mathfrak{p}}) \to \end{aligned}$$

where $H^3_{\mathrm{ord}}(\partial(\overline{\Gamma\backslash\mathbb{H}_2}), \tilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathcal{O}_{F,\mathfrak{p}})$ is still a free module of rank 1 over $\mathcal{O}_{F,\mathfrak{p}}$. Assume we find L-values $\Lambda(\tilde{f}, 14 + (41-1)\cdot a\cdot 41^b)/\tilde{\Omega}_+$ that are divisible by very high powers of 41. Then we hope that our philosophy "No exotic mixed Tate motives" implies that the Eisenstein class has a denominator which is divisible by a very high power. Since the torsion in $H^4_{c,\mathrm{ord}}(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_{\tilde{\lambda}} \otimes \mathcal{O}_{F,\mathfrak{p}})$ stays bounded, this divisibility implies some weaker but still high congruences between Siegel and elliptic modular forms.

This is in a certain sense much weaker than conjecture in [1-2-3]; on the other hand it is also much more general, it is an ℓ-adic version of the conjecture in contrast to a conjecture $\bmod \ell$.

At this point we raise the question to what extend such very high congruences also imply large unramified field extensions, or wether it implies the existence of element of high order in Selmer groups.

We did some computation for the modular form f_{22} and $\ell = 41$. For $d = 22$ we have $\ell | c_{\mathrm{global}}(6\gamma_\alpha + 13\gamma_\beta)$. (See [1-2-3]).The space of modular forms of weight 62 has dimension 4 and the Hecke operator T_2 has the characteristic polynomial

$$\begin{aligned} P(T) = T^4 &- 1\,146\,312\,000 T^3 - 6\,156\,169\,255\,669\,690\,368\, T^2 \\ &+ 2\,540\,887\,466\,526\,178\,560\,442\,368\,000\, T \\ &+ 3\,583\,176\,547\,297\,492\,565\,952\,659\,077\,522\,784\,256. \end{aligned}$$

Let $\mathcal{O}$ be the ring of integers of this field, then $\mathbb{Z}[T]/(P(T)) \subset \mathcal{O}$ is not a maximal order, even if we localize at ℓ. The polynomial splits completely in $\mathbb{Q}_\ell$, the number $a = 7\,929\,938\,323\,029$ is a root modulo ℓ^8 (we have to go that far in the approximation, this root is congruent to the root $b = 6\,275\,082\,596\,639$ modulo ℓ^2!) The root a (or better its ℓ-adic limit α) defines an embedding $\mathcal{O} \hookrightarrow \mathbb{Z}_\ell$ and hence an $\mathfrak{l}$ ordinary modular form f_{62} which is congruent to f_{22} modulo $\mathfrak{l}$ (see 4.3).

For this modular form f_{62} we can produce the list of values $c_{\mathrm{global}}(\nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta + (\ell-1)z_\alpha\gamma_\alpha + (\ell-1)z_\beta\gamma_\beta) \in \mathbb{Q}_\ell$.

The characteristic polynomial splits completely in $\mathcal{O}$ and if we choose a prime *41* then we find a modular form f_{62} with coefficients in $\mathbb{Z}_{41}$ which is congruent to $f_{22} \bmod 41$. This form can also be viewed as a form with coefficients in $\mathbb{R}$ and we can compute the special values of its L-function and hence the values of c_{global}. The computations become a little bit messy, especially we had some difficulties to compute real approximations of rational numbers to such a precision such that these rational numbers could be identified. But we could verify

$$41 | c_{\mathrm{global}}(6\gamma_\alpha + 13\gamma_\beta + 40\gamma_\beta),$$

as expected.

After all these speculations it becomes clear that the problem to understand the ℓ-ordinary torsion is fundamental. Especially it seems to be important to know whether we can find explicit bounds for that part of the torsion which consists of eigenclasses for the Hecke algebra and where the eigenvalues are eigenvalues of Eisenstein classes mod ℓ (Eisenstein torsion classes).

Especially the Lemma 4.5 shows that we can find criteria that for a given $\eta = \chi_0^{[1]} = \nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta$ we do not have Eisenstein torsion classes at all, i.e., for all

$$\lambda \in \Lambda_{\chi_0} = \{\nu_\alpha\gamma_\alpha + \nu_\beta\gamma_\beta + (\ell-1)z_\alpha\gamma_\alpha + (\ell-1)z_\beta\gamma_\beta\}.$$

We resume the considerations at the end of 4.2. We consider the weight $\lambda_0 = 6\gamma_\alpha + 13\gamma_\beta$ for Sl_3. With a little bit of luck we should be able to verify such a criterion for the cohomology $H^\bullet_{*,\mathrm{ord}}(S, \tilde{\mathcal{M}}_{\lambda_0} \otimes \mathbb{Z}_\ell)$ where $\ell = 41$. In any case it should be possible to check this on a computer. Then this should in fact imply that again

$$\begin{aligned} &c_{\mathrm{global}}(6\gamma_\alpha + 13\gamma_\beta + (\ell-1)(z_\alpha\gamma_\alpha + z_\beta\gamma_\beta) \\ &\qquad \equiv c_{\mathrm{global}}(6\gamma_\alpha + 13\gamma_\beta) + a_1\ell z_\alpha + b_1\ell z_\beta \mod \ell^2. \end{aligned}$$

If this is true then our computations yielded the value $a_1 = 32 \bmod \ell$ and then we have the ordinary form $f_{22+40\cdot 11} = f_{462}$ for which we should get

$$c_{\mathrm{global}}(6\gamma_\alpha + 455\gamma_\beta)) \equiv 0 \mod \ell^2.$$

This should yield a congruence $\bmod 41^2$ between f_{462} and a Siegel modular form in $S_{444,10}$.

I included these experimental computations because I know that Fritz was very interested in this kind of questions. I was hoping that we could discuss them sometimes in the future and I am very sad that this will not happen anymore.

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