QUANTUM K-THEORY OF GRASSMANNIANS

ANDERS S. BUCH AND LEONARDO C. MIHALCEA

ABSTRACT. We show that (equivariant) K-theoretic 3-point Gromov-Witten invariants of genus zero on a Grassmann variety are equal to triple intersections computed in the (equivariant) K-theory of a two-step flag manifold, thus generalizing an earlier result of Buch, Kresch, and Tamvakis. In the process we show that the Gromov-Witten variety of curves passing through 3 general points is irreducible and rational. Our applications include Pieri and Giambelli formulas for the quantum K-theory ring of a Grassmannian, which determine the multiplication in this ring. We also compute the dual Schubert basis for this ring, and show that its structure constants satisfy S_3-symmetry. Our formula for Gromov-Witten invariants can be partially generalized to cominuscule homogeneous spaces by using a construction of Chaput, Manivel, and Perrin.

1. INTRODUCTION

The study of the (small) quantum cohomology ring began with Witten [56] and Kontsevich [37] more than a decade ago, and has by now evolved into a subject with deep ramifications in algebraic geometry, representation theory and combinatorics, see e.g. [2, 19, 17, 14, 30, 50, 48, 38] and references therein.

A result of Buch, Kresch, and Tamvakis [10] reduced the computation of the (3-point, genus 0) Gromov-Witten invariants of a Grassmann variety to a computation in the ordinary cohomology of certain two-step flag manifolds. The Gromov-Witten invariants in question have an enumerative interpretation: they count rational curves meeting general translates of Schubert varieties. The identity from [10] was proved by establishing a set-theoretic bijection between the curves counted by a Gromov-Witten invariant and the points of intersection of three Schubert varieties in general position in a two-step flag manifold.

The Gromov-Witten invariants used to define more general quantum cohomology theories, such as equivariant quantum cohomology [21, 29] or quantum K-theory [22, 39], lack such an enumerative interpretation. For example, the K-theoretic Gromov-Witten invariants are equal to the sheaf Euler characteristic of Gromov-Witten varieties of rational curves of fixed degree meeting three general Schubert varieties. Gromov-Witten varieties are subvarieties of Kontsevich's moduli space of stable maps, and have been studied by Lee and Pandharipande [47, 40].

Despite lacking an enumerative interpretation, it turns out that the more general Gromov-Witten invariants satisfy the same identity as the one given earlier in [10]: an equivariant K-theoretic Gromov-Witten invariant on a Grassmannian is equal

Date: April 18, 2009.

2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 19E08, 14M15, 14N15, 14E08.

The first author was supported in part by NSF Grant DMS-0603822.

to a quantity computed in the ordinary equivariant K-theory of a two-step flag variety. This is our first main result.

There are two key elements in its proof. The first is a commutative diagram involving a variety $\mathrm{B}\ell_d$, which dominates both Kontsevich's moduli space of stable maps and a triple Grassmann-bundle over the two-step flag variety from [10]. Intuitively, for small degrees d, the variety $\mathrm{B}\ell_d$ is the blow-up of the moduli space along the locus of curves which have kernels and spans (in the sense of [8], see §4.2 below) of unexpected dimensions. Our commutative diagram combined with geometric properties of Kontsevich's moduli space makes it possible to translate the computation of a Gromov-Witten invariant from the moduli space to the Grassmann bundle, where classical intersection theory provides the final answer.

This approach suffices to compute all the equivariant Gromov-Witten invariants, and the (equivariant) K-theoretic invariants for small degrees. To compute K-theoretic invariants of large degrees d, we need the second key element. We will show that the Gromov-Witten variety of curves of degree d passing through 3 general points is irreducible and rational. This gives a partial answer to a question posed by Lee and Pandharipande in [40] (see §2 below).

In addition to Grassmannians of type A, the formula for Gromov-Witten invariants given in [10] was also obtained for Lagrangian Grassmannians and maximal orthogonal Grassmannians. Chaput, Manivel, and Perrin later gave a type-independent construction which generalized the formula to all cominuscule spaces. Our formula for Gromov-Witten invariants holds partially in this generality. To be precise, we will compute all the (3 point, genus zero) equivariant Gromov-Witten invariants on any cominuscule homogeneous space, as well as all the (equivariant) K-theoretic invariants for small degrees.

Our main motivation for this paper was to determine the structure of the *quantum K-theory ring* of a Grassmann variety. This ring was introduced by Givental and Lee, motivated by a study of the relationship between Gromov-Witten theory and integrable systems [22, 24, 39]. We describe the ring structure in terms of a *Pieri rule* that shows how to multiply arbitrary Schubert classes with special Schubert classes, corresponding to the Chern classes of the universal quotient bundle. Since the special Schubert classes generate the quantum K-theory ring, this gives a complete combinatorial description of its structure. As an application, we prove a Giambelli formula that expresses any Schubert class as a polynomial in the special classes; this formula greatly simplifies the computation of products of arbitrary Schubert classes. We also prove that the structure constants of the quantum K-theory ring satisfy S_3-symmetry in the sense that they are invariant under permutations of indexing partitions, and that the basis of Schubert structure sheaves can be dualized by multiplying all classes with a constant element. We remark that unlike ordinary quantum cohomology, the structure constants in quantum K-theory are not single Gromov-Witten invariants (see [22] and §5 below). This poses additional combinatorial difficulties for proving our Pieri formula.

For complete flag manifolds G/B, formulas for similar purposes have been conjectured by Lenart and Maeno [42] and by Lenart and Postnikov [43], based on a combinatorial point of view. However, due to the lack of functoriality in quantum cohomology, the connection to our results is unclear.

This paper is organized as follows. In section 2 we prove that Gromov-Witten varieties of high degree for Grassmannians of type A are rational. Section 3 sets up notation for equivariant K-theory and proves a criterion ensuring that the push-forward of the Grothendieck class of a variety is equal to the Grothendieck class of its image. Section 4 contains a general discussion of Gromov-Witten invariants of various types, and proves our formula for equivariant K-theoretic Gromov-Witten invariants of Grassmannians. In Section 5 we define the quantum K-theory ring of a Grassmann variety, state our Pieri formula, derive a Giambelli type formula, prove the S_3-symmetry property for the structure constants, and dualize the Schubert basis. We also discuss consequences and applications to computing structure constants and K-theoretic Gromov-Witten invariants, and provide the multiplication tables for the equivariant quantum K-theory rings of $\mathbb{P}^1$ and $\mathbb{P}^2$. Section 6 contains the proof of our Pieri formula, and section 7 generalizes our formula for Gromov-Witten invariants to cominuscule spaces.

1.1. **Acknowledgments.** This paper continues the above mentioned work with Andrew Kresch and Harry Tamvakis; we are grateful for the ideas and insights that have undoubtedly floated between the two projects. During the preparation of this paper we also benefited from helpful discussions and comments of many other people, including Arend Bayer, Prakash Belkale, Izzet Coskun, Ionut Ciocan-Fontanine, Johan de Jong, Friedrich Knop, Shrawan Kumar, and Chris Woodward. Special thanks are due to Ravi Vakil who made valuable comments on an earlier proof of the rationality of Gromov-Witten varieties. The second author thanks the Max Planck Institut für Mathematik for a providing a stimulating environment during the early stages of this work.

2. Rationality of Gromov-Witten varieties

Let $X = G/P$ be a homogeneous space defined by a complex connected semi-simple linear algebraic group G and a parabolic subgroup P. An N-pointed *stable map* (of genus zero) to X is a morphism of varieties $f : C \to X$, where C is a tree of projective lines, together with N distinct non-singular marked points of C, ordered from 1 to N, such that any component of C that is mapped to a single point in X contains at least three special points, where special means marked or singular [36]. The *degree* of f is the homology class $f_*[C] \in H_2(X;\mathbb{Z})$.

A fundamental tool in Gromov-Witten theory is Kontsevich's moduli space $\overline{\mathcal{M}}_{0,N}(X,d)$, which parametrizes all N-pointed stable maps to X of degree d. This space is equipped with *evaluation maps* $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,N}(X,d) \to X$ for $1 \leq i \leq N$, where ev_i sends a stable map f to its image of the i-th marked point in its domain C. We let $\mathrm{ev} = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_N : \overline{\mathcal{M}}_{0,N} \to X^N := X \times \cdots \times X$ denote the total evaluation map. For $N \geq 3$, there is also a forgetful map $\rho : \overline{\mathcal{M}}_{0,N}(X,d) \to \overline{\mathcal{M}}_{0,N} := \overline{\mathcal{M}}_{0,N}(\mathrm{point},0)$ which sends a stable map to its domain (after collapsing unstable components). The (coarse) moduli space $\overline{\mathcal{M}}_{0,N}(X,d)$ is a normal projective variety with at worst finite quotient singularities, and its dimension is given by

$$\dim \overline{\mathcal{M}}_{0,N}(X,d) = \dim(X) + \int_d c_1(T_X) + N - 3\,. \tag{1}$$

We refer to the notes [19] for the construction of this space. It has been proved by Kim and Pandharipande [31] and by Thomsen [53] that the Kontsevich space $\overline{\mathcal{M}}_{0,N}(X,d)$ is irreducible. Kim and Pandharipande also showed that $\overline{\mathcal{M}}_{0,N}(X,d)$ is rational.

Recall that a *Schubert variety* in X is an orbit closure for the action of a Borel subgroup $B \subset G$. A collection of Schubert varieties $\Omega_1, \dots, \Omega_N$ can be moved in *general position* by translating them with general elements of G. Given such a collection and a degree $d \in H_2(X)$, there is a *Gromov-Witten variety* defined by

$$GW_d(\Omega_1, \dots, \Omega_N) = \mathrm{ev}^{-1}(\Omega_1 \times \cdots \times \Omega_N) \ \subset \overline{\mathcal{M}}_{0,N}(X,d)\,. \tag{2}$$

When this intersection is finite, its cardinality is the *Gromov-Witten invariant* associated to the corresponding Schubert classes. In general, its Euler characteristic defines a *K-theoretic Gromov-Witten invariant* [39].

Lee and Pandharipande asked which Gromov-Witten varieties are rational in [40]. They also announced that for a fixed degree $d \in H_2(\mathbb{P}^1)$, the Gromov-Witten variety $GW_d(P_1, \dots, P_N) \subset \overline{\mathcal{M}}_{0,N}(\mathbb{P}^1, d)$ is rational for only finitely many integers N, where $P_1, \dots, P_N \in \mathbb{P}^1$ are general points. Pandharipande had earlier shown that the variety $GW_d(P_1, \dots, P_{3d-2}) \subset \overline{\mathcal{M}}_{0,3d-2}(\mathbb{P}^2, d)$ is a non-singular curve of positive genus for $d \geq 3$ [47]. In preparation for our computation of K-theoretic Gromov-Witten invariants in section 4 we will prove that, if X is a Grassmannian of type A and $N \geq 3$ is a fixed integer, then $GW_d(\Omega_1, \dots, \Omega_N) \subset \overline{\mathcal{M}}_{0,N}(X,d)$ is rational for all but finitely many degrees d. Our formulas for Gromov-Witten invariants can in turn be used to locate non-rational 3-pointed Gromov-Witten varieties of positive dimension, see Example 5.11.

Let $X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n : \dim V = m\}$ be the Grassmannian of m-dimensional vector subspaces of $\mathbb{C}^n$. This variety has a tautological subbundle $\mathcal{S} \subset \mathcal{O}_X^{\oplus n} = X \times \mathbb{C}^n$ given by $\mathcal{S} = \{(V,u) \in X \times \mathbb{C}^n : u \in V\}$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Notice that $H_2(X) = \mathbb{Z}$, so the degree of a stable map to X can be identified with a non-negative integer.

Theorem 2.1. *Let $P \in X^N$ and $Q \in \overline{\mathcal{M}}_{0,N}$ be general points, with $N \geq 3$, and let $d \geq 0$. Then the intersection $\mathrm{ev}^{-1}(P) \bigcap \rho^{-1}(Q) \subset \overline{\mathcal{M}}_{0,N}(X,d)$ is either empty or an irreducible rational variety.*

Proof. Since the point $Q \in \overline{\mathcal{M}}_{0,N}$ is general, it consists of N distinct points $(x_1 : y_1), \dots, (x_N : y_N)$ in $\mathbb{P}^1$. Write $d = mp + r$ where $0 \leq r < m$, and set $s = m - r$. Define the vector bundle $\mathcal{E} = \mathcal{O}(-p)^{\oplus s} \oplus \mathcal{O}(-p-1)^{\oplus r}$ on $\mathbb{P}^1$. Then the inverse image $\rho^{-1}(Q) \subset \overline{\mathcal{M}}_{0,N}(X,d)$ has a dense open subset of maps $f : \mathbb{P}^1 \to X$ for which $f^*(\mathcal{S}) \cong \mathcal{E}$. Any map in this subset is given by an injective element in $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{O}^{\oplus n})$, and two injective elements define the same map if and only if they differ by an automorphism of $\mathcal{E}$. The marked points in the domain of each map is given by Q.

We have $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{O}^{\oplus n}) = (\mathbb{C}^n)^{s(p+1)} \oplus (\mathbb{C}^n)^{r(p+2)} = \{(u_{ij}, v_{ij})\}$, where $u_{ij} \in \mathbb{C}^n$ is defined for $1 \leq i \leq s$ and $0 \leq j \leq p$, and $v_{ij} \in \mathbb{C}^n$ is defined for $1 \leq i \leq r$ and $0 \leq j \leq p+1$. An injective element (u_{ij}, v_{ij}) defines the map $f : \mathbb{P}^1 \to X$ which sends $(x:y) \in \mathbb{P}^1$ to the span of the vectors $\sum_{j=0}^{p} x^j y^{p-j}\, u_{ij}$ for $1 \leq i \leq s$ and $\sum_{j=0}^{p+1} x^j y^{p+1-j}\, v_{ij}$ for $1 \leq i \leq r$. The automorphisms of $\mathcal{E}$ are given by $\mathrm{Aut}(\mathcal{E}) = \mathrm{GL}(s) \times \mathrm{GL}(r) \times \mathrm{Mat}(r,s) \times \mathrm{Mat}(r,s)$, and composition with the element $(a,b,c,c') \in \mathrm{Aut}(\mathcal{E})$ is given by $(a,b,c,c').(u_{ij}, v_{ij}) = (u'_{ij}, v'_{ij})$ where $u'_{ij} = \sum_{l=1}^{s} a_{il} u_{lj}$ and $v'_{ij} = \sum_{l=1}^{r} b_{il} v_{lj} + \sum_{l=1}^{s} c_{il} u_{lj} + \sum_{l=1}^{s} c'_{il} u_{l,j-1}$. Here we write $u_{ij} = 0$ if $j \in \{-1, p+1\}$.

It follows from this description that a dense open subset U of the injective elements of $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{O}^{\oplus n})$ modulo the action of $\mathrm{Aut}(\mathcal{E})$ have unique representatives of the form (u_{ij}, v_{ij}) where $u_{ip} = e_i + \widetilde{u}_{ip}$ with $\widetilde{u}_{ip} \in 0^s \oplus \mathbb{C}^{n-s}$; $v_{i,p+1} = e_{s+i} + \widetilde{v}_{i,p+1}$ with $\widetilde{v}_{i,p+1} \in 0^m \oplus \mathbb{C}^{n-m}$; and $v_{ip} \in 0^s \oplus \mathbb{C}^{n-s}$. This shows that $U \subset \rho^{-1}(Q)$ is isomorphic to an open subset of an affine space, so $\rho^{-1}(Q)$ is rational. The evaluation maps $\mathrm{ev}_k : U \to X$ are then given by $\mathrm{ev}_k(u_{ij}, v_{ij}) = f(x_k : y_k)$ where $f : \mathbb{P}^1 \to X$ is defined as above.

It follows from Kleiman's transversality theorem [32, Thm. 2] that all components of $\mathrm{ev}^{-1}(P) \bigcap \rho^{-1}(Q)$ meet U. If we write $P = (V_1, \dots, V_N)$ with $V_i \in X$, then we conclude that $\mathrm{ev}^{-1}(P) \bigcap \rho^{-1}(Q)$ is birational to the set of points $(u_{ij}, v_{ij}) \in U$ which satisfy that $f(x_k : y_k) = V_k$ for each k. In other words, we require that $\sum_{j=0}^{p} x_k^j y_k^{p-j} u_{ij} \in V_k$ for $1 \le i \le s$ and $\sum_{j=0}^{p+1} x_k^j y_k^{p+1-j} v_{ij} \in V_k$ for $1 \le i \le r$. Since this amounts to a set of affine equations on $\{(u_{ij}, v_{ij})\}$, we conclude that $\mathrm{ev}^{-1}(P) \bigcap \rho^{-1}(Q)$ is either empty or rational, as claimed. □

The following corollary will be used to compute K-theoretic Gromov-Witten invariants of large degrees.

Corollary 2.2. *Let $V_1, V_2, V_3 \in X$ be general points. Then $GW_d(V_1, V_2, V_3)$ is rational for all degrees $d \ge \max(m, n-m)$.*

Proof. Since $\overline{\mathcal{M}}_{0,3}$ has only one point, the intersection of Theorem 2.1 is equal to $GW_d(V_1, V_2, V_3)$. We must show that this variety is not empty. We may assume that $m \le n-m \le d$. Then the dimension of the vector space $W = V_1 + V_2 \subset \mathbb{C}^n$ is $2m$. Choose $V_3' \in \mathrm{Gr}(m, W) \subset \mathrm{Gr}(m, \mathbb{C}^n)$ such that $V_3' \cap V_1 = V_3' \cap V_2 = 0$ and $V_3' \cap V_3 = W \cap V_3$. Using e.g. [10, Prop. 1] we can find a rational map $f_1 : \mathbb{P}^1 \to \mathrm{Gr}(m, W)$ of degree m such that V_1, V_2, and V_3' are contained in the image of f_1. Note that $A = V_3 \cap V_3'$ has dimension $m - d_2$ and $B = V_3 + V_3'$ has dimension $m + d_2$ where $d_2 = \min(n-2m, m)$. Another application of [10, Prop. 1] now shows that V_3 and V_3' are contained in the image of a rational map $f_2 : \mathbb{P}^1 \to \mathrm{Gr}(d_2, B/A) \subset X$ of degree d_2. If we let C be the union of the domains of f_1 and f_2, with the points mapping to V_3' identified, then f_1 and f_2 define a stable map $C \to X$ of degree $m + d_2 \le n - m \le d$ whose image contains V_1, V_2, and V_3. If necessary, we can add extra components to C to obtain a stable map of degree d. This constructs a point of $GW_d(V_1, V_2, V_3)$ and finishes the proof. □

We finally give a "converse" to Lee and Pandharipande's announcement from [40, p. 1379]. Notice that for sufficiently large degrees d, the map $(\mathrm{ev}, \rho) : \overline{\mathcal{M}}_{0,N}(X, d) \to X^N \times \overline{\mathcal{M}}_{0,N}$ is surjective. Theorem 2.1 implies that its fibers are irreducible for all points in a dense subset of $X^N \times \overline{\mathcal{M}}_{0,N}$, and using the Stein factorization we deduce that all fibers are connected.

Corollary 2.3. *Let $\Omega_1, \Omega_2, \dots, \Omega_N \subset X$ be Schubert varieties in general position. Choose d large enough so that the map $\overline{\mathcal{M}}_{0,N}(X, d) \to X^N \times \overline{\mathcal{M}}_{0,N}$ is surjective. Then the Gromov-Witten variety $GW_d(\Omega_1, \dots, \Omega_N)$ is birational to an affine bundle over $\Omega_1 \times \cdots \times \Omega_N \times \overline{\mathcal{M}}_{0,N}$. In particular, $GW_d(\Omega_1, \dots, \Omega_N)$ is rational.*

Proof. We first show that $GW_d(\Omega_1, \dots, \Omega_N)$ is an irreducible variety. The Kleiman-Bertini theorem [32, Remark 7] implies that this variety is locally irreducible, so it suffices to prove that it is connected. This follows because all fibers of the proper surjective map $(\mathrm{ev}, \rho) : GW_d(\Omega_1, \dots, \Omega_N) \to \Omega_1 \times \cdots \times \Omega_N \times \overline{\mathcal{M}}_{0,N}$ are connected. Finally, to see that $GW_d(\Omega_1, \dots, \Omega_N)$ is birational to an affine bundle, we observe that the construction used to prove Theorem 2.1 also parametrizes an open subset of $\overline{\mathcal{M}}_{0,N}(X, d)$ in terms of local coordinates on $X^N \times \overline{\mathcal{M}}_{0,N}$. □

Remark 2.4. In section 7 we will examine K-theoretic Gromov-Witten invariants for cominuscule homogeneous spaces, at which point a cominuscule analogue of Corollary 2.2 would be desirable. We have work in progress showing that certain Gromov-Witten varieties are unirational, which suffices for our purposes. This includes 3-points Gromov-Witten varieties for maximal orthogonal Grassmannians and 2-point Gromov-Witten varieties for Lagrangian Grassmannians. We can also show that 3-point Gromov-Witten varieties for Lagrangian Grassmannians have a rational component, but not that they are irreducible. These developments will be explained elsewhere.

3. Direct images of Grothendieck classes

In this section we prove some facts about equivariant K-theory in preparation for our computation of Gromov-Witten invariants. Our main references are Chapter 5 in [13] and Section 15.1 in [18]. To honor the assumptions in the first reference, we will assume that all varieties are quasi-projective over $\mathbb{C}$.

Let G be a complex linear algebraic group and let X be a (quasi-projective) G-variety. An *equivariant sheaf* on X is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ together with a given isomorphism $I : a^*\mathcal{F} \cong p_X^*\mathcal{F}$, where $a : G \times X \to X$ is the action and $p_X : G \times X \to X$ the projection. This isomorphism must satisfy that $(m \times \mathrm{id}_X)^* I = p_{23}^* I \circ (\mathrm{id}_G \times a)^* I$ as morphisms of sheaves on $G \times G \times G$, where m is the group operation on G and p_{23} is the projection to the last two factors of $G \times G \times X$.

The *equivariant K-homology group* $K_G(X)$ is the Grothendieck group of equivariant sheaves on X, i.e. the free Abelian group generated by isomorphism classes $[\mathcal{F}]$ of equivariant sheaves, modulo relations saying that $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ if there exists an equivariant exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. This group is a module over the *equivariant K-cohomology ring* $K^G(X)$, defined as the Grothendieck group of equivariant vector bundles on X. Both the multiplicative structure of $K^G(X)$ and the module structure of $K_G(X)$ are given by tensor products. The *Grothendieck class* of X is the class $[\mathcal{O}_X] \in K^G(X)$ of its structure sheaf. If X is non-singular, then the implicit map $K^G(X) \to K_G(X)$ that sends a vector bundle to its sheaf of sections is an isomorphism; this follows because every equivariant sheaf on X has a finite resolution by equivariant vector bundles [13, 5.1.28].

Given an equivariant morphism of G-varieties $f : X \to Y$, there is a ring homomorphism $f^* : K^G(Y) \to K^G(X)$ defined by pullback of vector bundles. If f is proper, then there is also a pushforward map $f_* : K_G(X) \to K_G(Y)$ defined by $f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]$. This map is a homomorphism of $K^G(Y)$-modules by the projection formula [28, Ex. III.8.3]. The Godement resolution can be used to obtain equivariant structures on the higher direct image sheaves of $\mathcal{F}$. Both pullback and pushforward are functorial with respect to composition of morphisms.

Recall that the variety X has *rational singularities* if there exists a desingularization $\pi : \widetilde{X} \to X$ for which $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$. When this is true, these identities hold for any desingularization, and X is normal (see e.g the proof of [28, Cor. 11.4]). If X is a G-variety, then it follows from [55, Thm. 7.6.1] or [3, Thm. 13.2] that X has an *equivariant desingularization*. More precisely, there exists an equivariant projective birational morphism $\pi : \widetilde{X} \to X$ from a non-singular G-variety $\widetilde{X}$ (see e.g. [49, §4]). This implies that $\pi_*[\mathcal{O}_{\widetilde{X}}] = [\mathcal{O}_X] \in K_G(X)$. More generally, if $f : X \to Y$ is any equivariant proper birational map of G-varieties with rational singularities, then the composition $f\pi : \widetilde{X} \to Y$ is a desingularization of

Y, so we obtain $f_*[\mathcal{O}_X] = f_*\pi_*[\mathcal{O}_{\widetilde{X}}] = [\mathcal{O}_Y] \in K_G(Y)$ by functoriality. We need the following generalization.

Theorem 3.1. *Let $f : X \to Y$ be a surjective equivariant map of projective G-varieties with rational singularities. Assume that the general fiber of f is rational, i.e. $f^{-1}(y)$ is an irreducible rational variety for all closed points in a dense open subset of Y. Then $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_G(Y)$.*

We will deduce this statement from the following result of Kollár [35, Thm. 7.1].

Theorem 3.2 (Kollár)**.** *Let $\psi : X \to Y$ be a surjective map between projective varieties, with X smooth and Y normal. Assume that the geometric generic fiber $F = X \times_Y \operatorname{Spec} \overline{\mathbb{C}(Y)}$ is connected. Then the following are equivalent:*
(i) $R^i\psi_*\mathcal{O}_X = 0$ for all $i > 0$;
(ii) Y has rational singularities and $H^i(F, \mathcal{O}_F) = 0$ for all $i > 0$.

To show that Kollár's theorem applies to our situation, we need the following two lemmas; they are most likely known, but since we lack a reference, we supply their proofs.

Lemma 3.3. *Let $\varphi : X \to Y$ be a dominant morphism of irreducible varieties, and assume that $\varphi^{-1}(y)$ is connected for all closed points y in a dense open subset of Y. Then the geometric generic fiber $X \times_Y \operatorname{Spec} \overline{\mathbb{C}(Y)}$ is connected.*

Proof. We may assume that $X = \operatorname{Spec}(S)$ and $Y = \operatorname{Spec}(R)$ are both affine. If the geometric generic fiber is disconnected, then there exists non-zero elements $f, g \in S \otimes_R \overline{K(R)}$ such that $f + g = 1$, $f^2 = f$, $g^2 = g$, and $fg = 0$ (cf. [28, Ex. 2.19]). These elements f and g will be contained in $S \otimes_R R'$ for some finitely generated $\mathbb{C}$-algebra R' with $R \subset R' \subset \overline{K(R)}$. Now consider the diagram:

$$\begin{array}{ccc} \operatorname{Spec}(S) & \xrightarrow{\varphi} & \operatorname{Spec}(R) \\ \uparrow & & \uparrow \\ \operatorname{Spec}(S \otimes_R R') & \longrightarrow & \operatorname{Spec}(R') \end{array}$$

By Grothendieck's generic freeness lemma, we may assume that $S = \bigoplus_i Rs_i$ is a free R-module, after replacing R with a localization R_h. Write $f = \sum s_i \otimes f_i$ and $g = \sum s_i \otimes g_i$ with $f_i, g_i \in R'$, and choose i and j such that $h = f_i g_j \in R'$ is non-zero. Then the images of f and g in $S \otimes_R R'/P$ are non-zero for each (closed) point $P \in \operatorname{Spec}(R'_h)$, which implies that the fiber $\varphi^{-1}(P \cap R) = \operatorname{Spec}(S \otimes_R R'/P)$ is disconnected. This is a contradiction because $\operatorname{Spec}(R'_h) \to \operatorname{Spec}(R)$ is a dominant morphism, so its image contains a dense open subset of $\operatorname{Spec}(R)$. □

We note that Lemma 3.3 (and its proof) is valid for varieties over any algebraically closed field. The same argument also shows that, if the general fiber of φ is integral, then so is the geometric generic fiber.

Lemma 3.4. *Let $f : X \to Y$ be a surjective projective morphism of irreducible varieties of characteristic zero, with Y normal. Assume that $f^{-1}(y)$ is connected for all closed points y in a dense open subset of Y. Then $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.*

Proof. By the proof of [28, III.11.5], f has a Stein factorization $f = gf'$, where $f' : X \to Y'$ is projective with $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$ and $g : Y' \to Y$ is finite. Since the general fiber of f is connected and the characteristic is zero, the map $g : Y' \to Y$ must be birational. But then g is an isomorphism by Zariski's Main Theorem. $\square$

Proof of Theorem 3.1. Let $\pi : \widetilde{X} \to X$ be an equivariant projective desingularization of X. Since X has rational singularities we know that $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $\pi_*[\mathcal{O}_{\widetilde{X}}] = [\mathcal{O}_X]$, so it is enough to show that $\psi_*[\mathcal{O}_{\widetilde{X}}] = [\mathcal{O}_Y]$ where $\psi = f\pi$. Lemma 3.4 implies that $f_* \mathcal{O}_X = \mathcal{O}_Y$, so $\psi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_Y$. By Kollár's Theorem 3.2, it is therefore enough to prove that the geometric generic fiber $F = \widetilde{X} \times_Y \operatorname{Spec} \overline{\mathbb{C}(Y)}$ is connected and $H^i(F, \mathcal{O}_F) = 0$ for $i > 0$.

By [28, III.10.7] we can find a dense open subset $U \subset Y$ such that $\psi : \psi^{-1}(U) \to U$ is smooth. Since the fibers of ψ are connected by [28, III.11.3], it follows that $\psi^{-1}(y)$ is a non-singular rational projective variety for every closed point y in a dense open subset of Y. In particular, Lemma 3.3 implies that F is connected.

To obtain the vanishing of cohomology, let $y \in Y$ be a closed point such that $\widetilde{X}_y = \psi^{-1}(y)$ is a non-singular rational projective variety. Then we have $H^0(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}_y}) = \mathbb{C}$ and $H^i(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}_y}) = 0$ for all $i > 0$ [27, p. 494]. It now follows from [28, III.12.11] that $R^i \psi_*(\mathcal{O}_{\widetilde{X}})$ is zero in an open neighborhood of y, and that $H^i(\widetilde{X}_z, \mathcal{O}_{\widetilde{X}_z}) = 0$ for all points z in this neighborhood, for $i > 0$. Taking z to be the generic point of Y, we obtain $H^i(F', \mathcal{O}_{F'}) = 0$ for $i > 0$, where $F' = \widetilde{X} \times_Y \operatorname{Spec} \mathbb{C}(Y)$. Finally, since $\operatorname{Spec} \overline{\mathbb{C}(Y)} \to \operatorname{Spec} \mathbb{C}(Y)$ is a flat morphism, it follows from [28, III.9.3] that $H^i(F, \mathcal{O}_F) = 0$ for $i > 0$, as required. $\square$

We also need the following consequence of the projection formula. Recall that $K^G(X)$ and $K_G(X)$ can be identified if X is non-singular.

Lemma 3.5. *Let $q_i : X_i \to Y$ be flat proper equivariant maps of non-singular G-varieties for $1 \leq i \leq n$, and let $\alpha_i \in K^G(X_i)$. Set $P = X_1 \times_Y \cdots \times_Y X_n$ with diagonal G-action and projections $e_i : P \to X_i$, and set $\psi = q_i e_i : P \to Y$. Then $\psi_*(e_1^* \alpha_1 \cdot e_2^* \alpha_2 \cdots e_n^* \alpha_n) = q_{1,*}\alpha_1 \cdot q_{2,*}\alpha_2 \cdots q_{n,*}\alpha_n \in K^G(Y)$.*

Proof. Let $P' = X_1 \times_Y \cdots \times_Y X_{n-1}$ with projections $e'_i : P' \to X_i$ and set $\psi' = q_i e'_i$. Then we have a fiber square with flat, proper, G-equivariant maps:

$$\begin{array}{ccc} P & \xrightarrow{e_n} & X_n \\ {\scriptstyle q'_n}\downarrow & & \downarrow{\scriptstyle q_n} \\ P' & \xrightarrow[\psi']{} & Y \end{array}$$

It follows from [28, III.9.3] or [13, 5.3.15] that $q'_{n,*} e_n^* \alpha_n = {\psi'}^* q_{n,*} \alpha_n \in K^G(P')$. By induction on n we therefore obtain

$$\begin{aligned} \psi_*(e_1^*\alpha_1 \cdots e_n^*\alpha_n) &= \psi'_* q'_{n,*}({q'}_n^*({e'}_1^*\alpha_1 \cdots {e'}_{n-1}^*\alpha_{n-1}) \cdot e_n^*\alpha_n) \\ &= \psi'_*({e'}_1^*\alpha_1 \cdots {e'}_{n-1}^*\alpha_{n-1} \cdot q'_{n,*} e_n^*\alpha_n) \\ &= \psi'_*({e'}_1^*\alpha_1 \cdots {e'}_{n-1}^*\alpha_{n-1} \cdot {\psi'}^* q_{n,*}\alpha_n) \\ &= \psi'_*({e'}_1^*\alpha_1 \cdots {e'}_{n-1}^*\alpha_{n-1}) \cdot q_{n,*}\alpha_n \\ &= q_{1,*}\alpha_1 \cdots q_{n-1,*}\alpha_{n-1} \cdot q_{n,*}\alpha_n \end{aligned}$$

as required. $\square$

We finally need the following facts about the pushforward and pullback of Schubert classes between homogeneous spaces. Let G be a complex connected semisimple linear algebraic group, T a maximal torus, and P and Q Borel subgroups, such that $T \subset P \subset Q \subset G$. Let $f : G/Q \to G/P$ be the projection.

Lemma 3.6. (a) *If $\Omega \subset G/Q$ is any T-stable Schubert variety, then $f_*([\mathcal{O}_\Omega]) = [\mathcal{O}_{f(\Omega)}] \in K_T(G/P)$.* (b) *If $\Omega \subset G/P$ is any T-stable Schubert variety, then $f^*([\mathcal{O}_\Omega]) = [\mathcal{O}_{f^{-1}(\Omega)}] \in K^T(G/P)$.*

Part (a) of this lemma is known from [5, Thm.3.3.4(a)], and part (b) is true because f is a flat morphism.

4. Gromov-Witten invariants

4.1. **Definitions.** We start our discussion of Gromov-Witten invariants by recalling the definitions and some general facts. Let $X = G/P$ be a homogeneous space and let $d \in H_2(X)$ be a degree. Given K-theory classes $\alpha_1, \dots, \alpha_N \in K^\circ(X)$, Lee and Givental define a K-theoretic Gromov-Witten invariant by [39, 22]

$$I_d(\alpha_1, \dots, \alpha_N) \;=\; \chi(\mathrm{ev}_1^*(\alpha_1) \cdots \mathrm{ev}_N^*(\alpha_N)) \;\in \mathbb{Z}\,, \tag{3}$$

where χ denotes Euler characteristic, i.e. proper pushforward along the structure morphism $\rho : \overline{\mathcal{M}}_{0,N}(X,d) \to \{\text{point}\}$. If the classes α_i are structure sheaves of closed subvarieties $\Omega_1, \dots, \Omega_N \subset X$ in general position such that $\sum \operatorname{codim}(\Omega_i) = \dim \overline{\mathcal{M}}_{0,N}(X,d)$, then the invariant $I_d(\mathcal{O}_{\Omega_1}, \dots, \mathcal{O}_{\Omega_d})$ is equal to the cohomological invariant

$$I_d([\Omega_1], \dots, [\Omega_d]) = \int_{\overline{\mathcal{M}}_{0,N}(X,d)} \mathrm{ev}_1^*[\Omega_1] \cdots \mathrm{ev}_N^*[\Omega_N]\,,$$

which in turn is equal to the number $\#GW_d(\Omega_1, \dots, \Omega_N)$ of stable maps $C \to X$ of degree d for which the i-th marked point of C is mapped into Ω_i (see [19, Lemma 14]). In general, the K-theoretic invariant $I_d(\mathcal{O}_{\Omega_1}, \dots, \mathcal{O}_{\Omega_N})$ is equal to the Euler characteristic of the structure sheaf of the Gromov-Witten variety $GW_d(\Omega_1, \dots, \Omega_N)$, which is the definition of K-theoretic Gromov-Witten invariants used in [40]. This can be seen by applying the K-theoretic (relative) Kleiman-Bertini theorem of Sierra [51, Thm. 2.2] (see also [46] for the non-relative case). Let $G^N = G \times G \times \cdots \times G$ act componentwise on X^N and consider the diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,N}(X,d) & \xrightarrow{\ \mathrm{ev}\ } & X^N & & \\ & & {\scriptstyle f}\uparrow & & \\ & & G^N \times W & \xrightarrow{\ \pi\ } & G^N \end{array}$$

where $W = \Omega_1 \times \cdots \times \Omega_N \subset X^N$ and f is defined by the action. If we let G^N act on the first factor of $G^N \times W$, then the map f is equivariant, and therefore flat since the action on X^N is transitive. Sierra's theorem implies that $Tor_i^{X^N}(\mathcal{O}_{\overline{\mathcal{M}}}, \mathcal{O}_{g.W}) = 0$ for some $g \in G^N$ and all $i > 0$. Replacing W with $g.W$, we obtain (cf. [18, Ex. 15.1.8])

$$\textstyle\prod_{i=1}^N \mathrm{ev}_i^*[\mathcal{O}_{\Omega_i}] = \mathrm{ev}^*[\mathcal{O}_W] = [\mathcal{O}_{\overline{\mathcal{M}}} \otimes_{\mathcal{O}_{X^N}} \mathcal{O}_W] = [\mathcal{O}_{GW_d(\Omega_1,\dots,\Omega_N)}]$$

in $K_\circ(\overline{\mathcal{M}})$, which finally implies that $I_d(\mathcal{O}_{\Omega_1}, \dots, \mathcal{O}_{\Omega_N}) = \chi(\mathcal{O}_{GW_d(\Omega_1,\dots,\Omega_n)})$. Notice that in contrast to cohomological Gromov-Witten invariants, the K-theoretic invariants need not vanish for large degrees $d \in H_2(X)$.

More generally, fix a maximal torus $T \subset G$. Then T acts on X and $\overline{\mathcal{M}}_{0,N}(X,d)$, and the evaluation maps are equivariant. The *equivariant K-theoretic Gromov-Witten invariant* given by classes $\alpha_1,\dots,\alpha_N \in K^T(X)$ is defined as the virtual representation

$$I_d^T(\alpha_1,\dots,\alpha_N) \;=\; \chi^T_{\overline{\mathcal{M}}}(\mathrm{ev}_1^*(\alpha_1)\cdots\mathrm{ev}_N^*(\alpha_N)) \;\in K^T(\text{point})\,, \tag{4}$$

where $\chi^T_{\overline{\mathcal{M}}}$ is the equivariant pushforward along ρ. If we write $T = (\mathbb{C}^*)^n$, then the virtual representations of T form the Laurent polynomial ring $K^T(\text{point}) = \mathbb{Z}[L_1^{\pm 1},\dots,L_n^{\pm 1}]$, where $L_i \cong \mathbb{C}$ is the representation with character $(t_1,\dots,t_n) \mapsto t_i$. This ring is contained in the ring of formal power series $\mathbb{Z}[\![y_1,\dots,y_n]\!]$ in the variables $y_i = 1 - L_i^{-1} \in K^T(\text{point})$. If we misuse notation and write y_i also for the equivariant Chern class of L_i, then the T-equivariant cohomology ring of a point is the ring $H_T^*(\text{point}) = \mathbb{Z}[y_1,\dots,y_n]$.

The study of equivariant Gromov-Witten invariants was pioneered by Givental [21] (see also [23]), who defined the cohomological invariants

$$I_d^T(\beta_1,\dots,\beta_N) \;=\; \int^T_{\overline{\mathcal{M}}_{0,3}(X,d)} \mathrm{ev}_1^*(\beta_1)\cdots\mathrm{ev}_N^*(\beta_N) \;:=\; \rho_*^T(\mathrm{ev}_1^*(\beta_1)\cdots\mathrm{ev}_N^*(\beta_N))$$

in $H_T^*(\text{point})$ for equivariant cohomology classes $\beta_1,\dots,\beta_N \in H_T^*(X)$. These invariants can be obtained as the leading terms of K-theoretic invariants as follows. For any T-variety Y we let $\mathrm{ch}_T : K^T(Y) \to \widehat{H}_T(Y) := \prod_{i=0}^{\infty} H_T^{2i}(Y;\mathbb{Q})$ be the equivariant Chern character, see [16, Def. 3.1]. By the equivariant Hirzebruch formula [16, Cor. 3.1] we then have

$$\mathrm{ch}_T\left(I_d^T(\alpha_1,\dots,\alpha_N)\right) = \rho_*^T\left(\mathrm{ev}_1^*(\mathrm{ch}_T(\alpha_1))\cdots\mathrm{ev}_N^*(\mathrm{ch}_T(\alpha_N))\cdot\mathrm{Td}^T(\overline{\mathcal{M}})\right) \tag{5}$$

where $\mathrm{Td}^T(\overline{\mathcal{M}}) \in \widehat{H}_T(\overline{\mathcal{M}})$ is the equivariant Todd class of the (singular) variety $\overline{\mathcal{M}}_{0,N}(X,d)$. Let $\Omega_1,\dots,\Omega_N \subset X$ be T-stable closed subvarieties. By using that $\mathrm{ch}_T(y_i) = 1 - \exp(-y_i) = y_i + \text{higher terms} \in \widehat{H}_T(\text{point})$ and $\mathrm{ch}_T(\mathcal{O}_{\Omega_i}) = [\Omega_i]+\text{higher terms in } \widehat{H}_T(X)$ [18, Thm. 18.3], we deduce from (5) that the term of (lowest) degree $\sum \mathrm{codim}(\Omega_i) - \dim \overline{\mathcal{M}}_{0,N}(X,d)$ in the K-theoretic invariant $I_d^T(\mathcal{O}_{\Omega_1},\dots,\mathcal{O}_{\Omega_N}) \in K^T(\text{point}) \subset \mathbb{Z}[\![y_1,\dots,y_n]\!]$ is equal to the cohomological invariant $I_d^T([\Omega_1],\dots,[\Omega_N])$.

We note that when $\sum \mathrm{codim}(\Omega_i) = \dim \overline{\mathcal{M}}_{0,N}(X,d)$ we have

$$I_d(\mathcal{O}_{\Omega_1},\dots,\mathcal{O}_{\Omega_N}) = I_d([\Omega_1],\dots,[\Omega_N]) = I_d^T([\Omega_1],\dots,[\Omega_N]) \;\in \mathbb{Z}\,,$$

but the equivariant K-theoretic invariant $I_d^T(\mathcal{O}_{\Omega_1},\dots,\mathcal{O}_{\Omega_N}) \in K^T(\text{point})$ may not be an integer. In general, the ordinary K-theoretic invariant $I_d(\mathcal{O}_{\Omega_1},\dots,\mathcal{O}_{\Omega_N}) \in \mathbb{Z}$ is the total dimension of the virtual representation $I_d^T(\mathcal{O}_{\Omega_1},\dots,\mathcal{O}_{\Omega_N}) \in K^T(\text{point})$.

Example 4.1. Let $X = \mathrm{Gr}(3,6)$ be the Grassmannian of 3-planes in $\mathbb{C}^6$, and let $T = (\mathbb{C}^*)^6$ act on X through the coordinatewise action on $\mathbb{C}^6$. Let $\Omega \subset X$ be the Schubert variety defined by the partition $\lambda = (2,1)$, i.e. $\Omega = \{V \in X \mid V \cap \mathbb{C}^2 \neq 0 \text{ and } \dim(V \cap \mathbb{C}^4) \geq 2\}$. Then $I_0([\Omega],[\Omega],[\Omega]) = I_0^T([\Omega],[\Omega],[\Omega]) = I_0(\mathcal{O}_\Omega,\mathcal{O}_\Omega,\mathcal{O}_\Omega) = 2$, whereas $I_0^T(\mathcal{O}_\Omega,\mathcal{O}_\Omega,\mathcal{O}_\Omega) = 1 + [\mathbb{C}_\chi]$ where the character χ is defined by $\chi(t_1,t_2,t_3,t_4,t_5,t_6) = (\frac{t_1 t_2}{t_5 t_6})^3$.

The non-equivariant cohomological (3-point) invariants of a (generalized) flag manifold have been computed by exploiting the fact that they are the structure constants for the quantum cohomology. We will not attempt to survey the subject, but the reader can consult [2, 14, 17, 20, 57] and references therein. The equivariant cohomological invariants, which appear as structure constants in equivariant quantum cohomology, have been computed in [45, 44], and more recent algorithms can also be found in [38].

If the cohomology of the variety X is generated by divisors (such as $\mathbb{P}^r$ or a full flag manifold), then Lee and Pandharipande's reconstruction theorem from [40] can be used to compute the N-pointed K-theoretic invariants starting from the 1-pointed invariants. For projective spaces a formula is known for the J-function, which encodes all the 1-pointed invariants [40, §2.2]. This yields a complete algorithm to compute the K-theoretic invariants in this case.

We will proceed to express the (equivariant) K-theoretic Gromov-Witten invariants of Grassmannians as triple intersections on two-step flag manifolds, thus generalizing the identity proved in [10].

4.2. **Grassmannians of type A.** Let $X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n : \dim V = m\}$ be the Grassmann variety of m-planes in $\mathbb{C}^n$. This variety has dimension mk, where $k = n - m$; the dimension of the associated Kontsevich moduli space $M_d := \overline{\mathcal{M}}_{0,3}(X,d)$ is equal to $\dim X + nd$. Following [8], we define the *kernel* of a 3-pointed stable map $f : C \to X$ to be the intersection of the m-planes $V \subset \mathbb{C}^n$ in its image, and we define the *span* of f as the linear span of these subspaces.

$$\mathrm{Ker}(f) = \bigcap_{V \in f(C)} V \quad ; \quad \mathrm{Span}(f) = \sum_{V \in f(C)} V \ \subset \mathbb{C}^n \,. \tag{6}$$

Given a degree $d \geq 0$ we set $a = \max(m-d, 0)$ and $b = \min(m+d, n)$. If $f : C \to X$ is a stable map of degree d, then its kernel and span satisfy the dimension bounds

$$\dim \mathrm{Ker}(f) \geq a \quad \text{and} \quad \dim \mathrm{Span}(f) \leq b \,. \tag{7}$$

This was proved in [8, Lemma 1] when $C = \mathbb{P}^1$, and in general it follows from this case by induction on the number of components of C. We will prove in Corollary 4.5 below that $f \mapsto \dim \mathrm{Ker}(f)$ is an upper semicontinuous function on M_d, while $f \mapsto \dim \mathrm{Span}(f)$ is lower semicontinuous. Since it is easy to construct stable maps $f : \mathbb{P}^1 \to X$ for which the bounds (7) are satisfied with equality (see [10, Prop. 1]), it follows that (7) is satisfied with equality for all stable maps in a dense open subset of M_d.

Define the two-step flag variety $Y_d = \mathrm{Fl}(a,b;n) = \{A \subset B \subset \mathbb{C}^n : \dim A = a, \dim B = b\}$ and the three-step flag variety $Z_d = \mathrm{Fl}(a,m,b;n) = \{A \subset V \subset B \subset \mathbb{C}^n : \dim A = a, \dim V = m, \dim B = b\}$. Let $p : Z_d \to X$ and $q : Z_d \to Y_d$ be the natural projections. Our main result for Grassmannians of type A is the following theorem.

Theorem 4.2. *For equivariant K-theory classes $\alpha_1, \alpha_2, \alpha_3 \in K^T(X)$ we have*

$$I_d^T(\alpha_1, \alpha_2, \alpha_3) = \chi^T_{Y_d}\left(q_* p^*(\alpha_1) \cdot q_* p^*(\alpha_2) \cdot q_* p^*(\alpha_3)\right).$$

This generalizes [10, Thm. 1] which gives this identity for non-equivariant cohomological Gromov-Witten invariants. We note that the cohomological invariants vanish for degrees d larger than $\min(m,k)$, but this is not true for the K-theoretic invariants. The definition of a and b given above is required to correctly compute the K-theoretic invariants for such degrees.

Let $\Omega \subset X$ be a Schubert variety. Like in [10, §2] we define a modified Schubert variety in Y_d by

$$\widetilde{\Omega} = q(p^{-1}(\Omega)) = \{(A,B) \in Y_d \mid \exists V \in \Omega : A \subset V \subset B\}\,.$$

We then have $q_* p^*([\mathcal{O}_\Omega]) = [\mathcal{O}_{\widetilde{\Omega}}] \in K_\circ(Y_d)$ by Lemma 3.6, and the cohomology class $q_* p^*([\Omega])$ is equal to $[\widetilde{\Omega}] \in H^*(Y_d)$ if $\operatorname{codim}(\widetilde{\Omega}) = \operatorname{codim}(\Omega) - d^2$ and is zero otherwise. If Ω is T-stable, then the same identities hold in equivariant K-theory and cohomology. We remark that the codimension of $\widetilde{\Omega}$ equals $\operatorname{codim}(\Omega) - d^2$ if and only if the Young diagram defining Ω contains a $d \times d$ rectangle. In particular we must have $d \leq \min(m,k)$. The role of the $d \times d$ rectangle was first discovered in [58], and the geometric interpretation was given in [10]. We derive the following corollary.

Corollary 4.3. *Let $\Omega_1, \Omega_2, \Omega_3 \subset X$ be Schubert varieties. Then we have*

$$I_d(\mathcal{O}_{\Omega_1}, \mathcal{O}_{\Omega_2}, \mathcal{O}_{\Omega_3}) = \chi_{Y_d}([\mathcal{O}_{\widetilde{\Omega}_1}] \cdot [\mathcal{O}_{\widetilde{\Omega}_2}] \cdot [\mathcal{O}_{\widetilde{\Omega}_3}])$$

and, if $d \leq \min(m,k)$, then

$$I_d([\Omega_1],[\Omega_2],[\Omega_3]) = \begin{cases} \int_{Y_d} [\widetilde{\Omega}_1] \cdot [\widetilde{\Omega}_2] \cdot [\widetilde{\Omega}_3] & \textit{if } \operatorname{codim}(\widetilde{\Omega}_i) = \operatorname{codim}(\Omega_i) - d^2 \ \forall\, i, \\ 0 & \textit{otherwise.} \end{cases}$$

If the $\Omega_1, \Omega_2, \Omega_3$ are T-stable, then these identities hold equivariantly as well.

The second identity was proved in [10] by putting the Schubert varieties in general position, and showing that the map $f \mapsto (\operatorname{Ker}(f), \operatorname{Span}(f))$ gives a bijection between the set of rational curves counted by the invariant $I_d([\Omega_1],[\Omega_2],[\Omega_3])$ and the set of points in the intersection $\widetilde{\Omega}_1 \cap \widetilde{\Omega}_2 \cap \widetilde{\Omega}_3$. This approach will not suffice for the general case, for example because the equivariant and K-theoretic invariants do not have an enumerative interpretation. Instead, we will give a cohomological proof of Theorem 4.2. We need some notation. For *arbitrary* integers a, b with $0 \leq a \leq m \leq b \leq n$ we define the subset

$$M_d(a,b) = \{(A,B,f) \in \operatorname{Fl}(a,b;n) \times M_d \mid A \subset \operatorname{Ker}(f) \text{ and } \operatorname{Span}(f) \subset B\}\,.$$

Lemma 4.4. *$M_d(a,b)$ is an irreducible closed subset of $\operatorname{Fl}(a,b;n) \times M_d$. Furthermore, when equipped with the reduced scheme structure, $M_d(a,b)$ is a projective variety with at worst finite quotient singularities.*

Proof. Let $\mathcal{A} \subset \mathcal{B} \subset \mathbb{C}^n \times Y$ be the tautological flag on $Y = \operatorname{Fl}(a,b;n)$. Then $M_d(a,b) = \{(y,f) \in Y \times M_d \mid \mathcal{A}(y) \subset \operatorname{Ker}(f) \text{ and } \operatorname{Span}(f) \subset \mathcal{B}(y)\}$. Let $U \subset Y$ be an open subset over which the tautological flag is isomorphic to the trivial flag $A_0 \times U \subset B_0 \times U \subset \mathbb{C}^n \times U$ given by some point $(A_0, B_0) \in Y$. Let $\operatorname{pr}_1 : Y \times M_d \to Y$ be the first projection. It follows from [19, p. 12] that the subset of stable maps $f : C \to X$ in M_d for which $f(C) \subset X' := \operatorname{Gr}(m-a, B_0/A_0)$ is closed and isomorphic to $\overline{\mathcal{M}}_{0,3}(X',d)$ with its reduced structure. Since the condition $f(C) \subset X'$ is equivalent to demanding that $A_0 \subset \operatorname{Ker}(f)$ and $\operatorname{Span}(f) \subset B_0$, we deduce that $M_d(a,b) \cap \operatorname{pr}_1^{-1}(U) = U \times \overline{\mathcal{M}}_{0,3}(X',d)$ is closed and reduced in $\operatorname{pr}_1^{-1}(U)$. The lemma follows from this. □

Corollary 4.5. *The function $f \mapsto \dim \operatorname{Ker}(f)$ is upper semicontinuous on M_d, and $f \mapsto \dim \operatorname{Span}(f)$ is lower semicontinuous.*

Proof. The set of stable maps f in M_d for which $\dim \operatorname{Ker}(f) \geq a$ and $\dim \operatorname{Span}(f) \leq b$ is the image of the projective variety $M_d(a,b)$ under the projection $\operatorname{Fl}(a,b;n) \times M_d \to M_d$. $\square$

From now on we set $a = \max(m-d, 0)$ and $b = \min(m+d, n)$ as in the statement of Theorem 4.2. In this case we write $\mathrm{B}\ell_d = M_d(a,b) = \{(A,B,f) \in Y_d \times M_d \mid A \subset \operatorname{Ker}(f) \text{ and } \operatorname{Span}(f) \subset B\}$. We suspect that this variety is isomorphic to the blowup of M_d along the closed subset where the kernel and span fail to have the expected dimensions (a,b), but we have not found a proof. We also define the variety $Z_d^{(3)} = \{(A, V_1, V_2, V_3, B) \mid (A,B) \in Y_d, V_i \in X, A \subset V_i \subset B\}$. Now construct the following commutative diagram, which is the heart of the proof of Theorem 4.2.

(8)
$$\begin{array}{ccccc}
\mathrm{B}\ell_d & & \xrightarrow{\quad \pi \quad} & & M_d \\
\downarrow{\scriptstyle \phi} & & & & \downarrow{\scriptstyle \mathrm{ev}_i} \\
Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\
 & & \downarrow{\scriptstyle q} & & \\
 & & Y_d & &
\end{array}$$

Here π is the projection to the second factor of $Y_d \times M_d$, and the other maps are given by $\phi(A,B,f) = (A, \mathrm{ev}_1(f), \mathrm{ev}_2(f), \mathrm{ev}_3(f), B)$ and $e_i(A, V_1, V_2, V_3, B) = (A, V_i, B)$. All the maps in the diagram are T-equivariant.

Lemma 4.6. *The map $\pi : B\ell_d \to M_d$ is birational.*

Proof. This map is surjective by the dimension bounds (7). Furthermore, for a dense open subset of stable maps f in M_d the dimension bounds (7) are satisfied with equality, which implies that $\pi^{-1}(f) = (\operatorname{Ker}(f), \operatorname{Span}(f), f)$. $\square$

Proposition 4.7. *We have $\phi_*[\mathcal{O}_{B\ell_d}] = [\mathcal{O}_{Z_d^{(3)}}]$ in $K^T(Z_d^{(3)})$.*

Proof. For any point $(A,B) \in Y_d$ we have $(qe_i\phi)^{-1}(A,B) = \overline{\mathcal{M}}_{0,3}(X', d)$ where $X' = \operatorname{Gr}(m-a, B/A)$, and $\phi^{-1}(A, V_1, V_2, V_3, B)$ is the set of stable maps $f : C \to X'$ that send the three marked points to V_1/A, V_2/A, V_3/A.

If $d \leq \min(m,k)$ then $X' = \operatorname{Gr}(d, 2d)$. Since the Gromov-Witten invariant $I_d(\text{point}, \text{point}, \text{point})$ on X' is equal to one, it follows that the general fiber of ϕ is a single point. This Gromov-Witten invariant can be computed with Bertram's structure theorems [2] or by using [10, Prop. 1]. We conclude that ϕ is a birational isomorphism, and the proposition follows because both $\mathrm{B}\ell_d$ and $Z_d^{(3)}$ have rational singularities.

For arbitrary degrees d, the proposition follows from Theorem 3.1, since the general fibers of ϕ are irreducible rational varieties by Corollary 2.2. $\square$

Proof of Theorem 4.2. By using Lemma 4.6 and Proposition 4.7, it follows from the projection formula that

$$\begin{aligned}
\chi_{M_d}(\mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdot \mathrm{ev}_3^*(\alpha_3)) &= \chi_{\mathrm{B}\ell_d}(\pi^* \mathrm{ev}_1^*(\alpha_1) \cdot \pi^* \mathrm{ev}_2^*(\alpha_2) \cdot \pi^* \mathrm{ev}_3^*(\alpha_3)) \\
&= \chi_{Z_d^{(3)}}(e_1^* p^*(\alpha_1) \cdot e_2^* p^*(\alpha_2) \cdot e_3^* p^*(\alpha_3)) \,.
\end{aligned}$$

Since $Z_d^{(3)} = Z_d \times_{Y_d} Z_d \times_{Y_d} Z_d$, the rest follows from Lemma 3.5. $\square$

Remark 4.8. The first author has conjectured a puzzle-based combinatorial formula for the structure constants of the equivariant cohomology of two-step flag varieties [15, §I.7], which generalizes results of Knutson and Tao [34, 33]. In view of Corollary 4.3, this conjecture specializes to a Littlewood-Richardson rule for the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$ of any Grassmannian. We refer to [10, §2.4] for the translation.

5. Quantum K-theory of Grassmannians

5.1. **Definitions.** In this section we apply Theorem 4.2 to compute the quantum K-theory of the Grassmannian $X = \mathrm{Gr}(m,n)$. Recall that a partition is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0)$. A partition can be identified with its Young diagram, which has λ_i boxes in row i. If (the Young diagram of) λ is contained in another partition ν, then ν/λ denotes the skew diagram of boxes in ν which are not in λ. The Schubert varieties in X are indexed by partitions contained in the rectangular partition $(k)^m = (k,k,\dots,k)$ with m rows and $k = n-m$ columns. The Schubert variety for λ relative to the Borel subgroup of $\mathrm{GL}(n)$ stabilizing a complete flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$ is defined by

$$X_\lambda = \{V \in X \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}\,.$$

The codimension of X_λ in X is equal to the weight $|\lambda| = \sum \lambda_i$. The Schubert classes $[X_\lambda]$ Poincaré dual to the Schubert varieties form a $\mathbb{Z}$-basis for the cohomology ring $H^*(X)$. The ordinary (small) quantum cohomology ring of X is an algebra over the polynomial ring $\mathbb{Z}[q]$, which as a module is defined by $\mathrm{QH}(X) = H^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$. The ring structure is given by

$$[X_\lambda] \star [X_\mu] = \sum_{\nu, d \geq 0} I_d([X_\lambda],[X_\mu],[X_{\nu^\vee}])\, q^d\, [X_\nu] \tag{9}$$

where the sum is over all partitions $\nu \subset (k)^m$ and non-negative degrees d, and $\nu^\vee = (k-\nu_m, \dots, k-\nu_1)$ is the Poincaré dual partition of ν.

The Grothendieck ring of X has a basis consisting of the Schubert structure sheaves $\mathcal{O}_\lambda := [\mathcal{O}_{X_\lambda}]$, $K(X) = \bigoplus_\lambda \mathbb{Z} \cdot \mathcal{O}_\lambda$. The determinant of the tautological subbundle on X defines the class $t = 1 - \mathcal{O}_{(1)}$ in $K(X)$. Define the K-theoretic dual Schubert class for λ by $\mathcal{O}_\lambda^\vee = t \cdot \mathcal{O}_{\lambda^\vee}$. By [7, §8] we have the Poincaré duality identity $\chi_X(\mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) = \delta_{\lambda,\nu}$.

The quantum K-theory ring of X is *not* obtained by replacing the cohomological Gromov-Witten invariants with K-theoretic invariants in (9), since this does not lead to an associative ring. Instead we need the following definition of structure constants, which comes from Givental's paper [22]. Given three partitions λ, μ, ν, define the constant

$$N_{\lambda,\mu}^{\nu,d} = \sum_{d_0,\dots,d_r,\kappa_1,\dots,\kappa_r} (-1)^r\, I_{d_0}(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_{\kappa_1}^\vee) \left(\prod_{i=1}^{r-1} I_{d_i}(\mathcal{O}_{\kappa_i}, \mathcal{O}_{\kappa_{i+1}}^\vee) \right) I_{d_r}(\mathcal{O}_{\kappa_r}, \mathcal{O}_\nu^\vee) \tag{10}$$

where the sum is over all sequences of non-negative integers $(d_0, \dots, d_r)$, $r \geq 0$, such that $\sum d_i = d$ and $d_i > 0$ for $i > 0$, and all partitions $\kappa_1, \dots, \kappa_r$. The two-point invariants can be obtained using the identity $I_d(\alpha_1, \alpha_2) = I_d(\alpha_1, \alpha_2, 1)$, which holds because the general fiber of the forgetful map $\overline{\mathcal{M}}_{0,3}(X,d) \to \overline{\mathcal{M}}_{0,2}(X,d)$ is rational, see [22, Cor. 1] or Theorem 3.1.

The constants $N^{\nu,d}_{\lambda,\mu}$ can also be defined by the (equivalent) inductive identity

$$N^{\nu,d}_{\lambda,\mu} = I_d(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu^\vee) - \sum_{\kappa, e>0} N^{\kappa,d-e}_{\lambda,\mu} \cdot I_e(\mathcal{O}_\kappa, \mathcal{O}_\nu^\vee)\,. \tag{11}$$

The degree zero constants $N^{\nu,0}_{\lambda,\mu}$ are the ordinary K-theoretic Schubert structure constants, i.e. $\mathcal{O}_\lambda \cdot \mathcal{O}_\mu = \sum_\nu N^{\nu,0}_{\lambda,\mu}\, \mathcal{O}_\nu$ in $K(X)$.

The K-theoretic quantum ring of X is the $\mathbb{Z}[\![q]\!]$-algebra given by $\mathrm{QK}(X) = K(X) \otimes_Z \mathbb{Z}[\![q]\!] = \bigoplus_\lambda \mathbb{Z}[\![q]\!]\, \mathcal{O}_\lambda$ as a module, and the algebra structure is defined by

$$\mathcal{O}_\lambda \star \mathcal{O}_\mu = \sum_{\nu, d\geq 0} N^{\nu,d}_{\lambda,\mu}\, q^d\, \mathcal{O}_\nu\,. \tag{12}$$

It was proved by Givental that this product is associative [22].

Remark 5.1. In the definition of the ring $\mathrm{QK}(X)$ we have replaced the polynomial ring $\mathbb{Z}[q]$ with the power series ring $\mathbb{Z}[\![q]\!]$, since the structure constants $N^{\nu,d}_{\lambda,\mu}$ might be non-zero for arbitrarily high degrees d. In fact, the invariants $I_d(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu)$ are equal to 1 for all sufficiently large degrees d. However, we will see in Corollary 5.8 that $N^{\nu,d}_{\lambda,\mu}$ is zero when $d > \ell(\lambda)$, so in fact we only work with polynomials in q. It appears to be an open question if this also occurs for other homogeneous spaces G/P.

Remark 5.2. Similarly to ordinary K-theory, $\mathrm{QK}(X)$ admits a topological filtration by ideals defined by $F_j\,\mathrm{QK}(X) = \bigoplus_{|\lambda|+ni\geq j} \mathbb{Z}\cdot q^i \mathcal{O}_\lambda$, and the associated graded ring is the ordinary quantum ring $\mathrm{QH}(X)$.

Remark 5.3. Implicit in Givental's proof that the K-theoretic quantum product is associative is the fact that the sum in (10) can be interpreted as a difference between two Euler characteristics. More precisely, let $\mathcal{D}$ be the closure of the locus of maps in $\overline{\mathcal{M}}_{0,3}(X,d)$ for which the domain has two components, the first and second marked points belong to one of these components, and the third marked point belongs to the other component. This subvariety is a union of boundary divisors in $\overline{\mathcal{M}}_{0,3}(X,d)$ (see e.g. [19, §6]) and these divisors have normal crossings, up to a finite group quotient (cf. Thm. 3 in *loc. cit.*). Then (10) can be rewritten as:

$$N^{\nu,d}_{\lambda,\mu} = \chi_{\overline{\mathcal{M}}_{0,3}(X,d)}(\mathrm{ev}_1^*(\mathcal{O}_\lambda)\cdot \mathrm{ev}_2^*(\mathcal{O}_\mu)\cdot \mathrm{ev}_3^*(\mathcal{O}_\nu^\vee)) - \chi_{\mathcal{D}}(\mathrm{ev}_1^*(\mathcal{O}_\lambda)\cdot \mathrm{ev}_2^*(\mathcal{O}_\mu)\cdot \mathrm{ev}_3^*(\mathcal{O}_\nu^\vee))\,.$$

This definition extends in an obvious manner to give the structure constants for the quantum K-theory of any homogeneous space $Y = G/P$. It is interesting to ask if $\mathrm{ev}_*[\mathcal{O}_{\overline{\mathcal{M}}_{0,3}(Y,d)}] = [\mathcal{O}_{Y\times Y\times Y}] = \mathrm{ev}_*[\mathcal{O}_{\mathcal{D}}]$ for sufficiently large degrees d. This would imply that K-theoretic quantum products are finite.

5.2. **Pieri formula.** Our main result about the quantum K-theory of Grassmannians is a Pieri formula for multiplying with the special classes $\mathcal{O}_i = \mathcal{O}_{(i)}$ given by partitions with a single part. It generalizes Bertram's Pieri formula [2] for the ordinary quantum cohomology $H^*(X)$ as well as Lenart's Pieri formula in ordinary K-theory [41, Thm. 3.4]. Lenart's formula states that $N^{0,\nu}_{i,\lambda}$ is non-zero only if ν/λ is a horizontal strip, in which case

$$N^{\nu,0}_{i,\lambda} = (-1)^{|\nu/\lambda|-i}\binom{r(\nu/\lambda)-1}{|\nu/\lambda|-i} \tag{13}$$

where $r(\nu/\lambda)$ is the number of non-empty rows in the skew diagram ν/λ.

Define the *outer rim* of the partition λ to be the set of boxes in its Young diagram that have no boxes strictly to the South-East. Any product of the form $\mathcal{O}_i \star \mathcal{O}_\lambda$ in $\mathrm{QK}(X)$ is determined by the following theorem combined with (13).

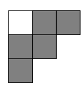
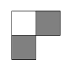
Theorem 5.4. *The constants $N^{\nu,d}_{i,\lambda}$ are zero for $d \geq 2$. Furthermore, $N^{\nu,1}_{i,\lambda}$ is non-zero only if $\ell(\lambda) = m$ and ν can be obtained from λ by removing a subset of the boxes in the outer rim of λ, with at least one box removed from each row. When these conditions hold, we have*

$$N^{\nu,1}_{i,\lambda} = (-1)^e \binom{r}{e}$$

where $e = |\nu| + n - i - |\lambda|$ and r is the number of rows of ν that contain at least one box from the outer rim of λ, excluding the bottom row of this rim.

This result will be proved in the next section.

Example 5.5. On $X = \mathrm{Gr}(3,6)$ we have $N^{(2,1),1}_{2,(3,2,1)} = -2$, since $e = 1$ and $r = 2$. In fact, the partitions $\lambda = (3,2,1)$ and $\nu = (2,1)$ look as follows when the boxes inside the outer rim of λ are shaded:

$\lambda =$  and $\nu =$ 

This gives the negative coefficient of the product $\mathcal{O}_2 \star \mathcal{O}_{3,2,1} = \mathcal{O}_{3,3,2} + q\,\mathcal{O}_2 + q\,\mathcal{O}_{1,1} - 2\,q\,\mathcal{O}_{2,1}$ in $\mathrm{QK}(X)$.

5.3. **Giambelli formula.** As a first application of the Pieri formula we derive a Giambelli formula that expresses K-theoretic quantum Schubert classes as polynomials in the special classes $\mathcal{O}_i$, $1 \leq i \leq k$. Let $c(\nu/\lambda)$ denote the number of non-empty columns of the skew diagram ν/λ. Given a partition μ of length ℓ, we let $\widehat{\mu} = (\mu_1 - 1, \ldots, \mu_\ell - 1)$ be the partition obtained by removing the first column from μ.

Theorem 5.6. *Let a be an integer and μ a partition such that $\mu_1 \leq a \leq k$ and $0 < \ell(\mu) < m$. In the quantum K-theory ring $\mathrm{QK}(X)$ we have*

$$\mathcal{O}_{a,\mu} = \sum_{p \geq a, \nu \subset \mu} (-1)^{|\mu/\nu|} \binom{p - a - 1 + c(\nu/\widehat{\mu})}{p - a - |\mu/\nu|} \mathcal{O}_p \star \mathcal{O}_\nu \,,$$

where the sum is over all integers $p \geq a$ and partitions ν contained in μ such that μ/ν is a vertical strip.

Proof. Let $\mathcal{G}_\lambda$ denote the stable Grothendieck polynomial for the partition λ, see [7]. The K-theoretic Jacobi-Trudi formula of [6, Thm. 6.1] states that

$$\mathcal{G}_{a,\mu} = \mathcal{G}_a \cdot \mathcal{G}_\mu + \sum_{s \geq 1, t \geq 0} (-1)^s \binom{s - 1 + t}{t} \mathcal{G}_{a+s+t} \cdot \mathcal{G}_{\mu /\!\!/ (1^s)} \,,$$

where $\mathcal{G}_{\mu /\!\!/ (1^s)} = \sum_\nu d^\mu_{(1^s),\nu} \mathcal{G}_\nu$ is defined in terms of the coproduct coefficients of the ring of stable Grothendieck polynomials. According to [7, Cor. 7.1] we have

$$\mathcal{G}_{\mu /\!\!/ (1^s)} = \sum_\nu (-1)^{s - |\mu/\nu|} \binom{c(\nu/\widehat{\mu})}{s - |\mu/\nu|} \mathcal{G}_\nu$$

where the sum is over all partitions $\nu \subset \mu$ such that μ/ν is a vertical strip, and $c(\nu/\widehat{\mu})$ is the number of non-empty columns in the skew diagram $\nu/\widehat{\mu}$.

By using the identity $\sum_{t\geq 0}\binom{a}{t}\binom{b}{c-t} = \binom{a+b}{c}$, these formulas combine to give the identity

$$\mathcal{G}_{a,\mu} = \sum_{p\geq a,\nu\subset\mu} (-1)^{|\mu/\nu|}\binom{p-a-1+c(\nu/\widehat{\mu})}{p-a-|\mu/\nu|}\mathcal{G}_p\cdot\mathcal{G}_\nu\,, \tag{14}$$

where the sum is over all integers $p \geq a$ and partitions ν contained in μ such that μ/ν is a vertical strip.

Since the stable Grothendieck polynomials represent K-theoretic Schubert classes on Grassmannians, we may replace each stable Grothendieck polynomial $\mathcal{G}_\lambda$ in (14) by the corresponding class $\mathcal{O}_\lambda$ in $K(X)$. Finally, since the partitions ν in (14) satisfy $\ell(\nu) \leq \ell(\mu) \leq m-1$, it follows from Theorem 5.4 that the ordinary K-theory product $\mathcal{O}_p\cdot\mathcal{O}_\nu$ agrees with the quantum product $\mathcal{O}_p\star\mathcal{O}_\nu$. The theorem follows from this. □

Corollary 5.7. *The class $\mathcal{O}_\lambda \in \mathrm{QK}(X)$ can be expressed as a polynomial $P_\lambda = P_\lambda(\mathcal{O}_1,\dots,\mathcal{O}_k)$ in the special classes $\mathcal{O}_i$. The coefficients of this polynomial are integers, and each monomial involves at most $\ell(\lambda)$ special classes.*

Proof. The polynomials P_λ can be defined by induction on $\ell(\lambda)$ by setting $P_i = \mathcal{O}_i$ for $1 \leq i \leq k$ and

$$P_\lambda = \sum_{p,\nu}(-1)^{|\mu/\nu|}\binom{p-\lambda_1-1+c(\nu/\widehat{\mu})}{p-\lambda_1-|\mu/\nu|}\,\mathcal{O}_p\,P_\nu$$

when $\ell(\lambda)\geq 2$. The sum is over $\lambda_1 \leq p \leq k$ and partitions ν contained in $\mu = (\lambda_2,\dots,\lambda_m)$ such that μ/ν is a vertical strip. Notice that $\ell(\nu) < \ell(\lambda)$, so the polynomials P_ν have already been defined. □

By using the polynomials P_λ, it becomes straightforward to compute the structure constants $N^{\nu,d}_{\lambda,\mu}$ of $\mathrm{QK}(X)$. In fact, we have $\mathcal{O}_\lambda\star\mathcal{O}_\mu = P_\lambda(\mathcal{O}_1,\dots,\mathcal{O}_k)\star\mathcal{O}_\mu$, and the latter product can be computed by letting P_λ act on $\mathcal{O}_\mu$, with the action determined by the Pieri formula of Theorem 5.4. Since multiplication by a single special class can result in at most the first power of q, we deduce that all exponents of q in the product $\mathcal{O}_\lambda\star\mathcal{O}_\mu$ are smaller than or equal to $\ell(\lambda)$.

Corollary 5.8. *The structure constant $N^{\nu,d}_{\lambda,\mu}$ is zero when $d > \ell(\lambda)$.*

The definition of the constants $N^{\nu,d}_{\lambda,\mu}$ in terms of Gromov-Witten invariants can be turned around to give the identity

$$I_d(\mathcal{O}_\lambda,\mathcal{O}_\mu,\mathcal{O}_\nu^\vee) \;=\; \sum_{\kappa,0\leq e\leq d} N^{\kappa,d-e}_{\lambda,\mu}\, I_e(\mathcal{O}_\kappa,\mathcal{O}_\nu^\vee)\,. \tag{15}$$

Here we have used the convention that a two-point Gromov-Witten invariant of degree zero is defined by the Poincaré pairing $I_0(\alpha_1,\alpha_2) = \chi_X(\alpha_1\cdot\alpha_2)$. By linearity we similarly have that

$$I_d(\mathcal{O}_\lambda,\mathcal{O}_\mu,\mathcal{O}_\nu) \;=\; \sum_{\kappa,0\leq e\leq d} N^{\kappa,d-e}_{\lambda,\mu}\, I_e(\mathcal{O}_\kappa,\mathcal{O}_\nu)\,. \tag{16}$$

Notice that all the required structure constants can be obtained by computing the single product $\mathcal{O}_\lambda\star\mathcal{O}_\mu$ in $\mathrm{QK}(X)$. Furthermore, it follows from Corollary 6.2

below that each two-point invariant $I_d(\mathcal{O}_\kappa, \mathcal{O}_\nu^\vee)$ is equal to one if ν is obtained from κ by removing its first d rows and columns, and is zero otherwise. By using [4, Thm. 4.2.1] it also follows that $I_d(\mathcal{O}_\kappa, \mathcal{O}_\nu)$ is equal to one if $\kappa_i + \nu_{m+d+1-i} \leq k+d$ for $d < i \leq m$, and is zero otherwise. The identities (15) and (16) therefore give alternative and very practical ways to compute K-theoretic Gromov-Witten invariants on Grassmannians.

Example 5.9. We compute the quantum product $\mathcal{O}_{2,1} \star \mathcal{O}_{2,1}$ on $X = \mathrm{Gr}(2,4)$. The Giambelli formula gives $\mathcal{O}_{2,1} = \mathcal{O}_1 \star \mathcal{O}_2$, so the product can be obtained as $\mathcal{O}_{2,1} \star \mathcal{O}_{2,1} = \mathcal{O}_1 \star (\mathcal{O}_2 \star \mathcal{O}_{2,1}) = \mathcal{O}_1 \star q\,\mathcal{O}_1 = q\,\mathcal{O}_2 + q\,\mathcal{O}_{1,1} - q\,\mathcal{O}_{2,1} \in \mathrm{QK}(X)$. Using (15) we obtain from this that $I_1(\mathcal{O}_{2,1}, \mathcal{O}_{2,1}, \mathcal{O}_{2,1}^\vee) = N_{(2,1),(2,1)}^{(2,1),1} = -1$ and (16) gives $I_1(\mathcal{O}_{2,1}, \mathcal{O}_{2,1}, \mathcal{O}_1) = 1 + 1 - 1 = 1$. The full multiplication table for $\mathrm{QK}(\mathrm{Gr}(2,4))$ looks as follows.

$$
\begin{aligned}
&\mathcal{O}_1 \star \mathcal{O}_1 = \mathcal{O}_{1,1} + \mathcal{O}_2 - \mathcal{O}_{2,1} && \mathcal{O}_{1,1} \star \mathcal{O}_1 = \mathcal{O}_{2,1} \\
&\mathcal{O}_{1,1} \star \mathcal{O}_{1,1} = \mathcal{O}_{2,2} && \mathcal{O}_2 \star \mathcal{O}_1 = \mathcal{O}_{2,1} \\
&\mathcal{O}_2 \star \mathcal{O}_{1,1} = q && \mathcal{O}_2 \star \mathcal{O}_2 = \mathcal{O}_{2,2} \\
&\mathcal{O}_{2,1} \star \mathcal{O}_1 = \mathcal{O}_{2,2} + q - q\,\mathcal{O}_1 && \mathcal{O}_{2,1} \star \mathcal{O}_{1,1} = q\,\mathcal{O}_1 \\
&\mathcal{O}_{2,1} \star \mathcal{O}_2 = q\,\mathcal{O}_1 && \mathcal{O}_{2,1} \star \mathcal{O}_{2,1} = q\,\mathcal{O}_{1,1} + q\,\mathcal{O}_2 - q\,\mathcal{O}_{2,1} \\
&\mathcal{O}_{2,2} \star \mathcal{O}_1 = q\,\mathcal{O}_1 && \mathcal{O}_{2,2} \star \mathcal{O}_{1,1} = q\,\mathcal{O}_2 \\
&\mathcal{O}_{2,2} \star \mathcal{O}_2 = q\,\mathcal{O}_{1,1} && \mathcal{O}_{2,2} \star \mathcal{O}_{2,1} = q\,\mathcal{O}_{2,1} \\
&\mathcal{O}_{2,2} \star \mathcal{O}_{2,2} = q^2 &&
\end{aligned}
$$

We finally pose the following conjecture, which has been verified for all Grassmannians $\mathrm{Gr}(m,n)$ with $n \leq 13$.

Conjecture 5.10. *The structure constants $N_{\lambda,\mu}^{\nu,d}$ have alternating signs in the sense that $(-1)^{|\nu|+nd-|\lambda|-|\mu|}\, N_{\lambda,\mu}^{\nu,d} \geq 0$.*

Lenart and Maeno have posed a similar conjecture for the quantum K-theory of complete flag varieties G/B [42]. Their conjecture is also based on computer evidence, although these computations are based on other conjectures.

We note that the Gromov-Witten invariants $I_d(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu^\vee)$ do not have alternating signs for $d > 0$, although the degree zero invariants $I_0(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu^\vee) = N_{\lambda,\mu}^{\nu,0}$ do have alternating signs [7]. A concrete example on $\mathrm{Gr}(2,4)$ is $I_d(\mathcal{O}_2, \mathcal{O}_2, \mathcal{O}_1^\vee) = I_d(\mathcal{O}_2, \mathcal{O}_{2,1}, \mathcal{O}_1^\vee) = 1$. The invariants $I_d(\mathcal{O}_\lambda, \mathcal{O}_\mu, \mathcal{O}_\nu)$ also do not have predictable signs even for $d = 0$, see [7, §8].

Example 5.11. On $X = \mathrm{Gr}(4,8)$ we have $I_2(\mathcal{O}_{4,3,2,1}, \mathcal{O}_{4,3,2,1}, \mathcal{O}_{4,3,2,1}) = 2$. The corresponding Gromov-Witten variety has dimension 2 and is not rational.

5.4. **Symmetry and duality.** As a further application of our Pieri rule, we prove some nice properties of the ring $\mathrm{QK}(X)$. Our first result shows that the structure constants $N_{\lambda,\mu}^{\nu^\vee,d}$ satisfy S_3-symmetry, i.e. these constants are invariant under arbitrary permutations of the partitions λ, μ, ν. We remark that this symmetry is not at all clear from geometry. For example, the two terms in the expression for the structure constants in Remark 5.3 do not satisfy S_3-symmetry individually.

Theorem 5.12. *For any degree d and partitions λ, μ, ν contained in the $m \times k$ rectangle we have $N_{\lambda,\mu}^{\nu,d} = N_{\lambda,\nu^\vee}^{\mu^\vee,d}$.*

Proof. This identity is immediate from Theorem 5.4 if $\lambda = (p)$ has a single part. In fact, ν is obtained by removing boxes from the outer rim of μ if and only if $\mu^\vee$ is obtained by removing boxes from the outer rim of $\nu^\vee$; and ν contains a box from the i-th row of the outer rim of μ if and only if $\mu^\vee$ contains a box from the $(m-i)$-th row of the outer rim of $\nu^\vee$ $(1 \leq i \leq m-1)$.

For a partition ν and class $\alpha \in \mathrm{QK}(X)$, let $\langle \alpha, \mathcal{O}_\nu \rangle \in \mathbb{Z}[\![q]\!]$ denote the coefficient of $\mathcal{O}_\nu$ in the $\mathbb{Z}[\![q]\!]$-linear expansion of α. It is enough to show that the set

$$S = \{\alpha \in \mathrm{QK}(X) \mid \langle \alpha \star \mathcal{O}_\mu, \mathcal{O}_\nu \rangle = \langle \alpha \star \mathcal{O}_{\nu^\vee}, \mathcal{O}_{\mu^\vee} \rangle \text{ for all partitions } \mu \text{ and } \nu\}$$

is equal to $\mathrm{QK}(X)$. This follows because S is a $\mathbb{Z}[\![q]\!]$-submodule of $\mathrm{QK}(X)$ that contains the special classes $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and is closed under multiplication. In fact, for $\alpha_1, \alpha_2 \in S$ we have $\langle \alpha_1 \star \alpha_2 \star \mathcal{O}_\mu, \mathcal{O}_\nu \rangle = \sum_\lambda \langle \alpha_1 \star \mathcal{O}_\mu, \mathcal{O}_\lambda \rangle \langle \alpha_2 \star \mathcal{O}_\lambda, \mathcal{O}_\nu \rangle = \sum_\lambda \langle \alpha_2 \star \mathcal{O}_{\nu^\vee}, \mathcal{O}_{\lambda^\vee} \rangle \langle \alpha_1 \star \mathcal{O}_{\lambda^\vee}, \mathcal{O}_{\mu^\vee} \rangle = \langle \alpha_2 \star \alpha_1 \star \mathcal{O}_{\nu^\vee}, \mathcal{O}_{\mu^\vee} \rangle$, so $\alpha_1 \star \alpha_2 \in S$. □

A skew diagram ν/λ is called a *rook strip* if each row and column contains at most one box. We need the following special case of the Pieri rule.

Lemma 5.13. *For any partition μ contained in the $m \times k$ rectangle we have*

$$\mathcal{O}_1 \star \mathcal{O}_\mu = \sum_\lambda (-1)^{|\lambda/\mu|} \mathcal{O}_\lambda + q \sum_\nu (-1)^{|\nu/\widehat{\widehat{\mu}}|} \mathcal{O}_\nu \,.$$

Here $\widehat{\widehat{\mu}}$ denotes the result of removing the first row and first column from μ. The first sum is over all partitions $\lambda \supsetneq \mu$ for which λ/μ is a rook strip. The second sum is empty unless $\mu_1 = k$ and $\ell(\mu) = m$, in which case it includes all partitions $\nu \supset \widehat{\widehat{\mu}}$ for which $\nu_1 = k-1$, $\ell(\nu) = m-1$, and $\nu/\widehat{\widehat{\mu}}$ is a rook strip.

Let $\chi^q : \mathrm{QK}(X) \to \mathbb{Z}[\![q]\!]$ denote the $\mathbb{Z}[\![q]\!]$-linear extension of the Euler characteristic, which maps each Schubert class $\mathcal{O}_\lambda$ to 1. Define the element $t_q = \frac{1-\mathcal{O}_1}{1-q} \in \mathrm{QK}(X)$. We finally show that $t_q \star \mathcal{O}_{\lambda^\vee}$ is the $\mathbb{Z}[\![q]\!]$-linear dual basis element of $\mathcal{O}_\lambda$.

Theorem 5.14. *For any partitions λ and ν contained in the $m \times k$ rectangle we have $\chi^q(\mathcal{O}_\lambda \star t_q \star \mathcal{O}_{\nu^\vee}) = \delta_{\lambda,\nu}$.*

Proof. Let $R = (k)^m$ denote the $m \times k$ rectangle considered as a partition. It follows from Lemma 5.13 that $\chi^q((1-\mathcal{O}_1) \star \mathcal{O}_\lambda) = (1-q)\, \delta_{\lambda,R}$. We deduce that

$$\begin{aligned}\chi^q((1-\mathcal{O}_1) \star \mathcal{O}_\lambda \star \mathcal{O}_{\nu^\vee}) &= \sum_{\mu,d} N^{\mu,d}_{\lambda,\nu^\vee}\, q^d\, \chi^q((1-\mathcal{O}_1) \star \mathcal{O}_\mu) = \sum_d N^{R,d}_{\lambda,\nu^\vee}\, q^d\, (1-q) \\ &= \sum_d N^{\nu,d}_{\lambda,\emptyset}\, q^d\, (1-q) = \delta_{\lambda,\nu}\, (1-q)\,,\end{aligned}$$

as required. The third equality follows from Theorem 5.12. □

It follows from Theorem 5.14 that the structure constants of $\mathrm{QK}(X)$ can be expressed in the following form, which makes the S_3-symmetry apparent:

$$\sum_{d \geq 0} N^{\nu^\vee,d}_{\lambda,\mu}\, q^d \;=\; \chi^q(t_q \star \mathcal{O}_\lambda \star \mathcal{O}_\mu \star \mathcal{O}_\nu)\,. \tag{17}$$

Example 5.15. On $X = \mathrm{Gr}(5,10)$ we have $\chi^q(t_q \star \mathcal{O}^3_{(5,4,3,2,1)}) = 14\,q^2 + q^3$, so the sum (17) may have more than one non-zero term. However, the sum is always finite by Corollary 5.8.

In the special case of ordinary K-theory, the S_3-symmetry of the structure constants $N^{\nu^\vee,0}_{\lambda,\mu}$ follows from a Puzzle version of the Littlewood-Richardson rule for $K(X)$ [54, Thm. 4.6]. The $q=0$ case of (17) was proved in [9, §2], which provides an alternative proof. Theorem 5.14 shows that the phenomenon that the Schubert basis of $K(X)$ can be dualized by multiplying all structure sheaves with a constant element carries over to quantum K-theory. It would be interesting to find an explanation in terms of generating functions. The S_3-symmetry of Theorem 5.12 might hint towards the existence of a puzzle rule for the structure constants $N^{\nu,d}_{\lambda,\mu}$ of $\mathrm{QK}(X)$, but so far we have not been able to find a working set of puzzle pieces.

5.5. **Equivariant quantum K-theory of $\mathbb{P}^1$ and $\mathbb{P}^2$.** Let $T \subset \mathrm{GL}_n$ be the torus of diagonal matrices, and let $\mathcal{O}_\lambda \in K_T(X)$ denote the equivariant class of the Schubert variety X_λ in $X = \mathrm{Gr}(m,n)$ relative to the standard T-stable flag $F_\bullet$ defined by $F_i = \mathbb{C}^i \oplus 0^{n-i}$. The T-equivariant quantum ring $\mathrm{QK}_T(X)$ is obtained by using equivariant Gromov-Witten invariants in the definition (11) of the structure constants $N^{\nu,d}_{\lambda,\mu}$, where $\mathcal{O}_\lambda^\vee$ denotes the equivariant Poincaré dual class of $\mathcal{O}_\lambda$. We include here the multiplication tables for the equivariant quantum K-theory rings of $\mathbb{P}^1 = \mathrm{Gr}(1,2)$ and $\mathbb{P}^2 = \mathrm{Gr}(1,3)$. By Theorem 4.2, the required Gromov-Witten invariants can be computed in $K_T(\text{point})$, $K_T(\mathbb{P}^1)$, and $K_T(\mathbb{P}^2)$ (see also Theorem 6.1 below). We note that Graham and Kumar have given explicit formulas for multiplication in $K_T(\mathbb{P}^n)$ [25, §6.3]. Let $\varepsilon_i : T \to \mathbb{C}^*$ be the character defined by $\varepsilon_i(t_1,\dots,t_n) = t_i$, and write $e^{\varepsilon_i} = [\mathbb{C}_{\varepsilon_i}] \in K_T(\text{point})$.

The multiplicative structure of $\mathrm{QK}_T(\mathbb{P}^1)$ is determined by:

$$\mathcal{O}_1 \star \mathcal{O}_1 \;=\; (1-e^{\varepsilon_1-\varepsilon_2})\,\mathcal{O}_1 + e^{\varepsilon_1-\varepsilon_2}\,q$$

And the multiplicative structure of $\mathrm{QK}_T(\mathbb{P}^2)$ is determined by:

$$\begin{aligned}
\mathcal{O}_1 \star \mathcal{O}_1 \;&=\; (1-e^{\varepsilon_2-\varepsilon_3})\,\mathcal{O}_1 + e^{\varepsilon_2-\varepsilon_3}\,\mathcal{O}_2\\
\mathcal{O}_1 \star \mathcal{O}_2 \;&=\; e^{\varepsilon_1-\varepsilon_3}\,q + (1-e^{\varepsilon_1-\varepsilon_3})\,\mathcal{O}_2\\
\mathcal{O}_2 \star \mathcal{O}_2 \;&=\; (1-e^{\varepsilon_1-\varepsilon_2})\,(1-e^{\varepsilon_1-\varepsilon_3})\,\mathcal{O}_2 + e^{\varepsilon_1-\varepsilon_3}(1-e^{\varepsilon_1-\varepsilon_2})\,q + e^{\varepsilon_1-\varepsilon_2}\,q\,\mathcal{O}_1
\end{aligned}$$

Notice that the structure constants $N^{\nu,d}_{\lambda,\mu}$ appearing in these examples satisfy Griffeth-Ram positivity [26] in the sense that

$$(-1)^{|\nu|+nd-|\lambda|-|\mu|}\,N^{\nu,d}_{\lambda,\mu} \;\in\; \mathbb{N}[e^{\varepsilon_1-\varepsilon_2}-1,\dots,e^{\varepsilon_{n-1}-\varepsilon_n}-1]\,.$$

Using Theorem 4.2, we have verified that this positivity also holds for all Grassmannians $\mathrm{Gr}(m,n)$ with $n \leq 5$, and it is natural to conjecture that it holds in general. In the case of ordinary equivariant K-theory, this has been proved in [1].

6. Proof of the Pieri formula

6.1. **Special Gromov-Witten invariants.** To prove the Pieri formula, we start by establishing a formula for certain special Gromov-Witten invariants on Grassmannians. Fix a degree $d \geq 0$. As usual we set $a = \max(m-d,0)$, $b = \min(a+d,n)$, $Y_d = \mathrm{Fl}(a,b;n)$, and $Z_d = \mathrm{Fl}(a,m,b;n)$. We also define $X_d = \mathrm{Gr}(b,n)$. The following commutative diagram was exploited earlier to obtain a quantum Pieri formula for submaximal orthogonal Grassmannians [11]. It was also applied to Grassmannians of type A in [52]. All maps in the diagram are projections.

$$\begin{array}{ccccc} Z_d & \xrightarrow{p_1} & \mathrm{Fl}(m,b;n) & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} & & \\ Y_d & \xrightarrow{\pi} & X_d & & \end{array}$$

Given a partition λ, let $\widehat{\lambda}$ denote the partition obtained by removing the first d columns, i.e. $\widehat{\lambda}_i = \max(\lambda_i - d, 0)$. If X_λ is a Schubert variety in X, then $q'(p_2^{-1}(X_\lambda))$ is the Schubert variety in X_d defined by $\widehat{\lambda}$. It follows from Lemma 3.6 that $q'_* p_2^*(\mathcal{O}_\lambda) = \mathcal{O}_{\widehat{\lambda}} \in K(X_d)$. Similarly, if λ is a partition contained in the $b \times (n-b)$ rectangle, then $p_{2*} {q'}^*(\mathcal{O}_\lambda) = \mathcal{O}_{\overline{\lambda}}$, where $\overline{\lambda} = (\lambda_{d+1}, \ldots, \lambda_b)$ is the partition obtained by removing the top d rows of λ. We will occasionally write $\widehat{\lambda}(d) = \widehat{\lambda}$ and $\overline{\lambda}(d) = \overline{\lambda}$ to avoid ambiguity about how many rows or columns to remove.

Theorem 6.1. *Let λ be a partition with $\ell(\lambda) \le d$. For any classes $\alpha_1, \alpha_2 \in K(X)$ we have*

$$I_d(\mathcal{O}_\lambda, \alpha_1, \alpha_2) = \chi_{X_d}(\mathcal{O}_{\widehat{\lambda}(d)} \cdot q'_* p_2^*(\alpha_1) \cdot q'_* p_2^*(\alpha_2)) \,.$$

Proof. If $\Omega \subset \mathrm{Fl}(m,b;n)$ is any subset, then $p_1(q^{-1}(q(p_1^{-1}(\Omega)))) = q'^{-1}(q'(\Omega))$. By taking Ω to be a Schubert variety, it follows from Lemma 3.6 that $p_{1*} q^* q_* p_1^* [\mathcal{O}_\Omega] = {q'}^* q'_* [\mathcal{O}_\Omega]$. We deduce that for arbitrary classes $\beta_1, \beta_2 \in K(\mathrm{Fl}(m,b;n))$ we have

$$\begin{aligned} &\pi_*(q_* p_1^*(\beta_1) \cdot q_* p_1^*(\beta_2)) = \pi_* q_*(q^* q_* p_1^*(\beta_1) \cdot p_1^*(\beta_2)) = q'_* p_{1*}(q^* q_* p_1^*(\beta_1) \cdot p_1^*(\beta_2)) \\ &= q'_*(p_{1*} q^* q_* p_1^*(\beta_1) \cdot \beta_2) = q'_*({q'}^* q'_*(\beta_1) \cdot \beta_2) = q'_*(\beta_1) \cdot q'_*(\beta_2) \ \in K(X_d) \,. \end{aligned}$$

Since $\ell(\lambda) \le d$, it follows by checking Schubert conditions that $q(p^{-1}(X_\lambda)) = \pi^{-1}(X_{\widehat{\lambda}})$, where $p = p_2 p_1$ and the Schubert variety $X_{\widehat{\lambda}}$ lives in the Grassmannian X_d. This implies that $q_* p^*(\mathcal{O}_\lambda) = \pi^*(\mathcal{O}_{\widehat{\lambda}})$. We deduce from Theorem 4.2 that

$$\begin{aligned} I_d(\mathcal{O}_\lambda, \alpha_1, \alpha_2) &= \chi_{Y_d}(q_* p^*(\mathcal{O}_\lambda) \cdot q_* p^*(\alpha_1) \cdot q_* p^*(\alpha_2)) \\ &= \chi_{Y_d}(\pi^*(\mathcal{O}_{\widehat{\lambda}}) \cdot q_* p^*(\alpha_1) \cdot q_* p^*(\alpha_2)) \\ &= \chi_{X_d}(\mathcal{O}_{\widehat{\lambda}} \cdot q'_* p_2^*(\alpha_1) \cdot q'_* p_2^*(\alpha_2)) \end{aligned}$$

as required. □

Let $\widehat{\overline{\lambda}}(d)$ denote the result of removing the first d rows and the first d columns from λ.

Corollary 6.2. *For any partition μ contained in the $m \times k$ rectangle and any class $\alpha \in K(X)$ we have $I_d(\mathcal{O}_\mu, \alpha) = \chi_X(\mathcal{O}_{\widehat{\overline{\mu}}(d)}, \alpha)$.*

Proof. By taking λ to be the empty partition, we obtain

$$\begin{aligned} I_d(\mathcal{O}_\mu, \alpha) &= \chi_{X_d}(q'_* p_2^*(\mathcal{O}_\mu) \cdot q'_* p_2^*(\alpha)) = \chi_{X_d}(\mathcal{O}_{\widehat{\mu}} \cdot q'_* p_2^*(\alpha)) \\ &= \chi_{\mathrm{Fl}(m,b;n)}({q'}^*(\mathcal{O}_{\widehat{\mu}}) \cdot p_2^*(\alpha)) = \chi_X(p_{2*} {q'}^*(\mathcal{O}_{\widehat{\mu}}) \cdot \alpha) = \chi_X(\mathcal{O}_{\widehat{\overline{\mu}}} \cdot \alpha) \end{aligned}$$

as claimed. □

We note that Theorem 6.1 and Corollary 6.2 have straightforward generalizations to K-equivariant Gromov-Witten invariants, with the same proofs.

6.2. **Pieri coefficients.** In the following we set $\mathcal{O}_i = 1 \in K(X)$ for $i \leq 0$. For the statement of the next lemma we need to remark that the structure constants $N^{\nu,0}_{\lambda,\mu}$ of degree zero are independent of the Grassmannian on which they are defined: they appear in the multiplication of stable Grothendieck polynomials $\mathcal{G}_\lambda \cdot \mathcal{G}_\mu = \sum N^{\nu,0}_{\lambda,\mu} \mathcal{G}_\nu$. In particular, these coefficients are well defined when the partitions λ, μ, and ν are not contained in the $m \times k$ rectangle.

Lemma 6.3. *Let λ be a partition contained in the $m \times k$ rectangle and $0 \leq i \leq m$. Then we have*

$$\mathcal{O}_{i-d} \cdot \mathcal{O}_{\widehat{\lambda}(d)} - q'_* p_2^*(\mathcal{O}_i \cdot \mathcal{O}_\lambda) = \sum_{\ell(\mu)=m+1} N^{\mu,0}_{i,\lambda} \, \mathcal{O}_{\widehat{\mu}(d)}$$

in $K(X_d)$, where the sum is over all partitions μ with exactly $m+1$ rows and at most k columns.

Proof. By using that $\mathrm{Fl}(m, b; n)$ is a Grassmann bundle over X_d, this is a special case of [6, Cor. 7.4], cf. [7, §8]. Notice that Lenart's Pieri rule (13) implies that $N^{\mu,0}_{i,\lambda}$ is zero whenever $\ell(\mu) \geq m+2$. □

Corollary 6.4. *Let λ be contained in the $m \times k$ rectangle, $0 \leq i \leq m$, and $\alpha \in K(X)$. Then we have*

$$I_d(\mathcal{O}_i, \mathcal{O}_\lambda, \alpha) - I_d(\mathcal{O}_i \cdot \mathcal{O}_\lambda, \alpha) = \sum_{\ell(\mu)=m+1} N^{\mu,0}_{i,\lambda} \, \chi_X(\mathcal{O}_{\widehat{\widehat{\mu}}(d)} \cdot \alpha) \,.$$

Proof. By Theorem 6.1 and Lemma 6.3 we have

$$\begin{aligned} & I_d(\mathcal{O}_i, \mathcal{O}_\lambda, \alpha) - I_d(\mathcal{O}_i \cdot \mathcal{O}_\lambda, \alpha) \\ & = \chi_{X_d}(\mathcal{O}_{i-d} \cdot \mathcal{O}_{\widehat{\lambda}} \cdot q'_* p_2^*(\alpha)) - \chi_{X_d}(q'_* p_2^*(\mathcal{O}_i \cdot \mathcal{O}_\lambda) \cdot q'_* p_2^*(\alpha)) \\ & = \sum_{\ell(\mu)=m+1} N^{\mu,0}_{i,\lambda} \, \chi_{X_d}(\mathcal{O}_{\widehat{\mu}(d)} \cdot q'_* p_2^*(\alpha)) \,. \end{aligned}$$

Finally, the projection formula implies that $\chi_{X_d}(\mathcal{O}_{\widehat{\mu}(d)} \cdot q'_* p_2^*(\alpha)) = \chi_X(\mathcal{O}_{\widehat{\widehat{\mu}}(d)} \cdot \alpha)$. □

Proof of Theorem 5.4. The Pieri coefficients of degree one are given by

$$\begin{aligned} N^{\nu,1}_{i,\lambda} & = I_1(\mathcal{O}_i, \mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) - \sum_\kappa N^{\kappa,0}_{i,\lambda} \, I_1(\mathcal{O}_\kappa, \mathcal{O}_\nu^\vee) \\ & = I_1(\mathcal{O}_i, \mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) - I_1(\mathcal{O}_i \cdot \mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) \\ & = \sum_{\ell(\mu)=m+1} N^{\mu,0}_{i,\lambda} \, \chi_X(\mathcal{O}_{\widehat{\widehat{\mu}}(1)} \cdot \mathcal{O}_\nu^\vee) = \sum_{j=\nu_1+1}^{k} N^{(j,\nu+1^m),0}_{i,\lambda} \,. \end{aligned} \tag{18}$$

Here we used Cor. 6.4 and the Poincaré duality. Notice that this implies that

$$\sum_\nu N^{\nu,1}_{i,\lambda} \mathcal{O}_\nu = \sum_{\ell(\mu)=m+1} N^{\mu,0}_{i,\lambda} \, \mathcal{O}_{\widehat{\widehat{\mu}}(1)} \,.$$

It follows that for any degree $d \geq 0$ we have

$$\begin{aligned} I_d(\mathcal{O}_i, \mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) - I_d(\mathcal{O}_i \cdot \mathcal{O}_\lambda, \mathcal{O}_\nu^\vee) &= \sum_{\ell(\mu)=m+1} N_{i,\lambda}^{\mu,0}\, \chi_X(\mathcal{O}_{\widehat{\overline{\mu}}(d)} \cdot \mathcal{O}_\nu^\vee) \\ &= \sum_\kappa N_{i,\lambda}^{\kappa,1}\, \chi_X(\mathcal{O}_{\widehat{\overline{\kappa}}(d-1)}, \mathcal{O}_\nu^\vee) \\ &= \sum_\kappa N_{i,\lambda}^{\kappa,1}\, I_{d-1}(\mathcal{O}_\kappa, \mathcal{O}_\nu^\vee)\,. \end{aligned}$$

This identity implies that $N_{i,\lambda}^{\nu,d} = 0$ for $d \geq 2$ by induction on d.

We finally prove that the constants $N_{i,\lambda}^{\nu,1}$ of degree one are given by the signed binomial coefficients of Theorem 5.4. Define a *marked horizontal strip* for the pair (i, λ) to be a horizontal strip D of some shape μ/λ, for which $|D| - i$ of the non-empty rows of D are marked, excluding the bottom row. Then Lenart's Pieri formula (13) states that

$$N_{i,\lambda}^{\mu,0} = \sum_{D:\, \mathrm{shape}(D)=\mu/\lambda} (-1)^{|D|-i}$$

and we obtain from (18) that

$$N_{i,\lambda}^{\nu,1} = \sum_D (-1)^{|D|-i} \tag{19}$$

where this sum is over all marked horizontal strips D for (i, λ) of some shape $(j, \nu + 1^m)/\lambda$ with $\nu_1 + 1 \leq j \leq k$.

It turns out that the sum (19) does not change if we include only marked horizontal strips D of shape $(k, \nu + 1^m)/\lambda$ such that the top row of D is not marked. This follows from the sign reversing involution that sends any $D = (j, \nu + 1^m)/\lambda$ with $j < k$ and the top row unmarked to $D' = (j + 1, \nu + 1^m)/\lambda$ with the top row marked; and which sends $D' = (j, \nu + 1^m)/\lambda$ with the top row marked to $D = (j - 1, \nu + 1^m)/\lambda$ with the top row unmarked. Note that if the top row of $(j, \nu + 1^m)$ is marked, then this row is not empty and $j > \nu_1 + 1$.

We finally notice that $(k, \nu+1^m)/\lambda$ is a horizontal strip if and only if ν is obtained from λ by removing a subset of the boxes in the outer rim of λ, with at least one box removed in each of the m rows. And the i-th row of $(k, \nu+1^m)/\lambda$ is non-empty if and only if the $(i-1)$-st row of ν contains a box from the outer rim of λ. □

7. Gromov-Witten invariants of cominuscule varieties

In this last section we generalize our formula for Grassmannian Gromov-Witten invariants to work for all cominuscule homogeneous spaces, with the exception that K-theoretic invariants can be computed for "*small*" degrees only. The computation of Gromov-Witten invariants in Theorem 4.2 extends almost verbatim to Lagrangian and maximal orthogonal Grassmannians by using a case by case analysis as in [10]. However, we will utilize here the unified approach of Chaput, Manivel, and Perrin [12], which makes it possible to state and prove our result in a type independent manner.

A cominuscule variety is a homogeneous space $X = G/P$, where G is a simple complex linear algebraic group and $P \subset G$ is a parabolic subgroup corresponding to a cominuscule simple root α. The latter means that when the highest root is expressed as a linear combination of simple roots, the coefficient of α is one.

Since P is maximal we have $H_2(X) \cong \mathbb{Z}$, so the degree of a stable map to X can be identified with a non-negative integer. The family of cominuscule varieties include Grassmannians of type A, Lagrangian Grassmannians $\mathrm{LG}(m, 2m)$, maximal orthogonal Grassmannians $\mathrm{OG}(m, 2m)$, quadric hypersurfaces $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$, as well as two exceptional varieties called the Cayley plane and the Freudenthal variety. All minuscule varieties are also cominuscule. We refer to [10, 12] for more details.

Given two points $x, y \in X$, we let $d(x, y)$ denote the smallest possible degree of a stable map $f : C \to X$ with $x, y \in f(C)$. This definition of F. L. Zak [59] gives X the structure of a metric space. Let $X(d)_{x,y}$ denote the union of the images of all such stable maps f of degree $d = d(x, y)$:

$$X(d)_{x,y} \;=\; \bigcup_{\deg(f)=d\,;\,x,y\in f(C)} f(C) \;\subset\; X\,.$$

It was proved in [12] that this set is a Schubert variety in X. Write $X(d)$ for the abstract variety defined by $X(d)_{x,y}$. Let Y_d be a set parametrizing all varieties $X(d)_{x,y}$ for $x, y \in X$ with $d(x, y) = d$, and let G act on this set by translation. For an element $\omega \in Y_d$ we let $X_\omega \subset X$ denote the corresponding variety. We also set $d_{\max} = d_{\max}(X) = \max\{d(x, y) \mid x, y \in X\}$. A root theoretic interpretation of this number can be found in [12, Def. 3.15]. A degree d is *small* if $d \leq d_{\max}$. The following statement combines Prop. 3.16, Prop. 3.17, and Fact 3.18 of [12].

Proposition 7.1 (Chaput, Manivel, Perrin)**.** *Let $d \leq d_{\max}$ be a small degree.*

(a) *The metric $d(x, y)$ attains all values between 0 and $d_{\max}$.*
(b) *G acts transitively on the set of pairs $(x, y) \in X \times X$ with $d(x, y) = d$.*
(c) *Let $\omega \in Y_d$. The stabilizer $G_\omega \subset G$ of X_ω is a parabolic subgroup of G that acts transitively on X_ω.*
(d) *Given a stable map $f : C \to X$ of degree d, there exists a point $\omega \in Y_d$ such that $f(C) \subset X_\omega$. If $f \in \overline{\mathcal{M}}_{0,3}(X, d)$ is a general point, then ω is uniquely determined.*
(e) *Let $\omega \in Y_d$ and let $x, y, z \in X_\omega$ be three general points. Then there exists a unique stable map $f : C \to X$ of degree d that sends the three marked points to x, y, and z. Furthermore we have $f(C) \subset X_\omega$.*

Parts (b) and (c) of this proposition imply that Y_d is a homogeneous G-variety. The idea of Chaput, Manivel, and Perrin's construction is that the kernel-span pairs known from the classical types are replaced by points in the variety Y_d, and the condition that an m-plane in $\mathrm{Gr}(m, n)$ lies between a given kernel-span pair ω is replaced with the condition that a point of X belongs to X_ω. For the cominuscule Grassmannians of types A, C, and D, the varieties Y_d and $X(d)$ are given in the following table; a complete list can be found in [12, Prop. 3.16].

X	$d_{\max}$	Y_d	$X(d)$
$\mathrm{Gr}(m, n)$	$\min(m, n-m)$	$\mathrm{Fl}(m-d, m+d; n)$	$\mathrm{Gr}(d, 2d)$
$\mathrm{LG}(n, 2n)$	n	$\mathrm{IG}(n-d, 2n)$	$\mathrm{LG}(d, 2d)$
$\mathrm{OG}(n, 2n)$	$\lfloor \frac{n}{2} \rfloor$	$\mathrm{OG}(n-2d, 2n)$	$\mathrm{OG}(2d, 4d)$

Define the incidence variety $Z_d = \{(\omega, x) \in Y_d \times X \mid x \in X_\omega\}$. It follows from part (c) of the proposition that the diagonal action of G on this set is transitive. Furthermore, since X_ω is a Schubert variety and all Borel subgroups in G are conjugate, one can choose $\omega \in Y_d$ such that the G_ω and P both contain a common

Borel subgroup of G. Since this Borel subgroup will be contained in the stabilizer of the point $(\omega_0, P/P) \in Z_d$, it follows that Z_d is also a homogeneous space for G. We also need the varieties $M_d = \overline{\mathcal{M}}_{0,3}(X,d)$, $\mathrm{B}\ell_d = \{(\omega, f) \in Y_d \times M_d \mid \mathrm{Im}(f) \subset X_\omega\}$, and $Z_d^{(3)} = \{(\omega, x_1, x_2, x_3) \in Y_d \times X^3 \mid x_i \in X_\omega \text{ for } 1 \leq i \leq 3\}$. These spaces define a generalization of the diagram (8), where the maps π, ϕ, ev_i, e_i, p, and q are as before defined using evaluation maps and projections. If $T \subset G$ is a maximal torus, then all of these maps are T-equivariant. Part (d) of Proposition 7.1 implies that π is birational, and part (e) implies that ϕ is birational. The following theorem is proved exactly as Theorem 4.2.

Theorem 7.2. *Let $d \leq d_{\max}$ be a small degree for the cominuscule variety X, and let $\alpha_1, \alpha_2, \alpha_3 \in K^T(X)$. Then*

$$I_d^T(\alpha_1, \alpha_2, \alpha_3) = \chi_{Y_d}^T\left(q_* p^*(\alpha_1) \cdot q_* p^*(\alpha_2) \cdot q_* p^*(\alpha_3)\right).$$

It would be possible to compute K-theoretic Gromov-Witten invariants of large degrees if the following is true.

Conjecture 7.3. *Let X be a cominuscule variety that is not a Grassmannian of type A and let $d > d_{\max}$. If x_1, x_2, x_3 are general points in X, then the Gromov-Witten variety $GW_d(x_1, x_2, x_3)$ is rational.*

In fact, if $\mathrm{ev} : M_d \to X^3$ is the total evaluation map, then any Gromov-Witten invariant can be written as

$$\begin{aligned} I_d^T(\alpha_1, \alpha_2, \alpha_3) &= \chi_{M_d}^T(\mathrm{ev}_1^*(\alpha_1) \cdot \mathrm{ev}_2^*(\alpha_2) \cdot \mathrm{ev}_3^*(\alpha_3)) \\ &= \chi_{X^3}^T(e_1^*(\alpha_1) \cdot e_2^*(\alpha_2) \cdot e_3^*(\alpha_3) \cdot \mathrm{ev}_*[\mathcal{O}_{M_d}]) \end{aligned}$$

where $e_i : X^3 \to X$ is the i-th projection. If Conjecture 7.3 is true, then Theorem 3.1 implies that $\mathrm{ev}_*[\mathcal{O}_{M_d}] = [\mathcal{O}_{X^3}]$, and using Lemma 3.5 we obtain:

Consequence 7.4. *If X and d are as in Conjecture 7.3 and $\alpha_1, \alpha_2, \alpha_3 \in K^T(X)$, then $I_d^T(\alpha_1, \alpha_2, \alpha_3) = \chi_X^T(\alpha_1) \cdot \chi_X^T(\alpha_2) \cdot \chi_X^T(\alpha_3)$.*

In particular, it would follow that $I_d^T([\mathcal{O}_{\Omega_1}], [\mathcal{O}_{\Omega_2}], [\mathcal{O}_{\Omega_3}]) = 1$ for all T-stable Schubert varieties $\Omega_1, \Omega_2, \Omega_3$ and $d > d_{\max}$. As mentioned in Remark 2.4, we can prove that the 3-point Gromov-Witten varieties for maximal orthogonal Grassmannians are unirational, which suffices to establish Consequences 7.4 in this case. The cohomological invariants of arbitrary degrees are given by the following result.

Theorem 7.5. *Let X be a cominuscule variety and let $\beta_1, \beta_2, \beta_3 \in H_T^*(X)$. Then*

$$I_d^T(\beta_1, \beta_2, \beta_3) = \begin{cases} \int_{Y_d}^T q_* p^*(\beta_1) \cdot q_* p^*(\beta_2) \cdot q_* p^*(\beta_3) & \text{if } d \leq d_{\max} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is true if X is a Grassmannians of type A by Theorem 4.2, and if d is a small degree by Theorem 7.2. If X is not of type A and $d > d_{\max}$, then a case by case check shows that $\dim(M_d) > 3\dim(X)$, which implies that $\mathrm{ev}_*[M_d] = 0$. □

Remark 7.6. Let $\Omega \subset X$ be a Schubert variety and set $\widetilde{\Omega} = q(p^{-1}(\Omega)) \subset Y_d$. Then we have $q_* p^*([\Omega]) = [\widetilde{\Omega}]$ if (a translate of) Ω is contained in the dual Schubert variety of $X(d)$, and otherwise $q_* p^*([\Omega]) = 0$. This follows from [12, (6)] together with Lemma 3.6.

References

[1] D. Anderson, S. Griffeth, and E. Miller, *Positivity and Kleiman transversality in equivariant K-theory of homogeneous spaces*, arχiv:0808.2785.

[2] A. Bertram, *Quantum Schubert calculus*, Adv. Math. **128** (1997), no. 2, 289–305. MR MR1454400 (98j:14067)

[3] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR MR1440306 (98e:14010)

[4] M. Brion, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85. MR MR2143072 (2006f:14058)

[5] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005. MR MR2107324 (2005k:14104)

[6] A. S. Buch, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103. MR MR1932326 (2003m:14018)

[7] ______, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR MR1946917 (2003j:14062)

[8] ______, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235. MR MR1985005 (2004c:14105)

[9] ______, *Combinatorial K-theory*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 87–103. MR MR2143073 (2007a:14056)

[10] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915 (electronic). MR MR1992829 (2004h:14060)

[11] ______, *Quantum Pieri rules for isotropic Grassmannians*, arχiv:0809.4966.

[12] P.E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, preprint, 2006.

[13] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR MR1433132 (98i:22021)

[14] I. Ciocan-Fontanine, *On quantum cohomology rings of partial flag varieties*, Duke Math. J. **98** (1999), no. 3, 485–524. MR MR1695799 (2000d:14058)

[15] I. Coskun and R. Vakil, *Geometric positivity in the cohomology of homogeneous spaces and generalized schubert calculus*, math.AG/0610538.

[16] D. Edidin and W. Graham, *Riemann-Roch for equivariant Chow groups*, Duke Math. J. **102** (2000), no. 3, 567–594. MR MR1756110 (2001f:14018)

[17] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), no. 3, 565–596. MR MR1431829 (98d:14063)

[18] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR MR1644323 (99d:14003)

[19] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR MR1492534 (98m:14025)

[20] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), no. 4, 641–661. MR MR2072765 (2005d:14078)

[21] A. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663. MR MR1408320 (97e:14015)

[22] ______, *On the WDVV equation in quantum K-theory*, Michigan Math. J. **48** (2000), 295–304, Dedicated to William Fulton on the occasion of his 60th birthday. MR MR1786492 (2001m:14078)

[23] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda lattices*, Comm. Math. Phys. **168** (1995), no. 3, 609–641. MR MR1328256 (96c:58027)

[24] A. Givental and Y.-P. Lee, *Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups*, Invent. Math. **151** (2003), no. 1, 193–219. MR MR1943747 (2004g:14063)

[25] W. Graham and S. Kumar, *On positivity in T-equivariant K-theory of flag varieties*, arχiv:0801.2776v1.

[26] S. Griffeth and A. Ram, *Affine Hecke algebras and the Schubert calculus*, European J. Combin. **25** (2004), no. 8, 1263–1283.

[27] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original. MR MR1288523 (95d:14001)

[28] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[29] B. Kim, *On equivariant quantum cohomology*, Internat. Math. Res. Notices (1996), no. 17, 841–851. MR MR1420551 (98h:14013)

[30] ______, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. (2) **149** (1999), no. 1, 129–148. MR MR1680543 (2001c:14081)

[31] B. Kim and R. Pandharipande, *The connectedness of the moduli space of maps to homogeneous spaces*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 187–201. MR MR1882330 (2002k:14021)

[32] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297. MR MR0360616 (50 #13063)

[33] A. Knutson, *A conjectural rule for GL_n Schubert calculus*, preprint, 1999.

[34] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260. MR MR1997946 (2006a:14088)

[35] J. Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42. MR MR825838 (87c:14038)

[36] M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368. MR MR1363062 (97d:14077)

[37] M. Kontsevich and Yu. Manin, *Quantum cohomology of a product*, Invent. Math. **124** (1996), no. 1-3, 313–339, With an appendix by R. Kaufmann. MR MR1369420 (97e:14064)

[38] T. Lam and M. Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, arχiv:0705.1386.

[39] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. **121** (2004), no. 3, 389–424. MR MR2040281 (2005f:14107)

[40] Y.-P. Lee and R. Pandharipande, *A reconstruction theorem in quantum cohomology and quantum K-theory*, Amer. J. Math. **126** (2004), no. 6, 1367–1379. MR MR2102400 (2006c:14082)

[41] C. Lenart, *Combinatorial aspects of the K-theory of Grassmannians*, Ann. Comb. **4** (2000), no. 1, 67–82. MR MR1763950 (2001j:05124)

[42] C. Lenart and T. Maeno, *Quantum Grothendieck polynomials*, math.CO/0608232.

[43] C. Lenart and A. Postnikov, *Affine Weyl groups in K-theory and representation theory*, to appear in Int. Math. Res. Not., available at math.RT/0309207, 2005.

[44] C. L. Mihalcea, *On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms*, Duke Math. J. **140** (2007), no. 2, 321–350.

[45] ______, *Equivariant quantum Schubert calculus*, Adv. Math. **203** (2006), no. 1, 1–33. MR MR2231042 (2007c:14061)

[46] E. Miller and D. Speyer, *A Kleiman-Bertini theorem for sheaf tensor products*, math.AG:0601202v4.

[47] R. Pandharipande, *The canonical class of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ and enumerative geometry*, Internat. Math. Res. Notices (1997), no. 4, 173–186. MR MR1436774 (98h:14067)

[48] A. Postnikov, *Affine approach to quantum Schubert calculus*, Duke Math. J. **128** (2005), no. 3, 473–509. MR MR2145741 (2006e:05182)

[49] Z. Reichstein and B. Youssin, *Equivariant resolution of points of indeterminacy*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2183–2187 (electronic). MR MR1896397 (2003c:14017)

[50] K. Rietsch, *Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties*, J. Amer. Math. Soc. **16** (2003), no. 2, 363–392.

[51] S. Sierra, *A general homological Kleiman-Bertini Theorem*, 0705.0055v1.

[52] H. Tamvakis, *Quantum cohomology of isotropic Grassmannians*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 311–338. MR MR2166090 (2006k:14103)

[53] J. F. Thomsen, *Irreducibility of $\overline{M}_{0,n}(G/P, \beta)$*, Internat. J. Math. **9** (1998), no. 3, 367–376. MR MR1625369 (99g:14032)

[54] R. Vakil, *A geometric Littlewood-Richardson rule*, Ann. of Math. (2) **164** (2006), no. 2, 371–421, Appendix A written with A. Knutson. MR MR2247964 (2007f:05184)

[55] O. E. Villamayor U., *Patching local uniformizations*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 6, 629–677. MR MR1198092 (93m:14012)

[56] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310. MR MR1144529 (93e:32028)

[57] C. T. Woodward, *On D. Peterson's comparison formula for Gromov-Witten invariants of* G/P, Proc. Amer. Math. Soc. **133** (2005), no. 6, 1601–1609 (electronic). MR MR2120266 (2005j:14080)

[58] A. Yong, *Degree bounds in quantum Schubert calculus*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2649–2655 (electronic). MR MR1974319 (2004c:14102)

[59] F. L. Zak, *Tangents and secants of algebraic varieties*, American Math. Soc. , Providence, RI 1993.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: asbuch@math.rutgers.edu

Department of Mathematics, Duke University, P.O. Box 90320 Durham, NC 27708-0320

E-mail address: lmihalce@math.duke.edu