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WITTEN VOLUME FORMULAS FOR SEMI-SIMPLE LIE ALGEBRAS

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Abstract

In this paper we provide an algebraic derivation of the explicit Witten volume formulas for a few semi-simple Lie algebras by combining a combinatorial method with the ideas used by Gunnells and Sczech in the computation of higher-dimensional Dedekind sums.

1. Introduction

In [9] Witten related the volumes of the moduli spaces of representations of the fundamental groups of two dimensional surfaces to the special values of the following zeta function attached to complex semisimple Lie algebras $\mathfrak{g}$ at positive integers:

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} \frac{1}{(\dim\varphi)^s},$$

where φ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. By physics considerations Witten showed that for any positive integer m

$$\zeta_W(2m;\mathfrak{g}) = c(2m;\mathfrak{g})\pi^{2mr},$$

where $c(2m;\mathfrak{g}) \in \mathbb{Q}$ and r is the number of positive roots of $\mathfrak{g}$. Such formulas are now called Witten volume formulas.

The precise Witten volume formula for $\mathfrak{sl}(3)$ was obtained by Zagier [10] (and independently by Garoufalidis and Weinstein):

$$\zeta_W(2m;\mathfrak{sl}(3)) = \frac{4^{m+1}}{3} \sum_{\substack{0\le i\le 2m \\ i\equiv 0 \bmod 2}} \binom{4m-i-1}{2m-1} \zeta(i)\zeta(6m-i). \tag{1}$$

In [1], Gunnells and Sczech studied higher-dimensional Dedekind sums and established their reciprocity law. As one application they derived the Witten volume formula for $\mathfrak{sl}(4)$ precisely.

Matsumoto and his collaborators recently defined the multiple variable analogs of $\zeta_W(s;\mathfrak{g})$ and studied some of their analytical and arithmetical properties (see [2, 3, 4, 5, 7]):

$$\zeta_{\mathfrak{g}}(\{s_\alpha\}_{\alpha\in\Delta_+}) := \sum_{m_1,\dots,m_\ell=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1+\cdots+m_\ell\lambda_\ell\rangle^{-s_\alpha},$$

where for fixed set $\Delta=\{\alpha_1,\dots,\alpha_\ell\}$ of simple roots Δ_+ is the set of all positive roots of $\mathfrak{g}$, $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ is the coroot attached to α, and $\{\lambda_1,\dots,\lambda_\ell\}$ are the fundamental weights satisfying $\langle\alpha_i^\vee,\lambda_j\rangle=\delta_{i,j}$. By a simple computation, we have

$$\zeta_W(s;\mathfrak{g}) = M(\mathfrak{g})^s\zeta_{\mathfrak{g}}(s,\dots,s), \quad \text{where} \quad M(\mathfrak{g}) = \prod_{\alpha\in\Delta_+}\langle\alpha^\vee,\lambda_1+\cdots+\lambda_\ell\rangle.$$

With this multiple variable setup Matsumoto et al. recently were able to obtain more general formulas which include Witten volume formulas as special cases for Lie algebras such as $\mathfrak{so}(5)$, $\mathfrak{so}(7)$, $\mathfrak{sp}(4)$, $\mathfrak{sp}(6)$, $\mathfrak{sl}(5)$, and $\mathfrak{g}_2$. However, their computation involves complicated analytical tools.

In this paper, we combine our combinatorial method developed in [11, 12, 13] and the technique of Gunnells and Sczech to provide an algebraic proof of Witten volume formulas for the above mentioned Lie algebras. In theory, it can also be applied to other Lie algebras including the sporadic ones.

This paper is inspired by the work of P.E. Gunnells and R. Sczech [1]. I want to thank them for their detailed explanation of the part of their paper closely related to Witten zeta functions. I am also grateful to K. Matsumoto and the referee for pointing out a mistake in the first draft of the paper and providing some very helpful suggestions which improve the paper a lot. This work was partially supported by the Faculty Development Fund of Eckerd College and a Fellowship from the Max-Planck Institut für Mathematik.

2. Key Ideas

We briefly recall the setup in [1, Section 1]. Let L be a lattice of rank $\ell \geq 1$ and $L^* = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$. Denote by $\mathbf{0}$ the zero linear form in L^*. Let $r\geq\ell$. For $e=(e_1,\dots,e_r)\in\mathbb{N}^r$, $v\in L^*\otimes\mathbb{R}$ and $\sigma=(\sigma_1,\dots,\sigma_r)\in(L^*\setminus\{\mathbf{0}\})^r$. Gunnells and Sczech define the Dedekind sum as

$$D(L,\sigma,e,v) := \frac{1}{(2\pi\sqrt{-1})^{\mathrm{wt}(e)}} {\sum_{x\in L}}' \frac{\exp(2\pi\sqrt{-1}\langle x,v\rangle)}{\langle x,\sigma_1\rangle^{e_1}\cdots\langle x,\sigma_r\rangle^{e_r}},$$

where $\mathrm{wt}(e) = e_1 + \cdots + e_r$, $\langle\ ,\ \rangle : L \times L^* \to \mathbb{Z}$ is the pairing, and $\sum'$ means the terms with vanishing denominator are to be omitted. When $L = \mathbb{Z}^\ell$ we represent each $v \in L^* \otimes \mathbb{R}$ by a vector in $\mathbb{R}^\ell$ so that $\langle (m_1, \ldots, m_\ell), (v_1, \ldots, v_\ell) \rangle = m_1 v_1 + \cdots + m_\ell v_\ell$. As pointed out in [1], this series converges absolutely if all $e_j > 1$, but may only conditionally converge if $e_j = 1$ for some j. Further, the series can be converted to a finite sum if among all the σ_i there are exactly ℓ distinct linear forms, up to proportionality when restricted to L (see [1, §1] for more details). In particular, if $r = \ell = 2$ and $e = (1; 1)$ then this infinite sum is closely related to the classical Dedekind sum.

Let ℓ be the rank of the semisimple Lie algebra $\mathfrak{g}$, $r = |\Delta_+|$, and W its Weyl group. Define an $\ell \times r$ integral matrix $\sigma(\mathfrak{g})$ whose j-th column v_j provides the coefficients of $\alpha_j \in \Delta_+$ in terms of the fundamental roots in Δ. Let $e = (2m, \ldots, 2m) \in \mathbb{N}^r$. Then by [1, Prop. 8.4] we have

$$\zeta_W(2m, \mathfrak{g}) = (2\pi\sqrt{-1})^{2mr} \frac{M(\mathfrak{g})^{2m}}{|W|} D(\mathbb{Z}^\ell, \sigma(\mathfrak{g}), e, \mathbf{0}). \tag{2}$$

In [1] Gunnells and Sczech demonstrated how one can use the reciprocity law of higher-dimensional Dedekind sums to derived the Witten volume formulas of some Lie algebras. We can replace this tool by the following simple combinatorial lemma (see [8, p. 48]).

Lemma 1. *Let s, t be two positive integers. Let x and y be two non-zero real numbers such that $x + y \neq 0$. Then*

$$\frac{1}{x^s y^t} = \sum_{a=1}^{s} \binom{s+t-a-1}{t-1} \frac{1}{x^s (x+y)^{s+t-a}} + \sum_{b=1}^{t} \binom{s+t-b-1}{s-1} \frac{1}{y^t (x+y)^{s+t-b}}.$$

To demonstrate this idea we have the following key lemma to be used many times later in the paper. Given any $\ell \times r$ matrix $\sigma = (\sigma_1, \ldots, \sigma_r)$ we denote by $\big((\sigma_1)_{e_1}, \ldots, (\sigma_r)_{e_r}\big)$ the new matrix obtained by repeating each linear form σ_j exactly e_j times, $j = 1, \ldots, r$. For simplicity we further set

$$Z\big((\sigma_1)_{e_1}, \ldots, (\sigma_r)_{e_r}\big) = (2\pi\sqrt{-1})^{\mathrm{wt}(e)} D(\mathbb{Z}^\ell, (\sigma_1, \ldots, \sigma_r), (e_1, \ldots, e_r), \mathbf{0}).$$

For example if

$$M = \begin{pmatrix} 1 & 1 \\ 0_4 & 2_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}$$

then

$$Z(M) = -(2\pi)^6 D\left(\mathbb{Z}^2, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, (4, 2), \mathbf{0}\right).$$

Lemma 2. *Suppose $a, b, c, d, e, f \in \mathbb{Z}$ such that $\gcd(a, b) = \gcd(c, d) = \gcd(e, f) = 1$ and $u\lambda(a, b) + \lambda(c, d) = (e, f)$ for some nonzero constant λ and $|u| = 1, 2$ or $1/2$.*

Let $\delta = \left|\begin{smallmatrix} a & e \\ b & f \end{smallmatrix}\right|$. *If* $|\delta| = 1, 2$, *and* $|u\delta| = 1, 2$ *then for all positive integers* i, j *and* k *such that* $w = i + j + k$ *is even, we have*

$$Z\begin{pmatrix} a & c & e \\ b_i & d_j & f_k \end{pmatrix} = (2\pi\sqrt{-1})^w \sum_{l=0}^{i} \binom{i+j-l-1}{j-1} u^{i-l} \lambda^{i+j-l} \frac{B_l B_{w-l}}{l!(w-l)!} \alpha_{l,w}(\delta) \quad (3)$$

$$+ (2\pi\sqrt{-1})^w \sum_{l=0}^{j} \binom{i+j-l-1}{i-1} u^{i} \lambda^{i+j-l} \frac{B_l B_{w-l}}{l!(w-l)!} \alpha_{l,w}(u\delta), \quad (4)$$

where $\alpha_{l,w}(\pm 1) = 1$ *and* $\alpha_{l,w}(\pm 2) = 1 - 1/2^l - 1/2^{w-l} + 2/2^w$.

Proof. Clearly we have

$$\begin{vmatrix} a & e \\ b & f \end{vmatrix} = \delta \quad \Longrightarrow \quad \begin{vmatrix} c & e \\ d & f \end{vmatrix} = -u\delta. \quad (5)$$

For any pair of integers (x, y) let $(x, y)^\perp = \{(m_1, m_2) \in \mathbb{Z}^2 : xm_1 + ym_2 = 0\}$. By the combinatorial Lemma 1 we have

$$\begin{aligned} & Z\begin{pmatrix} a & c & e \\ b_i & d_j & f_k \end{pmatrix} \\ =& \sum_{l=1}^{i} \binom{i+j-l-1}{j-1} u^{i-l} \lambda^{i+j-l} \left[Z\begin{pmatrix} a & e \\ b_l & f_{w-l} \end{pmatrix} - Z_{(c,d)^\perp}\begin{pmatrix} a & e \\ b_l & f_{w-l} \end{pmatrix} \right] \\ +& \sum_{l=1}^{j} \binom{i+j-l-1}{i-1} u^{i} \lambda^{i+j-l} \left[Z\begin{pmatrix} c & e \\ d_l & f_{w-l} \end{pmatrix} - Z_{(a,b)^\perp}\begin{pmatrix} c & e \\ d_l & f_{w-l} \end{pmatrix} \right]. \quad (6) \end{aligned}$$

Here for any sub lattice L of $\mathbb{Z}^2$ the sum Z_L is the sum Z restricted to L. These restricted sums on the right hand side of (6) in fact exactly correspond to those appearing on the right hand of the reciprocity law [1, (15)]. If $\ell \geq 2$ and $w - \ell \geq 2$ then every Z sum on the right hand side of (6) converges absolutely. So the condition $\gcd(c, d) = 1$ implies that

$$\begin{aligned} Z_{(c,d)^\perp}\begin{pmatrix} a & e \\ b_l & f_{w-l} \end{pmatrix} =& \sum_{\substack{m_1, m_2 \in \mathbb{Z},\ cm_1 + dm_2 = 0 \\ am_1 + bm_2 \neq 0,\ em_1 + fm_2 \neq 0}} \frac{1}{(am_1 + bm_2)^l (em_1 + fm_2)^{w-l}} \\ =& \sum_{N \in \mathbb{Z}^*,\ m_1 = dN,\ m_2 = -cN} \frac{u^l \lambda^l}{(u\lambda am_1 + u\lambda bm_2)^l (em_1 + fm_2)^{w-l}} \\ =& \sum_{N \in \mathbb{Z}^*} \frac{u^l \lambda^l}{((e - \lambda c)dN - (f - \lambda d)cN)^l (edN - fcN)^{w-l}} \\ =& \frac{1}{(u\delta)^w} \sum_{N \in \mathbb{Z}^*} \frac{u^l \lambda^l}{N^w} \quad (7) \end{aligned}$$

by (5) since w is even. If $\ell = 1$ or $w - \ell = 1$ then we need to modify the above computation by restricting the sums to $-t < (am_1 + bm_2)(em_1 + fm_2) < t$ and then take the limit as $t \to \infty$. With this modification (7) becomes

$$\lim_{T\to\infty} \frac{1}{(u\delta)^w} \sum_{N\in\mathbb{Z}^*, |N|<T} \frac{u^l\lambda^l}{N^w},$$

which is absolutely convergent since $w \geq 4$. Hence by an easy binomial identity we always get

$$\sum_{l=0}^{i} \binom{i+j-l-1}{j-1} u^{i-l}\lambda^{i+j-l} Z_{(c,d)^\perp}\begin{pmatrix} a & e \\ b_l & f_{w-l}\end{pmatrix} = \binom{i+j-1}{j} \frac{u^i\lambda^{i+j}}{(u\delta)^w} \sum_{N\in\mathbb{Z}^*} \frac{1}{N^w}. \tag{8}$$

Notice that for every integral matrix $\sigma = \left(\begin{smallmatrix} x & y \\ s & t\end{smallmatrix}\right)$ of determinant δ and positive integers j, k with $j + k \geq 4$ we have by [1, (4)]

$$\begin{aligned} Z\begin{pmatrix} x & y \\ s_j & t_k\end{pmatrix} &= (2\pi\sqrt{-1})^{j+k} D\left(\mathbb{Z}^2, \sigma, (j,k), \mathbf{0}\right) \\ &= \frac{(2\pi\sqrt{-1})^{j+k}}{j!k!|\delta|} \sum_{z\in\mathbb{Z}^2/\sigma\mathbb{Z}^2} \mathcal{B}_j(\sigma^{-1}z)\mathcal{B}_k(\sigma^{-1}z), \end{aligned} \tag{9}$$

where $\mathcal{B}_j(x)$ are the Bernoulli polynomials. When $u\delta = \pm 1$ the quantity in (8) provides exactly the $l = 0$ term in the sum of (4) by the following formula known to Euler: for even positive integer w

$$\sum_{N\in\mathbb{Z}^*} \frac{1}{N^w} = 2\zeta(w) = -(2\pi\sqrt{-1})^w \frac{B_w}{w!}.$$

When $u\delta = \pm 2$ by the assumption $\gcd(c, d) = \gcd(e, f) = 1$ and (9) we get

$$Z\begin{pmatrix} c & e \\ d_l & f_{w-l}\end{pmatrix} = \frac{(2\pi\sqrt{-1})^w}{2} \frac{B_l B_{w-l}}{l!(w-l)!} \Big(1 + \Big(\frac{2}{2^l} - 1\Big)\Big(\frac{2}{2^{w-l}} - 1\Big)\Big). \tag{10}$$

When $l = 0$ we find that (10) is equal to $(2\pi\sqrt{-1})^w B_w/(2^w w!)$ and therefore (8) again provides exactly the $l = 0$ term in the sum of (4).

Similarly, the $l = 0$ term in the sum of (3) can be obtained by the sum of $Z_{(a,b)^\perp}\left(\begin{smallmatrix} c & e \\ d_l & f_{w-l}\end{smallmatrix}\right)$ in (6). This finishes the proof of the lemma. □

We can obtain (1) immediately by applying the lemma to

$$Z\begin{pmatrix} 1 & 0 & 1 \\ 0_{2m} & 1_{2m} & 1_{2m}\end{pmatrix}.$$

To aid our computation we represent the procedure in Lemma 2 by Figure 1: the left is self-evident while the right is more elegant with only the removed columns recorded. For example, in $1/xy(x+y) = 1/x(x+y)^2 + 1/y(x+y)^2$ we may think of x, y and $x+y$ as corresponding to the first three columns and say that $1/x(x+y)^2$ is obtained by merging the second column into the 3rd, so that the second column is now removed. This is denoted by the node 2 in the right picture. We can generalize Lemma 1 and apply it to any three linearly dependent columns of a general matrix. Furthermore, a circled column number between any two sub-nodes signifies the column into which the two nodes are merged. We call this tree a *computation tree*. Notice that $w = i + j + k$ and we have omitted the summation over l.

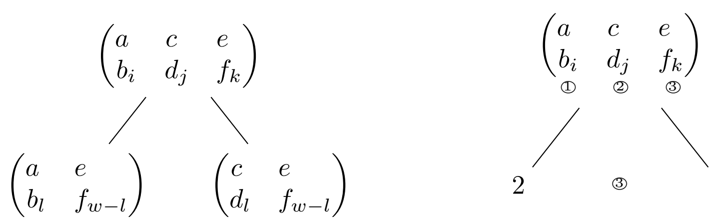


Figure 1: A computation tree of rank two.

We have the following partial generalization of Lemma 2 whose proof is left to the interested reader.

Lemma 3. *Let $\sigma = (\sigma_1, \dots, \sigma_r)$ be an $(r-1)\times r$ matrix with $r \ge 3$. Let $e_1, \dots, e_r \in \mathbb{N}$ and put $s = e_1 + e_2 + e_3$. Suppose $u = \pm 1$, $u\lambda\sigma_1 + \lambda\sigma_2 = \sigma_3$ for some non-zero constant λ and $\det(\sigma_2, \dots, \sigma_r) = \pm 1$. If each of σ_1 and σ_2 has at least one component equal to ± 1 then*

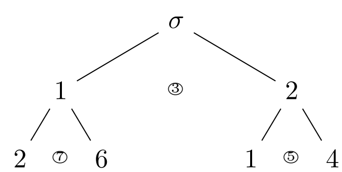
$$Z\big((\sigma_1)_{e_1}, \dots, (\sigma_r)_{e_r}\big) = (2\pi\sqrt{-1})^w \frac{B_{e_4} \dots B_{e_r}}{e_4! \dots e_r!} u^{e_1} \lambda^{e_1+e_2} \times$$
$$\times \left\{ \sum_{l=0}^{e_1} \binom{e_1+e_2-l-1}{e_2-1} \frac{B_l B_{s-l}}{u^l \lambda^l l!(s-l)!} + \sum_{l=0}^{e_2} \binom{e_1+e_2-l-1}{e_1-1} \frac{B_l B_{s-l}}{\lambda^l l!(s-l)!} \right\}.$$

Remark. We are not able to generalize Lemma 2 to arbitrary rank without the assumption that each of σ_1 and σ_2 has at least one component equal to ± 1. It seems that the naive generalization is incorrect.

Definition 4. If a node of binary tree has the following property then we say it is a *good parent*: every one of its descendants names one of their children, if it has any, the same as its only sibling. So a node with children but without grandchildren is always a good parent.

The definition is crucial for the following result which provides a possible simplification process to compute Z sums. Usually, when a node in a computation tree is not a good parent, the corresponding Z sum is more difficult to compute. For example, the initial node in the computation tree of the $\mathfrak{g}_2$ case in Section 4 is not a good parent. But the initial node of the $\mathfrak{so}(5)$ case in Section 3 is a good parent so its computation is much simpler.

Proposition 5. *Let $\sigma = (\sigma_1, \dots, \sigma_r)$ be an $\ell \times r$ matrix with $r \geq \ell + 1 \geq 2$. Suppose that every column has some component equal to ± 1. Let $e_1, \dots, e_r \in \mathbb{N}$ and put $s = e_1 + e_2 + e_3$. Assume σ has no grandchildren or σ is a good parent in its computation binary tree whose top part looks as follows:*



Here, $3, 5,$ and 7 (these numbers refer to the column numbers) may or may not be the same but $5 \neq 1, 2, 4$ and $7 \neq 1, 2, 6$. Suppose every node in the penultimate generation satisfies the conditions in Lemma 2 (resp. Lemma 3) if it has rank two (resp. greater than two). If $\lambda_1\sigma_1 + \lambda_2\sigma_2 = \sigma_3$. Then

$$\begin{aligned}
&Z\big((\sigma_1)_{e_2}, \dots, (\sigma_r)_{e_r}\big) \\
=&\lambda_1^{e_1}\lambda_2^{e_2}\sum_{i=0}^{e_2}\binom{e_1+e_2-i-1}{e_1-1}\frac{1}{\lambda_2^i}Z\big((\sigma_2)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big) \\
+&\lambda_1^{e_1}\lambda_2^{e_2}\sum_{i=0}^{e_1}\binom{e_1+e_2-i-1}{e_2-1}\frac{1}{\lambda_1^i}Z\big((\sigma_1)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big).
\end{aligned}$$

Proof. When $r = \ell + 1$ the proposition follows from Lemma 2 and Lemma 3 by our assumption since now the penultimate generation is exactly σ itself. Assume $r > \ell + 1$. By Lemma 1 it is clear that

$$\begin{aligned}
&Z\big((\sigma_1)_{e_1}, \dots, (\sigma_r)_{e_r}\big) \\
=&\lambda_1^{e_1}\lambda_2^{e_2}\sum_{i=1}^{e_2}\binom{e_1+e_2-i-1}{e_1-1}\frac{1}{\lambda_2^i}\Big[Z\big((\sigma_2)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big) - Z_{\sigma_1^\perp}\Big] \\
+&\lambda_1^{e_1}\lambda_2^{e_2}\sum_{i=1}^{e_1}\binom{e_1+e_2-i-1}{e_2-1}\frac{1}{\lambda_1^i}\Big[Z\big((\sigma_1)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big) - Z_{\sigma_2^\perp}\Big].
\end{aligned}$$

With fixed ℓ we now use induction on r to show that

$$\begin{aligned}
\lambda_2^{-i}Z_{\sigma_1^\perp}\big((\sigma_3)_s, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big) =&Z\big((\sigma_1)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big)\Big|_{i=0}, \\
\lambda_1^{-i}Z_{\sigma_2^\perp}\big((\sigma_3)_s, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big) =&Z\big((\sigma_2)_i, (\sigma_3)_{s-i}, (\sigma_4)_{e_4}, \dots, (\sigma_r)_{e_r}\big)\Big|_{i=0}, \quad (11)
\end{aligned}$$

which yields the lemma immediately by the simple combinatorial identities

$$\sum_{i=1}^{e_1}\binom{e_1+e_2-i-1}{e_2-1}=\binom{e_1+e_2-1}{e_2},\quad \sum_{i=1}^{e_2}\binom{e_1+e_2-i-1}{e_1-1}=\binom{e_1+e_2-1}{e_1}.$$

By the computation tree we may assume $\mu_1\sigma_1+\mu_2\sigma_4=\sigma_5$. Notice that for any $x\in\sigma_1^\perp$ we have

$$\begin{aligned}\lambda_2\langle\sigma_2,x\rangle=&\lambda_1\langle\sigma_1,x\rangle+\lambda_2\langle\sigma_2,x\rangle=\langle\sigma_3,x\rangle,\\ \mu_2\langle\sigma_4,x\rangle=&\mu_1\langle\sigma_1,x\rangle+\mu_2\langle\sigma_4,x\rangle=\langle\sigma_5,x\rangle.\end{aligned}$$

Hence (if $\textcircled{3}=\textcircled{5}$ then $e_5=0$)

$$\begin{aligned}&\lambda_2^{-i}Z_{\sigma_1^\perp}\big((\sigma_2)_i,(\sigma_3)_{s-i},(\sigma_4)_{e_4},\dots,(\sigma_r)_{e_r}\big)\\ =&Z_{\sigma_1^\perp}\big((\sigma_3)_s,(\sigma_4)_{e_4},\dots,(\sigma_r)_{e_r}\big)\\ =&\mu_2^{e_4}Z_{\sigma_1^\perp}Z\big((\sigma_3)_s,(\sigma_5)_{e_4+e_5},(\sigma_6)_{e_6},\dots,(\sigma_r)_{e_r}\big).\end{aligned}\tag{12}$$

On the other hand, by induction assumption we get

$$\begin{aligned}&Z\big((\sigma_1)_i,(\sigma_3)_{s-i},(\sigma_4)_{e_4},\dots,(\sigma_r)_{e_r}\big)\\ =&\mu_1^i\mu_2^{e_4}\sum_{j=0}^{e_4}\binom{e_4+i-j-1}{i-1}\frac{1}{\mu_2^j}Z\big((\sigma_3)_{s-i},(\sigma_4)_j,(\sigma_5)_{e_4+e_5+i-j},(\sigma_6)_{e_6},\dots,(\sigma_r)_{e_r}\big)\\ +&\mu_1^i\mu_2^{e_4}\sum_{j=0}^{i}\binom{e_4+i-j-1}{e_4-1}\frac{1}{\mu_1^j}Z\big((\sigma_1)_j,(\sigma_3)_{s-i},(\sigma_5)_{e_4+e_5+i-j},(\sigma_6)_{e_6},\dots,(\sigma_r)_{e_r}\big).\end{aligned}$$

Taking $i=0$ in this expression we see that the first sum is vacuous because of the binomial coefficient while the second sum is reduced to just one term:

$$\begin{aligned}&\mu_1^{i-j}\mu_2^{e_4}Z\big((\sigma_1)_j,(\sigma_3)_{s-i},(\sigma_5)_{e_4+e_5+i-j},(\sigma_6)_{e_6},\dots,(\sigma_r)_{e_r}\big)\Big|_{i=j=0}\\ &\qquad\qquad=\mu_2^{e_4}Z_{\sigma_1^\perp}\big((\sigma_3)_s,(\sigma_5)_{e_4+e_5},(\sigma_6)_{e_6},\dots,(\sigma_r)_{e_r}\big),\end{aligned}$$

by induction assumption. Thus equation (11) follows from (12). The proof of (11) is exactly the same. This concludes the proof of the proposition. □

3. The $\mathfrak{so}(5)$ Case

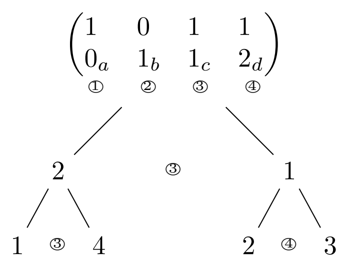
Let $m\in\mathbb{N}$ and $n=2m$. By the above we can write

$$\zeta_{\mathfrak{so}(5)}(n,n,n,n)=\sum_{a,b=1}^{\infty}\frac{1}{a^nb^n(a+b)^n(a+2b)^n}=\frac{(2\pi)^{8m}}{8}D(\mathbb{Z}^2,\sigma,(1,\dots,1),\mathbf{0})$$

where $(1,\dots,1)\in\mathbb{N}^{4n}$, the matrix $\sigma=\sigma(n,n,n,n)$ and

$$\sigma(a,b,c,d)=\begin{pmatrix}1 & 0 & 1 & 1\\ 0_a & 1_b & 1_c & 2_d\end{pmatrix}.$$

To prepare for the $\mathfrak{so}(7)$ case we first prove a generalization of the $\mathfrak{so}(5)$ case by the following computation tree



Theorem 6. *Let $a,b,c,d\in\mathbb{N}$ and suppose at most one of them is 1. Set $\binom{t}{-1}=0$ for all t and write $\beta_{j,w}=(2\pi\sqrt{-1})^w B_j B_{w-j}/(j!(w-j)!)$. If $w=a+b+c+d$ is even then*

$$Z\begin{pmatrix}1 & 0 & 1 & 1\\ 0_a & 1_b & 1_c & 2_d\end{pmatrix}$$
$$=\sum_{i=0}^{b}\binom{a+b-i-1}{a-1}\left(\sum_{j=0}^{i}\binom{w-d-j-1}{w-d-i-1}\beta_{j,w}+\sum_{j=0}^{w-d-i}\binom{w-d-j-1}{i-1}\beta_{j,w}\right)$$
$$+\sum_{i=0}^{a}\binom{a+b-i-1}{b-1}\left(\sum_{j=0}^{d}\binom{d+i-j-1}{i-1}\frac{\beta_{j,w}}{2^{d+i-j}}+\sum_{j=0}^{i}\binom{d+i-j-1}{d-1}\frac{\beta_{j,w}}{2^{d+i-j}}\right).$$

Proof. It is easy to check that all of the 2×2 minors of $\begin{pmatrix}1 & 0 & 1 & 1\\ 0_a & 1_b & 1_c & 2_d\end{pmatrix}$ have determinant ±1 or ±2. So we can apply Proposition 5 and get

$$Z\begin{pmatrix}1 & 0 & 1 & 1\\ 0_a & 1_b & 1_c & 2_d\end{pmatrix}=\sum_{i=0}^{b}\binom{a+b-i-1}{a-1}Z\begin{pmatrix}0 & 1 & 1\\ 1_i & 1_{w-d-i} & 2_d\end{pmatrix}$$
$$+\sum_{i=0}^{a}\binom{a+b-i-1}{b-1}Z\begin{pmatrix}1 & 1 & 1\\ 0_i & 1_{w-d-i} & 2_d\end{pmatrix}.$$

The theorem now follows from Lemma 2 directly. □

This implies the following as an immediate corollary.

Theorem 7. *Let n be a positive even integer and $w = 4n$. Set $\binom{t}{-1} = 0$ for all t and write $\tilde{\beta}_{j,w} = B_j B_{w-j}/(j!(w-j)!)$. Then*

$$\frac{8\zeta_W(n, \mathfrak{so}(5))}{6^n(2\pi)^{4n}} = \sum_{i=0}^{n} \binom{2n-i-1}{n-1} \Bigg[\sum_{j=0}^{3n-i} \binom{3n-j-1}{i-1} \tilde{\beta}_{j,w} + \sum_{j=0}^{i} \binom{3n-j-1}{3n-i-1} \tilde{\beta}_{j,w}$$
$$+ \sum_{j=0}^{n} \frac{1}{2^{n+i-j}} \binom{n+i-j-1}{i-1} \tilde{\beta}_{j,w} + \sum_{j=0}^{i} \frac{1}{2^{n+i-j}} \binom{n+i-j-1}{n-1} \tilde{\beta}_{j,w} \Bigg].$$

Remark. By exchanging the order of summation in the theorem we see that our formula agrees with that of Matsumoto et al. in [6, Theorem 8.1] by setting $s = 2n$ and $p = q = r = n$.

4. The $\mathfrak{g}_2$ Case

By definition we have $\zeta_W(s; \mathfrak{g}_2) = 120^s \zeta_{\mathfrak{g}_2}(s, \ldots, s)$ where

$$\zeta_{\mathfrak{g}_2}(s_1, \ldots, s_6) = \sum_{a,b=1}^{\infty} \frac{1}{a^{s_1} b^{s_2} (a+b)^{s_3} (a+2b)^{s_4} (a+3b)^{s_5} (2a+3b)^{s_6}}. \tag{13}$$

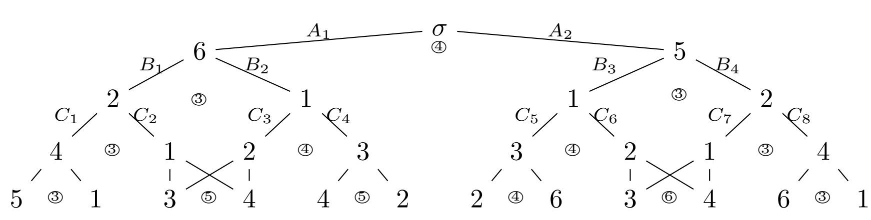
In the rest of this section we fix a positive even integer $n = 2m$. By (2) we have

$$\zeta_W(2m, \mathfrak{g}_2) = \frac{(2\pi)^{12m} 120^{2m}}{12} D(\mathbb{Z}^2, \sigma, (1, \ldots, 1), \mathbf{0})$$

where $(1, \ldots, 1) \in \mathbb{N}^{6n}$ and

$$\sigma = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0_n & 1_n & 1_n & 2_n & 3_n & 3_n \end{pmatrix}.$$
$$\quad ① \quad ② \quad ③ \quad ④ \quad ⑤ \quad ⑥$$

Similar to the case of $\mathfrak{so}(5)$ we can proceed using the following computation tree:



We thus get

$$Z\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0_n & 1_n & 1_n & 2_n & 3_n & 3_n \end{pmatrix} = \sum_{i=0}^{n} \binom{2n-i-1}{n-1} \frac{A_1(i) + A_2(i)}{3^{2n-i}}, \tag{14}$$

where

$$A_1(i) = Z\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0_n & 1_n & 1_n & 2_{3n-i} & 3_i \end{pmatrix}, \quad A_1(i) = Z\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0_n & 1_n & 1_n & 2_{3n-i} & 3_i \end{pmatrix}.$$

Note that σ is not a good parent so we have to check that the two perpendicular terms can indeed be absorbed into the summand when setting $i = 0$ in (14). This is not too difficult after finding out the explicit expressions of A_1 and A_2, both of which are good parents. Therefore it follows from Proposition 5 that

$$A_1(i) = \sum_{j=0}^{n} \binom{2n-j-1}{n-1} \Big(B_1(i,j) + B_2(i,j)\Big),$$
$$A_2(i) = \sum_{j=0}^{n} \binom{2n-j-1}{n-1} \Big(B_3(i,j) + B_4(i,j)\Big),$$

where

$$\begin{aligned} B_1(i,j) =& Z\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0_j & 1_{3n-j} & 2_{3n-i} & 3_i \end{pmatrix} \\ =& \sum_{k=0}^{j} \binom{3n-i+j-k-1}{3n-i-1} \frac{C_1(i,k)}{2^{3n-i+j-k}} \\ & \qquad + \sum_{k=0}^{3n-i} \binom{3n-i+j-k-1}{j-1} \frac{C_2(i,k)}{2^{3n-i+j-k}}, \end{aligned}$$

$$\begin{aligned} B_2(i,j) =& Z\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1_j & 1_{3n-j} & 2_{3n-i} & 3_i \end{pmatrix} \\ =& \sum_{k=0}^{3n-j} \binom{3n-k-1}{j-1} C_3(i,k) + \sum_{k=0}^{j} \binom{3n-k-1}{3n-j-1} C_4(i,k), \end{aligned}$$

$$\begin{aligned} B_3(i,j) =& Z\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1_j & 1_{3n-j} & 2_{3n-i} & 3_i \end{pmatrix} \\ =& \sum_{k=0}^{j} \binom{3n-k-1}{3n-j-1} C_3(i,k) + \sum_{k=0}^{3n-j} \binom{3n-k-1}{j-1} C_4(i,k), \end{aligned}$$

$$\begin{aligned} B_4(i,j) =& Z\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0_j & 1_{3n-j} & 2_{3n-i} & 3_i \end{pmatrix} \\ =& \sum_{k=0}^{3n-i} \binom{3n-i+j-k-1}{j-1} \frac{C_4(i,k)}{2^{3n-i+j-k}} \\ & \qquad + \sum_{k=0}^{j} \binom{3n-i+j-k-1}{3n-i-1} \frac{C_8(i,k)}{2^{3n-i+j-k}}, \end{aligned}$$

where, setting $w = 6n$, $\delta_2(l) = 1+2^{1-w}-2^{-l}-2^{l-w}$, and $\beta_{l,w} = (2\pi)^w B_l B_{w-l}/(w-l)!/l!$ and using Lemma 2 we have

$$\begin{aligned}
C_1(i,k) =& Z\begin{pmatrix} 1 & 1 & 1 \\ 0_k & 1_{6n-k-i} & 3_i \end{pmatrix} \\
=& \sum_{l=0}^{i} \binom{i+k-l-1}{k-1} \frac{2^k \delta_2(l)}{3^{i+k-l}} \beta_{l,w} + \sum_{l=0}^{k} \binom{i+k-l-1}{i-1} \frac{2^{k-l}}{3^{i+k-l}} \beta_{l,w}, \\
C_2(i,k) =& Z\begin{pmatrix} 1 & 1 & 1 \\ 1_{6n-k-i} & 2_k & 3_i \end{pmatrix} \\
=& \sum_{l=0}^{i} \binom{i+k-l-1}{k-1} (-1)^i 2^k \delta_2(l) \beta_{l,w} + \sum_{l=0}^{k} \binom{i+k-l-1}{i-1} (-1)^i 2^{k-l} \beta_{l,w}, \\
C_3(i,k) =& Z\begin{pmatrix} 1 & 1 & 1 \\ 1_k & 2_{6n-k-i} & 3_i \end{pmatrix} \\
=& \sum_{l=0}^{i} \binom{i+k-l-1}{k-1} \frac{(-1)^i \beta_{l,w}}{2^{k+i-l}} + \sum_{l=0}^{k} \binom{i+k-l-1}{i-1} \frac{(-1)^i \beta_{l,w}}{2^{k+i-l}}, \\
C_4(i,k) =& Z\begin{pmatrix} 0 & 1 & 1 \\ 1_k & 2_{6n-k-i} & 3_i \end{pmatrix} \\
=& \sum_{l=0}^{k} \binom{i+k-l-1}{i-1} (-1)^k \beta_{l,w} + \sum_{l=0}^{i} \binom{i+k-l-1}{k-1} (-1)^k \beta_{l,w}, \\
C_8(i,k) =& Z\begin{pmatrix} 1 & 1 & 2 \\ 0_k & 1_{6n-k-i} & 3_i \end{pmatrix} \\
=& \sum_{l=0}^{k} \binom{i+k-l-1}{i-1} \frac{\beta_{l,w}}{3^{k+i-l}} + \sum_{l=0}^{i} \binom{i+k-l-1}{k-1} \frac{\beta_{l,w}}{3^{k+i-l}}.
\end{aligned}$$

Putting everything together we finally arrive at

Theorem 8. *Let n be a positive even integer. Let $w = 6n$ and set $\binom{t}{-1} = 0$ for all t. Write $\beta_{l,w} = (2\pi)^w B_l B_{w-l}/(w-l)!/l!$. Then*

$$\frac{12\zeta_W(n,\mathfrak{g}_2)}{120^n} = \sum_{i=0}^{n} \binom{2n-i-1}{n-1} 3^{i-2n} \sum_{j=0}^{n} \binom{2n-j-1}{n-1} \times$$
$$\times \Bigg\{ \sum_{k=0}^{j} \binom{3n-i+j-k-1}{3n-i-1} 2^{i+k-3n-j} \sum_{t=1}^{2} \sum_{l=0}^{\lambda_t} \binom{i+k-l-1}{i+k-\lambda_t-1} \frac{1+2^k \delta_t(l)}{3^{i+k-l}} \beta_{l,w}$$
$$+ \sum_{k=0}^{3n-i} \binom{3n-i+j-k-1}{j-1} \sum_{t=1}^{2} \sum_{l=0}^{\lambda_t} \binom{i+k-l-1}{i+k-\lambda-1} \big((-1)^k + (-1)^i 2^k \delta_t(l)\big) \beta_{l,w}$$

$$+\sum_{k=0}^{3n-j}\binom{3n-k-1}{j-1}\sum_{\lambda=i,k}\sum_{l=0}^{\lambda}\binom{i+k-l-1}{i+k-\lambda-1}\big((-1)^k+(-1)^i 2^{l-i-k}\big)\beta_{l,w}$$

$$+\sum_{k=0}^{j}\binom{3n-k-1}{3n-j-1}\sum_{\lambda=i,k}\sum_{l=0}^{\lambda}\binom{i+k-l-1}{i+k-\lambda-1}\big((-1)^k+(-1)^i 2^{l-i-k}\big)\beta_{l,w}\Bigg\},$$

where $\lambda_1 = k$, $\lambda_2 = i$, $\delta_1(l) = 2^{-l}$, *and* $\delta_2(l) = 1 + 2^{1-6n} - 2^{-l} - 2^{l-6n}$.

Remark. Although we can not verify the agreement of our theorem with [5, Theorem 5.1] we are sure their result will follow by choosing another computation tree. However, we find our data for $\zeta_W(2m, \mathfrak{g}_2)$ $(m \le 10)$ agree with those in [5]. For example,

$$\zeta_W(2,\mathfrak{g}_2) = \frac{23}{297904566960}\pi^{12}$$

$$\zeta_W(4,\mathfrak{g}_2) = \frac{8165653}{1445838676129559305994400000}\pi^{24}$$

$$\zeta_W(6,\mathfrak{g}_2) = \frac{55940539974690617}{1318881563025306665441502148804584959636160000000}\pi^{36}$$

$$\zeta_W(8,\mathfrak{g}_2) = \frac{4734636546127925676801518 9}{148569762162395824473836871465267514811357530241219027 52\cdot 10^{11}}\pi^{48}$$

We have also verified numerically the correctness of these values by using the definition (13).

5. The $\mathfrak{so}(7)$ Case

By definition

$$\frac{\zeta_W(n;\mathfrak{so}(7))}{720^n} = \sum_{m_1,m_2,m_3=1}^{\infty}\left(\frac{1/\big(m_1m_2m_3(m_1+m_2)(m_2+m_3)(2m_2+m_3)\big)}{(2m_1+2m_2+m_3)(m_1+2m_2+m_3)(m_1+m_2+m_3)}\right)^n. \tag{15}$$

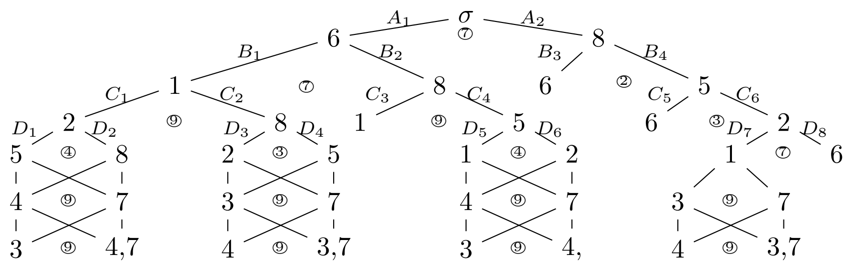
The corresponding matrix to $\mathfrak{so}(7)$ is

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 & 1 \\ 0_n & 0_n & 1_n & 0_n & 1_n & 1_n & 1_n & 1_n & 1_n \end{pmatrix}.$$
$$\quad ① \quad ② \quad ③ \quad ④ \quad ⑤ \quad ⑥ \quad ⑦ \quad ⑧ \quad ⑨$$

Given four column vectors $\sigma_1, \ldots, \sigma_4$ let $S(\sigma_1, \ldots, \sigma_4)$ be the set of the four possible choices of three columns. Then every triple of columns of the following are linearly dependent:

$$\{(1,5,9),(4,5,8)\} \cup S(1,6,7,8) \cup S(2,3,5,6) \cup S(3,4,7,9). \tag{16}$$

These fourteen dependencies are the only 3-column dependencies and will be used critically in the following computation tree of $\zeta_W(n,\mathfrak{so}(7))$ for even positive integer n.



Note that the column to which two sub-nodes are merged to may not be unique. But it should not affect the final result. For example, at the very beginning we can merge the 6th and 8th column to either the 1st or 7th column. We will choose the 7th column in the computation tree. On the other hand, if we go down the path $8-5-2-1-3$ then we can only merge the 4th and the 7th columns to the 9th column since the 3rd was removed already, and if we go down the path $8-5-2-1-7$ then we can only merge the 4th and 3rd columns.

By following the above computation tree we have

Theorem 9. *Let n be an even positive integer. Set $\binom{t}{-1}=0$ for all t. Then*

$$\frac{48\zeta_W(n,\mathfrak{so}(7))}{720^n(2\pi\sqrt{-1})^{9n}} = 2^n\sum_{i=0}^{n}\binom{2n-i-1}{n-1}\Bigg\{\sum_{j=0}^{i}\binom{n+i-j-1}{n-1}\bigg(\frac{B(j)}{2^i}+\frac{B''(i,j)}{(-1)^i}\bigg)$$
$$+\sum_{j=0}^{n}\binom{n+i-j-1}{i-1}\bigg(\frac{B'(j,n,j)}{2^i}+\frac{B'(i+n,j,j)}{(-1)^{i+j}}\bigg)\Bigg\},$$

where by writing $C'\{i,k\}=C'(n+i,3n+i-k,3n+i-k,2n+i-k)$

$$B(j)=\sum_{k=0}^{j}\binom{n+j-k-1}{n-1}C(j,j-k,k,j)+\sum_{k=0}^{n}\binom{n+j-k-1}{j-1}\frac{C(j,j-k,n,j)}{(-1)^k},$$
$$B'(a,b,j)=\sum_{k=0}^{b}\binom{n+j-k-1}{n+j-b-1}C(a,j-k,k,j)+\sum_{k=0}^{n+j-b}\binom{n+j-k-1}{b-1}C'(a,k,n,j),$$
$$B''(i,j)=\sum_{k=0}^{2n+i-j}\binom{2n+i-k-1}{j-1}\frac{2^{2n}C'\{i,k\}}{(-2)^{j+k-i}}+\sum_{k=0}^{j}\binom{2n+i-k-1}{2n+i-j-1}\frac{2^{2n}C''(i,k)}{(-1)^{j-i}},$$

and by setting $\mu = n+a-j$ *and* $\nu = n+b-c$

$$\begin{aligned}
C(a,b,c,j) &= \sum_{l=0}^{c} \binom{n+a-b-l-1}{n+a-b-c-1} \frac{D(a,b,n,j,l)}{(-1)^{n+a-b-c}} \\
&\quad + \sum_{l=0}^{n+a-b-c} \binom{n+a-b-l-1}{c-1} \frac{D(a,b,n,j,l)}{(-1)^{n+a-b-c-l}}, \\
C'(a,b,c,j) &= \sum_{l=0}^{\mu} \binom{\mu+\nu-l-1}{\nu-1} D(a,b,c,j,l) \\
&\quad + \sum_{l=0}^{\nu} \binom{\mu+\nu-l-1}{\mu-1} D(a,b,c,j,l), \\
C''(i,k) &= \sum_{l=0}^{k} \binom{n+k-l-1}{n-1} 2^n D'(i,k,l) \\
&\quad + \sum_{l=0}^{n} \binom{n+k-l-1}{k-1} 2^{n-l} D(i+l-k,n,3n+i-k,0,l),
\end{aligned}$$

where by setting $u = 4n-a$ *and* $v = 3n+a+b-c-j-l$

$$\begin{aligned}
D(a,b,c,j,l) &= \sum_{s=0}^{u} \binom{u+v-s-1}{v-1} \frac{E_c(l,s)}{(-1)^v} + \sum_{s=0}^{v} \binom{u+v-s-1}{u-1} \frac{E'_c(l,s)}{(-1)^{v-s}}, \\
D'(i,k,l) &= \sum_{s=0}^{4n+k-i-l} \binom{7n-l-s-1}{3n+i-k-1} \frac{E(l,s)}{2^{7n-l-s}} \\
&\quad + \sum_{s=0}^{3n+i-k} \binom{7n-l-s-1}{4n+k-i-l-1} \frac{E(l,s)}{2^{7n-l-s}}.
\end{aligned}$$

Here by setting $\beta'_{s,t} = -B_s B_t B_{9n-s-t}/(s!t!(9n-s-t)!)$, $\beta''_{a,b} = \beta'_{a,b}(1+(2^{1-a}-1)(2^{1-b}-1))/2$, *and* $\beta'''_{a,b,c} = \beta'_{a,b}(1+(2^{1-a}-1)(2^{1-b}-1)(2^{1-c}-1))/2$, *we have*

$$\begin{aligned}
E_c(l,s) &= \sum_{t=0}^{c} \binom{c+s-t-1}{s-1} \frac{\beta'_{l,t}}{2^{c+s-t}} + \sum_{t=0}^{s} \binom{c+s-t-1}{c-1} \frac{\beta'_{l,t}}{2^{c+s-t}}, \\
E'_c(l,s) &= \sum_{t=0}^{c} \binom{c+s-t-1}{s-1} \beta'_{l,t} + \sum_{t=0}^{s} \binom{c+s-t-1}{c-1} \beta'_{l,t}, \\
E(l,s) &= \sum_{t=0}^{n} \binom{n+s-t-1}{s-1} \beta'''_{l,t,9n-l-t} + \sum_{t=0}^{s} \binom{n+s-t-1}{n-1} \beta''_{l,t}.
\end{aligned}$$

For example, Maple computation shows that

$$\zeta_W(2,\mathfrak{so}(7)) = \frac{2^3 \cdot 19}{3^3 \cdot 7 \cdot 17!} \pi^{18},$$

$$\zeta_W(4,\mathfrak{so}(7)) = \frac{2^{12}\cdot 307\cdot 267743941589}{3\cdot 7\cdot 13\cdot 19\cdot 37!}\pi^{36},$$
$$\zeta_W(6,\mathfrak{so}(7)) = \frac{2^{21}\cdot 2053\cdot 9079132487\cdot 265178091767}{3\cdot 7\cdot 11\cdot 19\cdot 54!}\pi^{54},$$
$$\zeta_W(8,\mathfrak{so}(7)) = \frac{2^{29}\cdot 241\cdot 40670746903\cdot 362090344315673194559227058461 57}{3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 74!}\pi^{72},$$
$$\zeta_W(10,\mathfrak{so}(7)) = \frac{2^{37}\cdot 61\cdot 45197\cdot 3920899\cdot 3246046224154033\cdot a}{3^3\cdot 7\cdot 11\cdot 19\cdot 31\cdot 89!}\pi^{90},$$

where $a = 2020970252683932958095026589 29$. When taking the sum over the range of $|m_1|, |m_2|, |m_3| \le 100$ in (15) we find that $\zeta_W(2,\mathfrak{so}(7))$ is correct up to at least 19 digits, $\zeta_W(4,\mathfrak{so}(7))$ up to 42 digits, $\zeta_W(6,\mathfrak{so}(7))$ up to 64 digits, $\zeta_W(8,\mathfrak{so}(7))$ and $\zeta_W(10,\mathfrak{so}(7))$ up to at least 80 digits.

6. The $\mathfrak{sp}(6)$ Case

By definition

$$\frac{\zeta_W(n;\mathfrak{sp}(6))}{720^n} = \sum_{m_1,m_2,m_3=1}^{\infty}\left(\frac{1/\big(m_1m_2m_3(m_1+m_2)(m_2+m_3)(m_2+2m_3)\big)}{(m_1+m_2+m_3)(m_1+m_2+2m_3)(m_1+2m_2+2m_3)}\right)^n.$$

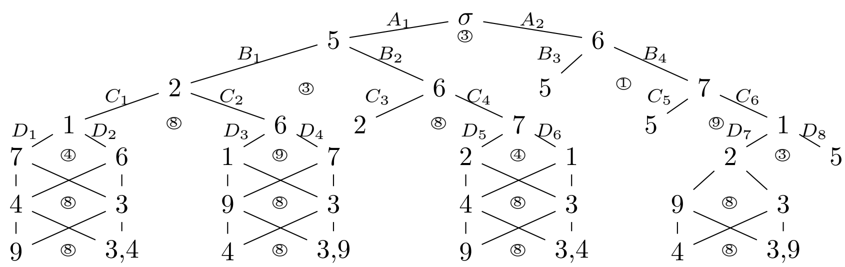
It turns out that even though $\mathfrak{so}(7)$ and $\mathfrak{so}(6)$ are not isomorphic Lie algebras, the computation of $\zeta_W(n,\mathfrak{sp}(6))$ is almost exactly the same as that of $\zeta_W(n,\mathfrak{so}(7))$. If we consider the matrix corresponding to $\mathfrak{sp}(6)$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0_n & 0_n & 1_n & 0_n & 1_n & 2_n & 2_n & 2_n & 1_n \end{pmatrix}$$
$$\quad ① \quad ② \quad ③ \quad ④ \quad ⑤ \quad ⑥ \quad ⑦ \quad ⑧ \quad ⑨$$

we find the following fourteen 3-column dependencies:

$$\{(2,7,8),(4,6,7)\}\cup S(2,3,5,6)\cup S(1,5,7,9)\cup S(3,4,8,9),$$

which are similar to the case of $\mathfrak{so}(7)$. In fact, these combinations can be obtained exactly from (16) by the permutation (12)(398657). Applying this permutation to the binary tree of $\mathfrak{so}(7)$ we get



Using this tree we can easily get

Theorem 10. *Let n be an even positive integer. Set $\binom{t}{-1} = 0$ for all t. Then*

$$\frac{48\zeta_W(n,\mathfrak{sp}(6))}{720^n(2\pi\sqrt{-1})^{9n}} = \sum_{i=0}^{n}\binom{2n-i-1}{n-1}\left\{\sum_{j=0}^{i}\binom{n+i-j-1}{n-1}\left(\frac{B(j)}{2^{n+i-j}} + \frac{2^i B''(i,j)}{(-2)^j}\right)\right.$$
$$\left. + \sum_{j=0}^{n}\binom{n+i-j-1}{i-1}\left(\frac{(-1)^j B'(n,n,j)}{2^{n+i-j}} + 2^i B'(j,2n+i-j,n)\right)\right\},$$

where

$$B(j) = \sum_{k=0}^{j}\binom{n+j-k-1}{n-1}C(j,k) + \sum_{k=0}^{n}\binom{n+j-k-1}{j-1}C'(2n+j-k,k,n)$$
$$\frac{B'(a,b,j)}{(-1)^j} = \sum_{k=0}^{a}\binom{a+j-k-1}{j-1}C'(a+j,b,k) + \sum_{k=0}^{j}\binom{a+j-k-1}{a-1}\frac{C''(a+j,b,0,k)}{(-1)^k}$$
$$B''(i,j) = \sum_{k=0}^{2n+i-j}\binom{2n+i-k-1}{j-1}C''(n,k,2n+i-k,n) + \sum_{k=0}^{j}\binom{2n+i-k-1}{2n+i-j-1}C'''(i,k),$$

and

$$(-1)^k C(j,k) = \sum_{l=0}^{n}\binom{n+k-l-1}{k-1}D(j,k,l) + \sum_{l=0}^{k}\binom{n+k-l-1}{n-1}(-1)^l D(j,k,l)$$
$$C'(a,b,k) = \sum_{l=0}^{k}\binom{b+k-l-1}{b-1}\frac{D'(a,b,k,l)}{2^{b+k-l}} + \sum_{l=0}^{b}\binom{b+k-l-1}{k-1}\frac{D'(a,b,k,l)}{2^{b+k-l}}$$
$$C''(a,b,c,k) = \sum_{l=0}^{k}\binom{b+k-l-1}{b-1}D''(a,b,c,k,l) + \sum_{l=0}^{b}\binom{b+k-l-1}{k-1}D''(a,b,c,k,l)$$
$$C'''(i,k) = \sum_{l=0}^{k}\binom{n+k-l-1}{n-1}D'(k,3n-i,k,l)$$

$$+\sum_{l=0}^{n}\binom{n+k-l-1}{k-1}(-1)^l D'(k,3n-i,k,l),$$

and

$$\begin{aligned}
D(j,k,l) &= \sum_{s=0}^{2n+k-l}\binom{6n+k-l-j-s-1}{4n-j-1}\frac{E_0(l,s)}{2^{j-4n}}\\
&\quad+\sum_{s=0}^{4n-j}\binom{6n+k-l-j-s-1}{2n+k-l-1}\frac{(-1)^s E_0'(l,s)}{2^{j+s-4n}}\\
D'(a,b,k,l) &= \sum_{s=0}^{n+b+k-l}\binom{7n-a+k-l-s-1}{6n-a-b-1}E_3(l,s)\\
&\quad+\sum_{s=0}^{6n-a-b}\binom{7n-a+k-l-s-1}{n+b+k-l-1}E(l,s)\\
D''(a,b,c,k,l) &= \sum_{s=0}^{n+b+k-l}\binom{7n+k-l-a-c-s-1}{6n-a-b-c-1}2^{6n-a-b-c}E_c(l,s)\\
&\quad+\sum_{s=0}^{6n-a-b-c}\binom{7n+k-l-a-c-s-1}{n+b+k-l-1}2^{6n-a-b-c-s}E_c'(l,s).
\end{aligned}$$

Here by setting $\beta'_{s,t} = -B_sB_tB_{9n-s-t}/(s!t!(9n-s-t)!)$ *and* $\beta^{(4)}_{a,b,c} = \beta'_{a,b}(1+(2^{1-b}-1)(2^{1-c}-1))/2$ *we have*

$$\begin{aligned}
E_c(l,s) &= \sum_{t=0}^{s}\binom{8n-l-c-t-1}{8n-l-c-s-1}\frac{2^{l+c}\beta'_{l,t}}{2^{8n-t}}+\sum_{t=0}^{8n-l-c-s}\binom{8n-l-c-t-1}{s-1}\frac{2^{l+c}\beta'_{l,t}}{2^{8n-t}}\\
E_c'(l,s) &= \sum_{t=0}^{s}\binom{n+c+s-t-1}{n+c-1}\beta'_{l,t}+\sum_{t=0}^{n+c}\binom{n+c+s-t-1}{s-1}\beta'_{l,t}\\
E(l,s) &= \sum_{t=0}^{n}\binom{n+s-t-1}{s-1}(-1)^t 2^s\beta^{(4)}_{l,t,9n-l-t}+\sum_{t=0}^{s}\binom{n+s-t-1}{n-1}2^{s-t}\beta'_{l,t}.
\end{aligned}$$

For example, Maple computation shows that

$$\begin{aligned}
\zeta_W(2,\mathfrak{sp}(6)) &= \frac{720^2\cdot 2^3\cdot 19}{3^3\cdot 7\cdot 17!}\pi^{18} = \frac{19}{16209713520}\pi^{18},\\
\zeta_W(4,\mathfrak{sp}(6)) &= \frac{720^4\cdot 2^{10}\cdot 104701\cdot 3140775089}{3\cdot 7\cdot 13\cdot 19\cdot 37!}\pi^{36},\\
\zeta_W(6,\mathfrak{sp}(6)) &= \frac{720^6\cdot 2^{17}\cdot 3774593\cdot 20951970345196831001}{3\cdot 7\cdot 11\cdot 19\cdot 54!}\pi^{54},\\
\zeta_W(8,\mathfrak{sp}(6)) &= \frac{720^8\cdot 2^{23}\cdot 2343331477562563285766267904404545351\cdot a}{3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 74!}\pi^{72},\\
\zeta_W(10,\mathfrak{sp}(6)) &= \frac{720^{10}\cdot 2^{30}\cdot 5892949721278651106889655941202462587 6607\cdot b}{3^3\cdot 7\cdot 11\cdot 19\cdot 31\cdot 89!}\pi^{90},
\end{aligned}$$

where $a = 757 \cdot 769 \cdot 16651$, $b = 15403412981713647521$. Furthermore, the value $\zeta_W(2, \mathfrak{sp}(6))$ is equal to $\zeta_W(2, \mathfrak{so}(7))$ as pointed out in [6, Remark 9.1]. The numerical value of $\zeta_W(4, \mathfrak{sp}(6))$ agrees with that given in loc. cit.

7. The $\mathfrak{sl}(5)$ Case

By definition

$$\frac{\zeta_W(n; \mathfrak{sl}(5))}{288^n} = \sum_{m_1,\ldots,m_4=1}^{\infty} \left(\frac{1/\big((m_1+m_2)(m_1+m_2+m_3)(m_1+m_2+m_3+m_4)\big)}{m_1 m_2 m_3 m_4 (m_2+m_3)(m_3+m_4)(m_2+m_3+m_4)} \right)^n .$$

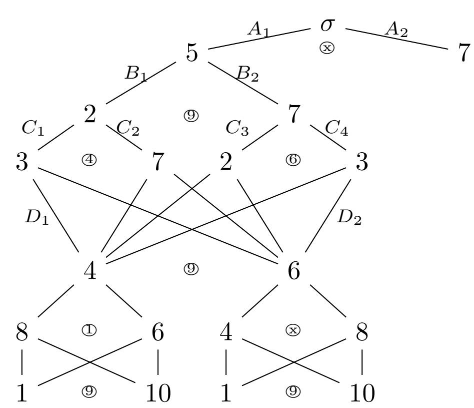
The corresponding matrix is

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0_n & 0_n & 0_n & 1_n & 0_n & 0_n & 1_n & 0_n & 1_n & 1_n \end{pmatrix}.$$
$$\quad ① \quad ② \quad ③ \quad ④ \quad ⑤ \quad ⑥ \quad ⑦ \quad ⑧ \quad ⑨ \quad \otimes$$

Here $\otimes$ is the 10th column. The set of 3-column dependencies is

$$\{(1,2,5),(1,6,8),(1,9,10),(2,3,6),(2,7,9),$$
$$(3,4,7),(3,5,8),(4,6,9),(4,8,10),(5,7,10)\}.$$

So we have the following computation tree



By symmetry $A_1 = A_2$ so we get

Theorem 11. *Let n be a positive even integer and set $w = 10n$. Set $\binom{t}{-1} = 0$ for all t. Define $\beta_{s,t,k} = 0$ if $s = 1$ or $t = 1$ or $k = 1$ and define $\beta_{s,t,k} =$*

$B_s B_t B_k B_{w-s-t-k}/(s!t!k!(w-s-t-k)!)$ *for all other nonnegative integers* s, t, k. *Then*

$$\frac{60\zeta_W(n,\mathfrak{sl}(5))}{288^n(2\pi)^{10n}} = \sum_{i=0}^{n}\binom{2n-i-1}{n-1}\left(\sum_{j=0}^{i}\binom{n+i-j-1}{n-1}B_1(i,j) + \sum_{j=0}^{n}\binom{n+i-j-1}{i-1}B_2(i,j)\right)$$

where for $\alpha = 1, 2$, *by setting* $d(j,k,l) = j-k-l-1$ *we have*

$$B_\alpha(i,j) = \sum_{k=0}^{j}\binom{n+j-k-1}{n-1}C_\alpha(i,j,k) + \sum_{k=0}^{n}\binom{n+j-k-1}{j-1}C_\alpha(i,j,k),$$

$$C_\alpha(i,j,k) = \sum_{l=0}^{n}\binom{3n+d(j,k,l)}{n-l}D_\alpha(i,k,l) + \sum_{l=0}^{2n+j-k}\binom{3n+d(j,k,l)}{n-1}D_{3-\alpha}(i,k,l),$$

$$D_\alpha(i,k,l) = \sum_{s=0}^{l}\binom{n+l-s-1}{n-1}E_\alpha(i,k,l,s) + \sum_{s=0}^{n}\binom{n+l-s-1}{l-1}E_\alpha(i,k,l,s),$$

and

$$E_1(i,k,l,s) = \sum_{t=0}^{2n+l-s}\binom{5n-t+d(l,s,i)}{3n-i-1}\beta_{s,t,k} + \sum_{t=0}^{3n-i}\binom{5n-t+d(l,s,i)}{2n+l-s-1}\beta_{s,t,k},$$

$$E_2(i,k,l,s) = \sum_{t=0}^{n}\binom{5n-t+d(l,s,i)}{n-t}\beta_{s,t,k} + \sum_{t=0}^{4n+l-i-s}\binom{5n-t+d(l,s,i)}{n-1}\beta_{s,t,k}.$$

For example, we have

$$\zeta_W(2,\mathfrak{sl}(5)) = \frac{1}{650970015609375}\pi^{20} = \frac{2^{16}\cdot 13}{3^2\cdot 5^3\cdot 7\cdot 11\cdot 18!}\pi^{20},$$

$$\zeta_W(4,\mathfrak{sl}(5)) = \frac{2^{38}\cdot 1523\cdot 2625375581}{3^2\cdot 5^2\cdot 7\cdot 11\cdot 41!}\pi^{40},$$

$$\zeta_W(6,\mathfrak{sl}(5)) = \frac{2^{57}\cdot 30677\cdot 208290556562765478732300 1}{3^2\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 31\cdot 61!}\pi^{60},$$

$$\zeta_W(8,\mathfrak{sl}(5)) = \frac{2^{79}\cdot 3^2\cdot 11\cdot 85081\cdot 1361779882876127669651\cdot 728520415874861}{5^2\cdot 7\cdot 17\cdot 82!}\pi^{80},$$

$$\zeta_W(10,\mathfrak{sl}(5)) = \frac{2^{98}\cdot 29\cdot 13^2\cdot 2143\cdot\cdot 4306678311496751027\cdot a}{3^2\cdot 5^2\cdot 11\cdot 17\cdot 101!}\pi^{100},$$

where $a = 20122334697956045252180319412759141 3$. The value of $\zeta_W(2,\mathfrak{sl}(5))$ is also given by [6, (7.63)].

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