THE $\bar{\mu}$–INVARIANT OF SEIFERT FIBERED HOMOLOGY SPHERES AND THE DIRAC OPERATOR

DANIEL RUBERMAN AND NIKOLAI SAVELIEV

ABSTRACT. We derive a formula for the $\bar{\mu}$–invariant of a Seifert fibered homology sphere in terms of the η–invariant of its Dirac operator. As a consequence, we obtain a vanishing result for the index of certain Dirac operators on plumbed 4-manifolds bounding such spheres.

1. INTRODUCTION

The $\bar{\mu}$–invariant is an integral lift of the Rohlin invariant for plumbed homology 3-spheres defined by Neumann [8] and Siebenmann [19]. It has played an important role in the study of homology cobordisms of such homology spheres. Fukumoto and Furuta [4] and Saveliev [17] showed that the $\bar{\mu}$–invariant is an obstruction for a Seifert fibered homology sphere to have finite order in the integral homology cobordism group Θ_H^3; this fact allowed them to make progress on the question of the splittability of the Rohlin homomorphism $\rho : \Theta_H^3 \to \mathbb{Z}_2$. Ue [22] and Stipsicz [20] studied the behavior of $\bar{\mu}$ with respect to rational homology cobordisms.

In the process, the $\bar{\mu}$–invariant has been interpreted in several different ways : as an equivariant Casson invariant in [2], as a Lefschetz number in instanton Floer homology in [14] and [16], and as the correction term in Heegaard Floer theory in [20] and [21].

More recently, $\bar{\mu}$ appeared in our paper [7] in connection with a Seiberg-Witten invariant λ_{SW} of a homology $S^1 \times S^3$. We conjectured that

$$\lambda_{SW}(X) = -\bar{\mu}(Y) \tag{1}$$

for any Seifert fibered homology sphere $Y = \Sigma(a_1, \ldots, a_n)$ and the mapping torus X of a natural involution on Y viewed as a link of a complex surface singularity. This conjecture will be explained in detail and proved in Section 7. For the purposes of this introduction, we will only mention that its proof will rely on the following identity.

Theorem 1.1. *Let $Y = \Sigma(a_1, \ldots, a_n)$ be a Seifert fibered homology sphere oriented as the link of complex surface singularity and endowed with a natural metric realizing the Thurston geometry on Y; see* [18]. *Then*

$$\frac{1}{2}\, \eta_{\mathrm{Dir}}(Y) + \frac{1}{8}\, \eta_{\mathrm{Sign}}(Y) = -\bar{\mu}(Y), \tag{2}$$

where $\eta_{\mathrm{Dir}}(Y)$ and $\eta_{\mathrm{Sign}}(Y)$ are the η–invariants of, respectively, the Dirac operator and the odd signature operator on Y.

In addition, identity (2) will be used to extend the vanishing result of Kronheimer for the index of the chiral Dirac operator on the E_8 manifold bounding $\Sigma(2, 3, 5)$; see [6, Lemma 2.2] and [3, Proposition 8]. Let Y be a Seifert fibered homology sphere as above, and X a plumbed manifold with boundary Y and a Riemannian metric which is a product near the boundary. Associated with X is the integral Wu class $w \in H_2(X; \mathbb{Z})$ which will be described in detail in Section 6.

Theorem 1.2. *Let $D_L^+(X)$ be the* spin^c *Dirac operator on X with $c_1(L)$ dual to the class $w \in H_2(X; \mathbb{Z})$, and with the Atiyah–Patodi–Singer boundary condition. Then $\mathrm{ind}\, D_L^+(X) = 0$. In particular, if X is spin then w vanishes and $\mathrm{ind}\, D^+(X) = 0$.*

The next four sections of the paper will be devoted to the proof of Theorem 1.1. We will proceed by expressing both sides of (2) in terms of Dedekind–Rademacher sums and by comparing the latter expressions using the reciprocity law and some elementary calculations. Theorem 1.2 will be proved in Section 6, and Conjecture (1) in Section 7. Our notation and conventions for Seifert fibered homology spheres will follow [15].

2. The η–invariants

Let $p > 0$ and $q > 0$ be pairwise relatively prime integers, and x and y arbitrary real numbers. The Dedekind–Rademacher sums were defined in [13] by the formula

$$s(q, p; x, y) = \sum_{\mu \bmod p} \left(\left(\frac{\mu + y}{p}\right)\right) \left(\left(\frac{q(\mu + y)}{p} + x\right)\right)$$

where, for any real number r, we set $\{r\} = r - [r]$ and

$$((r)) = \begin{cases} 0, & \text{if } r \in \mathbb{Z}, \\ \{r\} - 1/2, & \text{if } r \notin \mathbb{Z}. \end{cases}$$

It is clear that $s(q, p; x, y)$ only depends on $x, y \bmod 1$. When both x and y are integers, we get back the usual Dedekind sums

$$s(q, p) = \sum_{\mu \bmod p} \left(\left(\frac{\mu}{p}\right)\right) \left(\left(\frac{q\mu}{p}\right)\right). \tag{3}$$

The left-hand side of (2) was expressed by Nicolaescu [12] in terms of Dedekind–Rademacher sums. Note that since the a_i are coprime, at most one of them is even; if that occurs then we will choose the even one to be a_1.

Odd case: if all $a_1, \ldots, a_n$ are odd then, according to the formula (1.9) of [12], we have

$$\frac{1}{2}\, \eta_{\mathrm{Dir}}(Y) + \frac{1}{8}\, \eta_{\mathrm{Sign}}(Y) = -\frac{1}{8\, a_1 \cdots a_n} +$$

$$+ \frac{1}{8} + \frac{1}{2} \sum_{i=1}^{n} s(a_1 \cdots a_n/a_i, a_i) + \sum_{i=1}^{n} s(a_1 \ldots a_n/a_i, a_i; 1/2, 1/2). \tag{4}$$

Even case: if a_1 is even then, according to the formula (1.6) of [12],

$$\frac{1}{2}\,\eta_{\mathrm{Dir}}(Y) + \frac{1}{8}\,\eta_{\mathrm{Sign}}(Y) =$$

$$= \frac{1}{8} + \frac{1}{2}\sum_{i=1}^{n} s(a_1 \cdots a_n/a_i, a_i) + \sum_{i=1}^{n} s(a_1 \ldots a_n/a_i, a_i; 1/2, 1/2). \quad (5)$$

3. The $\bar{\mu}$–invariant

Let Σ be a plumbed integral homology sphere, and let X be an oriented plumbed 4-manifold such that $\partial X = \Sigma$. The integral Wu class $w \in H_2(X; \mathbb{Z})$ is the unique homology class which is characteristic and whose coordinates are either 0 or 1 in the natural basis in $H_2(X; \mathbb{Z})$ represented by embedded 2-spheres. According to Neumann [8], the integer

$$\bar{\mu}(\Sigma) = \frac{1}{8}\left(\mathrm{sign}(X) - w \cdot w\right)$$

is independent of the choices in its definition and reduces modulo 2 to the Rohlin invariant of Σ. It is referred to as the $\bar{\mu}$–invariant.

Let us now restrict ourselves to the case of $Y = \Sigma(a_1, \ldots, a_n)$. Choose integers $b_1, \ldots, b_n$ so that

$$a_1 \cdots a_n \sum_{i=1}^{n} \frac{b_i}{a_i} = \sum_{i=1}^{n} b_i\, a_1 \cdots a_n/a_i = 1. \quad (6)$$

Note that each b_i is defined uniquely modulo a_i. Then we have the following formulas for the $\bar{\mu}$–invariant; see [8, Corollary 2.3] and [9, Theorem 6.2].

Odd case: if all $a_1, \ldots, a_n$ are odd then

$$-\bar{\mu}(Y) = \frac{1}{8} - \frac{1}{8}\sum_{i=1}^{n}\left(c(a_i, b_i) + \mathrm{sign}\, b_i\right).$$

Even case: if a_1 is even, choose b_i so that all $a_i - b_i$ are all odd (by replacing, if necessary, b_i by $b_i \pm a_i$ for each $i > 1$, and then adjusting b_1 accordingly). Then

$$-\bar{\mu}(Y) = \frac{1}{8} - \frac{1}{8}\sum_{i=1}^{n} c(a_i - b_i, a_i).$$

Here, the integers $c(q,p)$ are defined for coprime integer pairs (q,p) with q odd as follows. First, assume that both p and q are positive. Then

$$c(q,p) = -\frac{1}{p}\sum_{\xi^p=-1}\frac{(\xi+1)(\xi^q+1)}{(\xi-1)(\xi^q-1)} = \frac{1}{p}\sum_{\substack{k=1\\k\ \mathrm{odd}}}^{2p-1}\cot\left(\frac{\pi k}{2p}\right)\cot\left(\frac{\pi q k}{2p}\right)$$

The integers $c(q,p)$ show up in the book [5, Theorem 1, pp. 102–103] under the name $-t_p(1,q)$. We can use that theorem together with formula (6) on page 100 of [5] to write

$$c(q,p) = -4s(q,p) + 8s(q,2p). \tag{7}$$

Next, the above definition of $c(q,p)$ is extended to both positive and negative p and q by the formula $c(q,p) = \mathrm{sign}(pq)\,c(|q|,|p|)$. Using (7), we can write the above formulas for the $\bar{\mu}$–invariant in the following form.

Odd case: if all $a_1,\ldots,a_n$ are odd then

$$-\bar{\mu}(Y) = \frac{1}{8} - \frac{1}{8}\sum_{i=1}^{n}\mathrm{sign}\,b_i + \frac{1}{2}\sum_{i=1}^{n}\mathrm{sign}\,b_i \cdot s(a_i,|b_i|)$$
$$-\sum_{i=1}^{n}\mathrm{sign}\,b_i \cdot s(a_i,2|b_i|). \tag{8}$$

Even case: if a_1 is even and b_i are chosen so that $a_i - b_i$ are all odd, then

$$-\bar{\mu}(Y) = \frac{1}{8} + \frac{1}{2}\sum_{i=1}^{n}s(a_i - b_i, a_i) - \sum_{i=1}^{n}s(a_i - b_i, 2a_i) \tag{9}$$

In the latter formula, we used a natural extension of the Dedekind sum $s(q,p)$ to the negative values of q as an odd function in q; it is still given by the formula (3). We will continue to assume, however, that p in $s(q,p)$ is positive.

4. The odd case

In this section, we will show that the right hand sides of (4) and (8) are equal to each other, thus proving the formula (2) in the case when all $a_1, \ldots, a_n$ are odd.

Lemma 4.1. *For any integers $a > 0$ and b, c such that $bc = 1 \bmod a$ we have $s(c, a) = s(b, a)$.*

Proof. Observe that $bc = 1 \bmod a$ implies that b and a are coprime hence

$$s(c, a) = \sum_{\mu \bmod a} \left(\!\!\left(\frac{\mu}{a}\right)\!\!\right) \left(\!\!\left(\frac{\mu c}{a}\right)\!\!\right) = \sum_{\mu \bmod a} \left(\!\!\left(\frac{\mu b}{a}\right)\!\!\right) \left(\!\!\left(\frac{\mu bc}{a}\right)\!\!\right)$$

$$= \sum_{\mu \bmod a} \left(\!\!\left(\frac{\mu b}{a}\right)\!\!\right) \left(\!\!\left(\frac{\mu}{a}\right)\!\!\right) = s(b, a).$$

$\square$

Lemma 4.2. *For any coprime positive integers a and b such that a is odd,*

$$\frac{1}{2} s(a, b) - s(a, 2b) \; = \; -s(a, b; 0, 1/2) - \frac{1}{2} s(a, b).$$

Proof. The proof goes by splitting the summation over $\mu \bmod 2b$ in $s(a, 2b)$ into two summations, one over even $\mu = 2\nu$, and the other over odd $\mu = 2\nu + 1$. More precisely,

$$s(a, 2b) = \sum_{\mu \bmod 2b} \left(\!\!\left(\frac{\mu}{2b}\right)\!\!\right) \left(\!\!\left(\frac{a\mu}{2b}\right)\!\!\right)$$

$$= \sum_{\nu \bmod b} \left(\!\!\left(\frac{\nu}{b}\right)\!\!\right) \left(\!\!\left(\frac{a\nu}{b}\right)\!\!\right) + \sum_{\nu \bmod b} \left(\!\!\left(\frac{2\nu + 1}{2b}\right)\!\!\right) \left(\!\!\left(\frac{a(2\nu + 1)}{2b}\right)\!\!\right)$$

$$= \sum_{\nu \bmod b} \left(\!\!\left(\frac{\nu}{b}\right)\!\!\right) \left(\!\!\left(\frac{a\nu}{b}\right)\!\!\right) + \sum_{\nu \bmod b} \left(\!\!\left(\frac{\nu + 1/2}{b}\right)\!\!\right) \left(\!\!\left(\frac{a(\nu + 1/2)}{b}\right)\!\!\right)$$

$$= s(a, b) + s(a, b; 0, 1/2).$$

The statement of the lemma now follows. $\square$

Applying Lemma 4.1 with $a = a_i$, $b = b_i$ and $c = a_1 \cdots a_n / a_i$, and Lemma 4.2 with $a = a_i$ and $b = |b_i|$ respectively to the formulas (4) and (8), we see

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that all we need to do is verify the following identity

$$-\frac{1}{8\,a_1\cdots a_n}+\frac{1}{2}\sum_{i=1}^{n}s(b_i,a_i)+\sum_{i=1}^{n}s(a_1\cdots a_n/a_i,a_i;1/2,1/2)=$$

$$-\sum_{i=1}^{n}\operatorname{sign} b_i\cdot\left(\frac{1}{8}+s(a_i,|b_i|;0,1/2)+\frac{1}{2}s(a_i,|b_i|)\right).\quad(10)$$

Use the reciprocity laws (see for instance Appendix in [12]) to obtain

$$s(a_i,|b_i|;0,1/2)=-s(|b_i|,a_i;1/2,0)+\frac{2b_i^2-a_i^2-1}{24\,a_i|b_i|}$$

and

$$s(a_i,|b_i|)=-s(|b_i|,a_i)-\frac{1}{4}+\frac{a_i^2+b_i^2+1}{12\,a_i|b_i|}.$$

Substituting the latter two formulas into (10) and keeping in mind that

$$\sum_{i=1}^{n}\frac{b_i}{a_i}=\frac{1}{a_1\cdots a_n}$$

because of (6), we reduce verification of (10) to proving the following lemma
(we write $\operatorname{sign} b_i\cdot s(|b_i|,a_i;1/2,0)=s(b_i,a_i;1/2,0)$).

Lemma 4.3. $\quad s(b_i,a_i;1/2,0)=s(a_1\cdots a_n/a_i,a_i;1/2,1/2).$

Proof. One can easily see that the identity that needs to be verified,

$$\sum_{\mu\bmod a_i}\left(\!\!\left(\frac{\mu}{a_i}\right)\!\!\right)\left(\!\!\left(\frac{b_i\mu}{a_i}+\frac{1}{2}\right)\!\!\right)=$$

$$\sum_{\nu\bmod a_i}\left(\!\!\left(\frac{\nu+1/2}{a_i}\right)\!\!\right)\left(\!\!\left(\frac{(\nu+1/2)a_1\cdots a_n/a_i}{a_i}+\frac{1}{2}\right)\!\!\right),$$

follows by substitution $\nu=b_i\,\mu+(a_i-1)/2\bmod a_i$. $\qquad\square$

5. The even case

In this section, we will prove the equality of the right hand sides of the
formulas (5) and (9) and hence prove (2) in the even case.

Lemma 5.1. $\quad s(a_i-b_i,a_i)=-s(b_i,a_i).$

Proof. Since $((x))$ is an odd function in x, we have

$$s(a_i - b_i, a_i) = \sum_{\mu \bmod a_i} \left(\!\!\left(\frac{\mu}{a_i}\right)\!\!\right) \left(\!\!\left(\frac{\mu(a_i - b_i)}{a_i}\right)\!\!\right)$$

$$= \sum_{\mu \bmod a_i} \left(\!\!\left(\frac{\mu}{a_i}\right)\!\!\right) \left(\!\!\left(\frac{-\mu b_i}{a_i}\right)\!\!\right) = -s(b_i, a_i).$$

$\square$

Using Lemma 4.1 and Lemma 5.1, we reduce our task to showing that, for every $i = 1, \ldots, n$,

$$s(a_1 \cdots a_n / a_i, a_i; 1/2, 1/2) + s(a_i - b_i, 2a_i) + s(b_i, a_i) = 0. \qquad (11)$$

Lemma 5.2. *For any coprime integers $a > 0$ and $c > 0$, we have*

$$s(c, a; 1/2, 1/2) + s(a - c, 2a) + s(c, a) = 0.$$

Proof. Like in the proof of Lemma 4.2, we will break the summation over μ mod $2a$ in $s(a - c, 2a)$ into two summations, one over $\mu = 2\nu$ and the other over $\mu = 2\nu + 1$. More precisely,

$$s(a - c, 2a) = \sum_{\mu \bmod 2a} \left(\!\!\left(\frac{\mu}{2a}\right)\!\!\right) \left(\!\!\left(\frac{(a - c)\mu}{2a}\right)\!\!\right)$$

$$= -\sum_{\nu \bmod a} \left(\!\!\left(\frac{\nu}{a}\right)\!\!\right) \left(\!\!\left(\frac{c\nu}{a}\right)\!\!\right) - \sum_{\nu \bmod a} \left(\!\!\left(\frac{2\nu + 1}{2a}\right)\!\!\right) \left(\!\!\left(\frac{c(2\nu + 1)}{2a} + \frac{1}{2}\right)\!\!\right)$$

$$= -s(c, a) - s(c, a; 1/2, 1/2).$$

$\square$

We will apply the above lemma with $a = a_i$ and $c = a_1 \cdots a_n / a_i$ to obtain $s(a_1 \cdots a_n / a_i, a_i; 1/2, 1/2) + s(a_i - a_1 \cdots a_n / a_i, 2a_i) + s(a_1 \cdots a_n / a_i, a_i) = 0$. Using Lemma 4.1 to replace $s(a_1 \cdots a_n / a_i, a_i)$ in the above formula by $s(b_i, a_i)$, we see that the proof of (11) will be complete after we prove the following formula.

Lemma 5.3. $\quad s(a_i - a_1 \cdots a_n / a_i, 2a_i) = s(a_i - b_i, 2a_i).$

Proof. This is immediate from Lemma 4.1 once we show that $(a_i - b_i)(a_i - a_1 \cdots a_n / a_i) = 1 \bmod 2a_i$. We will consider two separate cases. If $i = 1$ then a_1 is even and b_1 is odd. Multiply out to obtain $(a_1 - b_1)(a_1 - a_2 \cdots a_n) = a_1^2 + b_1 a_2 \cdots a_n - a_1(b_1 + a_2 \cdots a_n)$. Obviously, the first and the last summands are equal to zero modulo $2a_1$ because a_1 and $(b_1 + a_2 \cdots a_n)$ are even. Use the formula (6) to write $b_1 a_2 \cdots a_n = 1 - a_1(b_2 a_2 \cdots a_n + \ldots + b_n a_2 \cdots a_{n-1})$ and observe that the $b_2, \ldots, b_n$ are all even. This completes the proof in the case of $i = 1$.

Now suppose that $i \geq 2$. Since a_i and 2 are coprime, it is enough to check separately that $(a_i - b_i)(a_i - a_1 \cdots a_n / a_i)$ is 1 mod a_i and 1 mod 2. The former is clear from (6), and the latter follows from the observation that both $a_i - b_i$ and $a_i - a_1 \cdots a_n / a_i$ are odd. $\qquad\square$

6. Proof of Theorem 1.2

Endow $Y = \Sigma(a_1, \ldots, a_n)$ with a natural metric realizing the Thurston geometry on Y; see [18]. Let X be a plumbed manifold with boundary $\partial X = Y$ and with metric that restricts to the metric on Y and is a product near the boundary. If X is spin, the Atiyah–Patodi–Singer index theorem [1] asserts that

$$\frac{1}{2}\, \eta_{\mathrm{Dir}}(Y) + \frac{1}{8}\, \eta_{\mathrm{Sign}}(Y) \;=\; -\operatorname{ind} D^+(X) - \frac{1}{8}\, \mathrm{sign}(X). \qquad (12)$$

Here, we used the fact that the Dirac operator on Y has zero kernel; see Nicolaescu [10, Section 2.3]. On the other hand, it follows from the definition of the $\bar{\mu}$–invariant that $w = 0$ and hence

$$\bar{\mu}(Y) \;=\; \frac{1}{8}\, \mathrm{sign}(X).$$

The identity (2) then implies that $\operatorname{ind} D^+(X) = 0$. The special case of this when Y is the Poincaré homology sphere $\Sigma(2,3,5)$ and X is the negative definite E_8 manifold was proved by Kronheimer [6].

If X is not spin, for any choice of spin^c structure on X with determinant bundle L we have

$$\frac{1}{2}\,\eta_{\mathrm{Dir}}(Y) + \frac{1}{8}\,\eta_{\mathrm{Sign}}(Y) \;=\; -\operatorname{ind} D_L^+(X) - \frac{1}{8}\,\operatorname{sign}(X) + \frac{1}{8}\,c_1(L)^2.$$

(Compare with formula (1.37) in [11]). If the spin^c structure is such that $c_1(L)$ is dual to $w \in H_2(X;\mathbb{Z})$ then

$$\bar{\mu}(Y) = \frac{1}{8}\,(\operatorname{sign}(X) - w \cdot w) = \frac{1}{8}\,(\operatorname{sign}(X) - c_1(L)^2),$$

and (2) again implies that $\operatorname{ind} D_L^+(X) = 0$. This completes the proof of Theorem 1.2.

7. The invariant λ_{SW}

Let X be a homology $S^1 \times S^3$, by which we mean a closed oriented spin smooth 4-manifold with the integral homology of $S^1 \times S^3$. For a generic pair (g, β) consisting of a metric g on X and a perturbation $\beta \in \Omega^1(X, i\mathbb{R})$, the Seiberg–Witten moduli space $\mathcal{M}(X, g, \beta)$ has finitely many irreducible points. It is oriented by a choice of homology orientation, that is, a generator $1 \in H^1(X;\mathbb{Z})$. Let $\#\mathcal{M}(X, g, \beta)$ denote the signed count of the points in this space. To counter the dependence of $\#\mathcal{M}(X, g, \beta)$ on the choice of (g, β), we introduced in [7] a correction term, $w(X, g, \beta)$, and proved that the quantity

$$\lambda_{\mathrm{SW}}(X) \;=\; \#\mathcal{M}(X, g, \beta) - w(X, g, \beta)$$

is an invariant of X which reduces modulo 2 to its Rohlin invariant. The precise definition of the correction term is as follows.

Let $Y \subset X$ be a smooth connected 3-manifold dual to the generator $1 \in H^1(X;\mathbb{Z})$ and choose a smooth compact spin manifold Z with boundary Y. Cutting X open along Y we obtain a cobordism W from Y to itself, which we use to construct the periodic-end manifold

$$Z_+ = Z \cup W \cup W \ldots \cup W \cup \ldots$$

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The metric g and perturbation β extend to an end-periodic metric and, respectively, perturbation, on Z_+. This leads to the end-periodic perturbed Dirac operator $D^+(Z_+) + \beta$, where β acts via Clifford multiplication. We prove that $D^+(Z_+) + \beta$ is Fredholm in the usual Sobolev L^2-completion for generic (g, β). The correction term is then defined as

$$w(X, g, \beta) \;=\; \mathrm{ind}_{\mathbb{C}}\,(D^+(Z_+) + \beta) + \frac{1}{8}\,\mathrm{sign}\,(Z).$$

View $Y = \Sigma(a_1, \ldots, a_n)$ as a link of a complex surface singularity and let X be the mapping torus of the involution on Y induced by complex conjugation. The metric g realizing the Thurston geometry on Y is preserved by this involution and hence gives rise to a natural metric on X called again g. We showed in [7, Section 10] that the pair $(g, 0)$ is generic and that the space $\mathcal{M}(X, g, 0)$ is empty. One can easily see that the manifold Z_+ has a product end and hence the correction term can be computed as in (12) using the Atiyah–Patodi–Singer index theorem:

$$w(X, g, 0) \;=\; -\frac{1}{2}\,\eta_{\mathrm{Dir}}(Y) - \frac{1}{8}\,\eta_{\mathrm{Sign}}(Y).$$

The conjecture (1) now follows from Theorem 1.1.

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Department of Mathematics, MS 050

Brandeis University

Waltham, MA 02454

E-mail address: ruberman@brandeis.edu

Department of Mathematics

University of Miami

PO Box 249085

Coral Gables, FL 33124

E-mail address: saveliev@math.miami.edu