The Golod-Shafarevich inequality for Hilbert series of quadratic algebras and the Anick conjecture

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Abstract

We study the question on whether the famous Golod-Shafarevich estimate, which gives a lower bound for the Hilbert series of a (noncommutative) algebra, is attained. This question was considered by Anick in his 1983 paper 'Generic algebras and CW-complexes', Princeton Univ. Press., where he proved that the estimate is attained for the number of quadratic relations $d \leqslant \frac{n^2}{4}$ and $d \geqslant \frac{n^2}{2}$, and conjectured that it is the case for any number of quadratic relations. The particular point where the number of relations is equal to $\frac{n(n-1)}{2}$ was addressed by Vershik. He conjectured that a generic algebra with this number of relations is finite dimensional.

We prove that over any infinite field, the Anick conjecture holds for $d \geqslant \frac{4(n^2+n)}{9}$ and arbitrary number of generators, and confirm the Vershik conjecture over any field of characteristic 0. We give also a series of related asymptotic results.

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1 Introduction

Let $\mathcal{F}(n,\mathbb{K})=\mathbb{K}\langle x_1,\dots,x_n\rangle$ be a free associative algebra with n generators $x_1,\dots,x_n$, over a field $\mathbb{K}$. Recall that the free algebra carries the natural degree grading

$$\mathcal{F}(n,\mathbb{K})=\bigoplus_{k=0}^{\infty}\mathcal{F}_k(n,\mathbb{K}),\quad\text{where}\quad \mathcal{F}_k(n,\mathbb{K})=\mathtt{span}_{\mathbb{K}}\{x_{j_1}\dots x_{j_k}:1\leqslant j_1,\dots,j_k\leqslant n\}.$$

To define this grading we suppose that generators x_i all have degree one. We deal with *quadratic algebras generated in degree one*, that is, algebras given by homogeneous relations of degree 2:

$$R=\mathbb{K}\langle x_1,\dots,x_n\rangle\,/\,I,\quad\text{where}\quad I=\mathtt{Id}\,\{f_1,\dots,f_d\} \tag{1.1}$$

is the ideal generated by

$$f_1,\dots,f_d\in\mathcal{F}_2(n,\mathbb{K}):\quad f_j=\sum_{k,m=1}^{n}c_{j,k,m}x_kx_m,\quad c_{j,k,m}\in\mathbb{K}. \tag{1.2}$$

Since relations (1.2) are homogeneous, an algebra R inherits grading from the free algebra $\mathcal{F}(n,\mathbb{K})$:

$$R=\bigoplus_{k=0}^{\infty}R_k,\quad\text{where}\quad I=\bigoplus_{k=0}^{\infty}I_k,\quad\text{for}\quad I_k=I\cap\mathcal{F}_k(n,\mathbb{K})\quad\text{and}\quad R_k=\mathcal{F}_k(n,\mathbb{K})/I_k.$$

Recall also that the *Hilbert series* of R is a polynomial generating function associated to the sequence of dimensions of graded components $a_q = \dim_{\mathbb{K}} R_q$:

$$H_R(t) = \sum_{q=0}^{\infty} (\dim_{\mathbb{K}} R_q)\, t^q. \tag{1.3}$$

It belongs to the ring $\mathbb{Z}[[t]]$ of formal power series on one variable and we consider the following ordering on it. For two power series $a(t) = \sum_{j=0}^{\infty} a_j t^j$ and $b(t) = \sum_{j=0}^{\infty} b_j t^j$ (with real coefficients) we write $a(t) \geqslant b(t)$ if $a_j \geqslant b_j$ for any $j \in \mathbb{Z}_+$. For such a power series $a(t) = \sum_{j=0}^{\infty} a_j t^j$, we denote by $|a(t)|$ a series obtained from $a(t)$ by replacing all coefficients starting from the first negative one by zeros.

The famous result due to Golod and Shafarevich [4] gives a lower bound for the Hilbert series of algebras with n generators and d quadratic relations. (Throughout the text, whenever we are talking on the number of relations, we mean the number of linearly independent relations).

Theorem GS. *Let $\mathbb{K}$ be a field, $n \in \mathbb{N}$, $0 \leqslant d \leqslant n^2$ and R be a quadratic $\mathbb{K}$-algebra with n generators and d relations. Then $H_R(t) \geqslant |(1 - nt + dt^2)^{-1}|$.*

Let us note, that although we formulated above the Golod-Shafarevich estimate only for quadratic algebras, it is known for algebra with any, finite or infinite number of relations. Namely, it is as follows:

$$|(1 - nt + \sum_{i=2}^{\infty} r_i t^i)^{-1}| \leqslant H_A,$$

where r_i stands for the number of relations of degree i.

This estimate allowed to construct a counterexample for the Kurosh problem on the nilpotency of nil algebra and to the General Burnside Problem on the existence of a finitely generated infinite torsion p-group. It is also recognized due to other applications to p-groups and class field theory [5, 4].

It will be convenient for our purposes to state the Golod-Safaevich theorem also in terms of the numbers

$$h_q(\mathbb{K}, n, d) = \min_{R \in \mathcal{R}_{n,d}} \dim R_q, \tag{1.4}$$

where $\mathcal{R}_{n,d}$ is the set of all quadratic $\mathbb{K}$-algebras R with n generators and d relations. For $n \in \mathbb{N}$ and $0 \leqslant d \leqslant n^2$, consider the series

$$H^{\min}_{\mathbb{K},n,d}(t) = \sum_{q=0}^{\infty} h_q(\mathbb{K}, n, d) t^q. \tag{1.5}$$

Then Theorem GS admits the following equivalent form.

Theorem GS′. *Let $\mathbb{K}$ be a field, $n \in \mathbb{N}$ and $0 \leqslant d \leqslant n^2$. Then $H^{\min}_{\mathbb{K},n,d}(t) \geqslant |(1 - nt + dt^2)^{-1}|$.*

Note that *a priori* it is not clear why the algebra with the series $H^{\min}_{\mathbb{K},n,d}$ should exist in the class $R_{n,d}$. In fact, it is not difficult to show that not only it does exist, but it is in 'general position' in one or another sense. Usually by 'generic quadratic algebra' we mean generic in the sense of Zariski topology. Namely, we consider an algebra from $R_{n,d}$ as a point in $\mathbb{K}^{n^2 d}$, defined by the tuple of all coefficients of its defining relations. Then we say that a generic quadratic $\mathbb{K}$-algebra with n relations and d generators has a property $\mathcal{P}$, if the set $\{c_{j,k,m}\} \in \mathbb{K}^{n^2 d}$ of coefficient vectors for which the corresponding algebra R defined in (1.1) has property $\mathcal{P}$ contains a dense Zariski open subset of $\mathbb{K}^{n^2 d}$. The following proposition is a well-known fact, see, for instance [2, 3, 7].

Proposition 1.1. *Let $\mathbb{K}$ be an infinite field and $n \in \mathbb{N}$, $0 \leqslant d \leqslant n^2$. Then $\dim_{\mathbb{K}} R_q = h_q(\mathbb{K}, n, d)$ for a generic quadratic $\mathbb{K}$-algebra R with n generators and d relations. In particular, if $H^{\min}_{\mathbb{K},n,d}(t)$ is a polynomial, then $H_R(t) = H^{\min}_{\mathbb{K},n,d}(t)$ for a generic quadratic $\mathbb{K}$-algebra R with n generators and d relations.*

In the case $H^{\min}_{\mathbb{K},n,d}$ is not a polynomial, there are more subtleties involved in the question whether a generic algebra has this series. There are arguments (see [10]) showing that this is the case, when $\mathbb{K}$ is an uncountable algebraically closed field. In [3] we suggested a slightly modified notion of a 'generic' algebra, in the case $\mathbb{K} = \mathbb{R}$. Namely, we say that the generic in the Lebesgue sense algebra from $R_{n,d}$ has the property $\mathcal{P}$ if the set of algebras not having $\mathcal{P}$ has Lebesgue measure zero. We show that in this (weaker) sense a generic algebra has the series $H^{\min}$ even if it is infinite.

In his 1983 paper "Generic algebras and CW-complexes", Princeton Univ. Press, [1], Anick studied the behavior of Hilbert series of algebras given by relations and formulated the following conjecture.

Conjecture A. *For any infinite field $\mathbb{K}$, any $n, q \in \mathbb{N}$ and $0 \leqslant d \leqslant n^2$, a generic quadratic $\mathbb{K}$-algebra R with n generators and d relations $\dim R_q$ equals to the q^{th} coefficient of the series $|(1 - nt + dt^2)^{-1}|$. Equivalently, $H^{\min}_{\mathbb{K},n,d}(t) = |(1 - nt + dt^2)^{-1}|$.*

In other words, Conjecture A states that the lower estimate of the Hilbert series by Golod and Shafarevich is attained and a generic algebra has the minimal Hilbert series.

This question is very important to clarify in the light of key problems in the ring theory concerned with the behavior of nilpotent elements. Examples of such problems are the problem on the existence of simple nil ring, solved in affirmative by A.Smoktuniwicz [8], the Köthe conjecture, the Burnside type problem for finitely presented rings (see [9]).

Values of terms $h_q(\mathbb{K}, n, d)$ for $q = 0, 1, 2$ are obvious for an arbitrary algebra given by n generators and d relations: $h_0(\mathbb{K}, n, d) = 1$, $h_1(\mathbb{K}, n, d) = n$ and $h_2(\mathbb{K}, n, d) = n^2 - d$. Anick proved [1, 7] that his conjecture holds also for $q = 3$.

Theorem A. *Let $\mathbb{K}$ be any field, $n \in \mathbb{N}$ and $0 \leqslant d \leqslant n^2$. Then*

$$h_3(\mathbb{K}, n, d) = \begin{cases} 0 & \text{if } d \geqslant \frac{n^2}{2}; \\ n^3 - 2nd & \text{if } d < \frac{n^2}{2}. \end{cases} \tag{1.6}$$

Since the number in the right-hand side of (1.6) happens to coincide with the third coefficient near t^3 in $|(1 - nt + dt^2)^{-1}|$, Theorem A proves Conjecture A in the case $q = 3$ and in the case $d \geqslant \frac{n^2}{2}$. Conjecture A is also known to be true if $d \leqslant \frac{n^2}{4}$ [10, 7]. The region $\frac{n^2}{4} < d < \frac{n^2}{2}$ remained a white zone so far. Let us note that for $d > \frac{n^2}{4}$, the series $|(1-nt+dt^2)^{-1}|$ is a polynomial. Thus Conjecture A, if true, implies that a generic quadratic $\mathbb{K}$-algebra with infinite $\mathbb{K}$, n generators and $d > \frac{n^2}{4}$ relations is finite dimensional.

In [11] Vershik formulated a conjecture, which addresses a specific point of the 'difficult interval' $\frac{n^2}{4} < d < \frac{n^2}{2}$, $d = \frac{n(n-1)}{2}$, which is the number of relations defining the algebra of commutative polynomials or any PBW algebra.

Conjecture V. *Let $n \in \mathbb{N}$, $n \geqslant 3$. Then a generic quadratic $\mathbb{C}$-algebra with n generators and $\frac{n(n-1)}{2}$ relations is finite dimensional.*

As it is mentioned in [7] there was an attempt to prove this conjecture in [12], but the argument there was incorrect.

Our goal in this paper is to move the frame of the interval $(\frac{n^2}{4}, \frac{n^2}{2})$ which remains unknown since the Anick's 1983 paper. Namely, we prove the following.

Theorem 1.2. *For any infinite field, the Golod–Shafarevich estimate is attained for a generic quadratic algebra with n generators and $d \geqslant \frac{4(n^2+n)}{9}$ quadratic relations.*

Namely, the Hilbert series of the generic algebra is:

$$H(t) = |(1 - nt + dt^2)^{-1}| = 1 + nt + (n^2 - d)t^2 + (n^3 - 2nd)t^3.$$

The point $d = \frac{n(n-1)}{2}$ falls into the interval from the Theorem1.2, for big enough n, so we automatically get as a consequence an affirmative answer to the Vershik's question for $n \geqslant 17$. After some additional considerations, we get an affirmative answer for the Vershik's question for each $n \geqslant 3$ over a field of characteristic 0:

Theorem 1.3. *Let $\mathbb{K}$ be a field of characteristic 0 and $n \in \mathbb{N}$, $n \geqslant 3$. Then a generic quadratic $\mathbb{K}$-algebra R with n relations and $\frac{n(n-1)}{2}$ relations has the Hilbert series and the dimension given by the following formula*

$$H_R(t) = \begin{cases} 1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3 & \textit{if } n \geqslant 5; \\ 1 + 4t + 10t^2 + 16t^3 + t^4 & \textit{if } n = 4; \\ 1 + 3t + 6t^2 + 9t^3 + 9t^4 & \textit{if } n = 3, \end{cases} \qquad \dim_{\mathbb{K}} R = \begin{cases} \frac{3n(n+1)+2}{2} & \textit{if } n \geqslant 5; \\ 32 & \textit{if } n = 4; \\ 28 & \textit{if } n = 3. \end{cases} \tag{1.7}$$

We will formulate more explicit results later in the text. In order to illustrate them, we present their asymptotic versions straight away.

Let us note that the series of related questions on asymptotic characteristics of the Golod-Shafarevich inequality were considered in [6]

For a field $\mathbb{K}$, $n, q \in \mathbb{N}$ with $q \geqslant 2$, we denote

$$d(\mathbb{K}, n, q) = \min\{d \in \mathbb{N} : h_q(\mathbb{K}, n, d) = 0\}. \tag{1.8}$$

That is, $d(\mathbb{K}, n, q)$ is the minimal d for which there is a quadratic $\mathbb{K}$-algebra R with n generators and d relations satisfying $R_q = \{0\}$. Obviously, $d(\mathbb{K}, n, 2) = n^2$. Similarly

$$d(\mathbb{K}, n, \infty) = \min\Big\{d \in \mathbb{N} : \min_{q \in \mathbb{N}} h_q(\mathbb{K}, n, d) = 0\Big\}. \tag{1.9}$$

That is, $d(\mathbb{K}, n, \infty)$ is the minimal d for which there is a finite dimensional quadratic $\mathbb{K}$-algebra R with n generators and d relations. In order to formulate our asymptotic results we need the following lemma.

Lemma 1.4. *Let $\mathbb{K}$ be a field and $q \in \mathbb{N}$, $q \geqslant 2$ or $q = \infty$. Then the limit $\lim\limits_{n\to\infty} \frac{d(\mathbb{K},n,q)}{n^2} = \alpha(\mathbb{K}, q)$ does exist and*

$$\alpha(\mathbb{K}, q) = \lim_{n\to\infty} \frac{d(\mathbb{K}, n, q)}{n^2} = \inf\Big\{\frac{d(\mathbb{K}, n, q)}{n^2} : n \in \mathbb{N}\Big\}. \tag{1.10}$$

Moreover, $\{\alpha(\mathbb{K}, q)\}_{q\geqslant 3}$ is decreasing, $\alpha(\mathbb{K}, \infty) = \lim\limits_{q\to\infty} \alpha(\mathbb{K}, q) \geqslant \frac{1}{4}$ and $\alpha(\mathbb{K}, 3) = \frac{1}{2}$.

Theorem 1.5. *The equalities* $\alpha(\mathbb{K},4)=\frac{3-\sqrt{5}}{2}$ *and* $\alpha(\mathbb{K},5)=\frac{1}{3}$ *hold for any infinite field. Moreover,* $\frac{1}{4}\leqslant\alpha(\mathbb{K},\infty)\leqslant\alpha(\mathbb{K},6)\leqslant\frac{5}{16}$ *for any field* $\mathbb{K}$ *of characteristic* 0.

Corollary 1.6. *Let* $\mathbb{K}$ *be an infinite field and* $\varliminf\limits_{n\to\infty}\frac{d_n}{n^2}>\frac{3-\sqrt{5}}{2}$ *with* $n,d_n\in\mathbb{N}$ *and* $d_n\leqslant n^2$. *Then for any sufficiently large* n, *a generic quadratic* $\mathbb{K}$*-algebra with* n *generators and* d_n *relations has Hilbert series* $1+nt+(n^2-d_n)t^2+\max\{0,(n^3-2d_nn^2)\}t^3=|(1-nt+d_nt^2)^{-1}|$.

Proof. Indeed, by Theorem 1.5, the Hilbert series in question must be a polynomial of degree at most 3, whose specific shape is determined by Theorem A and Proposition 1.1. □

Corollary 1.7. *Let* $\mathbb{K}$ *be an infinite field and* $\varliminf\limits_{n\to\infty}\frac{d_n}{n^2}>\frac{1}{3}$ *with* $n,d_n\in\mathbb{N}$ *and* $d_n\leqslant n^2$. *Then for any sufficiently large* n, *the Hilbert series of a generic quadratic* $\mathbb{K}$*-algebra with* n *generators and* d_n *relations is a polynomial of degree at most* 4.

Corollary 1.8. *Let* $\mathbb{K}$ *be a field of characteristic* 0 *and* $\varliminf\limits_{n\to\infty}\frac{d_n}{n^2}>\frac{5}{16}$ *with* $n,d_n\in\mathbb{N}$ *and* $d_n\leqslant n^2$. *Then for any sufficiently large* n, *a generic quadratic* $\mathbb{K}$*-algebra with* n *generators and* d_n *relations has Hilbert series being a polynomial of degree at most* 5 *and therefore is finite dimensional.*

2 Notations and preliminary facts

Let $\mathbb{K}$ be a field, $n,d,q\in\mathbb{N}$, $1\leqslant d\leqslant n^2$, $q\geqslant 3$ and $\{c_{j,k,m}:1\leqslant j\leqslant d,1\leqslant k,m\leqslant n\}$ be variables taking values in $\mathbb{K}$. Consider the ideal I_c with $c=\{c_{j,k,m}\}$ in $\mathbb{K}\langle x_1,\dots,x_n\rangle$ generated by $f_1,\dots,f_d$, $f_j=\sum\limits_{k,m=1}^{n}c_{j,k,m}x_kx_m$ and the algebra $R_c=\mathbb{K}\langle x_1,\dots,x_n\rangle/I_c$. Clearly the q^{th} homogeneous component $(I_c)_q$ is spanned by $\mu f_j\nu$, where $1\leqslant j\leqslant d$ and μ,ν are two monomials in $\mathbb{K}\langle x_1,\dots,x_n\rangle$ with $\mathtt{deg}\,\mu+\mathtt{deg}\,\nu=q-2$. Hence $(I_c)_q$ is the image of the linear operator $L_c:\mathbb{K}^\Omega\to\mathcal{F}_q(n,\mathbb{K})$, where Ω is the set of triples (j,μ,ν) with $1\leqslant j\leqslant d$, μ and ν being monomials satisfying $\mathtt{deg}\,\mu+\mathtt{deg}\,\nu=q-2$ and L_c sends the standard basic vector $e_{j,\mu,\nu}$ to $\mu f_j\nu$. Then the dimension of $(I_c)_q$ equals to the rank $\mathtt{rk}\,L_c$ of L_c. Hence $\dim(R_c)_q=n^q-\dim(I_c)_q=n^q-r_c$, where $r_c=\mathtt{rk}\,L_c$. It immediately follows that

$$h_q(\mathbb{K},n,d)=n^q-r,\quad\text{where}\quad r=\max_c r_c.$$

Since the rank of a linear map equals to the maximal size of its square submatrix with non-zero determinant, there exist an $r\times r$ submatrix of the rectangular matrix of L_c, whose determinant $\delta(c)$ is non-zero for some c. On the other hand, $\delta(c)$ is a polynomial in $c_{j,k,m}$ with integer coefficients. Since a polynomial with integer coefficients over a field of characteristic 0 defines a zero function if and only if all its coefficients are zero, we see that the numbers $h_q(\mathbb{K},n,d)$ do not depend on the choice of $\mathbb{K}$ provided $\mathbb{K}$ has characteristic 0.

Now if p is a prime number and $\mathbb{K}=\mathbb{Z}_p$, then the fact that $\delta(c)$ is non-zero for some c implies that the coefficients of $\delta(c)$ are not all zeros as elements of $\mathbb{Z}_p$. Hence some of the coefficients of $\delta(c)$ considered as a polynomial with coefficients in $\mathbb{Z}$ are not multiples of p and therefore are non-zero. Hence $\delta(c)$ remains non-zero for some c after replacing the field $\mathbb{Z}_p$ by $\mathbb{Q}$. It follows that $h_q(\mathbb{Q},n,d)\leqslant h_q(\mathbb{Z}_p,n,d)$. Similar argument shows that if $h_q(\mathbb{K},n,d)$ does not depend on the choice of an infinite field $\mathbb{K}$ of a fixed positive characteristic p and that $h_q(\mathbb{K},n,d)\leqslant h_q(\mathbb{Z}_p,n,d)$ for any field $\mathbb{K}$ of characteristic p.

Next, if R is a quadratic $\mathbb{K}$-algebra with n generators $x_1,\dots,x_n$ and d relations, then the quotient R_0 of R by the ideal generated by $x_{n'+1},\dots,x_n$ is a quadratic $\mathbb{K}$-algebra with n' generators and d relations, whose homogeneous components are quotients of the homogeneous components of R. It follows that $h_q(\mathbb{K},n',d) \leqslant h_q(\mathbb{K},n,d)$ if $n' \leqslant n$. On the other hand adding new relations to an algebra can only decrease the dimension of its components. Hence $h_q(\mathbb{K},n,d) \geqslant h_q(\mathbb{K},n,d')$ if $d \leqslant d'$. Above observations are summarized in the following proposition.

Proposition 2.1. *For any field $\mathbb{K}$, $h_q(\mathbb{K},n,d)$ increase with respect to n and decrease with respect to d. Moreover, if $n \in \mathbb{N}$, $q,d \in \mathbb{Z}_+$ and $0 \leqslant d \leqslant n^2$ and p is a prime number, then $H^{\min}_{\mathbb{K},n,d}(t) = H^{\min}_{\mathbb{Q},n,d}(t) \leqslant H^{\min}_{\mathbb{Z}_p,n,d}(t)$ for any field $\mathbb{K}$ of characteristic zero and $H^{\min}_{\mathbb{K}_1,n,d}(t) = H^{\min}_{\mathbb{K}_2,n,d}(t) \leqslant H^{\min}_{\mathbb{Z}_p,n,d}(t)$ for any two infinite fields $\mathbb{K}_1$ and $\mathbb{K}_2$ of characteristic p.*

In what follows it is convenient to give an alternative definition of the numbers $h_q(\mathbb{K},n,d)$. We need the following notation. Let E be a vector space over a field $\mathbb{K}$. For $k \in \mathbb{Z}_+$, we denote the k^{th} tensor power of E by $E^{\otimes k}$. That is, $E^{\otimes 0} = \mathbb{K}$, $E^{\otimes 1} = E$ and $E^{\otimes k} = E \otimes \ldots \otimes E$ is the tensor product of k copies of E. If L is a subspace of $E^{\otimes 2} = E \otimes E$, then for $k \geqslant 2$ we can define the subspaces $\mathcal{E}_k(L,E)$ of $E^{\otimes k}$ inductively: $\mathcal{E}_2(L,E) = L$ and $\mathcal{E}_{k+1}(L,E) = E \otimes \mathcal{E}_k(L,E) \cap \mathcal{E}_k(L,E) \otimes E$. We can also write two explicit expressions for the space $\mathcal{E}_k(L,E)$:

$$\mathcal{E}_k(L,E) = \begin{cases} (E \otimes L^{\otimes(k-2)/2} \otimes E) \cap L^{\otimes(k/2)} & \text{if } k \text{ is even;} \\ (E \otimes L^{\otimes(k-1)/2}) \cap (L^{\otimes(k-1)/2} \otimes E) & \text{if } k \text{ is odd,} \end{cases} \tag{2.1}$$

$$\mathcal{E}_k(L,E) = \bigcap_{j=1}^{k-1} L^{k,j}, \quad \text{where } \ L^{k,j} = E^{\otimes(j-1)} \otimes L \otimes E^{\otimes(k-1-j)} \ \text{ for } 1 \leqslant j \leqslant k-1. \tag{2.2}$$

If E is an n-dimensional vector space over $\mathbb{K}$ with a fixed basis $\{e_1,\ldots,e_n\}$, we consider the symmetric bilinear form $[\cdot,\cdot]_j : E^{\otimes j} \times E^{\otimes j} \to \mathbb{K}$ such that

$$[e_{m_1} \otimes \ldots \otimes e_{m_j}, e_{r_1} \otimes \ldots \otimes e_{r_j}]_j = \delta_{m,r}, \quad \text{where } m,r \in \{1,\ldots,n\}^j.$$

For a subspace N of $E^{\otimes j}$ we write

$$N^{\perp} = \{f \in E^{\otimes j} : [\eta,f]_j = 0 \ \text{ for all } \ \eta \in N\}.$$

From the definition of $[\cdot,\cdot]_j$ it easily follows that the space $N^{\perp}$ is naturally isomorphic to the space of linear functionals on $E^{\otimes j}$ annihilating N. Hence

$$\dim N + \dim N^{\perp} = \dim E^{\otimes j} = n^j. \tag{2.3}$$

Moreover, one can easily see that $(N^{\perp})^{\perp} = N$ and therefore the set of $N^{\perp}$ for all d-dimensional subspaces N of $E^{\otimes j}$ coincides with the set of all $(n^j - d)$-dimensional subspaces of $E^{\otimes j}$.

Lemma 2.2. *Let $\mathbb{K}$ be a field, $n \in \mathbb{N}$, $1 \leqslant d \leqslant n^2$, E be an n-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1,\ldots,e_n\}$ and R be a quadratic $\mathbb{K}$-algebra with n generators $x_1,\ldots,x_n$ and d relations $f_s = \sum\limits_{1\leqslant a,b\leqslant n} c_{s,a,b}x_ax_b$. Let also M be the d-dimensional subspace of $E \otimes E$ spanned by $\sum\limits_{1\leqslant a,b\leqslant n} c_{s,a,b}e_a \otimes e_b$ for $1 \leqslant s \leqslant d$. Then $\dim R_q = \dim \mathcal{E}_q(M^{\perp},E)$ for any $q \geqslant 2$.*

Proof. Let $q \geqslant 2$. Under the linear isomorphism between $\mathcal{F}_q(\mathbb{K},n)$ which sends $x_{m_1}\ldots x_{m_q}$ to $e_{m_1} \otimes \ldots \otimes e_{m_q}$, the q^{th} homogeneous component I_q of the ideal I generated by $f_1,\ldots,f_d$ is mapped onto the subspace

$$\mathcal{M} = \sum_{j=1}^{q-1} M^{q,j}$$

of $E^{\otimes q}$, where $M^{q,j}$ are defined in (2.2). Using the last display, is straightforward to see that

$$\mathcal{M}^{\perp} = \bigcap_{j=1}^{q-1} (M^{q,j})^{\perp} = \bigcap_{j=1}^{q-1} (M^{\perp})^{q,j} = \mathcal{E}_q(M^{\perp}, E),$$

where the latter space is defined in (2.2). Now using (2.3), we see that

$$\dim R_q = n^q - \dim I_q = n^q - \dim \mathcal{M} = \dim \mathcal{E}_q(M^{\perp}, E),$$

which completes the proof. □

Next lemma relates the spaces $\mathcal{E}_q(L, E)$ and the numbers $h_q(\mathbb{K}, n, d)$.

Lemma 2.3. *Let $\mathbb{K}$ be a field, $q, n \in \mathbb{N}$, $d \in \mathbb{Z}_+$, $q \geqslant 3$, $0 \leqslant d \leqslant n^2$. Then*

$$h_q(\mathbb{K}, n, d) = \min\{\dim \mathcal{E}_q(L, E) : \dim L = n^2 - d\}, \tag{2.4}$$

where the minimum is taken over $(n^2 - d)$-dimensional subspaces L of $E \otimes E$ with E being an n-dimensional vector space over $\mathbb{K}$.

Proof. Follows immediately from the above lemma and the fact that the map $M \mapsto M^{\perp}$ is a bijection between the sets of d-dimensional and $(n^q - d)$-dimensional subspaces of $E^{\otimes q}$. □

3 Main lemma

Lemma 3.1. *Let $\mathbb{K}$ be a field, $n, q, m \in \mathbb{N}$, $q \geqslant 2$, E be an n-dimensional vector space and*

$$E_1 \subset E_2 \subset \ldots \subset E_{n-1} \subset E_n = E$$

is an increasing chain of subsets of E such that $\dim E_j = j$ for $1 \leqslant j \leqslant n$. Let also L be a subspace of $E \otimes E$, $n_1, \ldots, n_m \in \{1, \ldots, n\}$ and

$$G = \bigoplus_{j=1}^{m} E_{n_j} \quad \text{and} \quad L_G = \bigoplus_{1 \leqslant j,k \leqslant m} L_{j,k}, \quad \text{where} \quad L_{j,k} = (E_j \otimes E_k) \cap L.$$

Then

$$\dim G = \sum_{j=1}^{m} n_j, \quad \dim L_G = \sum_{j,k=1}^{m} \dim L_{n_j,n_k} \quad \text{and} \quad \dim \mathcal{E}_q(L_G, G) \leqslant m^q \dim \mathcal{E}_q(L, E). \tag{3.1}$$

Proof. The two equalities in (3.1) are trivial. Using the natural decomposition

$$G^{\otimes q} = \bigoplus_{a_1,\ldots,a_q=1}^{m} E_{n_{a_1}} \otimes \ldots \otimes E_{n_{a_q}}, \quad \text{we see that} \quad \mathcal{E}_q(L_G) = \bigoplus_{a_1,\ldots,a_q=1}^{m} F_a, \quad \text{where}$$

$$F_a = (L_{n_{a_1},n_{a_2}} \otimes E_{n_{a_3}} \otimes \ldots \otimes E_{n_{a_q}}) \cap (E_{n_{a_1}} \otimes L_{n_{a_2},n_{a_3}} \otimes E_{n_{a_4}} \otimes \ldots \otimes E_{n_{a_q}}) \cap \ldots$$
$$\cdots \cap (E_{n_{a_1}} \otimes \ldots \otimes E_{n_{a_{q-2}}} \otimes L_{n_{a_{q-1}},n_{a_q}}).$$

Clearly each F_a is isomorphic to a subspace of $\mathcal{E}_q(L, E)$ and therefore $\dim F_a \leqslant \dim \mathcal{E}_q(L, E)$ for any a. Now since $\mathcal{E}_q(L_G, G)$ is the sum of F_a and there are m^q multi-indices a, we get $\dim \mathcal{E}_q(L_G, G) \leqslant m^q \dim \mathcal{E}_q(L, E)$. □

Lemma 3.2. *Let $\mathbb{K}$ be a field, $n, m \in \mathbb{N}$, $q \geqslant 2$ and $d \in \mathbb{Z}_+$, $0 \leqslant d \leqslant n^2$. Then*

$$h_q(\mathbb{K}, mn, m^2 d) \leqslant m^q h_q(\mathbb{K}, n, d). \tag{3.2}$$

Proof. Let $h = h_q(\mathbb{K}, n, d)$ and E be an n-dimensional vector space over $\mathbb{K}$. By Lemma 2.3, there exists an $(n^2 - d)$-dimensional subspace L of $E \otimes E$ such that $\dim \mathcal{E}_q(L) = h$. Let G be the direct sum of m copies of E. Clearly $\dim G = nm$. Applying Lemma 3.1 with $n_1 = \ldots = n_m = n$, we find a subspace L_G of $G \otimes G$ of dimension $m^2(n^2 - d) = (mn)^2 - m^2 d$ such that $\dim \mathcal{E}(L_G, G) \leqslant m^q h$. By Lemma 2.3, $h_q(\mathbb{K}, mn, m^2 d) \leqslant m^q h$, which is the desired inequality. $\square$

Corollary 3.3. *Let $\mathbb{K}$ be a field, $n, m \in \mathbb{N}$, $q \geqslant 3$ and $d \in \mathbb{Z}_+$, $0 \leqslant d \leqslant n^2$. If $h_q(\mathbb{K}, n, d) = 0$, then $h_q(\mathbb{K}, mn, m^2 d) = 0$.*

Corollary 3.4. *Let $\mathbb{K}$ be a field, $n, m \in \mathbb{N}$ and $d \in \mathbb{Z}_+$, $0 \leqslant d \leqslant n^2$. If $H^{\min}_{\mathbb{K},n,d}(t) = |(1 - nt + dt^2)^{-1}|$, then $H^{\min}_{\mathbb{K},nm,m^2 d}(t) = |(1 - nmt + dm^2 t^2)^{-1}|$.*

Proof. Lemma 3.2 implies that $H^{\min}_{\mathbb{K},nm,m^2 d}(t) \leqslant H^{\min}_{\mathbb{K},n,d}(mt) = |(1 - nmt + dm^2 t^2)^{-1}|$. The opposite inequality follows from Theorem GS. $\square$

3.1 Proof of Lemma 1.4

Let $\mathbb{K}$ be a field and $3 \leqslant q \leqslant \infty$. For each $k \in \mathbb{N}$, let $d_k = d(\mathbb{K}, k, q)$ be the numbers defined by (1.8) and (1.9). By definition of d_n, $h_r(\mathbb{K}, n, d_n) = 0$ for some $r \in \mathbb{N}$, $3 \leqslant r \leqslant q$ (actually $r = q$ if $q < \infty$). Fix $n \in \mathbb{N}$ and let $k \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ and $j \in \{0, \ldots, n-1\}$ such that $k = mn - j$. By Proposition 1.1 and Corollary 3.3, $h_r(k, m^2 d_n) \leqslant h_r(\mathbb{K}, mn, m^2 d_n) = 0$. Hence $h_r(k, m^2 d_n) = 0$ and therefore $d_k \leqslant m^2 d_n$. Since $k = mn - j$ and $j \leqslant n - 1$, we have $m \leqslant \frac{k+n-1}{n}$. Thus $d_k \leqslant d_n \frac{(k+n-1)^2}{n^2}$ for any $k \in \mathbb{N}$. Equivalently,

$$\frac{d_k}{k^2} \leqslant \frac{d_n (k+n-1)^2}{n^2 k^2} \quad \text{for any} \quad k, n \in \mathbb{N}.$$

Passing to the limit as $k \to \infty$, we get $\varlimsup\limits_{k\to\infty} \frac{d_k}{k^2} \leqslant \frac{d_n}{n^2}$ for any $n \in \mathbb{N}$. Hence

$$\inf_{n\in\mathbb{N}} \frac{d_n}{n^2} \leqslant \varliminf_{n\to\infty} \frac{d_n}{n^2} \leqslant \varlimsup_{n\to\infty} \frac{d_n}{n^2} \leqslant \inf_{n\in\mathbb{N}} \frac{d_n}{n^2}.$$

That is, the limit $\lim\limits_{n\to\infty} \frac{d_n}{n^2}$ does exist and equals $\inf\limits_{n\in\mathbb{N}} \frac{d_n}{n^2}$.

From the definition of $d(\mathbb{K}, n, q)$ and Proposition 1.1 it immediately follows that $d(\mathbb{K}, n, q_1) \leqslant d(\mathbb{K}, n, q_2)$ if $q_1 \geqslant q_2$. Hence the sequence $\{\alpha(\mathbb{K}, q)\}_{q\geqslant 3}$ is decreasing. The equality $\alpha(\mathbb{K}, \infty) = \lim\limits_{q\to\infty} \alpha(\mathbb{K}, q)$ is also straightforward. The inequalities $\alpha(\mathbb{K}, 3) \geqslant \frac{1}{2}$ and $\alpha(\mathbb{K}, \infty) \geqslant \frac{1}{4}$ follow from Theorem GS. By Theorem A, $\alpha(\mathbb{K}, 3) \leqslant \frac{1}{2}$ and therefore $\alpha(\mathbb{K}, 3) = \frac{1}{2}$.

4 Hilbert series of degrees 3 and 4

In this section we compute $\alpha(\mathbb{K}, 4)$ and $\alpha(\mathbb{K}, 5)$ for any infinite field $\mathbb{K}$. We prove certain non-asymptotic estimates and use them to prove Theorem 1.3.

4.1 $\alpha(\mathbb{K},4)$ for any infinite field

We use the same idea as in the proof of Lemma 3.1. First, we need the following lemma.

Lemma 4.1. *Let $\mathbb{K}$ be any field, E be a vector space over $\mathbb{K}$ of dimension $n\in\mathbb{N}$, and $r\in\mathbb{N}$ be such that $1\leqslant r<n$ and $d^2+n^2d\leqslant n^4$, where $d=n^2-r^2$. Then there exist two subspaces L and M of $E\otimes E$ such that $(L\otimes L)\cap(E\otimes M\otimes E)=\{0\}$ and $\dim L=\dim M=d$.*

Proof. Pick a basis $\{e_1,\dots,e_n\}$ in E. Now we consider $M=\mathtt{span}\{e_j\otimes e_k:\max\{j,k\}>r\}$ and $L_0=\mathtt{span}(\{e_j\otimes e_k:1\leqslant j,k\leqslant r\}\cup\{e_j\otimes e_k+e_k\otimes e_j:1\leqslant j\leqslant r<k\leqslant n\})$. Clearly M and L_0 are subspaces of $E\otimes E$ of dimensions $d=n^2-r^2$ and rn respectively. The inequality $d^2+n^2d\leqslant n^4$ implies that $rn\geqslant d$ and therefore $\dim L_0\geqslant d$. It remains to show that $(L_0\otimes L_0)\cap(E\otimes M\otimes E)=\{0\}$. Indeed, then any d-dimensional subspace L of L_0 satisfies $(L\otimes L)\cap(E\otimes M\otimes E)=\{0\}$.

Let $\xi\in(L_0\otimes L_0)\cap(E\otimes M\otimes E)$. According to the definitions of L_0 and M,

$$\xi=\sum_{\substack{r<\max\{k,l\}\leqslant n\\ 1\leqslant j,m\leqslant n}} a_{j,k,l,m}e_j\otimes e_k\otimes e_l\otimes e_m=$$

$$=\sum_{1\leqslant j,k,l,m\leqslant r} b_{j,k,l,m}e_j\otimes e_k\otimes e_l\otimes e_m+\sum_{\substack{1\leqslant j,k\leqslant r\\ 1\leqslant l\leqslant r<m\leqslant n}} d_{j,k,l,m}(e_j\otimes e_k\otimes e_l\otimes e_m+e_j\otimes e_k\otimes e_m\otimes e_l)+$$

$$+\sum_{\substack{1\leqslant l,m\leqslant r\\ 1\leqslant j\leqslant r<k\leqslant n}} s_{j,k,l,m}(e_j\otimes e_k\otimes e_l\otimes e_m+e_k\otimes e_j\otimes e_l\otimes e_m)+$$

$$+\sum_{\substack{1\leqslant j\leqslant r<k\leqslant n\\ 1\leqslant l\leqslant r<m\leqslant n}} v_{j,k,l,m}(e_j\otimes e_k\otimes e_l\otimes e_m+e_j\otimes e_k\otimes e_m\otimes e_l+e_k\otimes e_j\otimes e_l\otimes e_m+e_k\otimes e_j\otimes e_m\otimes e_l),$$

where $a_{j,k,l,m}$, $b_{j,k,l,m}$, $c_{j,k,l,m}$, $d_{j,k,l,m}$, $s_{j,k,l,m}$ and $v_{j,k,l,m}$ are coefficients from $\mathbb{K}$.

If $j,k,l,m\leqslant r$, then the basic vector $e_j\otimes e_k\otimes e_l\otimes e_m$ appears in the above display only once and with the coefficient $b_{j,k,l,m}$. Hence $b_{j,k,l,m}=0$ for any j,k,l,m. If $j>r$ and $k,l,m\leqslant r$, then the basic vector $e_j\otimes e_k\otimes e_l\otimes e_m$ appears in the above display only once and with the coefficient $s_{k,j,l,m}$. Hence $s_{j,k,l,m}=0$ for any j,k,l,m. If $m>r$ and $j,k,l\leqslant r$, then the basic vector $e_j\otimes e_k\otimes e_l\otimes e_m$ appears in the above display only once and with the coefficient $d_{j,k,l,m}$. Hence $d_{j,k,l,m}=0$ for any j,k,l,m. If $j,m>r$ and $k,l\leqslant r$, then the basic vector $e_j\otimes e_k\otimes e_l\otimes e_m$ appears in the above display only once and with the coefficient $d_{k,j,l,m}$. Hence $d_{j,k,l,m}=0$ for any j,k,l,m. Thus the right-hand side of the above display vanishes and so does ξ. Hence $(L_0\otimes L_0)\cap(E\otimes M\otimes E)=\{0\}$. □

Just the same way as we speak of generic quadratic algebras with fixed number of relations and generators, we can speak of generic vector subspaces of given dimension in a fixed finite dimensional vector space over an infinite field. Let $\mathbb{K}$ be an infinite field and E be an n-dimensional vector space over $\mathbb{K}$. Using the argument exactly as in Section 2, one can easily show that if there exist d-dimensional subspaces L_0 and M_0 of $E\otimes E$ satisfying $(L_0\otimes L_0)\cap(E\otimes M_0\otimes E)=\{0\}$, then the equality $(L\otimes L)\cap(E\otimes M\otimes E)=\{0\}$ holds for generic d-dimensional subspaces L and M of $E\otimes E$. Similarly, if there exist d-dimensional subspaces L_0, N_0 and M_0 of $E\otimes E$ satisfying $(L_0\otimes N_0)\cap(E\otimes M_0\otimes E)=\{0\}$, then the equality $(L\otimes N)\cap(E\otimes M\otimes E)=\{0\}$ holds for generic d-dimensional subspaces L, N and M of $E\otimes E$. Thus Lemma 4.1 implies the following fact.

Lemma 4.2. *Let $\mathbb{K}$ be an infinite field, E be a vector space over $\mathbb{K}$ of dimension $n\in\mathbb{N}$, and $r\in\mathbb{N}$ be such that $1\leqslant r<n$ and $d^2+n^2d\leqslant n^4$, where $d=n^2-r^2$. Then $(L\otimes L)\cap(E\otimes M\otimes E)=\{0\}$ and $(L\otimes N)\cap(E\otimes M\otimes E)=\{0\}$ for generic d-dimensional subspaces L, N and M of $E\otimes E$.*

Proposition 4.3. *There exists a positive constant C such that*

$$\frac{3-\sqrt{5}}{2}n^2 < d(\mathbb{K},n,4) \leqslant \frac{3-\sqrt{5}}{2}n^2 + Cn^{3/2} \tag{4.1}$$

for any $n\in\mathbb{N}$ and any infinite field $\mathbb{K}$. In particular, $\alpha(\mathbb{K},4)=\frac{3-\sqrt{5}}{2}$ for any infinite field $\mathbb{K}$.

Proof. Let $n,d\in\mathbb{N}$ and $d\leqslant n^2$. Since the coefficient in $|(1-nt+dt^2)|$ in front of t^4 is $\max\{0,n^4-3n^2d+d^2\}$, it is positive if $d<\frac{3-\sqrt{5}}{2}n^2$. By Theorem GS, $h_4(\mathbb{K},n,d)>0$ if $d<\frac{3-\sqrt{5}}{2}n^2$. Hence $d(\mathbb{K},n,4)>\frac{3-\sqrt{5}}{2}n^2$ for any field $\mathbb{K}$.

Let now $\mathbb{K}$ be an infinite field and $n\in\mathbb{N}$. Choose $m\in\mathbb{N}$ such that $(m-1)^2<n\leqslant m^2$. Now let r be the unique integer such that $\frac{\sqrt{5}-1}{2}m<r<\frac{\sqrt{5}-1}{2}m+1$. Since $r^2>\frac{3-\sqrt{5}}{2}m^2$, we have $d<\frac{\sqrt{5}-1}{2}m^2$, where $d=m^2-r^2$. The latter inequality implies $d^2+m^2d<m^4$. Let $E_1,\dots,E_m$ be m-dimensional vector spaces over $\mathbb{K}$ and $E=E_1\oplus\ldots\oplus E_m$. Obviously $\dim E=m^2\geqslant n$. Consider the space

$$L=\bigoplus_{j,k=1}^m L_{j,k},$$

where $L_{j,k}$ is a d-dimensional subspace of $E_j\otimes E_k$ if $j\neq k$ and $L_{j,j}=\{0\}$ for $1\leqslant j\leqslant m$. Clearly $\dim L=(m^2-m)d$. According to (2.1), $\mathcal{E}_4(L,E)=(L\otimes L)\cap(E\otimes L\otimes E)$. Hence

$$\mathcal{E}_4(L,E)=\bigoplus_{j,k,l,s=1}^m M_{j,k,l,s}, \quad \text{where} \quad M_{j,k,l,s}=(L_{j,k}\otimes L_{l,s})\cap(E_j\otimes L_{k,l}\otimes E_s).$$

Since $L_{j,j}=0$, we have $M_{j,k,l,s}=\{0\}$ if $j=k$, or $k=l$, or $l=s$. If $j\neq k$, $k\neq l$ and $l\neq s$, then either (j,k), (k,l) and (l,s) are three different pairs or $(j,k)=(l,s)\neq(k,l)$. In any case Lemma 4.2 implies that $M_{j,k,l,s}=\{0\}$ for generic d-dimensional $L_{j,k}$ ($j\neq k$). According to the last display $\mathcal{E}_4(L,E)=\{0\}$ for generic d-dimensional $L_{j,k}$ ($j\neq k$). Thus, there exists a $d(m^2-m)$-dimensional subspace L of $E\otimes E$ such that $\mathcal{E}_4(L,E)=\{0\}$. By Lemma 2.3, $h_4(\mathbb{K},m^2,m^4-(m^2-m)d)=0$. Since $n\leqslant m^2$, we get $h_4(\mathbb{K},n,m^4-(m^2-m)d)=0$. Hence

$$d(\mathbb{K},n,4)\leqslant m^4-(m^2-m)d.$$

Since $d=m^2-r^2$ and $r<\frac{\sqrt{5}-1}{2}m+1$, we have $d>\frac{\sqrt{5}-1}{2}m^2-(\sqrt{5}-1)m-1$. Using this inequality together with $\sqrt{n}\leqslant m\leqslant\sqrt{n}+1$ and the fact that the functions $m\mapsto m^2-m$ and $m\mapsto\frac{\sqrt{5}-1}{2}m^2-(\sqrt{5}-1)m-1$ on $[1,\infty)$ are increasing, we see that the above display implies

$$d(\mathbb{K},n,4)\leqslant(\sqrt{n}+1)^4-(n-\sqrt{n})\Big(\frac{\sqrt{5}-1}{2}n-(\sqrt{5}-1)\sqrt{n}-1\Big)=$$
$$=\frac{3-\sqrt{5}}{2}n^2+\frac{5+3\sqrt{5}}{2}n^{3/2}+(8-\sqrt{5})n+3\sqrt{n}+1.$$

The above display immediately implies (4.1). □

4.2 An estimate of $d(\mathbb{K},n,4)$ for any field

We prove the following specific lemma in the appendix.

Lemma 4.4. *Let* $\mathbb{K}$ *be any field and*

$$R = \mathbb{K}\langle x_1, x_2, x_3\rangle / \mathtt{Id}\{x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 + x_3^2\}.$$

Then the Hilbert series of R *is* $1 + 3t + 5t^2 + 4t^3$. *In particular,* $h_4(\mathbb{K}, 3, 4) = 0$ *for any field* $\mathbb{K}$.

Corollary 4.5. *Let* $\mathbb{K}$ *be a field,* E *be a three-dimensional vector space over* $\mathbb{K}$ *with a basis* $\{e_1, e_2, e_3\}$ *and* L *be the* 5*-dimensional subspace of* $E \otimes E$ *spanned by* $e_2 \otimes e_1$, $e_3 \otimes e_2$, $e_3 \otimes e_1$, $e_1 \otimes e_1 - e_2 \otimes e_2$ *and* $e_1 \otimes e_1 - e_3 \otimes e_3$. *Then* $\mathcal{E}_4(L, E) = \{0\}$, *where* $\mathcal{E}_k(L, E)$ *are defined in (2.2).*

Proof. Let R be the algebra defined in Lemma 4.4. From Lemmas 4.4 and 2.2 it follows that $0 = \dim R_4 = \dim \mathcal{E}_4(M^\perp, E)$, where M is the 4-dimensional subspace of $E \otimes E$ spanned by the vectors $e_1 \otimes e_2$, $e_1 \otimes e_3$, $e_2 \otimes e_3$ and $e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$. It is straightforward to verify that $L = M^\perp$. Thus $\mathcal{E}_4(L, E) = \{0\}$. $\square$

Proposition 4.6. *Let* $\mathbb{K}$ *be a field. Then the numbers* $d(\mathbb{K}, n, 4)$ *defined in (1.8) satisfy the following inequality*

$$d(\mathbb{K}, n, 4) \leqslant d_n = \begin{cases} \frac{4n^2}{9} & \text{if } n = 3k,\ k \in \mathbb{N}; \\ \frac{4n^2+2n-2}{9} & \text{if } n = 3k+2,\ k \in \mathbb{Z}_+; \\ \frac{4n^2+4n-8}{9} & \text{if } n = 3k+1,\ k \in \mathbb{N}. \end{cases} \tag{4.2}$$

In any case $d(\mathbb{K}, n, 4) < \frac{4(n^2+n)}{9}$.

Proof. Let E_3 be a 3-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, e_2, e_3\}$ and $E_2 = \mathtt{span}\{e_1, e_2\}$. Consider the 5-dimensional subspace L of $E_3 \otimes E_3$ spanned by $e_2 \otimes e_1$, $e_3 \otimes e_2$, $e_3 \otimes e_1$, $e_1 \otimes e_1 - e_2 \otimes e_2$ and $e_1 \otimes e_1 - e_3 \otimes e_3$ and let $L_{2,2} = (E_2 \otimes E_2) \cap L$, $L_{2,3} = (E_2 \otimes E_3) \cap L$ and $L_{3,2} = (E_3 \otimes E_2) \cap L$. By Corollary 4.5, $\mathcal{E}_4(L, E_3) = \{0\}$. It is also easy to see that $L_{2,3} = L_{2,2}$ is spanned by $e_2 \otimes e_1$ and $e_1 \otimes e_1 - e_2 \otimes e_2$ and therefore $\dim L_{2,3} = \dim L_{2,2} = 2$. On the other hand $L_{3,2}$ is spanned by $e_2 \otimes e_1$, $e_3 \otimes e_2$, $e_3 \otimes e_1$ and $e_1 \otimes e_1 - e_2 \otimes e_2$ and therefore $\dim L_{3,2} = 4$.

Let $n = 3k$ with $k \in \mathbb{N}$. Then the direct sum G of k copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = \ldots = n_k = 3$, there exists a subspace L_G of $G \otimes G$ of dimension $k^2 \dim L = 5k^2 = \frac{5n^2}{9}$ such that $\mathcal{E}_4(L_G, G) = \{0\}$. By Lemma 2.3, $h_4(\mathbb{K}, n, \frac{4n^2}{9}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{4n^2}{9}$.

Let $n = 3k + 2$ with $k \in \mathbb{Z}_+$. Then the direct sum G of E_2 and k copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = 2$ and $n_2 = \ldots = n_{k+1} = 3$, there is a subspace L_G of $G \otimes G$ such that $\mathcal{E}_4(L_G, G) = \{0\}$ and $\dim L_G = k^2 \dim L + k(\dim L_{2,3} + \dim L_{3,2}) + \dim L_{2,2} = 5k^2 + 6k + 2 = \frac{5n^2-2n+2}{9}$. By Lemma 2.3, $h_4(\mathbb{K}, n, \frac{4n^2+2n-2}{9}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{4n^2+2n-2}{9}$.

Let $n = 3k + 1$ with $k \in \mathbb{N}$. Then the direct sum G of 2 copies of E_2 and $k - 1$ copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = n_2 = 2$ and $n_3 = \ldots = n_{k+1} = 3$, there is a subspace L_G of $G \otimes G$ such that $\mathcal{E}_4(L_G, G) = \{0\}$ and $\dim L_G = (k-1)^2 \dim L + 2(k-1)(\dim L_{2,3} + \dim L_{3,2}) + 4 \dim L_{2,2} = 5k^2 + 2k + 1 = \frac{5n^2-4n+8}{9}$. By Lemma 2.3, $h_4(\mathbb{K}, n, \frac{4n^2+4n-8}{9}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{4n^2+4n-8}{9}$. $\square$

Corollary 4.7. *Let* $\mathbb{K}$ *be any field. Then* $h_4(\mathbb{K}, n, \frac{n(n-1)}{2}) = 0$ *for any* $n \geqslant 17$ *and for* $n \in \{9, 12, 14, 15\}$.

Proof. Let $A = \{9, 12, 14, 15\} \cup \{n \in \mathbb{N} : n \geqslant 17\}$. By Proposition 4.6, $h_4(\mathbb{K}, n, d_n) = 0$, where d_n is defined in (4.2). It is straightforward to verify that $\frac{n(n-1)}{2} \geqslant d_n$ for $n \in A$. Hence $h_4(\mathbb{K}, n, \frac{n(n-1)}{2}) = 0$ for $n \in A$. $\square$

4.3 Proof of Theorem 1.2

Proposition 4.6 was the main step in the proof of Theorem 1.2. It ensures that for $d \geqslant \frac{4(n^2+n)}{9}$ the fourth component of the generic Hilbert series vanishes. Now we combine this with the already known due to Anick fact (Theorem A), that the third component of the generic series always coincides with the third component of the Golod-Shafarevich series (which vanishes for $d \geqslant \frac{n^2}{2}$). So, by looking at the third and fourth components of the Hilbert series we obtain the statement of Theorem 1.2.

4.4 $\alpha(\mathbb{K}, 5)$ for any field

The proof of the following Lemma is quite technical and is presented in the appendix.

Lemma 4.8. *Let $\mathbb{K}$ be a field and $R = \mathbb{K}\langle x_1, x_2, x_3\rangle/I$ with $I = \mathtt{Id}\{f_1, f_2, f_3\}$ and*

$$f_1 = x_3^2 - x_1x_2, \quad f_2 = x_3x_2 - x_2x_3 + x_2x_1 - x_1x_3 - x_1x_2 + x_1^2, \quad f_3 = x_3x_1 + x_2^2 - x_1^2.$$

Then the Hilbert series of R is $H_R(t) = 1 + 3t + 6t^2 + 9t^3 + 9t^4 = |(1 - 3t + 3t^2)^{-1}|$.

Corollary 4.9. *Let $\mathbb{K}$ be a field, E be a three-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, e_2, e_3\}$ and L be the 5-dimensional subspace of $E \otimes E$ spanned by Consider the 6-dimensional subspace L of $E_3 \otimes E_3$ spanned by $e_3 \otimes e_3 + e_1 \otimes e_2 + e_1 \otimes e_1 + e_2 \otimes e_2$, $e_2 \otimes e_3 + e_1 \otimes e_1 + e_2 \otimes e_2$, $e_1 \otimes e_3 + e_1 \otimes e_1 + e_2 \otimes e_2$, $e_2 \otimes e_1 - e_1 \otimes e_1 - e_2 \otimes e_2$, $e_3 \otimes e_2 - e_1 \otimes e_1 - e_2 \otimes e_2$ and $e_3 \otimes e_1 + e_1 \otimes e_1 - e_2 \otimes e_1$. Then $\mathcal{E}_5(L, E) = \{0\}$.*

Proof. Let R be the algebra defined in Lemma 4.8. From Lemmas 4.8 and 2.2 it follows that $0 = \dim R_5 = \dim \mathcal{E}_5(M^\perp, E)$, where M is the 3-dimensional subspace of $E \otimes E$ spanned by the vectors $e_3 \otimes e_3 - e_1 \otimes e_2$, $e_3 \otimes e_2 - e_2 \otimes e_3 + e_2 \otimes e_1 - e_1 \otimes e_3 - e_1 \otimes e_2 + e_1 \otimes e_1$ and $e_3 \otimes e_1 + e_2 \otimes e_2 - e_1 \otimes e_1$. It is straightforward to verify that $L = M^\perp$. Thus $\mathcal{E}_5(L, E) = \{0\}$. $\square$

Proposition 4.10. *Let $\mathbb{K}$ be any field. Then the numbers $d(\mathbb{K}, n, 5)$ defined in (1.8) satisfy the following inequality*

$$d(\mathbb{K}, n, 5) \leqslant \delta_n = \begin{cases} \frac{n^2}{3} & \text{if } n = 3k,\ k \in \mathbb{N}; \\ \frac{n^2+2n+1}{3} & \text{if } n = 3k+2,\ k \in \mathbb{Z}_+; \\ \frac{n^2+3n+1}{3} & \text{if } n = 3k+1,\ k \in \mathbb{N}. \end{cases} \tag{4.3}$$

In particular, $\alpha(\mathbb{K}, 5) = \frac{1}{3}$ for any field $\mathbb{K}$.

Proof. Let E_3 be a 3-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, e_2, e_3\}$, $E_2 = \mathtt{span}\{e_1, e_2\}$ and $E_1 = \mathtt{span}\{e_1\}$. Consider the 6-dimensional subspace L of $E_3 \otimes E_3$ defined in Corollary 4.9 and let $L_{j,k} = (E_j \otimes E_k) \cap L$ for $1 \leqslant j, k \leqslant 3$. By Corollary 4.9, $\mathcal{E}_5(L, E_3) = \{0\}$. Estimating from below the dimensions of $L_{j,k}$ by the number of basic vectors of L contained in $E_j \otimes E_k$, we get $\dim L_{3,3} = \dim L = 6$, $\dim L_{2,3} \geqslant 3$, $\dim L_{3,2} \geqslant 3$, $\dim L_{2,2} \geqslant 1$ and $\dim L_{3,1} \geqslant 1$.

Let $n = 3k$ with $k \in \mathbb{N}$. Then the direct sum G of k copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = \ldots = n_k = 3$, there exists a subspace L_G of $G \otimes G$ of dimension $k^2 \dim L = 6k^2 = \frac{2n^2}{3}$ such that $\mathcal{E}_5(L_G, G) = \{0\}$. By Lemma 2.3, $h_5(\mathbb{K}, n, \frac{n^2}{3}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{n^2}{3}$.

Let $n = 3k + 2$ with $k \in \mathbb{Z}_+$. Then the direct sum G of E_2 and of k copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = 2$ and $n_2 = \ldots = n_{k+1} = 3$, there exists a subspace L_G of $G \otimes G$ such that $\mathcal{E}_5(L_G, G) = \{0\}$ and $\dim L_G = k^2 \dim L + k(\dim L_{2,3} +$

$\dim L_{3,2}) + \dim L_{2,2} \geqslant 6k^2 + 6k + 1 = \frac{2n^2-2n-1}{3}$. By Lemma 2.3, $h_4(\mathbb{K}, n, \frac{n^2+2n+1}{3}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{n^2+2n+1}{3}$.

Let $n = 3k+1$ with $k \in \mathbb{N}$. Then the direct sum G of E_1 and k copies of E_3 has dimension n. By Lemma 3.1 with $n_1 = 1$ and $n_2 = \ldots = n_{k+1} = 3$, there exists a subspace L_G of $G \otimes G$ such that $\mathcal{E}_5(L_G, G) = \{0\}$ and $\dim L_G = k^2 \dim L + k(\dim L_{1,3} + \dim L_{3,1}) + \dim L_{1,1} \geqslant 6k^2 + k = \frac{2n^2-3n-1}{3}$. By Lemma 2.3, $h_5(\mathbb{K}, n, \frac{n^2+3n+1}{3}) = 0$. Hence $d(\mathbb{K}, n, 4) \leqslant \frac{n^2+3n+1}{3}$.

Thus we have verified (4.3), from which it follows that $\alpha(\mathbb{K}, 5) \leqslant \frac{1}{3}$. On the other hand, for $0 \leqslant d < \frac{n^2}{3}$, the coefficient in $|(1-nt+dt^2)^{-1}|$ in front of t^5 is $n^5 - 4n^3d + 3nd^2 > 0$. By Theorem GS, $h_5(\mathbb{K}, n, d) > 0$ for $d < \frac{n^2}{3}$. It follows that $\alpha(\mathbb{K}, 5) \geqslant \frac{1}{3}$. Thus $\alpha(\mathbb{K}, 5) = \frac{1}{3}$. $\square$

5 Further applications of Lemma 3.1

As we have seen in the last section, specific examples with 3 generators produce non-trivial estimates for $d(\mathbb{K}, n, 4)$ and $d(\mathbb{K}, n, 5)$ for any $\mathbb{K}$ and n. We proceed along the same lines. To this end we need more specific examples of Hilbert series of quadratic algebras. We produce them via lucky guesswork and application of the software package GRAAL ('Graded Algebras' by A.Verevkin and A.Kondratiev) to find the Hilbert series. This program uses the one-sided Gröbner basis technique to calculate the Hilbert series of $\mathbb{Z}_p$-algebras given by generators and relations.

Example 5.1. Let $R = \mathbb{Z}_2\langle x_1, \ldots, x_7\rangle / I$ with $I = \mathtt{Id}\{f_1, \ldots, f_{19}\}$ and

$$
\begin{array}{lll}
f_1 = x_1x_7, & f_2 = x_3x_7 + x_4x_6 + x_6x_2, & f_3 = x_5x_7 + x_6x_4 + x_3x_5 + x_2x_1 + x_4x_3,\\
f_4 = x_7x_1 + x_1x_6, & f_5 = x_7x_2 + x_6x_1 + x_1x_5, & f_6 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2,\\
f_7 = x_2x_7 + x_7x_3, & f_8 = x_6x_7 + x_3x_6 + x_4x_5 + x_5x_2, & f_9 = x_7x_5 + x_2x_6 + x_5x_3 + x_1x_4 + x_3x_2,\\
f_{10} = x_5x_7 + x_7x_6, & f_{11} = x_7x_6 + x_6x_2 + x_5x_1 + x_3x_4, & f_{12} = x_7x_4 + x_6x_3 + x_2x_5 + x_3x_2 + x_4x_1,\\
f_{13} = x_7x_5 + x_4x_7, & f_{14} = x_7x_2 + x_3x_6 + x_6x_4, & f_{15} = x_2x_7 + x_6x_5 + x_5x_4 + x_3x_1 + x_4x_2,\\
f_{16} = x_3x_7 + x_7x_4, & f_{17} = x_4x_7 + x_2x_6 + x_6x_3, & f_{18} = x_7x_3 + x_4x_6 + x_5x_2 + x_2x_4 + x_3x_1,\\
f_{19} = x_6x_7 + x_6x_4 + x_2x_6 + x_2x_5 + x_3x_5 + x_4x_5. & &
\end{array}
$$

Then the Hilbert series of R is $H_R(t) = 1 + 7t + 30t^2 + 77t^3 = |(1 - 7t + 19t^2)^{-1}|$.

Example 5.2. Let $R = \mathbb{Z}_2\langle x_1, x_2, x_3, x_4\rangle / I$ with $I = \mathtt{Id}\{f_1, \ldots, f_6\}$ and

$$
\begin{array}{lll}
f_1 = x_1x_2, & f_2 = x_1x_4 + x_4x_2 + x_2x_3, & f_3 = x_1x_3 + x_3x_4 + x_4x_1,\\
f_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2, & f_5 = x_3x_4 + x_4x_2 + x_2x_4, & f_6 = x_2x_3 + x_3x_1 + x_1x_3.
\end{array}
$$

Then the Hilbert series of R is $H_R(t) = 1 + 4t + 10t^2 + 16t^3 + t^4 = |(1 - 4t + 6t^2)^{-1}|$.

Example 5.3. Let $R = \mathbb{Z}_2\langle x_1, x_2, x_3, x_4\rangle / I$ with $I = \mathtt{Id}\{f_1, \ldots, f_5\}$ and

$$
\begin{array}{lll}
f_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2, & f_2 = x_1x_2 + x_2x_3 + x_3x_4, & f_3 = x_4x_1 + x_1x_3 + x_3x_2,\\
f_4 = x_1x_3 + x_3x_2 + x_2x_4, & f_5 = x_1x_4 + x_4x_3 + x_3x_2 + x_2x_4. &
\end{array}
$$

Then the Hilbert series of R is $H_R(t) = 1 + 4t + 11t^2 + 24t^3 + 41t^4 + 44t^5 = |(1 - 4t + 5t^2)^{-1}|$.

5.1 Proof of Theorem 1.5

The equalities $\alpha(\mathbb{K}, 4) = \frac{3-\sqrt{5}}{2}$ and $\alpha(\mathbb{K}, 5) = \frac{1}{3}$ follow from Propositions 4.3 and 4.10 respectively.

Let $\mathbb{K}$ be a field of characteristic 0. Example 5.3 shows that $d(\mathbb{Z}_2, 4, 6) \leqslant 5$. By Proposition 2.1 $d(\mathbb{K}, 4, 6) \geqslant 5$. By Lemma 1.4

$$\alpha(\mathbb{K}, \infty) \leqslant \alpha(\mathbb{K}, 6) = \lim_{n\to\infty} \frac{d(\mathbb{K}, n, 6)}{n^2} = \inf_{n\in\mathbb{N}} \frac{d(\mathbb{K}, n, 6)}{n^2} \leqslant \frac{d(\mathbb{K}, 4, 6)}{16} \leqslant \frac{5}{16}.$$

This completes the proof of Theorem 1.5.

5.2 Proof of Theorem 1.3

Throughout this section $\mathbb{K}$ is a field of characteristic 0, $n \geqslant 3$. Lemma 4.8 together with Theorem GS show that $H^{\min}_{\mathbb{K},3,3}(t) = 1+3t+6t^2+9t^3+9t^4 = |(1+3t-3t^2)^{-1}|$, which proves Theorem 1.3 for $n = 3$.

Example 5.2 and Proposition 2.1 show that $H^{\min}_{\mathbb{K},4,6}(t) \leqslant H^{\min}_{\mathbb{Z}_2,4,6}(t) \leqslant 1 + 4t + 10t^2 + 16t^3 + t^4 = |(1 - 4t + 6t^2)^{-1}|$. By Theorem GS, $H^{\min}_{\mathbb{K},4,6}(t) \geqslant |(1 - 4t + 6t^2)^{-1}|$. Hence $H^{\min}_{\mathbb{K},4,6}(t) = 1+4t+10t^2+16t^3+t^4 = |(1-4t+6t^2)^{-1}|$, which proves Theorem 1.3 for $n = 4$.

From now on $n \geqslant 5$. It suffices to prove that

$$h_4(\mathbb{K}, n, \delta_n) = 0, \quad \text{where } \delta_n = \frac{n(n-1)}{2}. \tag{5.1}$$

Indeed, Theorem A implies that $h_3(\mathbb{K}, n, \delta_n) = n^2$ and therefore $H^{\min}_{\mathbb{K},n,\delta_n}(t) = 1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3 = |(1 + nt - \delta_n t^2)^{-1}|$ provided (5.1) is true. Thus it remains to prove (5.1) for $n \geqslant 5$.

If $n \in A = \{9, 12, 14, 15\} \cup \{n \in \mathbb{N} : n \geqslant 17\}$, Corollary 4.7 implies that (5.1) is satisfied. It remains to consider $n \in \{5, 6, 7, 8, 10, 11, 13, 16\}$.

According to the remarks in the beginning of Section 2, the relations in Example 5.1 considered as relations in $\mathbb{K}\langle x_1, \dots, x_7\rangle$ define a quadratic $\mathbb{K}$-algebra R such that $H_R(t) \leqslant 1 + 7t + 30t^2 + 77t^3 = |(1 - 7t + 19t^2)^{-1}|$. On the other hand, by Theorem GS, $H_R(t) \geqslant |(1 - 7t + 19t^2)^{-1}|$. Hence

$$H_R(t) = 1 + 7t + 30t^2 + 77t^3.$$

Let E be a 7-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, \dots, e_7\}$ and $E_j = \mathtt{span}\,\{e_1, \dots, e_j\}$ for $1 \leqslant j \leqslant 7$. Let also M be the 19-dimensional subspace of $E \otimes E$ spanned by the elements obtained from the relations of R by replacing x_jx_k by $e_j \otimes e_k$. It is straightforward to verify that the 30-dimensional subspace $L = M^\perp$ of $E \otimes E$ is spanned by the following 30 vectors: $g_1 = e_1 \otimes e_2$, $g_2 = e_2 \otimes e_3$, $g_3 = e_1 \otimes e_3$, $g_4 = e_1 \otimes e_1 - e_2 \otimes e_2$, $g_5 = e_1 \otimes e_1 - e_3 \otimes e_3$, $g_6 = e_2 \otimes e_4 - e_3 \otimes e_1 + e_4 \otimes e_2$, $g_7 = e_1 \otimes e_4 - e_3 \otimes e_2 + e_4 \otimes e_1$, $g_8 = e_1 \otimes e_1 - e_4 \otimes e_4$, $g_9 = e_2 \otimes e_1 - e_4 \otimes e_3$, $g_{10} = e_1 \otimes e_1 - e_5 \otimes e_5$, $g_{11} = e_5 \otimes e_1 - e_3 \otimes e_4$, $g_{12} = e_5 \otimes e_4 - e_4 \otimes e_2$, $g_{13} = e_5 \otimes e_3 - e_3 \otimes e_2 + e_4 \otimes e_1$, $g_{14} = e_5 \otimes e_2 - e_4 \otimes e_5 - e_2 \otimes e_4 + e_3 \otimes e_5$, $g_{15} = e_2 \otimes e_5 - e_3 \otimes e_5 - e_3 \otimes e_2 + e_1 \otimes e_4$, $g_{16} = e_1 \otimes e_1 - e_6 \otimes e_6$, $g_{17} = e_6 \otimes e_1 - e_1 \otimes e_5$, $g_{18} = e_5 \otimes e_6$, $g_{19} = e_4 \otimes e_6 - e_6 \otimes e_2 + e_3 \otimes e_4 - e_2 \otimes e_4$, $g_{20} = e_6 \otimes e_4 - e_2 \otimes e_1 - e_3 \otimes e_5 - e_3 \otimes e_6 + e_5 \otimes e_2 - e_2 \otimes e_4$, $g_{21} = e_2 \otimes e_6 - e_1 \otimes e_4 - e_6 \otimes e_3 - e_3 \otimes e_5 + e_4 \otimes e_1$, $g_{22} = e_6 \otimes e_5 - e_4 \otimes e_2$, $g_{23} = e_1 \otimes e_1 - e_7 \otimes e_7$, $g_{24} = e_7 \otimes e_1 - e_1 \otimes e_6$, $g_{25} = e_3 \otimes e_7 - e_7 \otimes e_4 - e_4 \otimes e_6 + e_2 \otimes e_4 + e_4 \otimes e_1$, $g_{26} = e_5 \otimes e_7 - e_7 \otimes e_6 - e_2 \otimes e_1 + e_3 \otimes e_4$, $g_{27} = e_7 \otimes e_2 - e_6 \otimes e_1 - e_6 \otimes e_4 + e_3 \otimes e_5$, $g_{28} = e_2 \otimes e_7 - e_7 \otimes e_3 + e_2 \otimes e_4 - e_4 \otimes e_2$, $g_{29} = e_6 \otimes e_7 - e_3 \otimes e_5 - e_5 \otimes e_2 + e_2 \otimes e_4$ and $g_{30} = e_7 \otimes e_5 - e_4 \otimes e_7 + e_6 \otimes e_3 - e_4 \otimes e_1 - e_1 \otimes e_4$.

As usual, $L_{j,k} = L \cap (E_j \otimes E_k)$ for $2 \leqslant j, k \leqslant 7$. Denoting $d_j = \dim L_{j,j}$ for $1 \leqslant j \leqslant 7$, we easily obtain $d_7 = 30$, $d_6 = 22$, $d_5 = 15$ and $d_4 = 9$. Denoting $d_{j,k} = \dim L_{j,k} + \dim L_{k,j}$ for $1 \leqslant j < k \leqslant 7$ and estimating $\dim L_{l,m}$ from below by the number of basic vectors g_s in $E_l \otimes E_m$, we get $\dim d_{6,7} \geqslant 47$, $d_{5,6} \geqslant 33$ and $d_{4,6} \geqslant 22$.

By Lemma 2.2, the equality $R_4 = \{0\}$ implies

$$\mathcal{E}_4(L, E) = 0. \tag{5.2}$$

Condition (2.2) implies that $\mathcal{E}_4(L_{n,n}, E_n) = \{0\}$ for $1 \leqslant n \leqslant 7$. Using this equality and applying Lemma 2.3, we get $h_4(\mathbb{K}, n, n^2 - d_n) = 0$ for $1 \leqslant n \leqslant 7$. This equality for $n \in \{5, 6, 7\}$ gives $h_4(\mathbb{K}, 5, 10) = h_4(\mathbb{K}, 6, 14) = h_4(\mathbb{K}, 7, 19) = 0$. Since $\delta_5 = 10$, $\delta_6 = 15 > 14$ and $\delta_7 = 21 > 19$, (5.1) is satisfied for $n \in \{5, 6, 7\}$.

Using Lemma 3.1 with $m = 2$ and $n_1 = n_2 = 4$, we see that there exists a subspace N of $G \otimes G$ such that $\dim G = 8$, $\dim N \geqslant 4d_4 = 36$ and $\mathcal{E}_4(N, G) = 0$. By Lemma 2.3, $h_4(\mathbb{K}, 8, 28) = 0$, which is (5.1) for $n = 8$.

Using Lemma 3.1 with $m = 2$ and $n_1 = n_2 = 5$, we see that there exists a subspace N of $G \otimes G$ such that $\dim G = 10$, $\dim N \geqslant 4d_5 = 60$ and $\mathcal{E}_4(N, G) = 0$. By Lemma 2.3, $h_4(\mathbb{K}, 10, 40) = 0$. Hence $h_4(\mathbb{K}, 10, 45) = 0$ which is (5.1) for $n = 10$.

Using Lemma 3.1 with $m = 2$ and $n_1 = 5$ and $n_2 = 6$, we see that there exists a subspace N of $G \otimes G$ such that $\dim G = 11$, $\dim N \geqslant d_6 + d_5 + d_{5,6} \geqslant 22 + 15 + 33 = 70$ and $\mathcal{E}_4(N, G) = 0$. By Lemma 2.3, $h_4(\mathbb{K}, 11, 51) = 0$. Hence $h_4(\mathbb{K}, 11, 55) = 0$ which is (5.1) for $n = 11$.

Using Lemma 3.1 with $m = 2$ and $n_1 = 6$ and $n_2 = 7$, we see that there exists a subspace N of $G \otimes G$ such that $\dim G = 13$, $\dim N \geqslant d_7 + d_6 + d_{6,7} \geqslant 30 + 22 + 47 = 99$ and $\mathcal{E}_4(N, G) = 0$. By Lemma 2.3, $h_4(\mathbb{K}, 13, 70) = 0$. Hence $h_4(\mathbb{K}, 13, 78) = 0$ which is (5.1) for $n = 13$.

Using Lemma 3.1 with $m = 3$ and $n_1 = n_2 = 6$ and $n_3 = 4$, we see that there exists a subspace N of $G \otimes G$ such that $\dim G = 16$, $\dim N \geqslant 4d_6 + d_4 + 2d_{4,6} \geqslant 4 \cdot 22 + 9 + 2 \cdot 22 = 141$ and $\mathcal{E}_4(N, G) = 0$. By Lemma 2.3, $h_4(\mathbb{K}, 16, 115) = 0$. Hence $h_4(\mathbb{K}, 16, 120) = 0$ which is (5.1) for $n = 16$.

The proof of Theorem 1.3 is complete.

6 Appendix: Proof of Lemmas 4.4 and 4.8

In the proofs of both Lemmas 4.4 and 4.8 we use the non-commutative analog of the Buchberger algorithm of constructing a Gröbner basis.

6.1 Proof of Lemma 4.4

Ordering the variables as $x_1 > x_2 > x_3$ and considering the degree-lexicographical ordering on the monomials, one can easily see that the set

$$\{x_2x_3, x_1, x_3, x_1x_2, x_1^2 + x_2^2 + x_3^2, x_2^3 + x_3^2x_2, x_2^2x_1 + x_3^2x_1, x_3^2, x_2^2, x_3^2x_2x_1\}$$

is the reduced Gröbner basis of the ideal $I = \mathtt{Id}\{x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 + x_3^2\}$. By analyzing the above basis, one can easily verify that $\{x_3^2x_2 + I, x_3^2x_1 + I, x_3x_2^2 + I, x_3x_2x_1 + I\}$ is a linear basis in R_3 and that $R_4 = \{0\}$. Hence the Hilbert series of R is $1 + 3t + 5t^2 + 4t^3$. The proof is complete.

6.2 Proof of Lemma 4.8

We consider the ordering $x_1 > x_2 > x_3$ on the variables and the corresponding degree-lexicographical ordering on the monomials.

First, we assume that the characteristic of $\mathbb{K}$ is different from 2. Using the non-commutative analog of the Buchberger algorithm of constructing a Gröbner basis, we find the following homogeneous elements of I, written starting from the leading term (=highest monomial) and having the property that neither of the leading terms are subwords of the others:

$$g_1=x_1x_3-x_2x_1-x_2^2+x_2x_3-x_3x_1-x_3x_2+x_3^2,\quad g_2=x_1x_2-x_3^2,\quad g_3=x_1^2-x_1x_3+x_2x_1-x_2x_3+x_3x_2-x_3^2,$$
$$g_4=x_2^2x_3+x_2x_3x_1+2x_2x_3x_2-3x_2x_3^2-x_3x_1^2+2x_3x_1x_3-x_3x_2^2+x_3x_2x_3-x_3^2x_1+x_3^2x_2-x_3^3,$$
$$g_5=x_2^3+x_3x_2x_1-x_3^2x_1+x_3^2x_2,\quad g_6=x_2^2x_1+x_2x_3x_1+x_2x_3x_2-2x_2x_3^2-x_3x_1x_2+x_3x_1x_3+x_3x_2x_3-x_3^3,$$
$$g_7=2x_2x_3x_2x_1+x_3x_1^3-x_3x_1^2x_3+6x_3x_1x_2x_1-3x_3x_1x_2^2-x_3x_1x_2x_3-x_3x_1x_3x_1+3x_3x_1x_3x_2-$$
$$-2x_3x_1x_3^2-5x_3x_2x_1^2-4x_3x_2x_1x_2+6x_3x_2x_1x_3-2x_3x_2^2x_1+x_3x_2^3+2x_3x_2^2x_3-3x_3x_2x_3x_1+$$
$$+x_3x_2x_3x_2+3x_3x_2x_3^2+3x_32x_1^2+4x_3^2x_1x_2+x_3^2x_1x_3-8x_3^2x_2x_1-9x_3^2x_2^2+9x_3^2x_2x_3-5x_3^3x_2-3x_3^4,$$
$$g_8=x_2x_3x_2^2-x_3x_1^3-x_3x_1^2x_3-3x_3x_1x_2x_1+3x_3x_1x_2^2-x_3x_1x_2x_3+2x_3x_1x_3x_1+x_3x_1x_3x_2-x_3x_1x_3^2+2x_3x_2x_1^2-$$
$$-3x_3x_2x_1x_2-4x_3x_2x_1x_3+2x_3x_2^2x_1-x_3x_2^2x_3+x_3x_2x_3x_1+x_3x_2x_3x_2+x_3x_2x_3^2-$$
$$-3x_3^2x_1^2+3x_3^2x_1x_2+2x_3^2x_1x_3+x_3^2x_2x_1+4x_3^2x_2^2-4x_3^2x_2x_3+3x_3^3x_1-x_3^3x_2+2x_3^4,$$
$$g_9=2x_2x_3x_2x_3+x_3x_1^3+9x_3x_1x_2x_1-5x_3x_1x_2^2+2x_3x_1x_2x_3-x_3x_1x_3x_1+2x_3x_1x_3x_2-2x_3x_1x_32-10x_3x_2x_1^2-$$
$$-5x_3x_2x_1x_2+18x_3x_2x_1x_3-4x_3x_2^2x_1-2x_3x_2^3+4x_3x_2^2x_3-4x_3x_2x_3x_1-4x_3x_2x_3x_2+3x_3x_2x_3^2+$$
$$+7x_3^2x_1^2+5x_3^2x_1x_2-4x_3^2x_1x_3-16x_3^2x_2x_1-19x_3^2x_2^2+24x_3^2x_2x_3-4x_3^3x_1+4x_3^3x_2-13x_3^4,$$
$$g_{10}=2x_2x_3^2x_1+x_3x_1^3-x_3x_1^2x_3+4x_3x_1x_2x_1-x_3x_1x_2^2-x_3x_1x_2x_3-x_3x_1x_3x_1+3x_3x_1x_3x_2-2x_3x_1x_3^2-5x_3x_2x_1^2-$$
$$-6x_3x_2x_1x_2+6x_3x_2x_1x_3+x_3x_2^3+2x_3x_2^2x_3-3x_3x_2x_3x_1+x_3x_2x_3x_2+3x_3x_2x_3^2+3x_3^2x_1^2+$$
$$+6x_3^2x_1x_2+x_3^2x_1x_3-10x_3^2x_2x_1-9x_3^2x_2^2+9x_3^2x_2x_3+2x_3^3x_1-5x_3^3x_2-3x_3^4,$$
$$g_{11}=2x_2x_3^2x_2-x_3x_1^3-3x_3x_1x_2x_1+3x_3x_1x_2^2+x_3x_1x_3x_1-5x_3x_2x_1x_2+2x_3x_2^2x_1+x_3x_2x_32-x_3^2x_1^2+$$
$$+5x_3^2x_1x_2-2x_3^2x_2x_1-x_3^2x_2^2+4x_3^3x_1-x_3^4,$$
$$g_{12}=x_2x_3^3+x_3x_1x_2x_1+x_3x_1x_3x_2-x_3x_1x_3^2-2x_3x_2x_1^2-3x_3x_2x_1x_2+3x_3x_2x_1x_3+x_3x_2^2x_3-x_3x_2x_3x_1+x_3x_2x_3^2+$$
$$+x_3^2x_1^2+3x_3^2x_1x_2-4x_3^2x_2x_1-4x_3^2x_2^2+4x_3^2x_2x_3+x_3^3x_1-x_3^3x_2-2x_3^4,$$
$$g_{13}=x_3^2x_2x_3x_1,\quad g_{14}=x_3^2x_2x_3x_2,\quad g_{15}=x_3^2x_2x_3^2,\quad g_{16}=x_3^3x_2x_1,\quad g_{17}=x_3^3x_2^2,$$
$$g_{18}=x_3^3x_2x_3,\quad g_{19}=x_3^4x_1,\quad g_{20}=x_3^4x_2,\quad g_{21}=x_3^5.$$

It follows that the dimension of R_q does not exceed the number of monomials of degree q that do not contain a subword being the leading monomial of one of g_j. This observation gives $\dim R_3\leqslant 9$, $\dim R_4\leqslant 9$ and $\dim R_5=0$. Thus $H_R(t)\leqslant 1+3t+6t^2+9t^3+9t^4=|(1-3t+3t^2)^{-1}|$. On the other hand, Theorem GS implies that $H_R(t)\geqslant|(1-3t+3t^2)^{-1}|$. Hence $H_R(t)=1+3t+6t^2+9t^3+9t^4=|(1-3t+3t^2)^{-1}|$.

It remains to consider the case $\operatorname{char}\mathbb{K}=2$. Using the same algorithm, we find the following homogeneous elements of I, written starting from the leading term (=highest monomial) and having the property that neither of the leading terms are subwords of the others:

$$g_1=x_1^2+x_2^2+x_3x_1,\quad g_2=x_1x_2+x_3^2,\quad g_3=x_1x_3+x_2x_1+x_2^2+x_2x_3+x_3x_1+x_3x_2+x_3^2,$$
$$g_4=x_2^2x_1+x_2x_3x_1+x_2x_3x_2+x_3x_2x_1+x_3x_2^2+x_3^2x_1+x_3^2x_2+x_3^3,$$
$$g_5=x_2^3+x_3x_2x_1+x_3^2x_1+x_3^2x_2,\quad g_6=x_2^2x_3+x_2x_3x_1+x_2x_3^2+x_3x_2x_3+x_3^2x_2+x_3^3,$$
$$g_7=x_2x_3x_2x_1+x_2x_3^2x_1+x_3x_2x_3x_2+x_3^2x_2x_1+x_3^3x_1,$$
$$g_8=x_2x_3x_2^2+x_2x_3^2x_2+x_2x_3^3+x_3x_2x_3x_2+x_3x_2x_3^2+x_3^2x_2x_1+x_3^2x_2x_3+x_3^3x_2+x_3^4,$$
$$g_9=x_2x_3x_2x_3+x_2x_3^3+x_3x_2x_3x_1+x_3^2x_2^2+x_3^2x_2x_3+x_3^3x_1+x_3^4,$$
$$g_{10}=x_2x_3^2x_2+x_3x_2x_3x_2+x_3x_2x_3^2+x_3^2x_2x_1+x_3^2x_2^2+x_3^3x_1+x_3^4,$$
$$g_{11}=x_2x_3^3+x_3x_2x_3x_2+x_3x_2x_3^2+x_3^2x_2x_1+x_3^3x_1,\quad g_{12}=x_3x_2x_3x_1+x_3x_2x_3^2+x_3^2x_2x_1+x_3^2x_2^2,$$
$$g_{13}=x_3x_2x_3^2x_1,\quad g_{14}=x_3^2x_2x_3x_2,\quad g_{15}=x_3^2x_2x_3^2,\quad g_{16}=x_3^3x_2x_1,$$
$$g_{17}=x_3^3x_2^2,\quad g_{18}=x_3^3x_2x_3,\quad g_{19}=x_3^4x_1,\quad g_{20}=x_3^4x_2,\quad g_{21}=x_3^5.$$

Again, the dimension of R_q does not exceed the number of monomials of degree q that do not contain a subword being the leading monomial of one of g_j. This observation gives $\dim R_3\leqslant 9$, $\dim R_4\leqslant 9$ and $\dim R_5=0$. Thus $H_R(t)\leqslant 1+3t+6t^2+9t^3+9t^4=|(1-3t+3t^2)^{-1}|$. On the other hand, Theorem GS implies that $H_R(t)\geqslant|(1-3t+3t^2)^{-1}|$. Hence $H_R(t)=1+3t+6t^2+9t^3+9t^4=|(1-3t+3t^2)^{-1}|$. The proof is complete.

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