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# Involutive distributions of operator-valued evolutionary vector fields and their affine geometry

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We generalize the notion of a Lie algebroid over an infinite jet bundle by replacing the variational anchor with an $N$-tuple of differential operators the images of which in the Lie algebra of evolutionary vector fields of the jet space are subject to collective commutation closure. The linear space of such operators becomes an algebra with bidifferential structural constants, of which we study the canonical structure. In particular we show that these constants incorporate bidifferential analogues of Christoffel symbols.

## 1 Introduction

Lie algebroids [21] are an important and convenient construction that appear, e.g., in classical Poisson dynamics [2] or the theory of quantum Poisson manifolds [1, 22]. Essentially Lie algebroids extend the tangent bundle $TM$ over a smooth manifold $M$, retaining the information about the $C^\infty(M)$-module structure for its sections. In the paper [10] we defined Lie algebroids over the infinite jet spaces for mappings between smooth manifolds (e.g., from strings to space-time); the classical definition [21] is recovered by shrinking the source manifold to a point. A special case of Lie algebroids over spaces of finite jets for sections of the tangent bundle was firstly considered in [15]. Within the variational setup the anchors become linear matrix differential operators that map sections which belong to horizontal modules [13] to the generating sections $\varphi$ of evolutionary derivations $\partial_\varphi$ on the jet space; by assumption the images of such anchors are closed under commutation in the Lie algebra of evolutionary vector fields. The two main examples of variational anchors are the recursions with involutive images [8] and the Hamiltonian operators (see [12, 13, 19] and [8]) the domains of which consist of variational vectors and covectors, respectively.

In [8] we studied the *linear compatibility* of variational anchors, meaning that $N$ operators with a common domain span an $N$-dimensional linear space $\mathcal{A}$ such that each point $A_\lambda \in \mathcal{A}$ is itself an anchor with involutive image. For example Poisson-compatible Hamiltonian operators are linearly compatible and *vice versa* (Hamiltonian operators are *Poisson-compatible* if their linear combinations remain

Hamiltonian). The linear compatibility[1] allows us to reduce the case of many operators $A_1, \ldots, A_N$ to one operator $A_{\boldsymbol{\lambda}} = \sum \lambda_i \cdot A_i$ with the same properties.

In this paper we introduce a different notion of compatibility for the $N$ operators. Strictly speaking we consider the class of structures which is wider than the set of Lie algebroids over jet spaces, namely, we relax the assumption that each operator alone is a variational anchor, but instead we deal with $N$-tuples of total differential operators $A_1, \ldots, A_N$ the images of which are subject to the collective commutation closure: $\left[\sum_{i=1}^N \operatorname{im} A_i, \sum_{j=1}^N \operatorname{im} A_j\right] \subseteq \sum_{k=1}^N \operatorname{im} A_k$. This involutivity condition converts the linear space of operators to an algebra with bidifferential structural constants $\mathbf{c}_{ij}^k$, see (6) below. The Magri scheme [16] for the restriction of compatible Hamiltonian operators to the hierarchy of Hamiltonians yields an example of such an overlapping for $N = 2$ with $\mathbf{c}_{ij}^k \equiv 0$.

We study the standard decomposition of the structural constants $\mathbf{c}_{ij}^k$, which is similar to the previously known case (1) for $N = 1$ ([7,8,10]). From the bidifferential constants $\mathbf{c}_{ij}^k$ we extract the components $\Gamma_{ij}^k$ that act by total differential operators on both arguments at once. Our main result, Theorem 3, states that under a change of coordinates in the domain the symbols $\Gamma_{ij}^k$ are transformed by a proper analogue (11) of the classical rule $\Gamma \mapsto g\,\Gamma g^{-1} + \mathrm{d}g\, g^{-1}$ for the connection 1-forms $\Gamma$ and reparametrizations $g$. We note that the bidifferential symbols $\Gamma_{ij}^k$ are symmetric in their lower indices if the common domain of the $N$ operators $A_i$ consists of the variational covectors and hence its elements acquire their own odd grading.[2]

This note is organized as follows. In Section 2 we introduce operators with collective closure under commutation. For consistency we recall here the cohomological formulation [11] of the Magri scheme which gives us an example. In Section 3 we study the properties of the bidifferential constants that appear in such algebras of operators. The analogues of Christoffel symbols emerge here; as an example we calculate them for the symmetry algebra of the Liouville equation.

## 2 Compatible differential operators

We begin with some notation; for a more detailed exposition of the geometry of integrable systems we refer to [19] and [4,12,14,17]. In the sequel the ground field is the field $\mathbb{R}$ of real numbers and all mappings are $C^\infty$-smooth.

Let $\pi\colon E^{m+n} \xrightarrow[M^m]{} B^n$ be a vector bundle over an orientable $n$-dimensional manifold $B^n$ and, similarly, let $\xi\colon N^{d+n} \longrightarrow B^n$ be another vector[3] bundle

[1]When the set of admissible linear combinations $\{\boldsymbol{\lambda}\} \subsetneq \mathbb{R}^N$ has punctures near which the homomorphisms $A_{\boldsymbol{\lambda}}$ exhibit a nontrivial analytic behaviour, this concept reappears in the theory of continuous contractions of Lie algebras (see [18] and references therein).

[2]Throughout this paper we deal with a purely commutative setup, refraining from the treatment of supermanifolds. However, we emphasize that on a supermanifold the two notions of *parity* and *grading* (or *weight*) may be totally uncorrelated, see [22].

[3]For this paper the established term '*vector* bundle' is particularly unfortunate because in our

over $B^n$. Consider the bundle $\pi_\infty\colon J^\infty(\pi) \to B^n$ of infinite jets of sections for the bundle $\pi$ and take the pull-back $\pi_\infty^*(\xi)\colon N^{d+n} \times_{B^n} J^\infty(\pi) \to J^\infty(\pi)$ of the bundle $\xi$ along $\pi_\infty$. By definition the $C^\infty(J^\infty(\pi))$-module of sections $\Gamma(\pi_\infty^*(\xi)) = \Gamma(\xi) \otimes_{C^\infty(B^n)} C^\infty(J^\infty(\pi))$ is called *horizontal*, see [13] for further details.

For example let $\xi := \pi$. Then the variational vectors $\varphi \in \Gamma(\pi_\infty^*(\pi))$ are the generating sections of evolutionary derivations $\partial_\varphi$ on $J^\infty(\pi)$. For convenience we use the shorthand notation $\varkappa(\pi) \equiv \Gamma(\pi_\infty^*(\pi))$ and $\Gamma\Omega(\xi_\pi) \equiv \Gamma(\pi_\infty^*(\xi))$ in the general setup.

We consider firstly the case $N = 1$ for which there is only one total differential operator, $A\colon \Gamma\Omega(\xi_\pi) \to \varkappa(\pi)$, with involutive image

$$[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A. \tag{1}$$

The operator $A$ transfers the bracket in the Lie algebra $\mathfrak{g}(\pi) = \big(\varkappa(\pi), [\,,\,]\big)$ to the Lie algebraic structure $[\,,\,]_A$ on the quotient of its domain by the kernel. The standard decomposition of this bracket is [8, 10]

$$[\boldsymbol{p}, \boldsymbol{q}]_A = \partial_{A(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(\boldsymbol{p}) + \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A, \qquad \boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi). \tag{2}$$

The *linear compatibility* of operators (4), which means that their arbitrary linear combinations $A_{\boldsymbol{\lambda}} = \sum_i \lambda_i \cdot A_i$ satisfy (1), reduces the case of $N \geq 2$ operators to the previous case with $N = 1$ as follows.

**Theorem 1** ([8]). *The bracket $\{\{\,,\,\}\}_{A_{\boldsymbol{\lambda}}}$ induced by the combination $A_{\boldsymbol{\lambda}} = \sum_i \lambda_i \cdot A_i$ on the domain of the linearly compatible normal*[4] *operators $A_i$ is*

$$\{\{\,,\,\}\}_{\sum\limits_{i=1}^{N} \lambda_i A_i} = \sum_{i=1}^{N} \lambda_i \cdot \{\{\,,\,\}\}_{A_i}.$$

*The pairwise linear compatibility implies the collective linear compatibility of $A_1, \ldots, A_N$.*

**Proof.** Consider the commutator $\big[\sum_i \lambda_i A_i(\boldsymbol{p}), \sum_j \lambda_j A_j(\boldsymbol{q})\big]$, here $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$. On one hand it is equal to

$$\begin{aligned} &= \sum_{i\neq j} \lambda_i\lambda_j \big[A_i(\boldsymbol{p}), A_j(\boldsymbol{q})\big] \\ &+ \sum_i \lambda_i^2 A_i\big(\partial_{A_i(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A_i(\boldsymbol{q})}(\boldsymbol{p}) + \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_i}\big). \end{aligned} \tag{3}$$

main Example 1 the sections of such a bundle are variational *co*vectors and obey a nonvectorial transformation law.

[4]By definition, a total differential operator $A$ is *normal* if $A \circ \nabla = 0$ implies $\nabla = 0$; in other words it may be that $\ker A \neq 0$, but the kernel does not have any functional freedom for its elements, see [7].

On the other hand the linear compatibility of $A_i$ implies

$$= A_{\boldsymbol{\lambda}}\bigl(\partial_{A_{\boldsymbol{\lambda}}(\boldsymbol{p})}(\boldsymbol{q})\bigr) - A_{\boldsymbol{\lambda}}\bigl(\partial_{A_{\boldsymbol{\lambda}}(\boldsymbol{q})}(\boldsymbol{p})\bigr) + A_{\boldsymbol{\lambda}}\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_{\boldsymbol{\lambda}}}\bigr).$$

The entire commutator is quadratically homogeneous in $\boldsymbol{\lambda}$, whence the bracket $\{\{\,,\,\}\}_{A_{\boldsymbol{\lambda}}}$ is linear in $\boldsymbol{\lambda}$. From (3) we see that the individual brackets $\{\{\,,\,\}\}_{A_i}$ are contained in it. Therefore

$$\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_{\boldsymbol{\lambda}}} = \sum_{\ell} \lambda_\ell \cdot \{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_\ell} + \sum_{\ell} \lambda_\ell \cdot \gamma_\ell(\boldsymbol{p},\boldsymbol{q}),$$

where $\gamma_\ell\colon \Gamma\Omega(\xi_\pi)\times\Gamma\Omega(\xi_\pi)\to\Gamma\Omega(\xi_\pi)$.

We claim that all summands $\gamma_\ell(\cdot,\cdot)$, which do not depend upon $\boldsymbol{\lambda}$ at all, vanish. Indeed, assume the converse. Let there be $\ell\in[1,\ldots,N]$ such that $\gamma_\ell(\boldsymbol{p},\boldsymbol{q})\neq 0$; without loss of generality suppose that $\ell=1$. Then set $\boldsymbol{\lambda}=(1,0,\ldots,0)$, whence

$$\Bigl[\sum_i \lambda_i A_i(\boldsymbol{p}), \sum_j \lambda_j A_j(\boldsymbol{q})\Bigr] = \Bigl[(\lambda_1 A_1)(\boldsymbol{p}), (\lambda_1 A_1)(\boldsymbol{q})\Bigr] = (\lambda_1 A_1)\bigl(\lambda_1\gamma_1(\boldsymbol{p},\boldsymbol{q})\bigr)$$
$$+(\lambda_1 A_1)\Bigl(\partial_{(\lambda_1 A_1)(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{(\lambda_1 A_1)(\boldsymbol{q})}(\boldsymbol{p}) + \lambda_1\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_1}\Bigr) = \lambda_1 A_1\bigl(\lambda_1[\boldsymbol{p},\boldsymbol{q}]_{A_1}\bigr).$$

Consequently, $\gamma_\ell(\boldsymbol{p},\boldsymbol{q})\in\ker A_\ell$ for all $\boldsymbol{p}$ and $\boldsymbol{q}$. Now we use the assumption that each operator $A_\ell$ is normal. This implies that $\gamma_\ell=0$ for all $\ell$ which concludes the proof. ∎

Now we let $N>1$ and consider $N$-tuples of linear total differential operators

$$A_1,\ldots,A_N\colon \Gamma\Omega(\xi_\pi)\longrightarrow \varkappa(\pi), \tag{4}$$

the images of which in the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields on $J^\infty(\pi)$ are subject to collective closure of commutators.

**Definition 1.** We say that $N\geq 2$ total differential operators (4) are *strongly compatible* if the sum of their images is closed under commutation in the Lie algebra $\mathfrak{g}(\pi)=\bigl(\varkappa(\pi),[\,,\,]\bigr)$ of evolutionary vector fields,

$$\Bigl[\sum_i \operatorname{im} A_i, \sum_j \operatorname{im} A_j\Bigr] \subseteq \sum_k \operatorname{im} A_k, \qquad 1\leq i,j,k\leq N. \tag{5}$$

The involutivity (5) gives rise to the bidifferential operators

$$\mathbf{c}_{ij}^k\colon \Gamma\Omega(\xi_\pi)\times\Gamma\Omega(\xi_\pi)\to\Gamma\Omega(\xi_\pi)$$

through

$$\bigl[A_i(\boldsymbol{p}), A_j(\boldsymbol{q})\bigr] = \sum_k A_k\bigl(\mathbf{c}_{ij}^k(\boldsymbol{p},\boldsymbol{q})\bigr), \qquad \boldsymbol{p},\boldsymbol{q}\in\Gamma\Omega(\xi_\pi). \tag{6}$$

The structural constants $\mathbf{c}_{ij}^k$ absorb the bidifferential action on $\boldsymbol{p}$, $\boldsymbol{q}$ under commutation in the images of the operators.

**Remark 1.** If $N = 1$ and there is a unique operator $A\colon \Gamma\Omega(\xi_\pi) \to \varkappa(\pi)$ satisfying (1), then we recover the definition of the variational anchor in the Lie algebroid over the infinite jet space $J^\infty(\pi)$, see [10]. By construction, $\mathbf{c}_{11}^1 \equiv [\,,\,]_{A_1}$ if $N = 1$. However, for $N > 1$ we obtain a wider class of structures because we do not assume that the image of each operator $A_i$ alone is involutive. Therefore it may well occur that $\mathbf{c}_{ii}^k \neq 0$ for some $k \neq i$.

The Magri scheme [16] for the restriction of two compatible Hamiltonian operators $A_1$, $A_2$ onto the commutative hierarchy of the descendants $\mathcal{H}_i$ of the Casimirs $\mathcal{H}_0$ for $A_1$ gives us an example of (5) with $N = 2$ and $\mathbf{c}_{ij}^k \equiv 0$. We consider it in more detail; from now we standardly identify the Hamiltonian operators $A$ with the variational Poisson bivectors $\boldsymbol{A}$, see [13]. We recall that the variational Schouten bracket $[\![\,,\,]\!]$ of such bivectors satisfies the Jacobi identity

$$[\![[\![\boldsymbol{A}_1, \boldsymbol{A}_2]\!], \boldsymbol{A}_3]\!] + [\![[\![\boldsymbol{A}_2, \boldsymbol{A}_3]\!], \boldsymbol{A}_1]\!] + [\![[\![\boldsymbol{A}_3, \boldsymbol{A}_1]\!], \boldsymbol{A}_2]\!] = 0. \tag{7}$$

Hence the defining property $[\![\boldsymbol{A}, \boldsymbol{A}]\!] = 0$ for a Poisson bivector $\boldsymbol{A}$ implies that $\mathrm{d}_A = [\![\boldsymbol{A}, \cdot\,]\!]$ is a differential, giving rise to the Poisson cohomology $H_A^k$. Obviously the Casimirs $\mathcal{H}_0$ such that $[\![\boldsymbol{A}, \mathcal{H}_0]\!] = 0$ for a Poisson bivector $\boldsymbol{A}$ constitute the group $H_A^0$.

**Theorem 2 ([11,16]).** *Suppose $[\![\boldsymbol{A}_1, \boldsymbol{A}_2]\!] = 0$, $\mathcal{H}_0 \in H_{A_1}^0$ is a Casimir of $\boldsymbol{A}_1$ and the first Poisson cohomology w.r.t. $\mathrm{d}_{A_1} = [\![\boldsymbol{A}_1, \cdot\,]\!]$ vanishes. Then for any $k > 0$ there is a Hamiltonian $\mathcal{H}_k$ such that*

$$[\![\boldsymbol{A}_2, \mathcal{H}_{k-1}]\!] = [\![\boldsymbol{A}_1, \mathcal{H}_k]\!]. \tag{8}$$

*Put $\varphi_k := A_1\bigl(\delta/\delta u(\mathcal{H}_k)\bigr)$ such that $\partial_{\varphi_k} = [\![\boldsymbol{A}_1, \mathcal{H}_k]\!]$. The Hamiltonians $\mathcal{H}_i$, $i \geq 0$, pairwise Poisson commute w.r.t. either $A_1$ or $A_2$, the densities of $\mathcal{H}_i$ are conserved on any equation $u_{t_k} = \varphi_k$ and the evolutionary derivations $\partial_{\varphi_k}$ pairwise commute for all $k \geq 0$.*

**Standard proof of existence.** The main homological equality (8) is established by induction on $k$. Starting with a Casimir $\mathcal{H}_0$ we obtain

$$0 = [\![\boldsymbol{A}_2, 0]\!] = [\![\boldsymbol{A}_2, [\![\boldsymbol{A}_1, \mathcal{H}_0]\!]]\!] = -[\![\boldsymbol{A}_1, [\![\boldsymbol{A}_2, \mathcal{H}_0]\!]]\!] \mod [\![\boldsymbol{A}_1, \boldsymbol{A}_2]\!] = 0,$$

using the Jacobi identity (7). The first Poisson cohomology $H_{A_1}^1 = 0$ is trivial by an assumption of the theorem. Hence the closed element $[\![\boldsymbol{A}_2, \mathcal{H}_0]\!]$ in the kernel of $[\![\boldsymbol{A}_1, \cdot]\!]$ is exact: $[\![\boldsymbol{A}_2, \mathcal{H}_0]\!] = [\![\boldsymbol{A}_1, \mathcal{H}_1]\!]$ for some $\mathcal{H}_1$. For $k \geq 1$ we have

$$[\![\boldsymbol{A}_1, [\![\boldsymbol{A}_2, \mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{A}_2, [\![\boldsymbol{A}_1, \mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{A}_2, [\![\boldsymbol{A}_2, \mathcal{H}_{k-1}]\!]]\!] = 0$$

using (7) and by $[\![\boldsymbol{A}_2, \boldsymbol{A}_2]\!] = 0$. Consequently by $H_{A_1}^1 = 0$ we have that $[\![\boldsymbol{A}_2, \mathcal{H}_k]\!] = [\![\boldsymbol{A}_1, \mathcal{H}_{k+1}]\!]$, and we thus proceed infinitely. ∎

We see now that the inductive step — the existence of the $(k+1)$th Hamiltonian functional in involution — is possible if and only if $H_0$ is a Casimir,[5] and therefore the operators $A_1$ and $A_2$ are restricted onto the linear subspace which is spanned in the space of variational covectors by the Euler derivatives of the descendants of $\mathcal{H}_0$, i.e. of the Hamiltonians of the hierarchy. We note that the image under $A_2$ of a generic section from the domain of operators $A_1$ and $A_2$ cannot be resolved w.r.t. $A_1$ by (8). For example the first and second Hamiltonian structures for the KdV equation, which equal, respectively, $A_1 = \mathrm{d}/\mathrm{d}x$ and $A_2 = -\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x$, are not strongly compatible unless they are restricted onto some subspaces of their arguments. On the linear subspace of descendants of the Casimir $\int u\,\mathrm{d}x$ we have $\operatorname{im} A_2 \subset \operatorname{im} A_1$ and, since the image of the Hamiltonian operator $A_1$ is involutive, we conclude that $[\operatorname{im} A_1, \operatorname{im} A_2] \subset \operatorname{im} A_1$.

On the other hand the strong compatibility of the restrictions of Poisson-compatible operators $A_1$ and $A_2$ onto the hierarchy is valid since their images are commutative Lie algebras. Regarding the converse statement as a potential generator of multidimensional completely integrable systems we formulate the open problem: Is the strong compatibility of Poisson-compatible Hamiltonian operators achieved *only* for their restrictions onto the hierarchies of Hamiltonians in involution so that the bidifferential constants $\mathbf{c}_{ij}^k$ necessarily vanish? If so, this would have a remarkable similarity with the technique of the Bethe ansatz, one component of which is the extension of a commutative algebra of Hamiltonian operators on a Hilbert space to a bigger noncommutative algebra.

## 3 Bidifferential Christoffel symbols

Similarly to (2), we extract the total bidifferential parts of the structural constants $\mathbf{c}_{ij}^k$ in (6) and obtain

$$\mathbf{c}_{ij}^k = \partial_{A_i(\boldsymbol{p})}(\boldsymbol{q}) \cdot \delta_j^k - \partial_{A_j(\boldsymbol{q})}(\boldsymbol{p}) \cdot \delta_i^k + \Gamma_{ij}^k(\boldsymbol{p}, \boldsymbol{q}), \quad \boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi), \tag{9}$$

where $\Gamma_{ij}^k \in \mathcal{C}\mathrm{Diff}\big(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\big)$ and $\delta_i^k$, $\delta_j^k$ are the Kronecker delta symbols. By definition the three indices in $\Gamma_{ij}^k$ match the respective operators $A_i$, $A_j$, $A_k$ in (6). (The total number of the indices is much greater than three; moreover the proper upper or lower location of the omitted indices depends upon the (co)vector nature of the domain $\Gamma\Omega(\xi_\pi)$.) Obviously the convention

$$\Gamma_{11}^1 = \{\{\,,\,\}\}_{A_1}$$

holds if $N = 1$. At the same time for fixed $i$, $j$, $k$ the symbol $\Gamma_{ij}^k$ remains a (class of) matrix differential operator in each of its two arguments $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$.

[5] The Magri scheme starts from any two Hamiltonians $\mathcal{H}_{k-1}$, $\mathcal{H}_k$ that satisfy (8), but we operate with maximal subspaces of the space of functionals such that the sequence $\{\mathcal{H}_k\}$ cannot be extended with $k < 0$.

The symbol $\Gamma_{ij}^k$ represents a class of bidifferential operators because they are not uniquely defined. Indeed they are gauged by the conditions

$$\sum_{k=1}^{N} A_k\Big(\partial_{A_i(\boldsymbol{p})}(\boldsymbol{q})\delta_j^k - \partial_{A_j(\boldsymbol{q})}(\boldsymbol{p})\delta_i^k + \Gamma_{ij}^k(\boldsymbol{p},\boldsymbol{q})\Big) = 0, \quad \boldsymbol{p},\boldsymbol{q} \in \Gamma\Omega(\xi_\pi). \tag{10}$$

We let the r.h.s. of (10) be zero if the sum $\sum_\ell \operatorname{im} A_\ell$ of the images is indecomposable, which mean that no nontrivial sections commute with all the others: $\big[A_k(\boldsymbol{p}), \sum_{\ell=1}^N \operatorname{im} A_\ell\big] = 0$ implies that $\boldsymbol{p} \in \ker A_k$. For this it is sufficient that the sum of the images of $A_\ell$ in $\mathfrak{g}(\pi)$ be semisimple and the Whitehead lemma holds for it [5]. Otherwise the right-hand side of (10) belongs to the linear subspace of such nontrivial sections.

**Example 1 (see [9, 10]).** Consider the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$. The differential generators of its conservation laws are $w = u_x^2 - u_{xx} \in \ker \frac{\mathrm{d}}{\mathrm{d}y}\big|_{\mathcal{E}_{\text{Liou}}}$ and $\bar{w} = u_y^2 - u_{yy} \in \ker \frac{\mathrm{d}}{\mathrm{d}x}\big|_{\mathcal{E}_{\text{Liou}}}$. The operators[6] $\Box = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\Box} = u_y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}$ determine higher symmetries $\varphi, \bar{\varphi}$ of $\mathcal{E}_{\text{Liou}}$ by the formulas

$$\varphi = \Box\big(p(x,[w])\big), \quad \bar{\varphi} = \overline{\Box}\big(\bar{p}(y,[\bar{w}])\big)$$

for any variational covectors $p$, $\bar{p}$. The images of $\Box$ and $\overline{\Box}$ are closed w.r.t. the commutation; for instance the bracket (2) for $\Box$ contains $\{\{p,q\}\}_\Box = \frac{\mathrm{d}}{\mathrm{d}x}(p) \cdot q - p \cdot \frac{\mathrm{d}}{\mathrm{d}x}(q)$, and similarly for $\overline{\Box}$. The two summands in the symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Liou}} \simeq \operatorname{im}\Box \oplus \operatorname{im}\overline{\Box}$ commute between each other, $[\operatorname{im}\Box, \operatorname{im}\overline{\Box}] \doteq 0$ on $\mathcal{E}_{\text{Liou}}$. The operators $\Box$, $\overline{\Box}$ generate the bidifferential symbols

$$\Gamma_{\Box\Box}^{\Box} = \{\{\,,\,\}\}_\Box = \tfrac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbf{1} - \mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}x}, \quad \Gamma_{\overline{\Box}\,\overline{\Box}}^{\overline{\Box}} = \{\{\,,\,\}\}_{\overline{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}y} \otimes \mathbf{1} - \mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}y},$$
$$\Gamma_{\Box\overline{\Box}}^{\Box} = \tfrac{\mathrm{d}}{\mathrm{d}y} \otimes \mathbf{1}, \quad \Gamma_{\Box\overline{\Box}}^{\overline{\Box}} = -\mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}x}, \quad \Gamma_{\overline{\Box}\Box}^{\Box} = -\mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}y}, \quad \Gamma_{\overline{\Box}\Box}^{\overline{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbf{1},$$

where the notation is obvious. We note that $\Gamma_{\Box\overline{\Box}}^{\Box}(p,q) \doteq \Gamma_{\Box\overline{\Box}}^{\overline{\Box}}(p,q) \doteq \Gamma_{\overline{\Box}\Box}^{\Box}(q,p) \doteq \Gamma_{\overline{\Box}\Box}^{\overline{\Box}}(q,p) \doteq 0$ on $\mathcal{E}_{\text{Liou}}$ for any $p(x,[w])$ and $q(y,[\bar{w}])$.

The matrix operators $\Box$, $\overline{\Box}$ are well defined [7] for each 2D Toda chain $\mathcal{E}_{\text{Toda}}$ associated with a semisimple complex Lie algebra. They exhibit the same properties as above.

**Remark 2.** The operators $\Box$, $\overline{\Box}$ yield the involutive distributions of evolutionary vector fields that are tangent to the *integral manifolds*, the 2D Toda differential equations. Generally there is no Frobenius theorem for such distributions. Still, if the integral manifold exists and is an infinite prolongation of a differential equation $\mathcal{E} \subset J^\infty(\pi)$, then by construction this equation admits infinitely many symmetries of the form $\varphi = A_i(\boldsymbol{p})$ with free functional parameters $\boldsymbol{p} \in \Gamma\Omega(\xi_\pi)$. This property is close but not equivalent to the definition of systems of Liouville type (see [7, 9] and references therein).

[6] We denote the operators by $\Box$ and $\overline{\Box}$ following the notation of [7, 9], see also references therein.

The method by which we introduced the symbols $\Gamma^k_{ij}$ suggests that, under reparametrizations $g$ in the domain of the operators (4), they obey a proper analogue of the standard rule $\Gamma \mapsto g\,\Gamma\, g^{-1} + \mathrm{d}g \cdot g^{-1}$ for the connection 1-forms $\Gamma$. This is indeed so.

**Theorem 3 (Transformations of $\Gamma^k_{ij}$).** *Let $g$ be a reparametrization $\boldsymbol{p} \mapsto \tilde{\boldsymbol{p}} = g\boldsymbol{p}$, $\boldsymbol{q} \mapsto \tilde{\boldsymbol{q}} = g\boldsymbol{q}$ of sections $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$ in the domains*[7] *of strongly compatible operators (4). In this notation the operators $A_1, \ldots, A_N$ are transformed by the formula $A_i \mapsto \tilde{A}_i = A_i \circ g^{-1}\big|_{w=w[\tilde{w}]}$. Then the bidifferential symbols $\Gamma^k_{ij} \in \mathcal{C}\mathrm{Diff}\big(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\big)$ are transformed according to the rule*

$$\begin{aligned}\Gamma^k_{ij}(\boldsymbol{p},\boldsymbol{q}) \mapsto \tilde{\Gamma}^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{q}}) &= \big(g \circ \Gamma^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}\big)\big(g^{-1}\tilde{\boldsymbol{p}}, g^{-1}\tilde{\boldsymbol{q}}\big)\\ &\quad + \delta^{\tilde{k}}_{\tilde{\imath}} \cdot \partial_{\tilde{A}_{\tilde{\jmath}}(\tilde{\boldsymbol{q}})}(g)\big(g^{-1}\tilde{\boldsymbol{p}}\big) - \delta^{\tilde{k}}_{\tilde{\jmath}} \cdot \partial_{\tilde{A}_{\tilde{\imath}}(\tilde{\boldsymbol{p}})}(g)\big(g^{-1}\tilde{\boldsymbol{q}}\big). \qquad (11)\end{aligned}$$

**Proof.** Denote $A = A_i$ and $B = A_j$; without loss of generality we assume $i = 1$ and $j = 2$. We calculate the commutators of vector fields in the images of $A$ and $B$ using two systems of coordinates in the domain. We equate the commutators straighforwardly because the fibre coordinates in the images of the operators are not touched at all. So we have originally

$$\begin{aligned}\big[A(\boldsymbol{p}), B(\boldsymbol{q})\big] &= B\big(\partial_{A(\boldsymbol{p})}(\boldsymbol{q})\big) - A\big(\partial_{B(\boldsymbol{q})}(\boldsymbol{p})\big)\\ &\quad + A\big(\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q})\big) + B\big(\Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q})\big) + \sum_{k=3}^{N} A_k\big(\Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q})\big).\end{aligned}$$

On the other hand we substitute $\tilde{\boldsymbol{p}} = g\boldsymbol{p}$ and $\tilde{\boldsymbol{q}} = g\boldsymbol{q}$ into $\big[\tilde{A}(\tilde{\boldsymbol{p}}), \tilde{B}(\tilde{\boldsymbol{q}})\big]$ whence by the Leibnitz rule we obtain

$$\begin{aligned}\big[\tilde{A}(\tilde{\boldsymbol{p}}), \tilde{B}(\tilde{\boldsymbol{q}})\big] &= \tilde{B}\big(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(g)(\boldsymbol{q})\big) + \big(\tilde{B}\circ g\big)\big(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(\boldsymbol{q})\big)\\ &\quad - \tilde{A}\big(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(g)(\boldsymbol{p})\big) - \big(\tilde{A}\circ g\big)\big(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(\boldsymbol{p})\big)\\ &\quad + \big(A\circ g^{-1}\big)\big(\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big) + \big(B\circ g^{-1}\big)\big(\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big)\\ &\quad + \sum_{\tilde{k}=3}^{N}\big(A_{\tilde{k}}\circ g^{-1}\big)\big(\Gamma^{\tilde{k}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big).\end{aligned}$$

Therefore

$$\begin{aligned}\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) - \big(g^{-1}\circ\partial_{B(\boldsymbol{q})}(g)\big)(\boldsymbol{p}),\\ \Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) + \big(g^{-1}\circ\partial_{A(\boldsymbol{p})}(g)\big)(\boldsymbol{q}),\\ \Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{k}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) \qquad \text{for } k \geq 3.\end{aligned}$$

[7]Under an invertible change $\tilde{w} = \tilde{w}[w]$ of fibre coordinates (see Example 1) the variational covectors are transformed by the inverse of the adjoint linearization $g = \big[\big(\ell^{(w)}_{\tilde{w}}\big)^{\dagger}\big]^{-1}$ whereas for variational vectors, $g = \ell^{(w)}_{\tilde{w}}$ is the linearization.

Acting by $g$ upon these equalities and expressing $\boldsymbol{p} = g^{-1}\tilde{\boldsymbol{p}}$, $\boldsymbol{q} = g^{-1}\tilde{\boldsymbol{q}}$ we obtain (11) and conclude the proof. ■

**Remark 3.** Within the Hamiltonian formalism it is very productive to postulate that the arguments of Hamiltonian operators, the variational covectors, are *odd*,[8] see [22] and [13]. Indeed in this particular situation they can be conveniently identified with Cartan 1-forms times the pull-back of the volume form $\mathrm{dvol}\,(B^n)$ for the base of the jet bundle. We preserve this *grading* for such domains of operators (when $N = 1$, we referred to such operators in [10] as variational anchors of *second kind*). If moreover $\pi$ and $\xi$ are superbundles with Grassmann-valued sections, then the operators become bigraded [22]. Their proper grading is $-1$ because their images in $\mathfrak{g}(\pi)$ have grading zero, but the $\mathbb{Z}_2$-*parity*, if any, can be arbitrary.

**Corollary 1.** *For strongly compatible operators the domain $\Gamma\Omega(\xi_\pi)$ of which consists of variational covectors, the grading of the arguments equals* 1. *Therefore for any $i, j, k \in [1, \ldots, N]$ and for any $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$ we have that*

$$\Gamma^k_{ij}(\boldsymbol{p}, \boldsymbol{q}) = -\Gamma^k_{ji}(\boldsymbol{q}, \boldsymbol{p}) = (-1)^{|\boldsymbol{p}|_{gr}\cdot|\boldsymbol{q}|_{gr}} \cdot \Gamma^k_{ji}(\boldsymbol{q}, \boldsymbol{p}) \tag{12}$$

*due to the skew-symmetry of the commutators in* (5). *Hence the symbols $\Gamma^k_{ij}$ are symmetric in this case.*

**Proposition 1.** *If two normal operators $A_i$ and $A_j$ are simultaneously linear and strongly compatible, then their 'individual' brackets $\Gamma^i_{ii}$ and $\Gamma^j_{jj}$ are*

$$\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_i} = \Gamma^j_{ij}(\boldsymbol{p}, \boldsymbol{q}) + \Gamma^j_{ji}(\boldsymbol{p}, \boldsymbol{q}) \quad \textit{and} \quad \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_j} = \Gamma^i_{ij}(\boldsymbol{p}, \boldsymbol{q}) + \Gamma^i_{ji}(\boldsymbol{p}, \boldsymbol{q})$$

*for any $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$.*

**Proof.** For brevity denote $A = A_i$, $B = A_j$ and consider the linear combination $\mu A + \nu B$; by assumption its image is closed under commutation. By Theorem 1 we have

$$\begin{aligned}&(\mu A + \nu B)\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{\mu A+\nu B}\big)\\ &= \mu^2 A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big) + \mu\nu\cdot A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_B\big) + \mu\nu\cdot B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big) + \nu^2 B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big).\end{aligned}$$

On the other hand

$$\begin{aligned}&\Big[(\mu A + \nu B)(\boldsymbol{p}), (\mu A + \nu B)(\boldsymbol{q})\Big]\\ &= \mu^2\big[A(\boldsymbol{p}), A(\boldsymbol{q})\big] + \mu\nu\big[A(\boldsymbol{p}), B(\boldsymbol{q})\big] - \mu\nu\big[A(\boldsymbol{q}), B(\boldsymbol{p})\big] + \nu^2\big[B(\boldsymbol{p}), B(\boldsymbol{q})\big].\end{aligned}$$

Taking into account (9) and equating the coefficients of $\mu\nu$ we obtain

$$\begin{aligned}&A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_B\big) + B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big)\\ &= A\big(\Gamma^A_{AB}(\boldsymbol{p}, \boldsymbol{q})\big) + B\big(\Gamma^B_{AB}(\boldsymbol{p}, \boldsymbol{q})\big) - A\big(\Gamma^A_{AB}(\boldsymbol{q}, \boldsymbol{p})\big) - B\big(\Gamma^B_{AB}(\boldsymbol{q}, \boldsymbol{p})\big).\end{aligned}$$

Using the formulas $\Gamma^A_{AB}(\boldsymbol{q}, \boldsymbol{p}) = -\Gamma^A_{BA}(\boldsymbol{p}, \boldsymbol{q})$ and $\Gamma^B_{AB}(\boldsymbol{q}, \boldsymbol{p}) = -\Gamma^B_{BA}(\boldsymbol{p}, \boldsymbol{q})$, see (12), we isolate the arguments of the operators and obtain the assertion. ■

[8] Here we assume for simplicity that all fibre coordinates in $\pi$ are permutable.

## Conclusion

For every $\Bbbk$-vector space $V$ the space of endomorphisms $\mathrm{End}_{\Bbbk}(V)$ is a monoid with respect to the composition $\circ$. In this context one can study relations between recursion operators. For instance the structural relations for recursion operators of the Krichever–Novikov equations are described by hyperelliptic curves, see [3]. Likewise we have the relation $R_1 \circ R_2 - R_2 \circ R_1 = R_1^2$ between two recursions for the dispersionless 3-component Boussinesq system, see [6]. Simultaneously the space of endomorphisms carries the structure of a Lie algebra which is given by the formula $[R_i, R_j] = R_i \circ R_j - R_j \circ R_i$ for every $R_i, R_j \in \mathrm{End}_{\Bbbk}(V)$.

In this paper we proceed further and consider the class of structures on the linear spaces of total differential operators that generally do not in principle admit any associative composition. (The bracket of recursion operators that appears through (6) is different from the Richardson–Nijenhuis bracket [12], although we use similar geometric techniques.) The classification problem for such algebras of operators is completely open.

### Discussion

We performed all the reasonings for local differential operators in a purely commutative setup; all the structures were defined on the empty jet spaces. A rigorous extension of these objects to $\mathbb{Z}_2$-graded nonlocal operators on differential equations is a separate problem for future research. In addition the use of difference operators subject to (5) can be a fruitful idea *au début* for the discretization of integrable systems with free functional parameters in their symmetries (e.g., Toda-like difference systems [20]).

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