Generalized Bernstein–Reznikov integrals

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Abstract

We find a closed formula for the triple integral on spheres in $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ whose kernel is given by powers of the standard symplectic form. This gives a new proof to the Bernstein–Reznikov integral formula in the $n = 1$ case. Our method also applies for linear and conformal structures.

1 Triple product integral formula

We consider the symplectic form $[\ ,\ ]$ on $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ given by

$$[(x, \xi), (y, \eta)] := -\langle x, \eta\rangle + \langle y, \xi\rangle. \tag{1.1}$$

In this paper we prove a closed formula for the following triple integral:

Theorem 1.1. *Let $d\sigma$ be the Euclidean measure on the sphere S^{2n-1}. Then,*

$$\int_{S^{2n-1}\times S^{2n-1}\times S^{2n-1}} \big|[Y, Z]\big|^{\frac{\alpha-n}{2}} \big|[Z, X]\big|^{\frac{\beta-n}{2}} \big|[X, Y]\big|^{\frac{\gamma-n}{2}} d\sigma(X) d\sigma(Y) d\sigma(Z)$$
$$= \left(2\pi^{n-\frac{1}{2}}\right)^3 \frac{\Gamma\left(\frac{2-n+\alpha}{4}\right)\Gamma\left(\frac{2-n+\beta}{4}\right)\Gamma\left(\frac{2-n+\gamma}{4}\right)\Gamma\left(\frac{\delta+n}{4}\right)}{\Gamma(n)\Gamma\left(\frac{n-\lambda_1}{2}\right)\Gamma\left(\frac{n-\lambda_2}{2}\right)\Gamma\left(\frac{n-\lambda_3}{2}\right)}.$$

Here, $\alpha = \lambda_1 - \lambda_2 - \lambda_3$, $\beta = -\lambda_1 + \lambda_2 - \lambda_3$, $\gamma = -\lambda_1 - \lambda_2 + \lambda_3$, $\delta = -\lambda_1 - \lambda_2 - \lambda_3 = \alpha + \beta + \gamma$.

2000 Mathematics Subject Classifications: Primary 42C05; Secondary 11F67, 22E45, 33C20, 33C55.

The integral converges absolutely if and only if $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ lies in the following non-empty open region (see Proposition 6.8) defined by:

$$\begin{aligned} &\operatorname{Re}\alpha > n-2,\ \operatorname{Re}\beta > n-2,\ \operatorname{Re}\gamma > n-2 && (n \geq 2), \\ &\operatorname{Re}\alpha > n-2,\ \operatorname{Re}\beta > n-2,\ \operatorname{Re}\gamma > n-2,\ \operatorname{Re}\delta > -1 && (n = 1). \end{aligned}$$

The integral under consideration extends as a meromorphic function of λ_1, λ_2 and λ_3 ([3, Theorem 2], [15], see also [7]). A special case ($n = 1$) of Theorem 1.1 was previously established by J. Bernstein and A. Reznikov [5].

The strategy of our proof is to interpret the triple product integral as the trace of a certain integral operator, for which we will find an explicit formula of eigenvalues and their multiplicities. For this, our approach uses the Fourier transform in the ambient space, appeals to the classical Bochner identity, and finally reduces it to the special value of the hypergeometric function ${}_5F_4$. It gives a new proof even when $n = 1$.

Sections 2 and 3 are devoted to the proof of Theorem 1.1. In Section 4, we discuss analogous integrals of the triple product kernels involving $|x-y|^\lambda$ or $|\langle x, y\rangle|^\lambda$ instead of $\big|[x,y]\big|^\lambda$.

The underlying symmetries for Theorem 1.1 are given by the symplectic group $Sp(n, \mathbb{R})$ of any rank n, whereas those of Theorem 4.2 correspond to the rank one group $SO_0(m+1, 1)$. Even in rank one case, our methods give a new and simple proof of the original results due to Deitmar [6] for the case $|x-y|^\lambda$ (see Theorem 4.2). Section 5 highlights some general perspectives from representation theoretic point of view.

In Sections 2 and 6 we have made an effort, following questions of the referee, to explain the role of meromorphic families of homogeneous distributions and the precise condition for the absolute convergence of the triple integral, respectively.

Notations: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

2 Eigenvalues of integral transforms $\mathcal{T}_\mu$

We introduce a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$\mathcal{T}_\mu : C^\infty(S^{2n-1}) \to C^\infty(S^{2n-1})$$

defined by

$$(\mathcal{T}_\mu f)(\eta) := \int_{S^{2n-1}} f(\omega)\big|[\omega, \eta]\big|^{-\mu-n} d\sigma(\omega). \tag{2.1}$$

The integral (2.1) converges absolutely if $\operatorname{Re}\mu < -n+1$, and has a meromorphic continuation for $\mu \in \mathbb{C}$. If μ is real and sufficiently negative (e.g. $\mu < -n$), then the kernel function $K(\omega,\eta) := |[\omega,\eta]|^{-\mu-n}$ is square integrable on $S^{2n-1} \times S^{2n-1}$ and $K(\omega,\eta) = \overline{K(\eta,\omega)}$, and consequently, $\mathcal{T}_\mu$ becomes a self-adjoint, Hilbert–Schmidt operator on $L^2(S^{2n-1})$. In this section, we determine all the eigenvalues of $\mathcal{T}_\mu$ and the corresponding eigenspaces (see Theorem 2.1).

2.1 Harmonic polynomials on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$

First, let us remind the classic theory of spherical harmonics on real and complex vector spaces.

For $k \in \mathbb{N}$, we denote by $\mathcal{H}^k(\mathbb{R}^N)$ the vector space consisting of homogeneous polynomials $p(x_1,\ldots,x_N)$ of degree k such that $\displaystyle\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} p = 0$.

In the polar coordinates $x = r\omega$ ($r \geq 0$, $\omega \in S^{N-1}$), we have

$$\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = \frac{1}{r^2}\left(\left(r\frac{\partial}{\partial r}\right)^2 + (N-2)\left(r\frac{2}{\partial r}\right) + \Delta_{S^{N-1}}\right),$$

where $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on the unit sphere endowed with the standard Riemannian metric. In light that $r\frac{\partial}{\partial r}p = kp$ for a homogeneous function p of degree k, we see that the restriction $p|_{S^{N-1}}$ for $p \in \mathcal{H}^k(\mathbb{R}^N)$ belongs to the following eigenspace of $\Delta_{S^{N-1}}$:

$$V_k(S^{N-1}) := \{\varphi \in C^\infty(S^{N-1}) : \Delta_{S^{N-1}}\varphi = -k(k+N-2)\varphi\}. \tag{2.2}$$

Since homogeneous functions on $\mathbb{R}^N$ are determined uniquely by the restriction to S^{N-1}, we get an injective map

$$\mathcal{H}^k(\mathbb{R}^N) \to V_k(S^{N-1}), \quad p \mapsto p|_{S^{N-1}}$$

for each $k \in \mathbb{N}$. This map is also surjective (see [9, Introduction, Theorem 3.1] for example), and we shall identify $\mathcal{H}^k(\mathbb{R}^N)$ with $V_k(S^{N-1})$. Thus, we regard the following algebraic direct sum

$$\mathcal{H}(\mathbb{R}^N) := \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{R}^N)$$

as a dense subspace of $C^\infty(S^{N-1})$.

Analogously, we can define the space of harmonic polynomials on $\mathbb{C}^n$. For $\alpha, \beta \in \mathbb{N}$, we denote by $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ the vector space consisting of polynomials $p(Z, \bar{Z})$ on $\mathbb{C}^n$ subject to the following two conditions:

(1) $p(Z, \bar{Z})$ is homogeneous of degree α in $Z = (z_1, \ldots, z_n)$ and of degree β in $\bar{Z} = (\bar{z}_1, \ldots, \bar{z}_n)$.

(2) $\displaystyle\sum_{j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} p(Z, \bar{Z}) = 0.$

Then, $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ is a finite dimensional vector space. It is non-zero except for the case where $n = 1$ and $\alpha, \beta \geq 1$.

By definition, we have a natural linear isomorphism:

$$\mathcal{H}^k(\mathbb{R}^{2n}) \simeq \bigoplus_{\alpha+\beta=k} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n). \tag{2.3}$$

We shall see that $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ is an eigenspace of the operator $\mathcal{T}_\mu$ for any μ and for every α and β. To be more precise, we introduce a meromorphic function of μ by

$$A_k(\mu, \mathbb{C}^n) \equiv A_k(\mu) := \begin{cases} 0 & (k : \text{odd}), \\ 2\pi^{n-\frac{1}{2}} \dfrac{\Gamma(\frac{1-n-\mu}{2})\Gamma(\frac{k+n+\mu}{2})}{\Gamma(\frac{n+\mu}{2})\Gamma(\frac{k+n-\mu}{2})} & (k : \text{even}). \end{cases} \tag{2.4}$$

We shall use the notation $A_k(\mu, \mathbb{C}^n)$ when we emphasize the ambient space $\mathbb{C}^n$ (see (4.6)).

Theorem 2.1. *For $\alpha, \beta \in \mathbb{N}$,*

$$\mathcal{T}_\mu\Big|_{\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)} = (-1)^\beta A_{\alpha+\beta}(\mu)\,\mathrm{id}\,.$$

The rest of this section is devoted to the proof of Theorem 2.1.

2.2 Preliminary results on homogeneous distributions

In this section we collect some basic concepts and results on distributions in a way that we shall use later. See [7, Chapters 1 and 2], and also [12, Appendix].

A distribution F_ν depending on a complex parameter ν is defined to be meromorphic if for every test function φ, $\langle F_\nu, \varphi\rangle$ is a meromorphic function in ν. We say F_ν has a pole at $\nu = \nu_0$ if $\langle F_\nu, \varphi\rangle$ has a pole at $\nu = \nu_0$ for some φ. Then, taking its residue at ν_0, we get a new distribution

$$\varphi \mapsto \operatorname*{res}_{\nu=\nu_0} \langle F_\nu, \varphi\rangle,$$

which we denote by $\operatorname*{res}_{\nu=\nu_0} F_\nu$.

Suppose F is a distribution defined in a conic open subset in $\mathbb{R}^N$. We say F is homogeneous of degree λ if

$$\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} F = \lambda F \tag{2.5}$$

in the sense of distributions, or equivalently, $\langle F, \varphi\left(\frac{1}{a}\cdot\right)\rangle = a^{\lambda+N}\langle F, \varphi\rangle$ for any test function φ and $a > 0$.

Globally defined homogeneous distributions on $\mathbb{R}^N$ are determined by their restrictions to $\mathbb{R}^N \setminus \{0\}$ for generic degree:

Lemma 2.2. *Suppose f is a distribution on $\mathbb{R}^N$ which is homogeneous of degree λ. If $f|_{\mathbb{R}^N\setminus\{0\}} = 0$ and $\lambda \notin \{-N, -N-1, -N-2, \dots\}$, then $f = 0$ as distribution on $\mathbb{R}^N$.*

Proof. By the general structural theory on distributions, if $\operatorname{supp} f \subset \{0\}$ then f must be a finite linear combination of the Dirac delta function $\delta(x)$ and its derivatives. On the other hand, the degree of the delta function and its derivatives is one of $-N, -N-1, -N-2, \dots$. By our assumption on f, this does not happen. Hence, we conclude $f = 0$ as distribution on $\mathbb{R}^N$. $\square$

For a given function $p \in C^\infty(S^{N-1})$, we define its extension into a homogeneous function of degree λ by

$$p_\lambda(r\omega) := r^\lambda p(\omega), \quad (r > 0,\ \omega \in S^{N-1}).$$

We regard the locally integrable functions as distributions by multiplying the Lebesgue measure.

Lemma 2.3. *Let $p \in C^\infty(S^{N-1})$, then*

1) *p_λ is locally integrable on $\mathbb{R}^N$ if* $\operatorname{Re}\lambda > -N$.

2) *p_λ extends to a tempered distribution which depends meromorphically on $\lambda \in \mathbb{C}$. Its poles are simple and contained in the set $\{-N, -N-1, \dots\}$.*

3) *The distribution p_λ is homogeneous of degree λ in the sense of (2.5) if λ is not a pole.*

Proof. 1) Clear from the formula of the Lebesgue measure $dx = r^{N-1} dr d\sigma(\omega)$ in the polar coordinates $x = r\omega$ ($r > 0$, $\omega \in S^{N-1}$).

2) Take a test function $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Suppose first $\operatorname{Re}\lambda > -N$. Then, we can decompose $\langle p_\lambda, \varphi \rangle$ as

$$\langle p_\lambda, \varphi \rangle = \int_{|x|\le 1} p_\lambda(x)\varphi(x)dx + \int_{|x|>1} p_\lambda(x)\varphi(x)dx.$$

The second term extends holomorphically in the entire complex plane. Let us prove that the first term extends meromorphically in $\mathbb{C}$. For this, we fix $k \in \mathbb{N}$, and consider the Taylor expansion of φ:

$$\varphi(x) = \sum_{|\alpha|\le k} \frac{\varphi^{(\alpha)}(0)}{\alpha!} x^\alpha + \varphi_k(x),$$

where $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $\varphi_k(x) = O(|x|^{k+1})$. Accordingly, we have

$$\begin{aligned}
&\int_{|x|\le 1} p_\lambda(x)\varphi(x)dx \\
&= \sum_{|\alpha|\le k} \frac{\varphi^{(\alpha)}(0)}{\alpha!} \int_{S^{N-1}} p(\omega)\omega^\alpha d\sigma(\omega) \int_0^1 r^{\lambda+|\alpha|+N-1} dr + \int_{|x|\le 1} p_\lambda(x)\varphi_k(x)dx \\
&= \sum_{|\alpha|\le k} \frac{1}{\lambda+|\alpha|+N} \frac{\varphi^{(\alpha)}(0)}{\alpha!} \int_{S^{N-1}} p(\omega)\omega^\alpha d\sigma(\omega) + \int_{|x|\le 1} p_\lambda \varphi_k(x)dx.
\end{aligned}$$

The last term extends holomorphically to the open set $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -N-k\}$. Since k is arbitrary we see that $\langle p_\lambda, \varphi \rangle$ extends meromorphically to the entire complex plane, and all its poles are simple and contained in the set $\{-N, -N-1, -N-2, \dots\}$. Thus, the second statement is proved.

3) The differential equation $\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} p_\lambda(x) = \lambda p_\lambda(x)$ holds in the sense of distributions for $\operatorname{Re}\lambda \gg 0$. This equation extends to all complex λ except for poles because the distribution p_λ depends meromorphically on λ. $\square$

Example 2.4 ($k = 0$ case)**.** *For $k = 0$, $\mathcal{H}^k(\mathbb{R}^N)$ is one-dimensional, spanned by the constant function $\mathbf{1}$. We denote by r^λ the corresponding homogeneous distribution $\mathbf{1}_\lambda$.*

As we saw in the proof of Lemma 2.3, the distribution r^λ has a simple pole at $\lambda = -N$ and its residue is given by

$$\operatorname*{res}_{\lambda=-N} r^\lambda = \operatorname{vol}(S^{N-1})\delta(x) = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\delta(x). \tag{2.6}$$

Example 2.5 ($N = 1$ case)**.** *In the one dimensional case, S^{N-1} consists of two points, 1 and -1, and consequently, the homogeneous distribution p_λ is determined by the values $p_\lambda(1)$ and $p_\lambda(-1)$. From this viewpoint, we give a list of classical homogeneous distributions on $\mathbb{R}$.*

p_λ	x_+^λ	x_-^λ	$\lvert x\rvert^\lambda$	$\lvert x\rvert^\lambda \operatorname{sgn} x$	$(x+i0)^\lambda$	$(x-i0)^\lambda$
$p(1)$	1	0	1	1	1	1
$p(-1)$	0	1	1	-1	$e^{i\pi\lambda}$	$e^{-i\pi\lambda}$

Table 2.5.1: Homogeneous distributions on $\mathbb{R}$

The notation $(x \pm i0)^\lambda$ indicates that these distributions are obtained as the boundary values of holomorphic functions in the upper (or lower) half plane. For $\operatorname{Re}\lambda > -1$,

$$\lim_{\varepsilon\downarrow 0}(x \pm i\varepsilon)^\lambda = (x \pm i0)^\lambda$$

holds both in the ordinary sense and in distribution sense. The distributions $(x \pm i0)^\lambda$ extend holomorphically to all complex λ, whereas the poles of $x_\pm^\lambda$, $|x|^\lambda$, $|x|^\lambda \operatorname{sgn} x$ are located at $\{-1,-2,-3,\dots\}$, $\{-1,-3,-5,\dots\}$, $\{-2,-4,-6,\dots\}$, respectively.

For $\lambda \neq -1, -2, \ldots,$ any two in Table 2.5.1 form a basis in the space of homogeneous distributions of degree λ. For example, by a simple basis change one gets:

$$(x - i0)^\lambda = e^{-i\frac{\pi\lambda}{2}}(\cos\frac{\pi}{2}\lambda|x|^\lambda + i\sin\frac{\pi}{2}\lambda|x|^\lambda \operatorname{sgn} x). \tag{2.7}$$

2.3 Application of the Bochner identity

Let $\langle\ ,\ \rangle$ be the standard inner product on $\mathbb{R}^N$. We consider the Fourier transform $\mathcal{F} \equiv \mathcal{F}_{\mathbb{R}^N}$ on $\mathbb{R}^N$ normalized by

$$(\mathcal{F}f)(Y) := \int_{\mathbb{R}^N} f(X)e^{-2\pi i\langle X,Y\rangle}dX,$$

and we extend $\mathcal{F}$ to the space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions.

If $f \in \mathcal{S}'(\mathbb{R}^N)$ is homogeneous of degree λ, then its Fourier transform $\mathcal{F}f$ is homogeneous of degree $-\lambda - N$.

Example 2.6 ($N = 1$ case)**.**

1) $\mathcal{F}(x_+^\lambda)(y) = \dfrac{e^{-\frac{i\pi}{2}(\lambda+1)}\Gamma(\lambda+1)}{(2\pi)^{\lambda+1}}(y - i0)^{-\lambda-1}.$

2) $\mathcal{F}(|x|^\lambda)(y) = \dfrac{\Gamma(\frac{\lambda+1}{2})}{\pi^{\lambda+\frac{1}{2}}\Gamma(\frac{-\lambda}{2})}|y|^{-\lambda-1},$ *and*

$\mathcal{F}(|x|^\lambda \operatorname{sgn} x)(y) = \dfrac{-i\Gamma(\frac{\lambda+2}{2})}{\pi^{\lambda+\frac{1}{2}}\Gamma(\frac{1-\lambda}{2})}|y|^{-\lambda-1}\operatorname{sgn} y.$

These formulas may be found for instance in [7, Chapter II, §2.3], however, we shall give a brief proof because its intermediate step (e.g. (2.8) below) will be used later (see the proof of Proposition 2.13).

Proof of Example 2.6. 1) Suppose $\operatorname{Re}\lambda > -1$. Then x_+^λ is locally integrable on $\mathbb{R}$, and we have

$$\lim_{\varepsilon\downarrow 0} e^{-2\pi\varepsilon x}x_+^\lambda = x_+^\lambda$$

both in the ordinary sense and in the sense of distributions. Then, by Cauchy's integral formula and by the definition of the Gamma function, we get

$$\mathcal{F}(e^{-2\pi\varepsilon x}x_+^\lambda)(y) = \frac{e^{-\frac{\pi i}{2}(\lambda+1)}\Gamma(\lambda+1)}{(2\pi)^{\lambda+1}}(y - i\varepsilon)^{-\lambda-1}, \tag{2.8}$$

for $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0$ we get the desired identity for $\operatorname{Re}\lambda > -1$.

By the meromorphic continuation on λ, the first statement is proved.

2) Similarly to 1), we can obtain a closed formula for $\mathcal{F}(x_-^\lambda)(y)$. Then the second statement follows readily from the base change matrix for the three bases $\{x_+^\lambda, x_-^\lambda\}$, $\{|x|^\lambda, |x|^\lambda \operatorname{sgn} x\}$, and $\{(x+i0)^\lambda, (x-i0)^\lambda\}$ for homogeneous distributions on $\mathbb{R}$. (We also use the duplication formula of the Gamma function.) $\square$

We are ready to state the main result of this subsection. Let us define the following meromorphic function of λ by

$$B_N(\lambda, k) := \pi^{-\lambda-\frac{N}{2}} i^{-k} \frac{\Gamma(\frac{k+\lambda+N}{2})}{\Gamma(\frac{k-\lambda}{2})}.$$

Lemma 2.7. *For any $p \in \mathcal{H}^k(\mathbb{R}^N)$, we have the following identity*

$$\mathcal{F} p_\lambda = B_N(\lambda, k) p_{-\lambda-N} \tag{2.9}$$

as distributions on $\mathbb{R}^N$ that depend meromorphically on λ.

Example 2.8. *Since* (2.9) *is an identity for meromorphic distributions, we can pass to the limit, or compute residues at special values whenever it makes sense. For instance, let $k = 0$. Then, by* (2.6)*, the special value of* (2.9) *at $\lambda = 0$ yields*

$$\mathcal{F}(\mathbf{1}) = \lim_{\lambda \to 0} B_N(\lambda, 0) r^{-\lambda-N} = \delta(y).$$

In view of the identity $B_N(\lambda, 0) B_N(-\lambda - N, 0) = 1$, the residue of (2.9) *at $\lambda = -N$ yields*

$$\mathcal{F}(\delta(x)) = \mathbf{1}.$$

This, of course, is in agreement with the inversion formula for the Fourier transform.

Proof of Lemma 2.7. For $N = 1$, k equals either 0 or 1, and correspondingly, p_λ is a scalar multiple of $|x|^\lambda$ or $|x|^\lambda \operatorname{sgn} x$, respectively. Hence, Lemma 2.7 in the case $N = 1$ is equivalent to Example 2.6 2).

Let us prove (2.9) for $N \geq 2$ as the identity of distributions on $\mathbb{R}^N$. We shall first prove the identity (2.9) on $\mathbb{R}^N \setminus \{0\}$ in the non-empty domain:

$$-N < \operatorname{Re}\lambda < -\frac{1}{2}(N+1). \tag{2.10}$$

Since the both sides of (2.9) are homogeneous distributions of the same degree, this will imply that the identity (2.9) holds on $\mathbb{R}^N$ by Lemma 2.2. Further, since the both sides of (2.9) depend meromorphically on λ by Lemma 2.3, the identity (2.9) holds for all λ in the sense of distributions that depend meromorphically on λ.

The rest of this proof is devoted to show (2.9) on $\mathbb{R}^N \setminus \{0\}$ in the domain (2.10). For this, it is sufficient to prove that

$$\langle \mathcal{F} p_\lambda, gq \rangle = B_N(\lambda, k) \langle p_{-\lambda-N}, gq \rangle,$$

for any compactly supported function $g \in C_c^\infty(\mathbb{R}_+)$ and any $q \in \mathcal{H}^l(\mathbb{R}^N)$ ($l \in \mathbb{N}$) because the linear spans of such functions form a dense subspace in $C_c^\infty(\mathbb{R} \setminus \{0\})$. Here, gq stands for a function on $\mathbb{R}^N \setminus \{0\}$ defined by

$$(gq)(s\eta) = g(s)q(\eta) \quad (s > 0, \eta \in S^{N-1}).$$

By definition of the Fourier transform on $\mathcal{S}'(\mathbb{R}^N)$, $\langle \mathcal{F} p_\lambda, gq \rangle = \langle p_\lambda, \mathcal{F}(gq) \rangle$. Hence, what we need to prove is

$$\langle p_\lambda, \mathcal{F}(gq) \rangle = B_N(\lambda, k) \langle p_{-\lambda-N}, gq \rangle. \tag{2.11}$$

We note that both p_λ and $p_{-\lambda-N}$ are locally integrable functions on $\mathbb{R}^N$ under the assumption (2.10). To calculate the left-hand side of (2.11), we use the Bochner identity for $q \in \mathcal{H}^l(\mathbb{R}^N)$:

$$\int_{S^{N-1}} q(\omega) e^{-i\nu\langle \omega, \eta \rangle} d\omega = (2\pi)^{\frac{N}{2}} i^{-l} \nu^{1-\frac{N}{2}} J_{l+\frac{N}{2}-1}(\nu) q(\eta),$$

where $J_\mu(\nu)$ denotes the Bessel function of the first kind. Then, we get the following formula after a change of variables $x = 2\pi rs$:

$$\mathcal{F}(gq)(r\omega) = 2\pi i^{-l} r^{1-\frac{N}{2}} q(\omega) \int_0^\infty s^{\frac{N}{2}} g(s) J_{l+\frac{N}{2}-1}(2\pi rs) ds.$$

Hence, we have

$$\langle p_\lambda, \mathcal{F}(gq) \rangle = \int_0^\infty \int_{S^{N-1}} \Big(\int_0^\infty I(r,s) ds \Big) p(\omega) q(\omega) d\sigma(\omega) dr, \tag{2.12}$$

where we set

$$I(r,s) := 2\pi i^{-l} r^{\lambda+\frac{N}{2}} s^{\frac{N}{2}} g(s) J_{l+\frac{N}{2}-1}(2\pi rs).$$

At this point, we prepare the following:

Claim 2.9. *Assume that λ satisfies* (2.10) *and that g is compactly supported in $\mathbb{R}_+$.*

1) $I(r,s) \in L^1(\mathbb{R}_+ \times \mathbb{R}_+, drds)$.

2) $\displaystyle\int_0^\infty I(r,s)ds = B_N(\lambda, l) g(s) s^{-\lambda-1}$.

Proof of Claim 2.9. 1) Since the support of g is away from 0 and ∞, it follows from the asymptotic behaviour of the Bessel function $J_\mu(z)$ as $z \to 0$ and $z \to \infty$ that there exists a constant $c > 0$ such that

$$|I(r,s)| \leq \begin{cases} c\ r^{\operatorname{Re}\lambda + N + l - 1} & \text{as } r \to 0, \\ c\ r^{\operatorname{Re}\lambda + \frac{1}{2}(N-1)} & \text{as } r \to \infty. \end{cases}$$

By the assumption $-N < \operatorname{Re}\lambda < -\frac{1}{2}(N+1)$, we conclude $I(r,s)$ is an integrable function on $\mathbb{R}_+ \times \mathbb{R}_+$.

2) This is a direct consequence of the following classical formula of the Hankel transform [8, 6.561.14]

$$\int_0^\infty x^\mu J_\nu(x) dx = 2^\mu \frac{\Gamma(\frac{1+\nu+\mu}{2})}{\Gamma(\frac{1+\nu-\mu}{2})},$$

for $\operatorname{Re}(\mu+\nu) > -1$ and $\operatorname{Re}\mu < -\frac{1}{2}$. $\square$

Returning to the proof of Lemma 2.7, we can now apply Fubini's theorem for the right-hand side of (2.12) to get

$$\begin{aligned} \langle p_\lambda, \mathcal{F}(gq)\rangle &= \Big(\int_{S^{N-1}} p(\omega) q(\omega) d\sigma(\omega)\Big) \int_0^\infty \Big(\int_0^\infty I(r,s) ds\Big) dr \\ &= B_N(\lambda, l) \int_{S^{N-1}} p(\omega) q(\omega) d\sigma(\omega) \int_0^\infty g(s) s^{-\lambda-1} ds. \end{aligned}$$

We recall that $p \in \mathcal{H}^k(\mathbb{R}^N)$ and $q \in \mathcal{H}^l(\mathbb{R}^N)$, therefore the first factor is non-zero only if $k = l$. Then the right-hand side equals

$$= B_N(\lambda, k) \langle p_{-\lambda-N}, gq\rangle.$$

Hence (2.11) is proved in the non-empty open domain of λ satisfying the inequality (2.10). Therefore, the proof of Lemma 2.7 is completed. $\square$

2.4 Fourier transform of homogeneous functions

We consider the restriction of the Fourier transform $\mathcal{F}$ on $\mathbb{R}^N$ to the space of homogeneous functions. For $\mu \in \mathbb{C}$, we set

$$\begin{aligned} V_\mu &\equiv V_\mu(\mathbb{R}^N) \\ &:= \{f \in C^\infty(\mathbb{R}^N \setminus \{0\}) : f(tX) = |t|^{-\mu-\frac{N}{2}} f(X) \text{ for any } t \in \mathbb{R} \setminus \{0\}\}. \end{aligned} \tag{2.13}$$

Then, V_μ may be regarded as a subspace of the space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions for $\mu \neq \frac{N}{2}, \frac{N}{2}+2, \dots$. We note that $f \in V_\mu$ is determined by the restriction $f|_{S^{N-1}}$, and thus V_μ can be identified with the space of smooth even functions on S^{N-1}. In this subsection, we will prove:

Proposition 2.10. *Suppose $|\mu| \neq \frac{N}{2}, \frac{N}{2}+2, \dots$. Then the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$ induces a bijection between $V_{-\mu}$ and V_μ.*

For the proof of this proposition, we prepare some general result as follows. Let M be a compact smooth Riemannian manifold. We write Δ_M for the Laplace–Beltrami operator, $\mathrm{Ker}(\Delta_M - \lambda)$ for the eigenspace $\{f \in C^\infty(M) : \Delta_M f = \lambda f\}$, and τ for the Riemannian volume element. Then we can regard $C^\infty(M)$ as a subspace of $\mathcal{D}'(M)$, the space of distributions, by $f(x) \mapsto f(x)d\tau(x)$.

Lemma 2.11. *Suppose $A : C^\infty(M) \to \mathcal{D}'(M)$ is a linear map satisfying the following two properties:*

$$A \text{ acts as a scalar, say } a(\lambda) \in \mathbb{C}, \text{ on each eigenspace } \mathrm{Ker}(\Delta_M - \lambda). \tag{2.14}$$

$$\begin{aligned} &a(\lambda) \text{ is at most of polynomial growth, namely,} \\ &\text{there exist } C, N > 0 \text{ such that } |a(\lambda)| \leq C(1+|\lambda|)^N. \end{aligned} \tag{2.15}$$

Then, $A\psi \in C^\infty(M)$ for any $\psi \in C^\infty(M)$.

Remark 2.12. For a real analytic manifold M, an analogous statement holds for a linear map $A : \mathcal{A}(M) \to \mathcal{B}(M)$, where $\mathcal{A}$, $\mathcal{B}$ denote the sheaf of real analytic functions, (Sato's) hyperfunctions, respectively, if $a(\lambda)$ is at most of infra exponential growth (see [11, Section 2.3]).

Proof of Lemma 2.11. By Sobolev's lemma, a distribution T on M belongs to $C^\infty(M)$ if and only if $\Delta_M^l T$ (in the sense of distributions) belongs to $L^2(M)$ for any $l \in \mathbb{N}$. We will show that this is the case for $T = A\psi$ if $\psi \in C^\infty(M)$.

Let $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \dots$ the (negative) eigenvalues of Δ_M, repeated according to their multiplicites. We take an orthonormal basis $\{\varphi_j : j = 0, 1, 2, \dots\}$ in $L^2(M)$ consisting of real-valued eigenfunctions of Δ_M with eigenvalues λ_j. Then any distribution T on M can be expanded into a series of eigenfunctions (as a distribution):

$$T = \sum_j c_j \varphi_j, \quad c_j := \langle T, \varphi_j \rangle \ .$$

For $l \in \mathbb{N}$, the condition $\Delta_M^l T \in L^2(M)$ amounts to

$$\sum_j |c_j|^2 \lambda_j^{2l} < \infty \ ,$$

so that

$$T \in C^\infty(M) \Longleftrightarrow \sum_j |c_j|^2 (1 + |\lambda_j|)^{2l} < \infty \quad \text{for any } l \in \mathbb{N} \ .$$

Take any $\psi \in C^\infty(M)$, and expand it into a series of eigenfunctions

$$\psi = \sum_j b_j \varphi_j, \quad b_j = \langle \psi, \varphi_j \rangle_{L^2(M)} \ .$$

Applying the operator A, we get from (2.14)

$$A\psi = \sum_j b_j a(\lambda_j) \varphi_j \ .$$

For any $l \in \mathbb{N}$, using (2.15)

$$\sum_j |b_j a(\lambda_j)|^2 \lambda_j^{2l} \leq C \sum_j |b_j|^2 (1+|\lambda_j|)^{2N} \lambda_j^{2l} \leq C \sum_j |b_j|^2 (1+|\lambda_j|)^{2N+2l} < +\infty \, .$$

Thus $A\psi \in C^\infty(M)$. Hence Lemma 2.11 is proved. $\square$

We are ready to complete the proof of Proposition 2.10.

Proof of Proposition 2.10. For $f \in V_{-\mu}$, $\mathcal{F}f$ is a homogeneous distribution of degree $-\mu - \frac{N}{2}$. Therefore, to see that $\mathcal{F}f \in C^\infty(\mathbb{R}^N \setminus \{0\})$, it is sufficient to show that the restriction $\mathcal{F}f|_{S^{N-1}}$ is a smooth function on S^{N-1}. This follows from the general result (see Lemma 2.11) together with Lemma 2.7 and Stirling's formula on the asymptotic behaviour of the Gamma function. Hence, Proposition 2.10 is proved. $\square$

2.5 Operator $\mathcal{T}_\mu$ and Symplectic Fourier transform $\mathcal{F}_J$

The key idea to find eigenvalues of the integral transform $\mathcal{T}_\mu$ on $L^2(S^{2n-1})$ is to interpret it as the restriction of the *symplectic Fourier transform*, to be denoted by $\mathcal{F}_J$, on the ambient space $\mathbb{R}^{2n}$.

If $\operatorname{Re}\mu < -\frac{N}{2} + 1$, the following integral converges absolutely for any $h \in C^\infty(S^{N-1})$:

$$(\mathcal{Q}_\mu h)(\eta) := \int_{S^{N-1}} |\langle \omega, \eta \rangle|^{-\mu - \frac{N}{2}} h(\omega) d\omega. \tag{2.16}$$

Then $Q_\mu h$ extends meromorphically on $\mu \in \mathbb{C}$, whose poles are simple and contained in the set $\{1 - \frac{N}{2}, 3 - \frac{N}{2}, 5 - \frac{N}{2}, \dots\}$. Thus, we get a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$\mathcal{Q}_\mu : C^\infty(S^{N-1}) \to C^\infty(S^{N-1}).$$

We may regard $Q_\mu h$ as an even homogeneous function on $\mathbb{R}^N \setminus \{0\}$ of degree $-\mu - \frac{N}{2}$ by simply letting η be a variable in $\mathbb{R}^N \setminus \{0\}$. Then, $Q_\mu h \in V_\mu$. By Proposition 2.10, the Fourier transform $\mathcal{F}$ gives a bijection between $V_{-\mu}$ and V_μ for $|\mu| \neq \frac{N}{2}, \frac{N}{2} + 2, \dots$. On the other hand, V_μ can be identified with the space of smooth even functions on S^{N-1}. We notice that the latter space is independent of μ. Thus, we have the following diagram:

$$\begin{array}{ccccc}
\mathcal{F} & : & \mathcal{S}'(\mathbb{R}^N) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^N) \\
 & & \cup & \circlearrowleft & \cup \\
 & & V_{-\mu} & \xrightarrow{\sim} & V_\mu \\
 & & \cap & & \cap \\
\mathcal{Q}_\mu & : & C^\infty(S^{N-1}) & \longrightarrow & C^\infty(S^{N-1})
\end{array}$$

The lower diagram commutes up to a scalar constant. To make a precise statement, we define

$$
\begin{aligned}
C_N(\mu) &:= \frac{(2\pi)^{\mu+\frac{N}{2}}}{\Gamma(\mu+\frac{N}{2})\cos\frac{\pi}{2}(\mu+\frac{N}{2})} \\
&= \frac{2\pi^{\mu+\frac{N-1}{2}}\Gamma(\frac{2-N-2\mu}{4})}{\Gamma(\frac{N+2\mu}{4})}.
\end{aligned}
\tag{2.17}
$$

Then we have:

Proposition 2.13. *As operators that depend meromorphically on μ, $\mathcal{Q}_\mu$ satisfy the following identity:*

$$
\mathcal{Q}_\mu = C_N(\mu)\mathcal{F}\big|_{V_{-\mu}}.
$$

Proof. Any element in $V_{-\mu}$ is of the form

$$
h_{\mu-\frac{N}{2}}(r\omega) = r^{\mu-\frac{N}{2}}h(\omega) \quad (r>0,\ \omega\in S^{N-1}),
$$

for some $h \in C^\infty(S^{N-1})$ which is an even function, i.e., $h(\omega) = h(-\omega)$.

We shall prove

$$
Q_\mu h_{\mu-\frac{N}{2}} = C_N(\mu)\mathcal{F}h_{\mu-\frac{N}{2}} \tag{2.18}
$$

as distributions on $\mathbb{R}^N \setminus \{0\}$. For each fixed h, the both sides of (2.18) are distributions that depend meromorphically on μ. Therefore, it is sufficient to prove (2.18) for some non-empty open domain in μ, say,

$$
\operatorname{Re}\mu > -\frac{N}{2}. \tag{2.19}
$$

The inequality (2.19) implies that $h_{\mu-\frac{N}{2}} \in L^1_{loc}(\mathbb{R}^N)$, and we have

$$
\lim_{\varepsilon\downarrow 0} e^{-2\pi\varepsilon r}h_{\mu-\frac{N}{2}}(r\omega) = h_{\mu-\frac{N}{2}}(r\omega)
$$

as a locally integrable function, and also in $\mathcal{S}'(\mathbb{R}^N)$. Hence, taking the Fourier transform, we get

$$
\lim_{\varepsilon\downarrow 0}\mathcal{F}(e^{-2\pi\varepsilon r}h_{\mu-\frac{N}{2}}) = \mathcal{F}h_{\mu-\frac{N}{2}}
$$

in $\mathcal{S}'(\mathbb{R}^N)$.

Let us compute $\mathcal{F}(e^{-2\pi\varepsilon r}h_{\mu-\frac{N}{2}})$. Below, we use the Fourier transform for both $\mathbb{R}^N$ and $\mathbb{R}$, which will be denoted by $\mathcal{F}_{\mathbb{R}^N}$ and $\mathcal{F}_{\mathbb{R}}$ to avoid confusion. We note that $e^{-2\pi\varepsilon r}h_{\mu-\frac{N}{2}} \in L^1(\mathbb{R}^N)$ if $\varepsilon > 0$ and μ satisfies (2.19). Let $s > 0$ and $\eta \in S^{N-1}$. Then the Fourier transform can be computed by the Lebesgue integral:

$$\begin{aligned}
&\mathcal{F}_{\mathbb{R}^N}(e^{-2\pi\varepsilon r}h_{\mu-\frac{N}{2}})(s\eta) \\
&= \int_{S^{N-1}} \int_0^\infty e^{-2\pi\varepsilon r} r^{\mu+\frac{N}{2}-1} e^{-2\pi i r s\langle\omega,\eta\rangle} dr d\sigma(\omega) \\
&= \int_{S^{N-1}} \mathcal{F}_{\mathbb{R}}(r_+^{\mu+\frac{N}{2}-1})(s\langle\omega,\eta\rangle - i\varepsilon) d\sigma(\omega) \\
&= \frac{\Gamma(\mu+\frac{N}{2})e^{-\frac{\pi i}{2}(\mu+\frac{N}{2})}}{(2\pi)^{\mu+\frac{N}{2}}} \int_{S^{N-1}} (s\langle\omega,\eta\rangle - i\varepsilon)^{-\mu-\frac{N}{2}} h(\omega) d\sigma(\omega).
\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, we get

$$\mathcal{F}_{\mathbb{R}^N} h_{\mu-\frac{N}{2}}(s\eta) = \frac{\Gamma(\mu+\frac{N}{2})e^{-\frac{\pi i}{2}(\mu+\frac{N}{2})}}{(2\pi)^{\mu+\frac{N}{2}} s^{\mu+\frac{N}{2}}} \int_{S^{N-1}} (\langle\omega,\eta\rangle - i0)^{-\mu-\frac{N}{2}} h(\omega) d\sigma(\omega),$$

where $(\langle\omega,\eta\rangle - i0)^\lambda$ denotes the substitution of $x = \langle\omega,\eta\rangle$ into the distribution $(x - i0)^\lambda$ (see Example 2.5). Since h is an even function, the above integral amounts to

$$e^{i\frac{\pi}{2}(\mu+\frac{N}{2})} \cos\frac{\pi}{2}\left(\mu+\frac{N}{2}\right) \int_{S^{N-1}} |\langle\omega,\eta\rangle|^{-\mu-\frac{N}{2}} h(\omega) d\sigma(\omega)$$

by (2.7). Therefore, $\mathcal{F}_{\mathbb{R}^N} h_{\mu-\frac{N}{2}}(s\eta)$ equals

$$\begin{aligned}
&(2\pi)^{-\mu-\frac{N}{2}} s^{-\mu-\frac{N}{2}} \Gamma\left(\mu+\frac{N}{2}\right) \cos\frac{\pi}{2}\left(\mu+\frac{N}{2}\right) \int_{S^{N-1}} |\langle\omega,\eta\rangle|^{-\mu-\frac{N}{2}} h(\omega) d\sigma(\omega) \\
&= C_N(\mu)^{-1} s^{-\mu-\frac{N}{2}} (\mathcal{Q}_\mu h)(\eta).
\end{aligned}$$

Thus, Proposition 2.13 has been proved. $\square$

So far, N has been an arbitrary positive integer. Suppose now that N is an even integer, say, $N = 2n$. We introduce the symplectic Fourier transform defined by the formula:

$$(\mathcal{F}_J f)(Y) := \int_{\mathbb{R}^{2n}} f(X) e^{-2\pi i [X,Y]} dX.$$

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by $(x,\xi) \mapsto x + i\xi$. Correspondingly, the complex structure on $\mathbb{R}^{2n}$ is given by the linear transform

$$J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad J(x,\xi) := (-\xi, x).$$

Then the formula (1.1) is equivalent to

$$[X,Y] = \langle X, JY\rangle \quad (X,Y \in \mathbb{R}^{2n}),$$

and therefore, our $\mathcal{F}_J$ and the usual Fourier transform $\mathcal{F}_{\mathbb{R}^{2n}}$ are related by the formula:

$$(\mathcal{F}_J f)(Y) = \mathcal{F}_{\mathbb{R}^{2n}}(JY). \tag{2.20}$$

Likewise, the linear operators $\mathcal{T}_\mu$ (see (2.1)) and $\mathcal{Q}_\mu$ for $N = 2n$ (see (2.16)) are related by

$$\mathcal{T}_\mu f(Y) = \mathcal{Q}_\mu(JY).$$

Therefore, Proposition 2.13 leads us to:

Proposition 2.14. *Let $C_N(\mu)$ be the constant defined in* (2.17)*. Then,*

$$\mathcal{T}_\mu = C_{2n}(\mu)\mathcal{F}_J\big|_{V_{-\mu}}.$$

Remark 2.15. Since the symplectic Fourier transform $\mathcal{F}_J$ induces a bijection $\mathcal{F}_J\big|_{V_{-\mu}} : V_{-\mu} \xrightarrow{\sim} V_\mu$ for all $\mu \in \mathbb{C}$, Proposition 2.14 implies that $\mathcal{T}_\mu$ is also bijective as far as $C_{2n}(\mu) \neq 0, \infty$.

We note that $C_{2n}(\mu)$ has simple zeros at $\mu + n = 0, -2, -4, \ldots$. In this case, the kernel $\big|[X,Y]\big|^{-\mu-n}$ is a polynomial in Y of degree $-(\mu+n)$, and correspondingly, $(\mathcal{T}_\mu f)(Y)$ is also a polynomial of the same degree. Thus, $\operatorname{Image}\mathcal{T}_\mu$ is finite dimensional, and $\operatorname{Ker}\mathcal{T}_\mu$ is infinite dimensional.

On the other hand, $C_{2n}(\mu)$ has simple poles at $\mu + n = 1, 3, 5, \ldots$. This corresponds to the fact that the distribution $|x|^\lambda$ of one variable has simple poles at $\lambda = -1, -3, -5, \ldots$ (see [7]).

We are now ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose $p \in \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$. Since J acts on z_j $(1 \le j \le n)$ by $\sqrt{-1}$ and $\bar{z}_j$ by $-\sqrt{-1}$, we have

$$p(J\eta) = (-1)^{\frac{\alpha-\beta}{2}} p(\eta). \tag{2.21}$$

In view of Lemma 2.7, Proposition 2.14, and (2.20), the operator $\mathcal{T}_\mu$ acts on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ as a scalar

$$(-1)^{\frac{\alpha-\beta}{2}} C_{2n}(\mu) B_{2n}(\mu - n, \alpha+\beta).$$

This amounts to $(-1)^\beta A_{\alpha+\beta}(\mu)$, whence Theorem 2.1. $\square$

3 Proof of Theorem 1.1

3.1 Dimension formulas for spherical harmonics

This subsection summarizes some elementary results on the dimensions of harmonic polynomials in a way that we shall use later. They are more or less known, however, we give a brief account of them for the convenience of the reader.

Let $\mathcal{P}^k(\mathbb{R}^N)$ be the complex vector space of homogeneous polynomials in N variables of degree k. Its dimension is given by the binomial coefficient:

$$\dim \mathcal{P}^k(\mathbb{R}^N) = \binom{k+N-1}{k}.$$

In light of the linear bijection (see e.g. [9, pp. 17]):

$$\mathcal{H}^k(\mathbb{R}^N) \oplus \mathcal{P}^{k-2}(\mathbb{R}^N) \xrightarrow{\sim} \mathcal{P}^k(\mathbb{R}^N), \quad (p,q) \mapsto p(X) + |X|^2 q(X),$$

we get the dimension formula of $\mathcal{H}^k(\mathbb{R}^N)$:

$$\begin{aligned} \dim \mathcal{H}^k(\mathbb{R}^N) &= \dim \mathcal{P}^k(\mathbb{R}^N) - \dim \mathcal{P}^{k-2}(\mathbb{R}^N) \\ &= \frac{(k+N-3)!(2k+N-2)}{k!(N-2)!}. \end{aligned} \tag{3.1}$$

In the next subsection, we shall use the following recurrence formula:

Lemma 3.1. $\dim \mathcal{H}^k(\mathbb{R}^N) + \dim \mathcal{H}^{k-1}(\mathbb{R}^{N+1}) = \dim \mathcal{H}^k(\mathbb{R}^{N+1})$.

Proof. By the elementary combinatorial formula

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k},$$

we have

$$\dim \mathcal{P}^k(\mathbb{R}^N) + \dim \mathcal{P}^{k-1}(\mathbb{R}^{N+1}) = \dim \mathcal{P}^k(\mathbb{R}^{N+1}). \tag{3.2}$$

Taking the difference of (3.2) for k and $k-2$, and applying (3.1), we get Lemma 3.1. $\square$

To find the dimension formula of $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ one might apply the above method (see e.g. [16, Section 11.2.1]), but it would be more convenient for our purpose to use representation theory. There is a natural action of the unitary group $U(n)$ on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$. This representation is irreducible, and its highest weight is given by $(\alpha, 0, \ldots, 0, -\beta)$ in the standard coordinates of the Cartan subalgebra. By the Weyl character formula, we get

$$\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = \frac{(\alpha+\beta+n-1)}{(n-1)!(n-2)!} \prod_{i=2}^{n-1} (\alpha+i-1)(\beta+n-i).$$

If we use the Pochhammer symbol $(a)_l$ defined by

$$(a)_l := \frac{\Gamma(a+l)}{\Gamma(a)} = a(a+1)\cdots(a+l-1),$$

then we may express these dimensions as

$$\begin{aligned} \dim \mathcal{H}^k(\mathbb{R}^N) &= \frac{(k+1)_{N-3}(2k+N-2)}{\Gamma(N-1)}, \\ \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) &= \frac{(\alpha+\beta+n-1)(\alpha+1)_{n-2}(\beta+1)_{n-2}}{\Gamma(n)\Gamma(n-1)}. \end{aligned} \tag{3.3}$$

3.2 Alternating sum of $\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$

By the direct sum decomposition (2.3), the following identity is obvious:

$$\dim \mathcal{H}^k(\mathbb{R}^{2n}) = \sum_{\alpha+\beta=k} \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n).$$

However, what we need for the proof of Theorem 1.1 is an explicit formula for the alternating sum:

$$D(k) := \sum_{\alpha+\beta=k} (-1)^\beta \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n).$$

Clearly, $D(k) = 0$ for odd k because $\dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = \dim \mathcal{H}^{\beta,\alpha}(\mathbb{C}^n)$.

A closed formula of $D(k)$ for even k is the main issue of this subsection, and we establish the following relation:

Proposition 3.2.

$$D(2l) = \dim \mathcal{H}^l(\mathbb{R}^{n+1}) = \frac{(n-1)_l(\frac{n+1}{2})_l}{l!(\frac{n-1}{2})_l}. \tag{3.4}$$

Remark 3.3. The Pochhammer symbol $(a)_l$ may be regarded as a meromorphic function. Thus, the right-hand side of (3.4) can be regarded as a meromorphic function of n. In this sense, the right-hand side of (3.4) still makes sense for $n = 1$.

The rest of this subsection is devoted to the proof of Proposition 3.2. For this, we set

$$X^{(l)} := x^l + \frac{1}{x^l} \quad \text{for } l = 1, 2, \dots .$$

It is readily seen that $X^{(l)}$ is expressed as a monomial in

$$X := x + \frac{1}{x}$$

of degree l. For example,

$$X^{(1)} = X,\ X^{(2)} = X^2 - 2,\ X^{(3)} = X^3 - 3X, \dots . \tag{3.5}$$

For an arbitrary l, we have the following formula:

Lemma 3.4.

$$X^{(l)} = \sum_{j=0}^{[\frac{l}{2}]} (-1)^j \dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) X^{l-2j}. \tag{3.6}$$

Proof. We prove Lemma 3.4 by induction on l. The equation (3.6) holds for $l = 1, 2$ by (3.5). Suppose $l \geq 2$. We shall prove the equation (3.6) for $l + 1$. We use

$$\begin{aligned} X^{(l+1)} &= \left(x + \frac{1}{x}\right)\left(x^l + \frac{1}{x^l}\right) - \left(x^{l-1} + \frac{1}{x^{l-1}}\right) \\ &= XX^{(l)} - X^{(l-1)}. \end{aligned}$$

By substituting (3.6) for l and $l - 1$ into the right-hand side, we get

$$\begin{aligned} X^{(l+1)} = &\sum_{j=0}^{[\frac{l}{2}]} (-1)^j \dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) X^{l+1-2j} \\ &- \sum_{i=0}^{[\frac{l-1}{2}]} (-1)^i \dim \mathcal{H}^i(\mathbb{R}^{l+1-2i}) X^{l-1-2i} \\ = X^{l+1} &+ \sum_{j=1}^{[\frac{l+1}{2}]} \left((-1)^j (\dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) + \dim \mathcal{H}^{j-1}(\mathbb{R}^{l+3-2j})) X^{l+1-2j}\right). \end{aligned}$$

To see the second equality for odd l, we note that $\dim \mathcal{H}^d(\mathbb{R}^1) = 0$ for $d \geq 2$, and thus

$$\dim \mathcal{H}^j(\mathbb{R}^{l+2-2j}) = 0 \quad \text{for } j = \frac{l+1}{2}. \tag{3.7}$$

Applying the recurrence formula given in Lemma 3.1, we get (3.6) for $l+1$. By induction, we have proved Lemma 3.4. $\square$

Proof of Proposition 3.2. We take a maximal torus T of $U(n)$ and its coordinate $(x_1, \ldots, x_n)$ such that

$$T = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\},$$

and that the linear map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is represented as $J = (\sqrt{-1}, \ldots, \sqrt{-1}) \in T$. Then the character $\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(g)$ of the representation of $O(2n)$ on $\mathcal{H}^k(\mathbb{R}^{2n})$ takes the value

$$\sum_{\alpha+\beta=k} (-1)^{\frac{\alpha-\beta}{2}} \dim \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) = (-1)^{\frac{k}{2}} D(k)$$

at $g = J$.

By using this observation, we shall analyze the character $\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(g)$ as g approaches to the singular point $J \in T$.

Let

$$X_j^{(l)} := x_j^l + \frac{1}{x_j^l} \quad (1 \leq j \leq n,\ l \in \mathbb{N}),$$

and we set

$$s_k(x) := \det \begin{pmatrix} X_1^{(k+n-1)} & X_2^{(k+n-1)} & \cdots & X_n^{(k+n-1)} \\ X_1^{(n-2)} & X_2^{(n-2)} & \cdots & X_n^{(n-2)} \\ \vdots & \vdots & & \vdots \\ X_1^{(1)} & X_2^{(1)} & & X_n^{(1)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Then, by the Weyl character formula for the group $O(2n)$ and by using a trick which reduces the summation over the Weyl group for $O(2n)$ to that over the symmetric group $\mathcal{S}_n$ (see [14]), we have

$$\chi_{\mathcal{H}^k(\mathbb{R}^{2n})}(x) = \frac{s_k(x)}{s_0(x)} \quad \text{for } x \in T.$$

Since $X^{(l)} \equiv X^l \bmod \mathbb{Q}\text{-span}\langle 1, X, \dots, X^{l-1}\rangle$ an elementary property of the determinant shows:

$$s_k(x) = \det \begin{pmatrix} X_1^{(k+n-1)} & X_2^{(k+n-1)} & \cdots & X_n^{(k+n-1)} \\ X_1^{n-2} & X_2^{n-2} & \cdots & X_n^{n-2} \\ \vdots & \vdots & & \vdots \\ X_1 & X_2 & & X_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

As x_j goes to $\sqrt{-1}$, X_j tends to 0 $(1 \le j \le n)$. Therefore, we have

$$\begin{aligned} \chi^{2l}_{\mathcal{H}(\mathbb{R}^{2n})}(J) &= \lim_{X_1,\dots,X_N \to 0} \frac{s_{2l}(x)}{s_0(x)} \\ &= \text{the coefficient of } X^{n-1} \text{ in the expansion (3.6)} \\ &\quad\ \text{for } X^{(2l+n-1)} \\ &= (-1)^l \dim \mathcal{H}^l(\mathbb{R}^{n+1}). \end{aligned}$$

Here, we have used Lemma 3.4 for the last equality. Thus, we have proved

$$D(2l) = \dim \mathcal{H}^l(\mathbb{R}^{n+1}).$$

The second equality of (3.4) is immediate from (3.1). $\square$

3.3 Triple integral as a Trace

We are now ready to prove Theorem 1.1. As we remarked in Introduction, the both sides of Theorem 1.1 are meromorphic functions of λ_1, λ_2, and λ_3. Therefore, it is sufficient to prove the identity in Theorem 1.1 in an open set of the parameters $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$.

By the change of variables $\mu_j := \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 - n) - \lambda_j$ $(1 \le j \le 3)$, we first consider the case when $\operatorname{Re}\mu_1 \ll 0$, $\operatorname{Re}\mu_2 \ll 0$, and $\operatorname{Re}\mu_3 \ll 0$. Then, the operators $\mathcal{T}_{\mu_1}$, $\mathcal{T}_{\mu_2}$, and $\mathcal{T}_{\mu_3}$ are Hilbert–Schmidt operators on $L^2(S^{2n-1})$. In particular, the composition $\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}$ is of trace class, and its trace is given by

$$\begin{aligned} &\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) \\ &= \int_{(S^{2n-1})^3} \big|[X,Y]\big|^{-\mu_1-n} \big|[Y,Z]\big|^{-\mu_2-n} \big|[Z,X]\big|^{-\mu_3-n} d\sigma(X) d\sigma(Y) d\sigma(Z). \end{aligned}$$

On the other hand, the trace of the operator $\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}$ can be also computed by its eigenvalues. Therefore, by using Theorem 2.1, we have

$$\begin{aligned}
\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) &= \sum_{\alpha,\beta}\left(\prod_{j=1}^{3}(-1)^{\beta}A_{\alpha+\beta}(\mu_j)\right)\dim\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)\\
&= \sum_{k=0}^{\infty}\prod_{j=1}^{3}A_k(\mu_j)\left(\sum_{\alpha+\beta=k}(-1)^{3\beta}\dim\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)\right)\\
&= \sum_{l=0}^{\infty}D(2l)\prod_{j=1}^{3}A_{2l}(\mu_j).
\end{aligned}$$

Applying Proposition 3.2, we get

$$\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) = \sum_{l=0}^{\infty}A_{2l}(\mu_1)A_{2l}(\mu_2)A_{2l}(\mu_3)\dim\mathcal{H}^{l}(\mathbb{R}^{n+1}).$$

In light of the recurrence relation:

$$\frac{A_{2l+2}(\mu)}{A_{2l}(\mu)} = \frac{l+\frac{n+\mu}{2}}{l+\frac{n-\mu}{2}},$$

the meromorphic function $A_{2l}(\mu)$ can be expressed in terms of Pochhammer symbols as

$$A_{2l}(\mu) = \frac{(\frac{n+\mu}{2})_l}{(\frac{n-\mu}{2})_l}A_0(\mu),$$

where

$$A_0(\mu) = 2\pi^{n-\frac{1}{2}}\frac{\Gamma(\frac{1-n-\mu}{2})}{\Gamma(\frac{n-\mu}{2})}. \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3})\\
&= A_0(\mu_1)A_0(\mu_2)A_0(\mu_3)\sum_{l=0}^{\infty}\frac{(n-1)_l(\frac{n+1}{2})_l}{l!(\frac{n-1}{2})_l}\prod_{j=1}^{3}\frac{(\frac{n+\mu_j}{2})_l}{(\frac{n-\mu_j}{2})_l}\\
&= A_0(\mu_1)A_0(\mu_2)A_0(\mu_3)\,{}_5F_4\left(\begin{matrix} n-1 & \frac{n+1}{2} & \frac{n+\mu_1}{2} & \frac{n+\mu_2}{2} & \frac{n+\mu_3}{2} \\ & \frac{n-1}{2} & \frac{n-\mu_1}{2} & \frac{n-\mu_2}{2} & \frac{n-\mu_3}{2}\end{matrix};1\right).
\end{aligned}$$

Here ${}_5F_4$ is a generalized hypergeometric function.

A generalized hypergeometric function

$$
{}_pF_q\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p \\ & \beta_1 & \cdots & \beta_q\end{pmatrix}; z\Bigg)
$$

is called *well-poised* (see [1]) if $p = q + 1$ and

$$
1 + \alpha_1 = \alpha_2 + \beta_1 = \cdots = \alpha_p + \beta_q.
$$

In particular, our case is well-poised, and we can use the following *Dougall–Ramanujan identity* (see [loc. cit., pp. 25–26]):

$$
\begin{aligned}
&{}_5F_4\left(\begin{matrix}m-1 & \frac{m+1}{2} & -x & -y & -z \\ & \frac{m-1}{2} & x+m & y+m & z+m\end{matrix}; 1\right) \\
&= \frac{\Gamma(x+m)\Gamma(y+m)\Gamma(z+m)\Gamma(x+y+z+m)}{\Gamma(m)\Gamma(x+y+m)\Gamma(y+z+m)\Gamma(x+z+m)}.
\end{aligned}
$$

Together with (3.8), we get

$$
\operatorname{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3}) = \frac{(2\pi^{n-\frac{1}{2}})^3\Gamma(\frac{1-n-\mu_1}{2})\Gamma(\frac{1-n-\mu_2}{2})\Gamma(\frac{1-n-\mu_3}{2})\Gamma(\frac{-\mu_1-\mu_2-\mu_3-n}{2})}{\Gamma(n)\Gamma(-\frac{\mu_1+\mu_2}{2})\Gamma(-\frac{\mu_2+\mu_3}{2})\Gamma(-\frac{\mu_1+\mu_3}{2})}. \tag{3.9}
$$

Now, Theorem 1.1 follows by substituting $\mu_1 = -\frac{1}{2}(\alpha + n)$, $\mu_2 = -\frac{1}{2}(\beta + n)$, and $\mu_3 = -\frac{1}{2}(\gamma + n)$.

4 Other triple integral formulas

In this section, we discuss explicit formulas for the integrals of the triple product of powers of $|x - y|$ and $|\langle x, y\rangle|$ instead of those of the symplectic form $\big|[X, Y]\big|$.

4.1 Triple product of powers of $|x - y|$

In this subsection we consider a family of linear operators that depend meromorphically on $\mu \in \mathbb{C}$ by

$$
\mathcal{R}_\mu : C^\infty(S^m) \to C^\infty(S^m)
$$

defined by

$$(\mathcal{R}_\mu f)(\eta) = \int_{S^m} |\omega - \eta|^{-\mu-m} f(\omega) d\sigma(\omega). \tag{4.1}$$

The multiplier action of $\mathcal{R}_\mu$ on spherical harmonics is known (see e.g. [2]):

$$\mathcal{R}_\mu \Big|_{\mathcal{H}^k(\mathbb{R}^{m+1})} = \gamma_k(\mu)\,\mathrm{id}, \tag{4.2}$$

where $\gamma_k(\mu) \equiv \gamma_k(\mu, \mathbb{R}^{m+1})$ is given by

$$\gamma_k(\mu) = \frac{\Gamma(m+\frac{1}{2})\Gamma(-\frac{\mu}{2})\Gamma(k+\frac{m+\mu}{2})}{2^{\mu+1}\sqrt{\pi}\Gamma(\frac{\mu+m}{2})\Gamma(k+\frac{m-\mu}{2})}. \tag{4.3}$$

Then, by an argument parallel to Section 3.3, we can obtain a closed formula for the triple integral built on $\mathcal{R}_\mu$ (see Theorem 4.2 below). Instead of repeating similar computations, we pin down a comparison result between the two triple integral formulas by using Proposition 3.2. This comparison result explains the reason why the same method (e.g. Dougall–Ramanujan identity) is applicable, and seems interesting for its own sake.

Proposition 4.1.

$$\begin{aligned} &\mathrm{Trace}(\mathcal{R}_{\mu_1}\mathcal{R}_{\mu_2}\mathcal{R}_{\mu_3} : L^2(S^m) \to L^2(S^m)) \\ &= c\,\mathrm{Trace}(\mathcal{T}_{\mu_1}\mathcal{T}_{\mu_2}\mathcal{T}_{\mu_3} : L^2(S^{2m-1}) \to L^2(S^{2m-1})), \end{aligned} \tag{4.4}$$

where

$$c = \left(\frac{\Gamma(m+\frac{1}{2})}{2^2\pi^m}\right)^3 \prod_{j=1}^3 \frac{\Gamma(-\frac{\mu_j}{2})}{2^{\mu_j}\Gamma(\frac{-\mu_j-m+1}{2})}.$$

Proof. By (4.2) the left-hand side of (4.4) equals

$$\sum_{k=0}^\infty \left(\prod_{j=1}^3 \gamma_k(\mu_j, \mathbb{R}^{m+1})\right) \dim \mathcal{H}^k(\mathbb{R}^{m+1}). \tag{4.5}$$

Comparing (4.3) with Theorem 2.1 we get

$$\frac{\gamma_k(\mu, \mathbb{R}^{m+1})}{A_{2k}(\mu, \mathbb{C}^m)} = \frac{\Gamma(m+\frac{1}{2})\Gamma(-\frac{\mu}{2})}{2^{\mu+2}\pi^m\Gamma(\frac{-\mu-m+1}{2})}. \tag{4.6}$$

By (4.6) and (3.9), we see that (4.5) equals the right-hand side of (4.4). $\square$

The right-hand side in Proposition 4.1 was found in (3.9). Then, by a simple computation, we get

$$\begin{aligned}&\operatorname{Trace}(\mathcal{R}_{\mu_1}\mathcal{R}_{\mu_2}\mathcal{R}_{\mu_3})\\ &= \left(\frac{\Gamma(m+\frac{1}{2})}{2\pi^{\frac{1}{2}}}\right)^3 \frac{\Gamma(\frac{-\mu_1-\mu_2-\mu_3-m}{2})}{\Gamma(m)} \prod_{j=1}^{3} \frac{\Gamma(-\frac{\mu_j}{2})}{2^{\mu_j}\Gamma(\frac{\mu_j-(\mu_1+\mu_2+\mu_3)}{2})}.\end{aligned}$$

Finally, substituting $\mu_j = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3-m)-\lambda_j$ $(1 \le j \le 3)$, we have proved the following:

Theorem 4.2. *Let α, β, γ, and δ be as in Theorem 1.1*

$$\begin{aligned}&\int_{S^m\times S^m\times S^m} |Y-Z|^{\frac{\alpha-m}{2}}|Z-X|^{\frac{\beta-m}{2}}|X-Y|^{\frac{\gamma-m}{2}} d\sigma(X)d\sigma(Y)d\sigma(Z)\\ &= \left(\frac{\Gamma(m+\frac{1}{2})}{2^{1-\frac{m}{2}}\pi^{\frac{1}{2}}}\right)^3 \frac{1}{2^{\frac{\lambda_1+\lambda_2+\lambda_3}{2}}\Gamma(m)} \frac{\Gamma(\frac{\alpha+m}{4})\Gamma(\frac{\beta+m}{4})\Gamma(\frac{\gamma+m}{4})\Gamma(\frac{\delta+m}{4})}{\Gamma(\frac{m-\lambda_1}{2})\Gamma(\frac{m-\lambda_2}{2})\Gamma(\frac{m-\lambda_3}{2})}.\end{aligned}$$

We will give in Proposition 6.9 the domain for the absolute convergence of the above integral.

Remark 4.3. The formula in Theorem 4.2 was previously found by A. Deitmar [6] by a different method; namely it established a recurrence formula bridging $SO_o(\ell+1,1)$ to $SO_o(\ell-1,1)$ and used the Bernstein–Reznikov formula for $SO_o(2,1)$ and an analogous formula for $SO_o(3,1)$.

4.2 Triple product of powers of $|\langle x,y\rangle|$

In this subsection we consider the third case, namely, the linear operators $\mathcal{Q}_\mu : C^\infty(S^{N-1}) \to C^\infty(S^{N-1})$ defined by the kernel $|\langle x,y\rangle|^{-\mu-\frac{N}{2}}$ (see (2.16)) and the corresponding triple product integrals.

Here is the counterpart of Theorem 2.1 for $\mathcal{Q}_\mu$:

Proposition 4.4. $\mathcal{Q}_\mu\Big|_{\mathcal{H}^k(\mathbb{R}^N)} = 0$ *for odd k, and*

$$\mathcal{Q}_\mu\Big|_{\mathcal{H}^{2l}(\mathbb{R}^N)} = c_N(\mu,l)\,\mathrm{id},$$

where

$$c_N(\mu,l) = (-1)^l \frac{2\pi^{\frac{N-1}{2}}\Gamma(\frac{2-N-2\mu}{4})\Gamma(l+\frac{2\mu+N}{4})}{\Gamma(\frac{N+2\mu}{4})\Gamma(l+\frac{-2\mu+N}{4})}.$$

Proof. By Lemma 2.7 and Proposition 2.13, we have

$$c_N(\mu,l)=C_N(\mu)B_N(\mu-\frac{N}{2},2l).$$

$\square$

As in the previous cases, we have

$$\begin{aligned}&\mathrm{Trace}(\mathcal{Q}_{\mu_1}\mathcal{Q}_{\mu_2}\mathcal{Q}_{\mu_3}:L^2(S^{N-1})\to L^2(S^{N-1}))\\&=\sum_{l=0}^{\infty}\left(\prod_{j=1}^{3}c_N(\mu_j,l)\right)\dim\mathcal{H}^{2l}(\mathbb{R}^N).\end{aligned}\tag{4.7}$$

By substituting

$$\begin{aligned}c_N(\mu,l)&=(-1)^l c_N(\mu,0)\frac{(\frac{N+2\mu}{4})_l}{(\frac{N-2\mu}{4})_l},\\ \dim\mathcal{H}^{2l}(\mathbb{R}^N)&=\frac{(\frac{N}{2}-1)_l(\frac{N-1}{2})_l(\frac{N+2}{4})_l}{l!(\frac{1}{2})_l(\frac{N-2}{4})_l}\end{aligned}$$

into the right-hand side of (4.7), we see that (4.7) equals

$$\begin{aligned}&\left(\prod_{j=0}^{3}c_N(\mu_j,0)\right)\sum_{j=0}^{\infty}(-1)^l\prod_{j=1}^{3}\frac{(\frac{N+2\mu_j}{4})_l}{(\frac{N-2\mu_j}{4})_l}\frac{(\frac{N}{2}-1)_l(\frac{N-1}{2})_l(\frac{N+2}{4})_l}{l!(\frac{1}{2})_l(\frac{N-2}{4})_l}\\&=\prod_{j=0}^{3}c_N(\mu_j,0){}_6F_5\left(\begin{matrix}\frac{N}{2}-1 & \frac{N+2}{4} & \frac{N-1}{2} & \frac{N+2\mu_1}{4} & \frac{N+2\mu_2}{4} & \frac{N+2\mu_3}{4}\\ & \frac{N-2}{4} & \frac{1}{2} & \frac{N-2\mu_1}{4} & \frac{N-2\mu_2}{4} & \frac{N-2\mu_3}{4}\end{matrix};1\right).\end{aligned}$$

By using Whipple's transformation ([1, p.28]):

$$\begin{aligned}&{}_6F_5\left(\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e\\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix};-1\right)\\&=\frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\\&\quad\times{}_3F_2\left(\begin{matrix}1+a-b-c, & d, & e\\ & 1+a-b, & 1+a-c\end{matrix};1\right),\end{aligned}$$

we get

$$\operatorname{Trace}(\mathcal{Q}_{\mu_1}\mathcal{Q}_{\mu_2}\mathcal{Q}_{\mu_3}) = \frac{(2\pi^{\frac{N-3}{2}})^3\prod_{j=1}^3\Gamma(\frac{2-N-2\mu_j}{4})}{\Gamma(\frac{N}{2})\Gamma(-\frac{\mu_2+\mu_3}{2})\Gamma(\frac{N-2\mu_1}{4})} \times {}_3F_2\left(\begin{matrix}\frac{2-N-2\mu_1}{4} & \frac{N+2\mu_2}{4} & \frac{N+2\mu_3}{4} \\ & \frac{1}{2} & \frac{N-2\mu_1}{4}\end{matrix};1\right). \tag{4.8}$$

Hence we have proved:

Theorem 4.5. *We have the following identity as a meromorphic function of* (ν_1, ν_2, ν_3)*:*

$$\int_{S^{N-1}\times S^{N-1}\times S^{N-1}} |\langle y,z\rangle|^{-2\nu_1}|\langle z,x\rangle|^{-2\nu_2}|\langle x,y\rangle|^{-2\nu_3}d\sigma(x)d\sigma(y)d\sigma(z)$$
$$= \frac{(2\pi^{\frac{N-3}{2}})^3\prod_{j=1}^3\Gamma(\frac{1}{2}-\nu_j)}{\Gamma(\frac{N}{2})\Gamma(-\nu_2-\nu_3+\frac{N}{2})\Gamma(-\nu_1+\frac{N}{2})} \times {}_3F_2\left(\begin{matrix}\frac{1}{2}-\nu_1 & \nu_2 & \nu_3 \\ & \frac{1}{2} & -\nu_1+\frac{N}{2}\end{matrix};1\right).$$

We will give in Proposition 6.7 the precise region for the absolute convergence of the above integral.

5 Perspectives from representation theory

In this paper we have proved closed formulas for the triple integrals (see e.g. Theorem 1.1), based on a combination of methods from classical harmonic analysis. As we have seen, these methods allow us to establish explicit formulas for symplectic groups of any rank, and even in rank one case it gives a new proof of the original results due to Bernstein and Reznikov [5] and Deitmar [6].

So far we have avoided infinite dimensional representation theory, which was not used in our proof of main results. On the other hand, there are a number of interesting perspectives of these formulas, and also of the steps in its proof, that deserve comments.

One aspect of Theorem 1.1 is that the triple integral considered therein arises from a particular series of representations π_μ of the symplectic group $G = Sp(n,\mathbb{R})$ of rank n induced from a maximal parabolic subgroup $P \subset G$ and depending on a complex parameter μ. Section 5 highlights this point mostly.

Another aspect is that of analytic number theory, which was the main theme of [4, 5]. Motivated by the classical Rankin–Selberg method, authors considered a cocompact discrete subgroup of the rank one symplectic group and automorphic functions on the associated locally symmetric space. The product of two such functions may be decomposed in terms of a basis of automorphic functions and the corresponding coefficients are related to automorphic L-functions. The closed formula ($n = 1$ in Theorem 1.1) gave an estimate of their decay [5].

Yet another aspect of the above mentioned triple integral is that it arises also in pseudo-differential analysis of the phase space $\mathbb{R}^{2n}$. This phenomenon was treated in [13], where the symmetries of the Weyl operator calculus on the Hilbert space $L^2(\mathbb{R}^{2n})$ were considered.

5.1 Invariant trilinear forms

Now we focus on some links between the triple integrals discussed in Sections 1–4 and representation theory of semisimple Lie groups.

We begin with a construction of an invariant trilinear form based on the Knapp–Stein intertwining operators. Let G be a connected real semisimple Lie group and P an arbitrary parabolic subgroup. Let $P = MAN$ be a Langlands decomposition, $\mathfrak{a}$ and $\mathfrak{n}$ the Lie algebras of A and N respectively, and 2ρ the sum of roots of $\mathfrak{n}$ with respect to $\mathfrak{a}$. Take a Cartan involution θ of G stabilizing MA and set $K = \{g \in G : \theta(g) = g\}$.

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we define (possibly degenerate) principal series representations of G, to be denoted by π_λ, on the space of smooth sections for the G-equivariant line bundle $\mathcal{L}_{\lambda+\rho} = G \times_P \mathbb{C}_{\lambda+\rho}$ over the real flag variety G/P, equivalently on the vector space

$$V_\lambda^\infty \equiv V_\lambda := \{f \in C^\infty(G) : f(gman) = a^{-\lambda-\rho} f(g), \forall man \in P\}.$$

In our parametrization, $\mathcal{L}_{2\rho}$ is the volume bundle $\Lambda^{\dim G/P} T^*(G/P)$ over G/P. Similarly, the space of distribution sections for $\mathcal{L}_{\lambda+\rho}$ will be denoted by $V_\lambda^{-\infty}$. These representations are called *spherical* because V_λ contains a K-fixed vector $\mathbb{1}_\lambda$ which is defined by the formula: $\mathbb{1}_\lambda(kman) := a^{-\lambda-\rho}$ for $kman \in KP$.

Denote by $\overline{P} = MA\overline{N}$ the opposite parabolic subgroup to P. Assume that it satisfies the condition:

C1. P and $\overline{P}$ are conjugate in G.

Then there exists the G-intertwining operators $\mathcal{T}_\lambda : V_{-\lambda} \to V_\lambda$, referred to as the *Knapp–Stein intertwining operators* [10], that depend meromorphically on λ. They are given by the distribution-valued kernels $K_\lambda(x,y) \in V_\lambda^{-\infty} \otimes V_\lambda^{-\infty}$ such that $(\mathcal{T}_\lambda f)(x) = \langle f(y),\, K_\lambda(x,y)\rangle \in V_\lambda$ for $f \in V_{-\lambda}$. The Knapp–Stein kernel K_λ may be thought of as a distribution on $G \times G$ subject to the following invariance condition for $g \in G$ and $m_j a_j n_j \in MAN$ $(j = 1, 2)$:

$$K_\lambda(gxm_1a_1n_1, gym_2a_2n_2) = a_1^{-\lambda-\rho} a_2^{-\lambda-\rho} K_\lambda(x,y). \tag{5.1}$$

For $f_j \in V_{\lambda_j}$ $(j = 1, 2, 3)$, we set

$$\begin{aligned}&\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(f_1, f_2, f_3)\\ &:= \langle K_{\frac{1}{2}(\alpha-\rho)}(y,z) K_{\frac{1}{2}(\beta-\rho)}(z,x) K_{\frac{1}{2}(\gamma-\rho)}(x,y), f_1(x) f_2(y) f_3(z)\rangle,\end{aligned} \tag{5.2}$$

where $\alpha = \lambda_1 - \lambda_2 - \lambda_3$, $\beta = -\lambda_1 + \lambda_2 - \lambda_3$, $\gamma = -\lambda_1 - \lambda_2 + \lambda_3 \in \mathfrak{a}_{\mathbb{C}}^*$.

We have the following:

Proposition 5.1. *Assume P and $\overline{P}$ are conjugate in G. Then there exists a non–empty open region of $(\lambda_1, \lambda_2, \lambda_3) \in (\mathfrak{a}_{\mathbb{C}}^*)^3$ for which the integral (5.2) converges. It extends as a meromorphic function of λ_1, λ_2 and λ_3. Then, the resulting continuous trilinear form*

$$\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3} : V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3} \longrightarrow \mathbb{C} \tag{5.3}$$

is invariant with respect to the diagonal action of G:

$$\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(\pi_{\lambda_1}(g)\, f_1,\ \pi_{\lambda_2}(g)\, f_2,\ \pi_{\lambda_3}(g)\, f_3) = \mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(f_1,\ f_2,\ f_3).$$

Proof. We will give in Section 5.2 a sufficient condition on $(\lambda_1, \lambda_2, \lambda_3)$ for which the integral (5.2) converges absolutely. The meromorphic continuation can be justified by the Atiyah–Bernstein–Gelfand regularization of the integral (5.2) ([3], see also [7]). Parameters α, β and γ are chosen in such a way that the integrand in (5.2) is a section of the volume bundle of $(G/P)^3$. Whence the invariance follows. $\square$

The case when P is a minimal parabolic subgroup was considered in [6] for $G = SO_0(m+1, 1)$. We note that in this situation $\overline{P}$ is automatically conjugate to P.

Returning to our settings, we have an isomorphism of Lie algebras:

$$\mathfrak{sp}(1, \mathbb{R}) \simeq \mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R}),$$

each of which is the 'bottom' of different series of Lie algebras, namely $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{so}(n,1)$, and $\mathfrak{sl}(n,\mathbb{R})$. Bearing this in mind, we list the following three cases:

Case **Sp**. Theorem 1.1 corresponds to the evaluation of the trilinear form (5.3) on the K-fixed vector $\mathbb{1}_{\lambda_1} \otimes \mathbb{1}_{\lambda_2} \otimes \mathbb{1}_{\lambda_3}$ for the following particular pair: $G = Sp(n,\mathbb{R})$ and $P = MAN$ a maximal parabolic subgroup such that $M \simeq \mathbb{Z}/2\mathbb{Z} \times Sp(n-1,\mathbb{R})$ and N is the Heisenberg group in $2n-1$ variables. Notice that S^{2n-1} is a double covering of G/P. The representation space V_μ can be identified with $V_\mu(\mathbb{R}^{2n})$ introduced in (2.13). Then the kernel of the operator $\mathcal{T}_\mu$ introduced in (2.1) is $K_\mu(X,Y) = |[X,Y]|^{-\mu-n} \in V_\mu^{-\infty} \otimes V_\mu^{-\infty}$ which gives rise to the Knapp–Stein intertwining operator.

Case **SO**. Theorem 4.2 corresponds to the case where $G = SO_o(m+1,1)$ and P is a minimal parabolic subgroup. Through the identification $G/P \simeq S^m$ the Knapp–Stein intertwining operator is given by $\mathcal{R}_\mu$ (see (4.1)), and the triple integral in Theorem 4.2 corresponds to the evaluation of the trilinear form (5.2) on the K-fixed vector.

Case: GL. Yet another expression of the sphere S^{N-1} as a homogeneous space is given by G/P, where $= GL(N,\mathbb{R})$ and P is a maximal parabolic subgroup corresponding to the partition $N = 1 + (N-1)$. The operators $\mathcal{Q}_\mu$ introduced in (2.16) and involved in the Theorem 4.5 can also be interpreted as the Knapp–Stein integrals for representations induced from P and its opposite parabolic $\overline{P}$. Notice that the condition C1 fails for $N > 2$ and Proposition 5.1 does not apply.

What we have found in particular is the eigenvalues of operators $\mathcal{T}_\mu$, $\mathcal{Q}_\mu$ and $\mathcal{R}_\mu$ in terms of Gamma functions. The corresponding eigenspaces are irreducible representation spaces of the maximal compact subgroup K. Indeed, in all three cases the following condition holds:

C2. The space $K/(K \cap M)$ is a multiplicity–free space, in other words, $(K, K \cap M)$ is a Gelfand pair.

This implies that the representation space V_μ contains an algebraic direct sum of pairwise inequivalent irreducible representations of K as its dense subspace. Therefore the action of the operators $\mathcal{T}_\mu$ on each K-representation space is automatically a scalar multiple of the identity by Schur's lemma. For example in Case **Sp**, $K \simeq U(n)$, the corresponding restriction $\pi_\mu\big|_K$ is given by $\bigoplus_{\alpha,\beta\in\mathbb{N}} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$, and the eigenvalues are described in Theorem 2.1.

In Cases **SO** and **GL** the condition C2 is also satisfied. We can see this by a direct computation but also by the general observation that the unipotent radical N is abelian and consequently $(K, M \cap K)$ is a symmetric pair.

Another feature of our settings is the following condition:

C3. The diagonal action of G on $(G/P)^3$ admits an open orbit.

(In fact, there is only one such an open dense orbit except the case of $SL(2, \mathbb{R})$, where there are two open orbits.)

The condition C3 is connected to the upper bound of the number of linearly independent trilinear forms for generic λ_1, λ_2 and λ_3. If this number equals one then such an invariant trilinear form is proportional to the one constructed in Proposition 5.1 under the condition C1.

Case **Sp** ($n \geq 2$) is of a particular interest: the group G is of arbitrarily high rank, N is non-abelian, and $(K, M \cap K)$ is a non-symmetric pair. Nevertheless all the conditions C1, C2 and C3 are fulfilled. The corresponding trilinear form $\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}$ has recently arisen in a different context, namely in pseudo-differential analysis. More precisely, a new (non-perturbative) composition formula based on this trilinear form is established for the Weyl operator calculus on $L^2(\mathbb{R}^{2n})$ in [13], where a slightly different notation is adopted: $\mathbf{T}_{\lambda_1,\lambda_2,\lambda_3}(f_1, f_2, f_3) = \mathbf{J}^{0,0;0}_{-\lambda_1,-\lambda_2;\lambda_3}(f_1, f_2, f_3)$.

5.2 Convergence of the invariant triple integral

This subsection provides a sufficient condition for the convergence of the triple integral in Proposition 5.1.

We take $\Sigma^+(\mathfrak{g}; \mathfrak{a})$ to be the set of weights of $\mathfrak{n}$ with respect to $\mathfrak{a}$. The corresponding dominant Weyl chamber $\mathfrak{a}^*_+$ is defined by

$$\mathfrak{a}^*_+ := \{\nu \in \mathfrak{a}^* : \langle \nu, \alpha \rangle \geq 0 \quad \text{for any} \quad \alpha \in \Sigma^+(\mathfrak{g}; \mathfrak{a})\}.$$

According to the direct sum decomposition $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* + \sqrt{-1}\mathfrak{a}^*$, we write $\lambda = \operatorname{Re} \lambda + \sqrt{-1} \operatorname{Im} \lambda$ for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Then we have

Proposition 5.2. *Suppose we are in the setting of Proposition 5.1. If*

$$\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} \gamma \in -\rho - \mathfrak{a}^*_+, \tag{5.4}$$

then the integral (5.2) *converges absolutely.*

Here, $-\rho - \mathfrak{a}_+^*$ is a subset of $\mathfrak{a}^*$ given by $-\rho - \mathfrak{a}_+^* := \{-\lambda - \rho : \lambda \in \mathfrak{a}_+^*\}$.

The rest of this subsection is devoted to the proof of Proposition 5.2. We will show that the integral kernel

$$K_{\frac{1}{2}(\alpha-\rho)}(y,z)K_{\frac{1}{2}(\beta-\rho)}(z,x)K_{\frac{1}{2}(\gamma-\rho)}(y,z)$$

is bounded on the triple product manifold $K \times K \times K$ if the assumption (5.4) is satisfied. (As we shall see in Section 6 for specific cases, the condition (5.4) is not a necessary condition for the absolute convergence.)

Consider the multiplication map

$$N \times M \times A \times \overline{N} \to G, \quad (n, m, a, \overline{n}) \mapsto nma\overline{n}.$$

This is a diffeomorphism into an open dense subset $G' := NMA\overline{N}$ of G (the open *Bruhat cell*). We define the projection μ by

$$\mu : G' \to \mathfrak{a}, \quad nme^X\overline{n} \mapsto X.$$

We set $K' := G' \cap K$. Then we have

Lemma 5.3. *K' is dense in K. Further, if $\nu \in \mathfrak{a}_+^*$ then* $\inf_{k\in K'}\langle \nu, \mu(k)\rangle > -\infty$.

Proof. It follows from the Iwasawa decomposition $G = KA\overline{N}$ that any element of G' is written as $g' = ka\overline{n}$ ($k \in K$, $a \in A$, $\overline{n} \in \overline{N}$). Since G' contains the subgroup $A\overline{N}$, we get $k \in K'$. This leads us to the bijection $G'/MA\overline{N} \simeq K'/M$, which then is a dense subset of $G/MA\overline{N} \simeq K/M$. Hence, K' is dense in K.

In order to prove the second assertion, we may assume that G is a linear group contained in a connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$. Let $\overline{P_{\mathbb{C}}} = M_{\mathbb{C}}A_{\mathbb{C}}\overline{N_{\mathbb{C}}}$ be the complexified parabolic subgroup of $\overline{P}$. We take a θ-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$. Then, $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$.

We fix a positive set $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of the root system such that $\widetilde{\alpha}|_{\mathfrak{a}} \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$ for any $\widetilde{\alpha} \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Suppose $\widetilde{\lambda} \in \mathfrak{h}_{\mathbb{C}}^*$ is a dominant integral weight subject to the following condition:

$$\widetilde{\lambda}|_{\mathfrak{a}} = \lambda, \quad \widetilde{\lambda}|_{\mathfrak{t}} \equiv 0, \text{ and } \widetilde{\lambda} \text{ lifts to a holomorphic character of } M_{\mathbb{C}}A_{\mathbb{C}}. \tag{5.5}$$

Then we get a holomorphic character, to be denoted by $\mathbb{C}_\lambda$, of $\overline{P_{\mathbb{C}}}$ by extending trivially on $\overline{N_{\mathbb{C}}}$.

Let $\mathcal{L}_\lambda^{\mathbb{C}} := G_{\mathbb{C}} \times_{\overline{P_{\mathbb{C}}}} \mathbb{C}_\lambda$ be the $G_{\mathbb{C}}$-equivariant holomorphic bundle over $G_{\mathbb{C}}/\overline{P_{\mathbb{C}}}$ associated to the holomorphic character $\mathbb{C}_\lambda$ of $\overline{P_{\mathbb{C}}}$. Then, by the Borel–Weil theorem, the space $F_\lambda := \mathcal{O}(G_{\mathbb{C}}/\overline{P_{\mathbb{C}}}, \mathcal{L}_\lambda^{\mathbb{C}})$ of holomorphic sections for $\mathcal{L}_\lambda^{\mathbb{C}}$ gives an irreducible finite dimensional representation of $G_{\mathbb{C}}$ with highest weight $\widetilde{\lambda}$.

Let $f_\lambda \in F_\lambda$ be the highest weight vector normalized as $f_\lambda(e) = 1$. Then, we have $f_\lambda(nma\overline{n}) = a^{-\lambda}$ for $nma\overline{n} \in N_{\mathbb{C}} M_{\mathbb{C}} A_{\mathbb{C}} \overline{N_{\mathbb{C}}}$. In particular, if $g \in G'$, we get

$$f_\lambda(g) = e^{-\langle \lambda, \mu(g) \rangle}.$$

Since f_λ is a matrix coefficient, $f_\lambda|_K$ is a bounded function for any λ coming from the above $\widetilde{\lambda}$. In light that $\mathbb{R}_+$-span$\{\lambda \in \mathfrak{a}^* : \widetilde{\lambda} \text{ satisfies (5.5)}\}$ equals $\mathfrak{a}_+^*$, we have proved Lemma 5.3. $\square$

Let us complete the proof of Proposition 5.2. We recall how the Knapp–Stein integral operator [10] is given in the present context. Since we have assumed that the parabolic subgroup P is conjugate to $\overline{P}$, we can find $w \in K$ such that $w^{-1} N w = \overline{N}$. Then, we define a function K_λ defined on an open dense subset of $G \times G$ by

$$K_\lambda(g_1, g_2) = e^{\langle \lambda + \rho, \mu(g_1^{-1} g_2 w) \rangle} \quad \text{if } g_1^{-1} g_2 w \in G'.$$

It follows from Lemma 5.3 below that K_λ is bounded on $K' \times K'$ if $-(\nu + \rho) \in \mathfrak{a}_+^*$, and in particular, defines a locally integrable function on $G \times G$. The distribution kernel of the Knapp–Stein intertwining operator coincides with $K_\lambda(g_1, g_2)$ when λ stays in this range.

Return to the setting of Proposition 5.2, and assume the condition (5.4). Then, by Lemma 5.3, we see that $K_{\frac{1}{2}(\alpha-\rho)}(y, z)$ is bounded on $K \times K$, and likewise for $K_{\frac{1}{2}(\beta-\rho)}(z, x)$ and $K_{\frac{1}{2}(\gamma-\rho)}(x, y)$. Since the integral (5.2) is performed over the product of three copies of the compact manifold K/M ($\simeq G/P$), the integral (5.2) converges absolutely for any $f_j \in V_{\lambda_j}$ ($j = 1, 2, 3$). Hence, Proposition 5.2 has been proved. $\square$

6 Convergence of the triple integrals

In Section 5.2, we have given a sufficient condition for the absolute convergence of the invariant triple integral in the general setting. In this section, for the convenience of the reader, we give the *precise* region of the parameters for which the triple integrals in our main results converge absolutely. Section 6.1 provides a basic machinery for the convergence of the integral of the product of complex powers under a certain regularity assumption (6.1). This criterion gives immediately the precise region of the absolute convergence of the integral in Theorem 4.5 (see Proposition 6.7). Unfortunately, the regularity assumption (6.1) is fulfilled only for generic points for the triple integral in Theorem 1.1. This difficulty is overcome by additional local arguments in Section 6.2 (see Proposition 6.8).

6.1 Convergence under the regularity condition

Let M be a differentiable manifold, and $f_1, \ldots, f_r \in C^\infty(M)$. We shall always assume that the zero set $\{p \in M : f_j(p) = 0\}$ is non-empty for any j $(1 \le j \le r)$. For each point $p \in M$, we define a subset of the index set $\{1, \ldots, r\}$ by

$$I(p) := \{j : f_j(p) = 0\},$$

and a non-negative integer by

$$r(p) := \dim \mathbb{R}\text{-span}\{df_j(p) : j \in I(p)\}.$$

Clearly, we have

$$r(p) \le \# I(p) \ (\le r).$$

We fix a Radon measure on M which is equivalent to the Lebesgue measure on coordinating neighbourhoods (i.e. having the same sets of measure zero). Here is a basic lemma for the convergence of the integral of $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ on M.

Lemma 6.1. *Assume $f_1, \ldots, f_r \in C^\infty(M)$ satisfy the following regularity condition:*

$$r(p) = \# I(p) \quad \textit{for any } p \in M. \tag{6.1}$$

Let $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$. Then, $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ is locally integrable if and only if

$$\operatorname{Re} \lambda_j > -1 \quad \textit{for any } j \ (1 \le j \le r). \tag{6.2}$$

Remark 6.2. The local integrability does not depend on the choice of our measure on M.

Here is a prototype of Lemma 6.1:

Example 6.3. *Let $M = \mathbb{R}^n$. We fix $r \leq n$, and set $f_j(x) = x_j$ $(1 \leq j \leq r)$. Then $|x_1|^{\lambda_1} \cdots |x_r|^{\lambda_r}$ is locally integrable against the Lebesgue measure $dx_1 \cdots dx_n$ if and only if $\operatorname{Re} \lambda_j > -1$ $(1 \leq j \leq r)$.*

This assertion is obvious for $r = 1$, and the proof for general r is reduced to the $r = 1$ case. We observe that the regularity assumption (6.1) *is satisfied because $df_j = dx_j$ $(1 \leq j \leq r)$ are linearly independent at any point $p \in \mathbb{R}^n$.*

The proof of Lemma 6.1 is reduced to Example 6.3 as follows:

Proof of Lemma 6.1. Fix a point $p \in M$, and suppose $I(p) = \{j_1, \ldots, j_k\}$. Then, by the implicit function theorem, we can find differentiable functions $y_{k+1}, \ldots, y_n$ in a neighbourhood V_p of p such that $\{f_{j_1}, \ldots, f_{j_k}, y_{k+1}, \ldots, y_n\}$ forms coordinates of V_p.

Assume (6.2) is satisfied. Then it follows from Example 6.3 that the function $\prod_{j \in I(p)} |f_j|^{\lambda_j}$ is integrable near p. Multiplying it by the continuous function $\prod_{j \notin I(p)} |f_j|^{\lambda_j}$, we see that $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ is also integrable near p.

Conversely, assume $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ is locally integrable. We will show (6.2). Take a point $p \in M$ such that $f_1(p) = 0$. By using the above mentioned coordinates in V_p, we can find $q \in V_p$ such that $f_1(q) = 0$ and $f_j(q) \neq 0$ $(2 \leq j \leq r)$. Then the integrability of $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ near q is equivalent to that of $|f_1|^{\lambda_1}$. This implies $\operatorname{Re} \lambda_1 > -1$. Similarly, we get $\operatorname{Re} \lambda_j > -1$ for all j $(1 \leq j \leq r)$. Hence, Lemma 6.1 has been proved. $\square$

Next, we discuss the local integrability of the function $|f_1|^{\lambda_1} \cdots |f_r|^{\lambda_r}$ when the regularity condition (6.1) fails. The following two examples will be used to determine the range of parameters for which the triple product integral in Theorem 1.1 is absolutely convergent.

Example 6.4. *The function $h_\lambda(x, y) := |x|^{\lambda_1} |y|^{\lambda_2} |x - y|^{\lambda_3}$ is locally integrable on $\mathbb{R}^2$ if and only if $\operatorname{Re} \lambda_j > -1$ $(1 \leq j \leq 3)$ and $\operatorname{Re}(\lambda_1 + \lambda_2 + \lambda_3) > -2$.*

Example 6.5. *Let $p, q > 0$ and $p + q > 2$. Suppose $Q(z)$ is a quadratic form on $\mathbb{R}^{p+q}$ of signature (p, q). Then $|x|^{\lambda_1} |y|^{\lambda_2} |x + y + Q(z)|^{\lambda_3}$ is locally integrable on $\mathbb{R}^{p+q+2}$ if and only if $\operatorname{Re} \lambda_j > -1$ $(1 \leq j \leq 3)$.*

We observe that the regularity condition $r(p) = \#I(p)$ of Lemma 6.1 fails at the origin in both of these examples. This failure affects the condition on λ_j for the absolute convergence of the integral in Example 6.4, but does not affect in Example 6.5.

Prof of Example 6.4. Applying Lemma 6.1 to $\mathbb{R}^2 \setminus \{0\}$, we see that $h_\lambda(x,y)$ is locally integrable on $\mathbb{R}^2 \setminus \{0\}$ if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \le j \le 3)$.

To examine the integrability near the origin, we use the polar coordinate $(x,y) = (r\cos\theta, r\sin\theta)$. Then we have

$$h_\lambda(x,y)dxdy = r^{\lambda_1+\lambda_2+\lambda_3+1} h_\lambda(\cos\theta, \sin\theta) dr d\theta. \tag{6.3}$$

Since $\cos\theta$, $\sin\theta$, $\cos\theta - \sin\theta$ do not vanish simultaneously and have simple zero, (6.3) is integrable near the origin if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \le j \le 3)$ and $\operatorname{Re}(\lambda_1+\lambda_2+\lambda_3) > -2$. Therefore, Example 6.4 is proved. $\square$

In order to give a proof of Example 6.5, we prepare the following:

Claim 6.6. *Let $Q(z)$ be as in Example 6.5. Then there exists a continuous function $A(t,\delta)$ of two variables $t \in \mathbb{R}$ and $\delta \ge 0$ such that*

$$A(t,\delta) \sim c\,\delta^{p+q-2} \quad \text{as } t \to 0$$

for some positive constant c and that

$$\int_{|z| \le \delta} g(Q(z)) dz_1 \cdots dz_{p+q} = \int_{-\delta^2}^{\delta^2} g(t) A(t,\delta) dt.$$

Proof of Claim 6.6. Without loss of generality, we may and do assume that $Q(z)$ is of the standard form $Q(z) = z_1^2 + \cdots + z_p^2 - z_{p+1}^2 - \cdots - z_{p+q}^2$. Taking the double polar coordinates

$$z = (r\omega, s\eta),\ r, s, \ge 0,\ \omega \in S^{p-1},\ \eta \in S^{q-1},$$

we have

$$\int_{|z|<\delta} g(Q(z)) dz = \operatorname{vol}(S^{p-1}) \operatorname{vol}(S^{q-1}) \int_{B_+(\delta)} g(r^2 - s^2) r^{p-1} s^{q-1} dr ds, \tag{6.4}$$

where we set $B_+(\delta) := \{(r,s) \in \mathbb{R}^2 : r \ge 0, s \ge 0, r^2 + s^2 \le \delta^2\}$. By the change of variables $R := r^2 + s^2$, $t := r^2 - s^2$, the right-hand side of (6.4) amounts to

$$\int_{-\delta^2}^{\delta^2} g(t) A(t,\delta) dt,$$

where $A(t,\delta)$ is defined by

$$A(t,\delta) := \frac{\operatorname{vol}(S^{p-1}) \operatorname{vol}(S^{q-1})}{2^{\frac{p+q}{2}+1}} \int_{|t|}^{\delta^2} (R+t)^{\frac{p-2}{2}} (R-t)^{\frac{q-2}{2}} dR.$$

Putting $t = 0$, we have

$$A(0,\delta) = \frac{\delta^{p+q-2}\operatorname{vol}(S^{p-1})\operatorname{vol}(S^{q-1})}{2^{\frac{p+q}{2}+1}(p+q-2)}.$$

Thus, Claim 6.6 is shown. $\square$

We are ready to complete the proof of Example 6.5.

Proof of Example 6.5. The regularity condition (6.1) is fulfilled except for the origin. Applying Lemma 6.1 to $\mathbb{R}^{p+q+2} \setminus \{0\}$, we see that $|x|^{\lambda_1}|y|^{\lambda_2}|x + y + Q(z)|^{\lambda_3}$ is locally integrable on $\mathbb{R}^{p+q+2} \setminus \{0\}$ if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \le j \le 3)$.

What remains to prove is that this function is still integrable near the origin under the same assumption. By Claim 6.6 we can reduce the convergence of the integral to that of the three variables case, namely, it is sufficient to show that $|x|^{\lambda_1}|y|^{\lambda_2}|x + y + t|^{\lambda_3}A(t,\delta)$ is integrable against $A(t,\delta)dxdydt$ near $(0,0,0) \in \mathbb{R}^3$ for a fixed $\delta > 0$. Since $\{x, y, x + y + t\}$ meets the regularity condition (6.1), we can apply Lemma 6.1 again, and conclude that it is integrable if $\operatorname{Re}\lambda_j > -1$ $(1 \le j \le 3)$. Therefore, the proof of Example 6.5 has been completed. $\square$

6.2 Applications to the triple integrals

We apply Lemma 6.1 to find a condition for the convergence of the triple integrals in the previous sections. The first case is a direct consequence of Lemma 6.1:

Proposition 6.7 (see Theorem 4.5)**.** *Let $M := S^{N-1} \times S^{N-1} \times S^{N-1}$, and*

$$h_\lambda(x,y,z) := |\langle y,z\rangle|^{\lambda_1}|\langle z,x\rangle|^{\lambda_2}|\langle x,y\rangle|^{\lambda_3}.$$

Then $\displaystyle\int_M h_\lambda(x,y,z)d\sigma(x)d\sigma(y)d\sigma(z)$ converges if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \le j \le 3)$.

Proof of Proposition 6.7. Since $h_\lambda(x,y,z)$ is a homogeneous function of x (also, that of y and z), h_λ is integrable on M if and only if it is locally integrable on $\widetilde{M} := (\mathbb{R}^N \setminus \{0\}) \times (\mathbb{R}^N \setminus \{0\}) \times (\mathbb{R}^N \setminus \{0\})$.

We set the following functions of $(x,y,z) \in \widetilde{M}$ as $f_1 := \langle y,z\rangle$, $f_2 := \langle z,x\rangle$, $f_3 := \langle x,y\rangle$. Then we have

$$\begin{aligned} df_1 &= \phantom{\langle z,dx\rangle + {}} \langle z,dy\rangle + \langle y,dz\rangle,\\ df_2 &= \langle z,dx\rangle \phantom{{}+\langle z,dy\rangle} + \langle x,dz\rangle,\\ df_3 &= \langle y,dx\rangle + \langle x,dy\rangle. \end{aligned}$$

Let us verify that the functions f_1, f_2, and f_3 meet the regularity assumption (6.1) of Lemma 6.1 on $\widetilde{M}$.

Suppose $\#I(x,y,z) = 3$, namely,

$$\langle y,z\rangle = \langle z,x\rangle = \langle x,y\rangle = 0. \tag{6.5}$$

We will show $r(x,y,z) = 3$. If not, there would exist $(a,b,c) \neq (0,0,0)$ such that $a df_1 + b df_2 + c df_3 = 0$, namely,

$$bz + cy = 0,\ az + cx = 0,\ ay + bx = 0. \tag{6.6}$$

This would contradict to (6.5). Hence $r(x,y,z)$ must be equal to 3. Thus (6.1) holds when $\#I(x,y,z) = 3$. Similarly, (6.1) can be verified when $\#I(x,y,z) = 1$ or 2. Therefore, the assertion of Proposition 6.7 follows from Lemma 6.1. $\square$

In contrast with Proposition 6.7 for the triple product of complex powers of inner products, the regularity assumption (6.1) does not hold in the symplectic case. The next example discusses this situation.

Proposition 6.8 (see Theorem 1.1)**.** *Let $M = S^{2n-1} \times S^{2n-1} \times S^{2n-1}$, and*

$$h_\lambda(X,Y,Z) := |[Y,Z]|^{\lambda_1} |[Z,X]|^{\lambda_2} |[X,Y]|^{\lambda_3}.$$

Then the triple integral $\displaystyle\int_M h_\lambda(X,Y,Z)\,d\sigma(X)d\sigma(Y)d\sigma(Z)$ converges absolutely if and only if

$$\begin{aligned} &\operatorname{Re}\lambda_j > -1\ (1 \le j \le 3),\ \operatorname{Re}(\lambda_1+\lambda_2+\lambda_3) > -2 && \textit{for } n = 1,\\ &\operatorname{Re}\lambda_j > -1\ (1 \le j \le 3), && \textit{for } n \ge 2. \end{aligned}$$

Proof of Proposition 6.8. Since $h_\lambda(X,Y,Z)$ is a homogeneous function of X (and also, that of Y and Z), h_λ is integrable on M if and only if it is locally integrable on $\widetilde{M} := (\mathbb{R}^{2n} \setminus \{0\}) \times (\mathbb{R}^{2n} \setminus \{0\}) \times (\mathbb{R}^{2n} \setminus \{0\})$.

Similarly to the way of establishing (6.6) in the proof of Example 6.7, we see that the regularity assumption (6.1) for $(X, Y, Z) \in \widetilde{M}$ fails if and only if the three vectors $X, Y, Z \in \mathbb{R}^{2n} \setminus \{0\}$ are proportional to each other. Hence, h_λ is locally integrable on $\widetilde{M} \setminus \{(a\omega, b\omega, c\omega) : \omega \in S^{2n-1},\ a, b, c \in \mathbb{R} \setminus \{0\}\}$ if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \leq j \leq 3)$.

Let us find the condition of integrability of h_λ near the point $(X, Y, Z) = (a_0\omega_0, b_0\omega_0, c_0\omega_0)$ for some $a_0 > 0$, $b_0 \neq 0$, $c_0 \neq 0$, and $\omega_0 \in S^{N-1}$. For this we take coordinates as

$$\begin{aligned} X &= a\omega, \\ Y &= b\omega + xJ\omega + u, \\ Z &= c\omega + yJ\omega + v, \end{aligned}$$

where $a, b, c, x, y \in \mathbb{R}$, $\omega \in S^{2n-1}$, and $u, v \in (\mathbb{R}\text{-span}\{\omega, J\omega\})^\perp$ $(\simeq \mathbb{R}^{2n-2})$, and we consider the case where $a - a_0, b - b_0, c - c_0,\ x, y, u, v$ are near the origin. In view of the relations $[\omega, J\omega] = -1$ and $\langle \omega, u\rangle = \langle \omega, v\rangle = \langle J\omega, u\rangle = \langle J\omega, v\rangle = 0$, we have

$$\begin{aligned} &h_\lambda(X, Y, Z)dXdYdZ \\ &= |-by + cx + [u, v]|^{\lambda_1}|ay|^{\lambda_2}|ax|^{\lambda_3}a^{2n-1}dadbdcd\sigma(\omega)dxdydudv. \end{aligned} \tag{6.7}$$

Since we are dealing with the local integrability for $a, b, c \neq 0$, the main issue is the local integrability against $dxdydudv$.

First, suppose $n = 1$. Then (6.7) is locally integrable if and only if $|-by + cx|^{\lambda_1}|y|^{\lambda_2}|x|^{\lambda_3}dxdy$ is locally integrable on $\mathbb{R}^2$ for fixed $b, c \neq 0$. By Example 6.4, this is the case if and only if $\operatorname{Re}(\lambda_1 + \lambda_2 + \lambda_3) > -2$ in addition to $\operatorname{Re}\lambda_j > -1$ $(1 \leq j \leq 3)$.

Second, suppose $n \geq 2$. Then $[u, v]$ is a quadratic form on $\mathbb{R}^{4n-4}$ of signature $(2n-2, 2n-2)$. By Example 6.5, $|-by+cx+[u, v]|^{\lambda_1}|y|^{\lambda_2}|x|^{\lambda_3}dxdydudv$ is locally integrable on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2}$ if and only if $\operatorname{Re}\lambda_j > -1$ $(1 \leq j \leq 3)$ and this estimate is locally uniform with respect to a, b, c $(\neq 0)$ and $\omega \in S^{2n-1}$. Hence, the right-hand side of (6.7) is locally integrable if $\operatorname{Re}\lambda_j > -1$ $(1 \leq j \leq 3)$.

Thus, the proof of Proposition 6.8 is completed. $\square$

Proposition 6.9. *Let $M = S^m \times S^m \times S^m$ and*

$$h_\lambda(X, Y, Z) = |Y - Z|^{\lambda_1}|Z - X|^{\lambda_2}|Z - Y|^{\lambda_3}\ .$$

Then the triple integral $\int_M h_\lambda(X,Y,Z)d\sigma(X)d\sigma(Y)d\sigma(Z)$ *converges if and only if*

$$\mathrm{Re}\,\lambda_j > -m \quad (1 \le j \le 3), \quad \mathrm{Re}\,(\lambda_1+\lambda_2+\lambda_3) > -2m\ .$$

The proof follows the same lines as before.

Acknowledgement. The second author is partially supported by Grant-in-Aid for Scientific Research (B) (18340037), Japan Society for the Promotion of Science, and the Alexander von Humboldt Foundation.

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