ON NONDEGENERATE COUPLING FORMS

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Abstract. The aim of the present paper is to investigate new classes of symplectically fat fibre bundles. We prove a general existence theorem for fat vectors with respect to the canonical invariant connections. Based on this result we give new proofs of some constructions of symplectic structures. This includes twistor bundles and locally homogeneous complex manifolds. The proofs are conceptually simpler and allow for obtaining more general results.

1. Introduction

1.1. **Fat vectors.** Let $G \to P \to B$ be a principal bundle with a connection. Let θ and Θ be the connection one-form and the curvature form of the connection, respectively. Both forms have values in the Lie algebra $\mathfrak{g}$ of the group G. Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle\, , \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is ***fat***, if the two–form

$$(X, Y) \to \langle \Theta(X, Y), u\rangle$$

is nondegenerate for all *horizontal* vectors X, Y. Note that if a connection admits at least one fat vector then it admits the whole coadjoint orbit of fat vectors.

Let (M, ω) be a symplectic manifold with a Hamiltonian action of a group G and the moment map $\Psi : M \to \mathfrak{g}^*$. Consider the associated Hamiltonian bundle

$$(M, \omega) \to E := P \times_G M \to B.$$

Sternberg [23] constructed a certain closed two–form $\Omega \in \Omega^2(E)$ associated with the connection θ. It is called the ***coupling form*** and pulls back to the symplectic form on each fibre and it is degenerate in general. However, if the image of the moment map consists of fat vectors then the coupling form is nondegenerate, hence symplectic. This was observed by Weinstein in [27, Theorem 3.2] where he used this idea to give a new construction of symplectic manifolds. In the sequel, the bundles with a nondegenerate coupling form will be called ***symplectically fat.*** Let us state the result of Sternberg and Weinstein precisely.

Theorem 1.2 (Sternebrg-Weinstein)**.** *Let $G \to P \to B$ be a principal bundle. Let (M, ω) be a a symplectic manifold with a Hamiltonian G-action and the moment map $\Psi : M \to \mathfrak{g}^*$. If there exists a connection in the principal bundle such that all vectors in $\Psi(M) \subset \mathfrak{g}^*$ are fat, then the coupling form on the total space of the associated bundle*

$$F \to P \times_G F \to B$$

is symplectic.

Notice that, if the base is symplectic then, according to Thurston [24], the existence of a coupling form suffices to construct a fibrewise symplectic form. Thus the most important and interesting examples provided by fat bundles are the ones with nonsymplectic or highly connected bases, e.g. spheres.

Another interesting feature of fat bundles is that they provide a rich source of nontrivial symplectic characteristic classes. This follows from the functoriality of the coupling form. More precisely, if Ω is the coupling form for a Hamiltonian connection in a bundle

$$(M^{2n}, \omega) \to E \xrightarrow{\pi} B$$

then the fibre integrals $\mu_k := \pi_![\Omega^{n+k}] \in H^{2k}(B\,\mathrm{Ham}(M, \omega))$ define Hamiltonian characteristic classes. Thus, if the coupling form is nondegenerate then the fibre integral of the top power is a nonzero top cohomology class of the base. If the base is a sphere, then the corresponding fibre integral in the universal bundle is an indecomposable class. This is the first step to the understanding the ring structure of $H^*(B\,\mathrm{Ham}(M, \omega)))$.

Certain explicit examples of symplectically fat bundles are discussed by Guillemin, Lerman and Sternberg in [11, 16] (mostly fibrations of coadjoint orbits over coadjoint orbits) and by the first two authors in [25]. Some finiteness results for fat bundles were obtained by Derdziński and Rigas in [6] and Chaves in [5]. Notice, however, that they investigate quite strong fatness conditions as their motivation is to construct metrics of positive sectional curvature.

1.3. **The main results.**

(1) In Theorem 2.1, we establish an equivalence between the nondegeneracy of a coupling form and the existence of a fat vector.
(2) In Theorem 3.1, we describe fat vectors with respect to the canonical invariant connection in a principal bundle of the form

$$H \to G \to G/H.$$

This has been done by Lerman in [16] for compact semisimple Lie groups. A compact semisimple Lie group is a compact real form of a complex semisimple group. We observe that Lerman's proof works also for noncompact real forms. This gives a generalization of his result and has applications, going beyond the examples known from [11] and [16].

(3) An application of the above result is a certain duality between fat bundles over dual symmetric spaces. This duality is discussed in Section 7. Examples include pairs where one manifold is complex and not Kähler and the second is Kähler.

(4) In Theorem 4.7, we prove that the orthonormal frame bundle over a manifold of pinched curvature with sufficiently small pinching constant admits fat vectors. As a consequence, we get a conceptually simpler proof of Reznikov's construction of symplectic forms on twistor bundles [21]. In fact, the result of Reznikov is a consequence of the existence of a fat vector.

(5) The same method yields the structure of a symplectically fat fibre bundle on a locally homogeneous complex manifold in the sense of Griffiths and Schmid [9], see Theorem 5.7. A special case is also mentioned by Amorós et al. in [1, Remark 6.18]. We see that this is an exemplification of a general construction of a fat bundle.

(6) In Theorem 8.4, we show that the μ_n class is indecomposable in the cohomology ring $H^*(B\operatorname{Ham}(M,\omega))$ of the classifying space of the group of Hamiltonian diffeomorphisms of a certain coadjoint orbit of $SO(2n)$.

(7) In Section 6, we prove that the tautological bundle over an infinite dimensional space of symplectic configurations (in the sense of Gal and Kędra [8]) admits fat vectors. This provides a large class of examples of infinite dimensional symplectic manifolds.

Remark 1.4. Fat bundles have a physical meaning. The standard model of elementary particles is a geometry of a certain principal bundle over the spacetime. The coupling form was found by Sternberg as a description of the Yang-Mills field interacting with a particle arising from an irreducible representation of the structure group.

A reduction of the structure group in the standard model is known as breaking of symmetry. Our Theorem 3.1 describes, in a sense, an opposite phenomenon. More precisely, a fat vector for the canonical invariant connection in a bundle $H \to G \to G/H$ is an element $X \in \mathfrak{h} = \mathfrak{g}$ avoiding certain Weyl chambers. It provides a symplectic structure on the associated bundle $H/V \to G/V = G \times_H H/V \to G/H$. When

X approaches a forbidden Weyl chamber this symplectic structure degenerates. This is due to the fact that the isotropy subgroup $V \subset G$ becomes bigger and it is no longer a subgroup of H.

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2. Coupling forms and fat vectors

Let (M,ω) be a closed symplectic $2n$-manifold and let $(M,\omega) \xrightarrow{i} E \xrightarrow{\pi} B$ be a Hamiltonian bundle, that is, a bundle with structure group acting on M by Hamiltonian diffeomorphisms. A closed two form $\Omega \in \Omega^2(E)$ is called a ***coupling form*** if

(1) $i^*\Omega = \omega$ and
(2) $\pi_!\Omega^{n+1} = 0$, where $\dim M = 2n$.

We construct the coupling form associated to a connection following Guillemin-Lerman-Sternberg [11]. Let $G \to P \to B$ be the associated principal bundle with a connection $\theta \in \Omega^1(P,\mathfrak{g})$ with Hamiltonian reduced holonomy. Let $\Theta \in \Omega^2(P,\mathfrak{g})$ be the curvature two–form. The connection defines a horizontal distribution $\mathcal{H} \subset TE$. Let $\Psi : M \to \mathfrak{g}^*$ be the moment map for the Hamiltonian action $G \to \mathrm{Ham}(M,\omega)$. The coupling form Ω is defined so that the horizontal distribution $\mathcal{H}$ is Ω-orthogonal to the vertical distribution $\ker d\pi$ and

$$\Omega_{[p,x]}(X,Y) = \begin{cases} \omega_x(X,Y) & \text{if } X,Y \in \ker d\pi \\ \langle \Psi(x), \Theta_p(X^*,Y^*)\rangle & \text{if } X,Y \in \mathcal{H} \end{cases}$$

The vectors $X,Y \in T_{[p,x]}E$ are the images of $X^*,Y^* \in T_pP$ under the map $x : P \to E$ defined by a point $x \in M$ by the formula

$$x(p) := [x,p] \in E = P \times_G M.$$

The closeness of this form is proved on page 9 in [11] or on page 225 in McDuff-Salamon [18].

Conversely, given a coupling form $\Omega \in \Omega^2(E)$ the corresponding connection is defined by the following horizontal distribution

$$\mathcal{H} := \{ X \in TE \,|\, \Omega(X,V) = 0 \ \text{ for all } V \in \ker d\pi \}.$$

The curvature two-form on the associated principal bundle

$$\mathrm{Ham}(M,\omega) \to P \to B$$

is given by

$$\Theta_p(X^*, Y^*) := \Omega_{[p,-]}(X, Y) \in C^\infty(M),$$

for any horizontal vectors $X^*, Y^* \in T_pP$. Recall that the Lie algebra of the group of $\mathrm{Ham}(M, \omega)$ Hamiltonian diffeomorphisms is identified with the space $C^\infty(M)$ of smooth functions of the zero mean with respect to the volume form defined by the symplectic structure.

The evaluation at a point defines an embedding $M \to C^\infty(M)^*$ which is the moment map for the action of $\mathrm{Ham}(M, \omega)$ on (M, ω). A point $x \in M$ is called ***fat*** with respect to the Hamiltonian connection θ if the curvature two-form Θ evaluated at x is nondegenerate on the horizontal distribution. That is if the two-form

$$(X^*, Y^*) \mapsto \Omega_{[p,x]}(X, Y)$$

is nondegenerate on $\mathcal{H} \subset TP$.

It is now clear that the nondegeneracy of the coupling form is equivalent to the existence of a fat point. Due to the equivariance of the moment map, if there exist one fat point then every point in M is fat as M is a coadjoint orbit of $\mathrm{Ham}(M, \omega)$. This proves the first part of the following.

Theorem 2.1. *Let $(M, \omega) \to E \xrightarrow{\pi} B$ be a Hamiltonian bundle. It admits a connection $\mathcal{H}$ with a nondegenerate coupling form if and only if the associated connection θ in the principal bundle*

$$\mathrm{Ham}(M, \omega) \to P \to B$$

admits a fat point $x \in M \subset C^\infty(M)^* = \mathfrak{ham}(M, \omega)^*$.

If $H \subset \mathrm{Ham}(M, \omega)$ is the connected component of the holonomy group of the above connection then every element $u \in \mathfrak{h}^*$ in the image of the moment map $\mu : M \to \mathfrak{h}^*$ is fat. □

The second statement is implied by the following lemma whose proof is straightforward.

Lemma 2.2. *Let $H \to Q \to B$ be a reduction of a principal bundle $G \to P \to B$ equipped with the induced connection. A vector $u \in \mathfrak{h}^*$ is fat if and only if every $v \in \mathfrak{g}^*$ equal to u when restricted to $\mathfrak{h}$ is fat.* □

Example 2.3. Consider the bundle $\mathbb{CP}^1 \to \mathbb{CP}^3 \to \mathbb{HP}^1$. For the standard symplectic form ω_3 on $\mathbb{CP}^3$ and any positive real number $r > 0$ the form $\Omega_r := r \cdot \omega_3$ is a coupling form for the scaled standard symplectic form $r \cdot \omega_1$ on $\mathbb{CP}^1$. This coupling form induces a connection with holonomy group equal to $PSU(2)$. Since Ω_r is nondegenerate for

all $r > 0$, we get that every nonzero vector in $\mathfrak{psu}(2)^*$ is fat with respect to the induced connection on the principal bundle

$$PSU(2) \to P \to \mathbb{HP}^1 = S^4.$$

We used the fact that the manifolds $(\mathbb{CP}^1, r \cdot \omega_1)$ are all nonzero coadjoint orbits of $PSU(2)$ (cf. the remark at the bottom of page 224 of Weinstein [27]).

3. Fatness of the canonical connection in a principal bundle $H \to G \to G/H$

In the paper [16], Lerman investigated the fatness of the canonical invariant connection in a principal bundle

$$H \to G \to G/H,$$

where G is compact and semisimple, and G/H is a coadjoint orbit. We shall show in this section that his proof essentially works in a more general situation. Namely, that G is a semisimple Lie group and $H \subset G$ is a subgroup of maximal rank such that the Killing form for G is nondegenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of H.

Let us start with several known facts from Lie theory and introduce notation. We denote by $\mathfrak{g}$ the Lie algebra of a Lie group G. The symbol $\mathfrak{g}^c$ denotes the complexification. Let $\mathfrak{t}$ be a maximal abelian subalgebra in $\mathfrak{h}$. Then $\mathfrak{t}^c$ is a Cartan subalgebra in $\mathfrak{g}^c$. We denote by $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t}^c)$ the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$. Under these choices the root system for $\mathfrak{h}^c$ is a subsystem of Δ. Denote this subsystem as $\Delta(\mathfrak{h})$.

If the Killing form B is nondegenerate on $\mathfrak{h}$ then the subspace

$$\mathfrak{m} := \{X \in \mathfrak{g} \,|\, B(X, Y) = 0, \text{ for all } Y \in \mathfrak{h}\}$$

defines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

The decomposition is ad_H-invariant and the restriction of the Killing form to $\mathfrak{m}$ is nondegenerate (see Theorem 3.5 in Section X of [14]). The decomposition complexifies to $\mathfrak{g}^c = \mathfrak{h}^c \oplus \mathfrak{m}^c$. Thus, we have root decompositions:

$$\begin{aligned}
\mathfrak{g}^c &= \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha, \\
\mathfrak{h}^c &= \mathfrak{t}^c + \sum_{\alpha \in \Delta(\mathfrak{h})} \mathfrak{g}^\alpha, \\
\mathfrak{m}^c &= \sum_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} \mathfrak{g}^\alpha.
\end{aligned}$$

Since G is semisimple, the Killing form B defines an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ between the Lie algebra of G and its dual. If the Killing form is nondegenerate on $\mathfrak{h}$, the composition

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^* \to \mathfrak{h}^*$$

is an Ad_H-equivariant isomorphism. Let us denote this isomorphism by $u \mapsto X_u$. Let $C \subset \mathfrak{t}$ be the Weyl chamber and let C_α denote its wall determined by the root α.

Theorem 3.1. *Let G be a semisimple Lie group, and $H \subset G$ a compact subgroup of maximal rank. Suppose that the Killing form B of G is nondegenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup H. The following conditions are equivalent*

(1) *A vector $u \in \mathfrak{h}^*$ is fat with respect to the the canonical invariant connection in the principal bundle*

$$H \to G \to G/H.$$

(2) *The vector X_u does not belong to the set*

$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha.)$$

(3) *The isotropy subgroup $V \subset H$ of $u \in \mathfrak{h}^*$ with respect to the coadjoint action is the centralizer of a torus in G.*

Proof. The equivalence of the first and the second condition: The curvature form of the canonical connection in the given principal bundle has the form

$$\Omega(X^*, Y^*) = -\frac{1}{2}[X, Y]_{\mathfrak{h}}, \quad X, Y \in \mathfrak{m}$$

[13, Theorem 11.1]. Hence the fatness condition is expressed as the non-degeneracy of the form

$$(X, Y) \to B(X_u, [X, Y]_{\mathfrak{h}}). \quad (*)$$

Recall that here the pairing is given by the Killing form. Since $X_u \in \mathfrak{h}$, $B(X_u, \mathfrak{m}) = 0$ and we get

$$B(X_u, [X, Y]_{\mathfrak{h}}) = B(X_u, [X, Y]) = B([X_u, X], Y)$$

It follows from the hypothesis that B is non-degenerate on $\mathfrak{m}$ and the form (*) is nondegenerate if and only if $[X_u, X] \neq 0$. This is equivalent to

$$(\ker \mathrm{ad}_{X_u}) \cap \mathfrak{m} = \{0\}.$$

Without loss of generality we can assume that $X_u \in \mathfrak{t}$. Then the last equality is, after complexification, equivalent to the condition that

$$\alpha(X_u) \neq 0$$

for all roots $\alpha \in \Delta \backslash \Delta(\mathfrak{h})$ (see the root decomposition of $\mathfrak{m}^c$) which means that X_u does not belong to a wall C_α for $\alpha \in \Delta \backslash \Delta(\mathfrak{h})$. The general case (that is X_u is not necessarily in $\mathfrak{t}$) follows since $\mathfrak{h} = \cup_{h \in H} \operatorname{Ad}_h(\mathfrak{t})$.

The equivalence of the first and the third condition: Let $u \in \mathfrak{h}^*$ be fat. Then its isotropy subgroup $V \subset H$ is connected and has the Lie algebra $\mathfrak{v} = \{ X \in \mathfrak{h} \,|\, [X, X_u] = 0 \}$. Since the fatness of u implies

$$(\ker \operatorname{ad}_{X_u}) \cap \mathfrak{m} = \{0\}$$

(see the previous part), we get $\mathfrak{v} = \{ X \in \mathfrak{g} \,|\, [X, X_u] = 0 \}$ which means that V is the centralizer of the torus $S := \overline{\{\exp(tX_u)\}} \subset G$.

Now, suppose that the isotropy subgroup $V \subset H \subset G$ is the centralizer of a torus $S \subset G$. Let $X_u \in \mathfrak{g}$ be a generator of this torus. That is S is the closure of the one-parameter subgroup defined by X_u. Clearly, V is the coadjoint isotropy subgroup of $u := B(X_u, -)$. The associated symplectic form is the fibrewise symplectic structure on the associated bundle $H/V \to G/V \to G/H$. Since it is G-invariant, the associated connection is induced by the canonical invariant connection in the principal bundle $H \to G \to G/H$. □

Corollary 3.2. *Let K be a compact semisimple group and let $H \subset K$ be a closed subgroup. The canonical invariant connection in the bundle*

$$H \to K \to K/H$$

admits fat vectors if and only if $\operatorname{rank} K = \operatorname{rank} H$.

Proof. According to Theorem 3.1, if $\operatorname{rank} K = \operatorname{rank} H$ then there exist fat vectors (for example, those which lie in the interior of the Weyl chamber).

Let $u \in \mathfrak{h}^*$ be a fat vector. By perturbing u slightly, if necessary, we may assume that its isotropy subgroup is a maximal torus $T \subset H$. The associated bundle

$$H/T \to K \times_H H/T = K/T \to K/H$$

admits a fibrewise symplectic structure. For cohomological reasons the rank the torus T has to be of maximal rank in K. This implies that the ranks of K and H are equal. □

Remark 3.3. We see that the above argument shows that if rank $K >$ rank H, there is no chance for fatness even for other connections.

4. Fat vectors for orthonormal frame bundles

4.1. **Twistor bundles over spaces of constant nonzero curvature.** A twistor bundle over an even dimensional Riemannian manifold (M, g) is the bundle of complex structures in the tangent spaces T_pM. More precisely, it is a bundle associated with the orthonormal frame bundle to M with fibre $SO(2n)/U(n)$. It generalizes a construction of Penrose in dimension four [4, 20]

Reznikov proved that twistor bundles over manifolds with suitably pinched curvature admits fibrewise symplectic forms [21]. The following proposition proves his result for manifolds of constant non-zero curvature.

Proposition 4.2. *Let G be either $SO(2n+1)$ or $SO(2n,1)$ and let $SO(2n) \subset G$ be the obvious inclusion. Let $J \in \mathfrak{so}(2n)$ be a matrix with the blocks*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the diagonal and zeros elsewhere. Then the vector $u := B(J, -) \in \mathfrak{so}(2n)^*$ is fat with respect to the canonical connection in the bundle

$$SO(2n) \to G \to G/SO(2n).$$

Proof. Choose a maximal torus so that its Lie algebra $\mathfrak{t}$ consist of matrices with 2×2-blocks of the form

$$\begin{pmatrix} 0 & -t_i \\ t_i & 0 \end{pmatrix}$$

on the diagonal, where $t_i \in \mathbb{R}$ and $i = 1, \dots, n$. The roots for G are given by

$$t_i - t_j \text{ for } i \neq j, \quad \pm(t_i + t_j) \text{ for } i < j \quad \text{ and } \pm t_i$$

where $i, j = 1, \dots n$. The forbidden walls are defined by the roots $\pm t_i$. Since $J = (1, 1, \dots, 1)$ in the coordinates t_i, it belongs to no forbidden wall and according to Theorem 3.1 the corresponding vector $u \in \mathfrak{so}(2n)^*$ is fat. □

Corollary 4.3 (Reznikov)**.** *The twistor bundle over even dimensional sphere*

$$SO(2n)/U(n) \to SO(2n+1)/U(n) \to S^{2n}$$

is symplectically fat, i.e. it admits a fibrewise symplectic structure. □

Corollary 4.4 (Reznikov)**.** *Let $\Gamma \subset SO(2n,1)$ be a lattice trivially intersecting $SO(2n)$. The twistor bundle over an even dimensional hyperbolic manifold*

$$SO(2n)/U(n) \to \Gamma\backslash SO(2n,1)/U(n) \to \Gamma\backslash SO(2n,1)/SO(2n)$$

is symplectically fat, i.e. it admits a fibrewise symplectic structure.

Proof. It follows from Proposition 4.2 that the associated bundle

$$SO(2n)/U(n) \to SO(2n,1)/U(n) \to SO(2n,1)/SO(2n)$$

admits an invariant fibrewise symplectic structure. Hence it descends to a fibrewise symplectic structure after taking the quotient by the lattice Γ. □

Remark 4.5. Since the orbit $SO(2n)/U(n)$ is the minimal in the hierarchy of coadjoint orbits in the sense of [11, page 21], it follows that there are more fat vectors defining topologically distinct coadjoint orbits.

4.6. $SO(2n)$**-bundles over manifolds of pinched curvature.** The sectional curvature K_g of a Riemannian manifold (M,g) is called ε-***pinched*** if it satisfies the following inequality

$$1-\varepsilon \leq |K_g| \leq 1.$$

Theorem 4.7. *The orthonormal bundle $SO(2n) \to P \to M$ over Riemannian manifold M with $\frac{3}{2n+1}$-pinched curvature admits fat vectors $f \in \mathfrak{so}(2n)^*$.*

Proof. Let $J \in \mathfrak{so}(2n)$ be a matrix with the blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the diagonal and zeros elsewhere. The corresponding element $f \in \mathfrak{so}(2n)^*$ in the dual space is defined by $\langle f, A\rangle := \mathrm{Tr}(A \cdot J)$. We shall show that $f \in \mathfrak{so}(2n)^*$ is a fat vector.

Let g be a Riemannian metric with small enough pinching constant and let $\Theta \in \Omega^2(P, \mathfrak{so}(2n))$ denote the corresponding curvature two-form. Recall that the curvature tensor $R(X,Y)$ for vector fields X, Y on M is related to the curvature form of the Riemannian connection by the formula

$$R(X,Y)Z = u(\Theta(X^*,Y^*)u^{-1}(Z)), \text{ for} X,Y,Z \in T_{\pi(u)}M$$

where $\pi : P \to M$ is a bundle projection, and the point $u \in P$ is an orthogonal frame $u : \mathbb{R}^n \to T_{\pi(u)}M$. We need to show that

$$(X^*,Y^*) \mapsto \langle \Theta(X^*,Y^*), f\rangle$$

is a non-degenerate 2-form on the horizontal distribution. For any $X^*, Y^* \in \mathcal{H}_u$ which are the horizontal lifts of vectors $X, Y \in T_{\pi(u)}M$ we make the following computation.

$$\begin{aligned}\langle \Theta_u(X^*, Y^*), f\rangle &= \operatorname{Tr}(\Theta_u(X^*, Y^*) \cdot J) \\ &= \operatorname{Tr}(u \cdot \Theta_u(X^*, Y^*) \cdot J \cdot u^{-1}) \\ &= \operatorname{Tr}(R_{\pi(u)}(X, Y) \cdot u \cdot J \cdot u^{-1}).\end{aligned}$$

Let $J_u := u \cdot J \cdot u^{-1}$ denote the complex structure on $T_{\pi(u)}M$ corresponding to the frame u. It follows from the above calculation that the vector f is fat if and only if the two-from

$$(X^*, Y^*) \mapsto \operatorname{Tr}(R(X, Y) \cdot J_u)$$

is nondegenerate.

Let $X_1, J_uX_1, ..., X_n, J_uX_n$ be an orthonormal basis of the horizontal subspace $\mathcal{H}_u \cong T_{\pi(u)}M$. Then the inequality

$$\operatorname{Tr}(R(X_i, J_uX_i)J_u) \neq 0$$

for all $i = 1, ..., n$ implies that the desired nondegeneracy. This trace is calculated in a usual way, taking into consideration the orthogonality of vectors and the fact that $J_u^2 = -\operatorname{id}$. We have

$$\begin{aligned}|\operatorname{Tr}(R(X_i, J_uX_i) \cdot J_u)| &= \left|\sum_{j=1}^{n} g(R(X_i, J_uX_i)J_uX_j, X_j)\right| \\ &\geq (1-\varepsilon) - \left|\sum_{j\neq i} g(R(X_i, J_uX_i)J_uX_j, X_j)\right| \\ &\geq (1-\varepsilon) - \sum_{j\neq i} |g(R(X_i, J_uX_i)J_uX_j, X_j)| \\ &\geq (1-\varepsilon) - (n-1)\sum_{j\neq i}\frac{2}{3}\varepsilon \\ &\geq (1-\varepsilon) - (n-1)\frac{2}{3}\varepsilon \\ &= 1 - \frac{2n+1}{3}\varepsilon.\end{aligned}$$

In the calculation, we used the assumption of pinched sectional curvature:

$$|K(X_i, J_uX_i)| = |g(R(X_i, J_uX_i)J_uX_i, X_i)| \geq 1 - \varepsilon,$$

Berger's inequality

$$|g(R(X_i, X_j)X_k, X_l)| \leq \frac{2}{3}\varepsilon,$$

([2], inequality (7), p. 69) and the skew symmetric property of the curvature tensor (in the last two arguments). Finally, taking the pinching constant,

$$\varepsilon < \frac{3}{2n+1}$$

we get that the vector f is fat. □

Remark 4.8. Notice that Berger proves the above mentioned inequality for positively curved pinched manifolds. However, his calculation goes through almost verbatim in the negative pinching case.

Corollary 4.9. *Let $\xi \in \mathfrak{so}(2n)^*$ be a covector in a small neighbourhood of f and let M_ξ denote its coadjoint orbit. Then the associated bundle $M_\xi \to P \times_{SO(2n)} M_\xi \to M$ over a manifold of sufficiently pinched curvature admits a fibrewise symplectic form.* □

Remark 4.10. A Riemannian manifold with positive pinched curvature is known to be homeomorphic to a sphere. However, it is not known if exotic spheres admit metrics of positive curvature (see Berger [3, Section 12.2] for a survey).

In the negative curvature the situation is completely different. There are examples of negatively curved closed manifolds with arbitrarily small pinching constants. One source of examples is due to Farrell and Jones [7]. They prove that a connected sum of a hyperbolic manifold M with an exotic sphere admits a pinched negative (non-constant) curvature and it is not diffeomorphic to M. Another construction is due to Gromov and Thurston [10]. They construct negatively curved pinched metrics on certain branched coverings of hyperbolic manifolds.

Example 4.11. According to the above remark and Theorem 4.7 we get a rich family of symplectic structures on twistor bundles over negatively curved manifolds. Let M and $M\#\Sigma$ be homeomorphic but not diffeomorphic manifolds equipped with metrics with pinched negative curvature as constructed by Farrell and Jones. Are the total spaces of the associated twistor bundles diffeomorphic? And if so then are they symplectomorphic?

5. More examples

5.1. **Other fat bundles over non-symplectic bases.** The twistor bundle over a sphere is an interesting example because the base of

the bundle is not symplectic. The following proposition provides more examples if this kind. Its proof is straightforward.

Proposition 5.2. *Let K be a compact semisimple Lie group and let $H \subset K$ be its maximal rank subgroup with finite fundamental group. Then the bundle $H \to K \to K/H$ admits fat vectors with respect to the canonical connection and the base K/H is not symplectic.* □

5.3. **Non-homogeneous examples.** Let $T \to G \to G/T$ be a principal bundle with the canonical invariant connection θ, where the torus T is a subgroup of maximal rank. Let (M, ω) be a closed symplectic manifold endowed with a Hamiltonian torus action $T \to \mathrm{Ham}(M, \omega)$ with the moment map $\Psi : M \to \mathfrak{t}^*$. Since the torus T is an abelian group we can add a constant to the moment map and a resulting mapping will be equivariant. That is, we take $\Psi + a$, where $a \in \mathfrak{t}^*$ and define a two–form to be equal to ω on vertical vectors and on the horizontal distribution to be given by

$$\Omega_a[p, x](X, Y) := \langle \Psi(x) + a, \Theta_p(X^*.Y^*) \rangle.$$

By Theorem 3.1, all vectors away the walls of the Weyl chambers are fat. Choosing an element $a \in \mathfrak{t}^*$ such that the image of $\Psi + a$ does not intersect the walls of the Weyl chambers we obtain a fibrewise symplectic form on the associated bundle

$$M \to G \times_T M \to G/T.$$

If G is non-compact we can divide by a suitable lattice to obtain fat bundles over locally homogeneous spaces $\Gamma \backslash G/T$.

This construction is an application of Thurston's theorem because choosing an element $a \in \mathfrak{t}^*$ is equivalent to pulling back the symplectic form from the base G/T representing the class $a \in \mathfrak{t}^* \cong H^2(G/T; \mathbb{R})$.

5.4. **Locally homogeneous manifolds.** The next proposition is a corollary of Theorem 3.1. Its proof is analogous to the one of Corollary 4.4.

Proposition 5.5. *Let G be a semisimple Lie group, $K \subset G$ its maximal compact subgroup of maximal rank and $\Gamma \subset G$ a lattice trivially intersecting K. Let $V \subset K$ be a connected subgroup that is the centralizer of a torus in G. Then the bundle*

$$K/V \to \Gamma \backslash G/V \to \Gamma \backslash G/K$$

is Hamiltonian and it admits a fibrewise symplectic structure. □

In order to construct examples from the above proposition one needs to find a compact subgroup $V \subset G$ that is the centralizer of a torus in

G. A large family of examples is provided by ***locally homogeneous complex manifolds*** investigated by Griffiths and Schmid in [9]. These are manifolds of the form $\Gamma\backslash G/V$ where G is a non-compact real form of a complex semisimple group G^c and $V = G \cap P$, where $P \subset G^c$ is a parabolic subgroup. They are indeed complex because $G/V \subset G^c/B$ is an open subvariety of a complex projective variety G^c/B, and therefore, inherits the G-invariant complex structure.

The proof of the following lemma uses standard facts from Lie algebras which can be found for example in Chapter 6 of [26]. We also adopt the notation from this book.

Lemma 5.6. *Let G be a semisimple Lie group of non-compact type which is a real form of a complex semisimple Lie group G^c. Let P be a parabolic subgroup in G^c such that $V = P \cap G$ is compact. Then $V = Z_G(S)$ is the centralizer of a torus $S \subset G$.*

Proof. Let $\mathfrak{v} \subset \mathfrak{k} \subset \mathfrak{g} \subset \mathfrak{g}^c$ denote the Lie algebras corresponding to the groups in the statement of the lemma. Here $\mathfrak{k} \subset \mathfrak{g}$ is a maximal compact subalgebra. Let $\mathfrak{p} \subset \mathfrak{g}^c$ be the parabolic subalgebra corresponding to $P \subset G$.

Let Δ be a root system for $\mathfrak{g}^c$ and let $\Pi \subset \Delta$ be the subsystem of simple roots such that

$$\mathfrak{k}^c = \mathfrak{t} \oplus \sum_{\alpha \in [M_k]} \mathfrak{g}_\alpha$$

for some $M_k \subset \Pi$. This is possible due to the fact that $\mathfrak{k}^c$ is reductive in $\mathfrak{g}^c$. Now observe that

$$\mathfrak{v}^c = (\mathfrak{p} \cap \mathfrak{g})^c = \mathfrak{p} \cap \mathfrak{k}^c.$$

Since $\mathfrak{p}$ is a parabolic subalgebra we have

$$\mathfrak{p} = \mathfrak{t} \oplus \sum_{\alpha \in [M_p] \cup \Delta^+} \mathfrak{g}_\alpha$$

where $M_p \subset \Pi$ and Δ^+ is the set of positive roots with respect to Π. We then get

$$\mathfrak{v}^c = \mathfrak{t} \oplus \sum_{\alpha \in [M_p \cap M_k]} \mathfrak{g}_\alpha$$

which means that $\mathfrak{v}^c = \mathfrak{z}_{\mathfrak{g}^c}(\mathfrak{a})$ is the centralizer of an Abelian subalgebra $\mathfrak{a} \subset \mathfrak{t}$. There exists a vector $X \in \mathfrak{a}$ such that $\mathfrak{z}_{\mathfrak{g}^c}(\mathfrak{a}) = \mathfrak{z}_{\mathfrak{g}^c}(X)$ and this vector can be chosen to be *real*, that is $X \in \mathfrak{a} \cap \mathfrak{g}$. This implies

$$\mathfrak{v} = \mathfrak{v}^c \cap \mathfrak{g} = \mathfrak{z}_{\mathfrak{g}^c}(X) \cap \mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(X)$$

which proves that $V \subset G$ is a centralizer of the torus $S := \overline{\exp(tX)}$. $\square$

Notice that it follows from the above proof that $V = Z_G(S) = Z_K(S)$ where $K \subset G$ is the maximal compact subgroup of G corresponding to the subalgebra $\mathfrak{k}$. This implies that K/V is a Kähler manifold and the bundle

$$K/V \to G/V \to G/K$$

is a Hamiltonian fat bundle. Choosing an appropriate lattice $\Gamma \subset G$ we obtain the following result.

Theorem 5.7. *Let G be a noncompact real form of a complex semisimple Lie group G^c and let $P \subset G^c$ be a parabolic subgroup such that $V := P \cap G$ is compact. Let $K \subset G$ be a maximal compact subgroup containing V and let $\Gamma \subset G$ be a cocompact lattice trivially intersecting K. Then the bundle*

$$K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K$$

is Hamiltonian and admits a fibrewise symplectic structure. □

5.8. **A relation to a question of Weinstein.** Weinstein [27] was interested in constructing a simply connected symplectic and not Kähler manifold as a total space of a bundle. We don't know if there are simply connected symplectically fat bundles. However, the above theorem provides many examples of symplectically fat bundles which have non-Kähler fundamental groups. Hence they are not homotopy equivalent to Kähler manifolds. This idea was used first by Reznikov in [21] for twistor bundles over spaces of negative curvature.

Proposition 5.9. *Let $K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K$ be a fibre bundle as in Theorem 5.7. Suppose that G/K is not Hermitian symmetric. Then the bundle is symplectically fat and its total space is not homotopy equivalent to a Kähler manifold.*

Proof. In view of Theorem 5.7, it remains to prove that $\Gamma\backslash G/V$ is non-Kähler. The proof is analogous to the proof of Theorem 6.17 in [1]. Suppose that the total space is homotopy equivalent to a Kähler manifold M. Since the bundle is Hamiltonian with compact structure group the projection π induces a surjective homomorphism on homology. This follows from a general cohomological splitting for Hamiltonian bundles [15, Corollary 4.10].

Now, according to a theorem of Siu [1, Theorem 6.14], the composition of the homotopy equivalence $M \to \Gamma\backslash G/V$ and the bundle projection is homotopic to a holomorphic map for some invariant complex structure on G/K. This contradicts the assumption that G/K is not Hermitian symmetric. □

6. Infinite dimensional examples

Let (M,ω) and (W,ω_W) be symplectic manifolds and let $\mathrm{Symp}(M,W)$ denote the space of symplectic embeddings of M into W. The group of symplectic diffeomorphisms of (M,ω) acts freely from the right on this space of embeddings while symplectic diffeomorphisms of (W,ω_W) act from the right. We denote the quotient by $\mathrm{Conf}(M,W)$ and following Gal and Kędra [8] it is called the space of ***symplectic configurations of*** M ***in*** W. There is a principal bundle

$$\mathrm{Symp}(M,\omega)\to\mathrm{Symp}(M,W)\to\mathrm{Conf}(M,W).$$

It admits a symplectic connection $\mathcal{H}$ whose curvature two-form is given by the following formula [8, Section 4.2]

$$\Theta(X,Y)(f):=\{H_X,H_Y\}\circ f,$$

where $f\in\mathrm{Symp}(M,W)$ is a symplectic embedding and X,Y are horizontal vectors at f. Recall, that a vector tangent to the space of embedding is a section of the pull back bundle $f^*(TW)$. Such a section can be extended to a Hamiltonian vector field in a neighbourhood of $f(M)\subset W$. The functions $H_X,H_Y:W\to\mathbb{R}$ are the Hamiltonians of these extensions.

Proposition 6.1. *If the dimension of W is bigger than the dimension of M then the connection $\mathcal{H}$ is fat.*

Proof. This form is nondegenerate on the horizontal subspace at an embedding f if and only if for any nonzero function $H:W\to\mathbb{R}$ constant on $f(M)$ there exists a function $F:W\to\mathbb{R}$ also constant on $f(M)$ such that the Poisson bracket $\{H,F\}$ is nonzero on $f(M)$. The statement easily follows from the local expression of the Poisson bracket. □

Corollary 6.2. *The coupling form on the total space of the associated tautological bundle over symplectic configurations*

$$M\to E:=\mathrm{Symp}(M,W)\times_{\mathrm{Symp}(M,\omega)}M\to\mathrm{Conf}(M,W)$$

is nondegenerate. □

We obtain this way a large family of examples of infinite dimensional symplectic manifolds. Notice that the symplectic form can be calculated using the following formula

$$\Omega_{[f,x]}\left(\frac{d}{dt}[\varphi_t,x],\frac{d}{dt}[\psi_t,x]\right)=(\omega_W)_{f(x)}\left(\frac{d}{dt}\varphi_t(x)),\frac{d}{dt}\psi_t(x)\right)$$

where $\varphi_0 = \psi_0 = f$ and $\varphi_t = \psi_t \in \mathrm{Symp}(M, W)$. These examples are interesting because they admit a Hamiltonian action of the group of Hamiltonian diffeomorphisms of the manifold (W, ω_W).

Proposition 6.3. *The group* $\mathrm{Ham}(W, \omega_W)$ *acts on the total space* E *of the tautological bundle over the symplectic configurations from the left and the action preserves the symplectic form. Moreover, the moment map*

$$\Psi : \mathrm{Symp}(M, W) \times_{\mathrm{Symp}(M,\omega)} M \to \mathfrak{ham}(W, \omega_W)^* = C^\infty(W)^*$$

is defined by the following formula

$$\langle \Psi[f, x], F \rangle := F(f(x)).$$

Proof. Let $\Phi \in \mathrm{Ham}(W, \omega_W)$ and let $\widehat{\Phi} : E \to E$ be defined by $\widehat{\Phi}[f, x] := [\Phi \circ f, x]$. We need to show that $\widehat{\Phi}$ preserves the symplectic form Ω. This is the following calculation.

$$\begin{aligned}
\widehat{\Phi}^*\Omega_{[f,x]}\left(\frac{d}{dt}[\varphi_t, x], \frac{d}{dt}[\psi_t, x]\right) &= \Omega_{[\Phi\circ f,x]}\left(\frac{d}{dt}[\Phi \circ \varphi_t, x], \frac{d}{dt}[\Phi \circ \psi_t, x]\right) \\
&= (\omega_W)_{\Phi(f(x))}\left(\frac{d}{dt}\Phi(\varphi_t(x)), \frac{d}{dt}\Phi(\psi_t(x))\right) \\
&= (\Phi^*\omega_W)_{f(x)}\left(\frac{d}{dt}\varphi_t(x), \frac{d}{dt}\psi_t(x))\right) \\
&= (\omega_W)_{f(x)}\left(\frac{d}{dt}\varphi_t(x), \frac{d}{dt}\psi_t(x))\right) \\
&= \Omega_{[f,x]}\left(\frac{d}{dt}[\varphi_t, x], \frac{d}{dt}[\psi_t, x]\right)
\end{aligned}$$

Let $ev : E \to W$ be the evaluation map defined by $ev[f, x] = f(x)$. Given a function $F : W \to \mathbb{R}$ we denote by $\underline{F}$ the vector field on E generated by F. We have to check that

$$d(F \circ ev) = i_{\underline{F}}\Omega$$

for every function $F : W \to \mathbb{R}$. Let $f_t \in \mathrm{Symp}(M, W)$ be a path of embeddings with $f_0 = f$ and let X_F denote the Hamiltonian vector field on (W, ω_W) generated by the flow $\Phi_t \in \mathrm{Ham}(W, \omega_W)$ defined by the Hamiltonian F. We have the following computation.

$$\begin{aligned} d(F\circ ev)_{[f,x]}\left(\frac{d}{dt}[f_t,x]\right) &= \frac{d}{dt}F(ev[f_t,x]) \\ &= \frac{d}{dt}F(f_t(x)) \\ &= dF(f_t(x)) \\ &= (i_{X_F}\omega_W)_{f(x)}\left(\frac{d}{dt}f_t(x)\right) \\ &= (\omega_W)_{f(x)}\left(\frac{d}{dt}\Phi_t(f(x)),\frac{d}{dt}f_t(x)\right) \\ &= \Omega_{[f,x]}\left(\frac{d}{dt}[\Phi_t\circ f,x],\frac{d}{dt}[f_t,x]\right) \\ &= (i_{\underline{F}}\Omega)\left(\frac{d}{dt}[f_t,x]\right) \end{aligned}$$

□

7. A duality of fat bundles

Let G be a noncompact semisimple Lie group with a maximal compact subgroup $K\subset G$ and let $\Gamma\subset G$ be an irreducible cocompact lattice trivially intersecting K. Assume that G is a real from of a complex semisimple group G^c. Let $M\subset G^c$ be a maximal compact subgroup. It was observed by Okun [19] that the map $\Gamma\backslash G/K\to BK$ classifying the principal bundle

$$K\to\Gamma\backslash G\to\Gamma\backslash G/K=B\Gamma$$

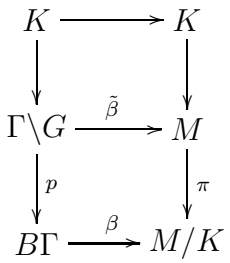
lifts to a map $\beta\colon\Gamma\backslash G/K\to M/K$ after passing to a sublattice of finite index if necessary. Moreover this map is tangential, that is the pull back of the tangent bundle of the target manifold is isomorphic to the tangent bundle of the source manifold. The homomorphism $H^*(M/K)\to H^*(B\Gamma)$ induced by the above tangential map was investigated by Matsushima in [17]. His main result says that this homomorphism is injective in all degrees and surjective in degrees smaller than the rank of G.

We want to apply this observation to a special case of those semisimple noncompact Lie groups G whose maximal compact subgroup $K\subset G$ is of maximal rank. We obtain the following duality of fat bundles.

Proposition 7.1. *Let $\Gamma\backslash G/K$ be a locally symmetric space of noncompact type and let M/K be its dual. Assume moreover that $K\subset G$ is a*

maximal compact subgroup of maximal rank. The following statements hold;

(1) *There is a pullback diagram of K-principal bundles*



(2) *Both bundles have the same nonempty sets of fat vectors.*
(3) *The pull back of the horizontal distribution $\mathcal{H}_M \subset TM$ is isomorphic (as a bundle) to the horizontal distribution $\mathcal{H}_{\Gamma\backslash G} \subset T(\Gamma\backslash G)$.*
(4) *The morphism $\tilde{\beta}$ does not preserve the connections.*

Proof.

(1) The existence of the pull back diagram is a direct application of the above mentioned result of Okun.
(2) It follows from Theorem 3.1 that the bundles have the same and nonempty set of fat vectors.
(3) Since the Okun map β is tangential we get $\beta^*(T(M/K)) \cong T(\Gamma\backslash G/K)$. On the other hand we have the following isomorphisms of bundles $\pi^*(T(M/K)) \cong \mathcal{H}_M$ and $p^*(T(\Gamma\backslash G/K)) \cong \mathcal{H}_{\Gamma\backslash G}$. Composing these isomorphisms we get the statement.
(4) Since M/K is simply connected and $B\Gamma$ is not, the Okun map has singularities. Hence the bundle morphism cannot preserve the connections.

□

The following corollary is straighforward. The third part follows from the uniqueness of the coupling class (see Section 8.1).

Corollary 7.2. *Let $\xi \in \mathfrak{k}^*$ be a fat vector for the connections in Proposition 7.1 and let $H \subset K$ denote its isotropy subgroup. Then the following statements hold;*

(1) *There is a pull back diagram of the associated bundles.*

$$\begin{array}{ccc} K/H & \longrightarrow & K/H \\ \downarrow & & \downarrow \\ \Gamma\backslash G/H & \xrightarrow{\widehat{\beta}} & M/H \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ B\Gamma & \xrightarrow{\beta} & M/K \end{array}$$

(2) *The map $\widehat{\beta}$ is tangential.*
(3) *The map $\widehat{\beta}$ preserves the cohomology classes of symplectic forms. In other words, it is a c-symplectic morphism.*

□

Example 7.3. Applying the above proposition to the twistor bundle over a hyperbolic manifold $X := \Gamma\backslash SO(2n,1)/SO(2n)$ we obtain the following pull back diagram.

$$\begin{array}{ccc} SO(2n)/U(n) & \longrightarrow & SO(2n)/U(n) \\ \downarrow & & \downarrow \\ \Gamma\backslash SO(2n,1)/U(n) & \longrightarrow & SO(2n+1)/U(n) \\ \downarrow & & \downarrow \\ X & \longrightarrow & S^{2n} \end{array}$$

The lattice Γ is a non-Kähler group (see Theorem 6.22 in [1]) and hence the symplectic manifold $\Gamma\backslash SO(2n,1)/SO(2n)$ is not Kähler. On the other hand the manifold $SO(2n+1)/U(n)$ is Kähler.

Example 7.4. Let G be a noncompact real form of a semisimple complex Lie group G^c. Let $P \subset G^c$ be a parabolic subgroup such that $H = G \cap P$ is compact. Let $\Gamma \subset G$ be a suitable cocompact and irreducible lattice. We obtain the pair of dual manifolds

$$\Gamma\backslash G/H \text{ and } M/H$$

and, according to Griffiths and Schmid the first one is complex. In general it is not Kähler. The second manifold is always Kähler.

8. Applications to Hamiltonian characteristic classes

8.1. **Hamiltonian characteristic classes defined by the coupling class.** The cohomology class of a coupling form is called the ***coupling class***. If either the fibre or the base is simply connected then the

coupling class is unique. This follows directly from the two conditions defining the coupling form and the Leray-Serre spectral sequence for the fibration. Since the classifying space of the group of Hamiltonian diffeomorphisms is simply connected, the coupling class can be defined universally as the cohomology class $\Omega \in H^2(M_{\mathrm{Ham}})$ where

$$(M,\omega) \to M_{\mathrm{Ham}} \xrightarrow{\pi} B\,\mathrm{Ham}(M,\omega)$$

is the universal Hamiltonian fibration for (M,ω). By integrating the powers of the coupling class we get Hamiltonian characteristic classes $\mu_k := \pi_!(\Omega^{n+k}) \in H^{2k}(B\,\mathrm{Ham}(M,\omega))$.

Let $(M,\omega) \to E \xrightarrow{p} B$ be a Hamiltonian bundle classified by a map $c : B \to B\,\mathrm{Ham}(M,\omega)$. The uniqueness of the coupling class and the functoriality of the fibre integration implies that the characteristic class $\mu_k(E) := c^*(\mu_k)$ is equal to the fibre integral $p_!(\Omega_E^{n+k})$, where Ω_E is the coupling class of the bundle E. The following proposition is straightforward (cf. Theorem 4.1 in Weinstein [27]).

Proposition 8.2. *Let $(M^{2n},\omega) \to E \to B$ be a Hamiltonian bundle over $2k$-dimensional base. If the coupling form Ω_E is nondegenerate then the characteristic class $\mu_k(E)$ is nonzero.* □

8.3. **Fat bundles over spheres.** Applying the last proposition to the examples of fat bundles over spheres we obtain the following result.

Theorem 8.4. *Let $SO(2n) \to P \to S^{2n}$ be the frame bundle. Let $\xi \in \mathfrak{so}(2n)^*$ be a fat vector with respect to the canonical connection (see Section 4.1) and let M_ξ denote its coadjoint orbit. Then the class $\mu_n \in H^*(B\,\mathrm{Ham}(M_\xi))$ is a nonzero indecomposable element.* □

Indecomposability means that the class is not a sum of products of classed of positive degree. Indeed, if the μ_k class was a sum of products its pull back evaluated over a sphere would be zero and this would contradict with the results of Section 4.1. In other words, the μ_k can be chosen to be a generator of the cohomology ring of the classifying space of the group of Hamiltonian diffeomorphisms of M_ξ.

8.5. **Hamiltonian actions of $SU(2)$.** The bundle $\mathbb{CP}^1 \to \mathbb{CP}^2 \to \mathbb{HP}^1 = S^4$ admits a fibrewise symplectic structure for every symplectic form on the fibre. Hence every nonzero vector in $\mathfrak{su}(2)$ is fat with respect to the induced connection in the associated principal bundle $SU(2) \to P \to S^4$, see Example 2.3. This proposition has been first proved by Reznikov in [22] and its generalization by Kędra and McDuff in [12].

Proposition 8.6. *Let $SU(2) \to \mathrm{Ham}(M,\omega)$ be a nontrivial Hamiltonian action. It induces a surjective homomorphism*

$$H^4(B\,\mathrm{Ham}(M,\omega);\mathbb{R}) \to H^4(BSU(2);\mathbb{R})$$

Proof. Consider the associated bundle

$$(M,\omega) \to E := P \times_{SU(2)} M \to S^4$$

where P is the above principal bundle endowed with a connection with respect to which all nonzero vectors are fat. Let $\mu : M \to \mathfrak{su}(2)^*$ be the moment map. The coupling form for the above fat connection is nondegenerate away the subset

$$P \times_{SU(2)} \mu^{-1}(0) \subset E.$$

Since the complement of this subset is connected and of full measure, we get that the integral of the top power of the coupling form over E is nonzero. Hence the fibre integral of the top power is nontrivial in $H^4(S^4;\mathbb{R})$ which proves the statement. □

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