ISOMORPHIC PROPERTIES OF INTERSECTION BODIES

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Abstract. We study isomorphic properties of two generalizations of intersection bodies - the class $\mathcal{I}_k^n$ of k-intersection bodies in $\mathbb{R}^n$ and the class $\mathcal{BP}_k^n$ of generalized k-intersection bodies in $\mathbb{R}^n$. In particular, we show that all convex bodies can be in a certain sense approximated by intersection bodies, namely, if K is any symmetric convex body in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$ then the outer volume ratio distance from K to the class $\mathcal{BP}_k^n$ can be estimated by

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) := \inf\{\left(\frac{|C|}{|K|}\right)^{\frac{1}{n}} : C \in \mathcal{BP}_k^n, K \subseteq C\} \leqslant c\sqrt{\frac{n}{k}\log\frac{en}{k}},$$

where $c > 0$ is an absolute constant.

Next we prove that if K is a symmetric convex body in $\mathbb{R}^n$, $1 \leqslant k \leqslant n-1$ and its k-intersection body $I_k(K)$ exists and is convex, then

$$d_{BM}(I_k(K), B_2^n) \leqslant c(k),$$

where $c(k)$ is a constant depending only on k, d_{BM} is the Banach-Mazur distance, and B_2^n is the unit Euclidean ball in $\mathbb{R}^n$. This generalizes a well-known result of Hensley and Borell. We conclude the paper with volumetric estimates for k-intersection bodies.

1. Introduction

Let K be an symmetric star body in $\mathbb{R}^n$. Following Lutwak ([35]), we say that a body $I(K)$ is the *intersection body of* K if the radius of $I(K)$ in every direction is equal to the volume of the central hyperplane section of K perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_{I(K)}(\xi) = |K \cap \xi^\perp|,$$

where $\xi^\perp$ is the central hyperplane perpendicular to ξ and $|\cdot|$ stands for the volume. A more general class of *intersection bodies* can be defined as the closure in the radial metric of the class of intersection bodies of star bodies.

Intersection bodies play an important role in the solution of the Busemann-Petty problem posed in [9] in 1956: suppose that K and L are origin symmetric convex bodies in $\mathbb{R}^n$ so that, for every $\xi \in S^{n-1}$,

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|.$$

Does it follow that $|K| \leq |L|$? The problem was completely solved in the end of the 90's, and the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution has appeared as a result of work of many mathematicians (see [12, Ch.8] or [25, Ch.5] for details). A connection between intersection bodies and the Busemann-Petty problem was established by Lutwak ([35]): the answer to the Busemann-Petty

problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

A more general concept of a k-intersection body was introduced in [28], [29]. For an integer k, $1 \leq k < n$ and star bodies K, D in $\mathbb{R}^n$, we say that D is the *k-intersection body of K* if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$|D \cap H^{\perp}| = |K \cap H|.$$

The k-intersection body of K is unique, but for $k > 1$ it does not always exist. If the k-intersection body of K exists, we denote it by $I_k(K)$. Taking the closure in the radial metric of the class of all D's that appear as k-intersection bodies of star bodies, we define the class of *k-intersection bodies*. The class of k-intersection bodies is related to a certain generalization of the Busemann-Petty problem in the same way as intersection bodies are related to the original problem (see [28] for details; this generalization offers a condition on the volume of sections that allows to compare the volumes of two bodies in arbitrary dimensions). We denote the class of k-intersection bodies in $\mathbb{R}^n$ by $\mathcal{I}_k^n$. In [29] the first named author also gave a Fourier characterization of $\mathcal{I}_k^n$: $K \in \mathcal{I}_k^n$ if and only if $\|\cdot\|_K^{-k}$ is a positive definite distribution in $\mathbb{R}^n$.

Another generalization of intersection bodies was introduced by Zhang ([50]). Let $1 \leqslant k \leqslant n-1$. The spherical Radon transform R_{n-k} is an operator acting from $C(S^{n-1})$ to the space $C(G_{n,n-k})$ of continuous functions on the Grassmanian:

$$R_{n-k}f(H) = \int_{S^{n-1}\cap H} f(x)dx, \qquad \forall f \in C(S^{n-1}),\ H \in Gr_{n,n-k}.$$

We say that a star body K is a *generalized k-intersection body* if there exists a linear positive functional μ on $R_{n-k}(C(S^{n-1}))$ such that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x)dx = \mu(R_{n-k}(f)).$$

Following the notation of [38] we denote the class of all generalized k-intersection bodies in $\mathbb{R}^n$ by $\mathcal{BP}_k^n$. A characterization of the class $\mathcal{BP}_k^n$ was obtained by Grinberg and Zhang ([17]) as a generalization of the corresponding result of Goodey and Weil for the original intersection bodies ([18]): a star body K belongs to the class $\mathcal{BP}_k^n$ if and only if it is the limit (in the radial metric) of k-radial sums of ellipsoids. The class of generalized intersection bodies is related to the so-called lower dimensional Busemann-Petty problem (LDBP-problem). Suppose that $1 \leq k < n$ and symmetric convex bodies K, L in $\mathbb{R}^n$ satisfy

$$|K \cap H| \leq |L \cap H|,$$

for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$. Does it follow that $|K| \leq |L|$? It was proved in [50] that the answer to this question is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is a generalized k-intersection body. Using this, Bourgain and Zhang ([6]) (see also [46]) proved that for the dimensions of sections $n-k > 3$ the answer to the LDBP-problem is negative. Another proof of this result was given later in [29]. The problem is still open in the cases where the dimension of sections $n-k = 2, 3$.

In the case $k = 1$ the classes $\mathcal{I}_1^n = \mathcal{BP}_1^n$ coincide with the class of original intersection bodies. Also in the case $k = n-1$ the two classes contain all star bodies in $\mathbb{R}^n$. It was proved in [31] (see also [25, Corollary 4.9]) that the class $\mathcal{I}_{n-3}^n$ contains all symmetric convex bodies in $\mathbb{R}^n$, and this is no longer true for the classes

$\mathcal{I}_k^n$ with $k < n-3$ ([28] or [25, Theorem 4.13]). It is known that $\mathcal{BP}_k^n \subseteq \mathcal{I}_k^n$ ([29]; see also [38] or [25, p.92] for simpler proofs). The latter two results immediately imply the negative answer to the LDBP-problem with the dimension of sections greater than 3. The answer to the problem with two- or three-dimensional sections would be positive, if the classes $\mathcal{BP}_k^n$ and $\mathcal{I}_k^n$ with $k = n-2$ or $k = n-3$ were equal. However, Milman ([39]) proved that $\mathcal{BP}_k^n$ is a proper subclass of $\mathcal{I}_k^n$ for $2 \leqslant k \leqslant n-2$. The example of Milman is not convex, so LDBP-problem is still open for two- and three-dimensional sections. Another open problem is whether the classes $\mathcal{I}_k^n$ increase with k. It was proved by Yaskin ([49]) that for $k_1 < k_2$ there exists a symmetric convex body that belongs to $\mathcal{I}_{k_2}^n$ but not to $\mathcal{I}_{k_1}^n$. However, the inclusion $\mathcal{I}_{k_1}^n \subset \mathcal{I}_{k_2}^n$ is known only in the case where k_1 divides k_2 (see [38]). For more results on these classes of bodies see [32] and references there.

In spite of all these results, the isomorphic properties of intersection bodies are not very well understood. The first result of this kind was established by Hensley and Borell (see [20], [4]): if K is symmetric and convex, then the Banach-Mazur distance $d_{BM}(I(K), B_2^n) \leqslant c$, where $c > 0$ is an absolute constant, which means that intersection bodies of convex bodies are isomorphic to ellipsoids (note that if K is a symmetric convex body, then the classical result of Busemann ([7]) guaranties that $I(K)$ is also symmetric and convex). Busemann ([8]) also showed an "isoperimetric" type inequality: if K is symmetric convex and $|K| = 1$, then $|I(K)| \leqslant |I(D_n)|$, where D_n is the Euclidean ball with volume 1. This result can be extended to a class more general than convex bodies (even for Borel measurable sets) (see [16], [13]). However, intersection bodies of convex bodies form only a small part of the class of convex intersection bodies.

As it was proved by the first named author ([30])(using the Fourier characterization of the intersection bodies), the unit ball of any finite dimensional subspace of $L_p,\ p \in (0, 2]$ is a k-intersection body for every k, and in particular all polar projection bodies (unit balls of subspaces of L_1) are intersection bodies. The class of intersection bodies is strictly larger than the class of polar of projection bodies; see [26] for examples. A long standing question is if the two classes are "isomorphic", i.e. whether for every intersection body I there exists a projection body Π, such that $d_{BM}(I, \Pi^\circ) \leqslant c$, where $c > 0$ is an absolute constant (this question is related to the 1970 problem of Kwapien ([33]) from the Banach space theory through the connection between intersection bodies and L_p-spaces; see [22]). A closely related result was proved in [22]: for any $q \in (0, 1)$ and any $k < n$, there exists a constant $c(q, k)$ such that for any convex k-intersection body K in $\mathbb{R}^n$ there exists a subspace of L_q whose unit ball D satisfies $d_{BM}(K, D) < c(q, k)$. Note that the constant $c(q, k)$ goes to infinity when q tends to 1, and, if not for that, the case $q = 1$ would imply the desired result for polar projection bodies.

In this article we prove several isomorphic results for intersection bodies and their generalizations. We have already mentioned the fact that the class $\mathcal{I}_{n-3}^n$ contains all symmetric convex bodies, but this is no longer the case for the classes $\mathcal{I}_k^n$ with $k < n-3$. We start with a result showing that k-intersection bodies with $k < n-3$ are still in some sense dense in the class of all convex bodies.

Let K be a symmetric convex body in $\mathbb{R}^n$ and let $\mathcal{A}$ be a class of star bodies. We define the outer volume ratio distance by

$$\text{o.v.r}(K, \mathcal{A}) := \inf\{\text{o.v.r.}(K, C),\ C \in \mathcal{A}\},$$

where

$$\text{o.v.r.}(K,C) := \inf\{\left(\frac{|TC|}{|K|}\right)^{\frac{1}{n}} : K \subseteq TC,\ T \in GL_n\}.$$

Theorem 1.1. *Let K be a symmetric convex body in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$. Then*

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) \leqslant c\sqrt{\frac{n\log\frac{en}{k}}{k}}, \tag{1.1}$$

where $c > 0$ is an absolute constant.

Recall that $\mathcal{BP}_k^n \subseteq \mathcal{I}_k^n$, so the result also applies to k-intersection bodies.

Our second result extends to the class of k-intersection bodies the classical result of Hensley and Borell that an intersection body of a convex body is isomorphic to an ellipsoid. Here one faces two additional difficulties. First the k-intersection body of a convex body does not necessarily exist and, secondly, even if it exists it may not be convex. So any result must take into account these two conditions as additional assumptions. We prove

Theorem 1.2. *Let K be a symmetric convex body in $\mathbb{R}^n$, $1 \leqslant k \leqslant n-1$ and assume that $I_k(K)$ exists and it is convex. Then*

$$d_{BM}(I_k(K), B_2^n) \leqslant c(k), \tag{1.2}$$

where $c(k)$ depends only on k.

Finally, we get some estimates for the volume radius of k-intersection bodies:

Theorem 1.3. *Let K be a symmetric star-shaped body in $\mathbb{R}^n$ with $|K| = |B_2^n|$. Assume that the k-intersection body of K exists. Then*

$$\left(\frac{|I_k(K)|}{|I_k(B_2^n)|}\right)^{\frac{1}{n}} \geqslant \frac{L_{B_2^n}}{L_K}, \tag{1.3}$$

with equality if and only if K is a symmetric ellipsoid. Here L_K stands for the isotropic constant of K.

Moreover, if $I_k(K)$ is a convex body, we have that

$$\left(\frac{|I_k(K)|}{|I_k(B_2^n)|}\right)^{\frac{1}{n}} \leqslant c\log\left(1 + d_{BM}(I_k(K), B_2^n)\right) \leqslant c\min\{\log n, k\log k\}, \tag{1.4}$$

where $c > 0$ is a universal constant.

The paper is organized as follows: In section 2 we introduce basic definitions and notation. In sections 3, 4 and 5 we give the proof of Theorems 1.1, 1.2 and 1.3 respectively. We provide some final remarks in section 6.

2. Notation and Definitions

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. We will write D_n for the euclidean ball of volume 1. ($D_n := |B_2^n|^{-\frac{1}{n}} B_2^n$). The Grassmannian manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. We denote GL_n the set of linear invertible transformations and SL_n for the measure preserving linear transformations.

A compact set K in $\mathbb{R}^n$ is called a star body if the origin is an interior point of K, every straight line passing through the origin crosses the boundary of K at exactly two points and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}, \qquad \forall x \in \mathbb{R}^n$$

is a continuous function on $\mathbb{R}^n$. The radial function of K is given by

$$\rho_K(x) = \max\{a > 0 : \ ax \in K.$$

Throughout this article we say that K is symmetric if $K = -K$. The radial metric on the class of star bodies is defined by

$$d_r(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

If K, L are star bodies and $k \in \mathbb{N}$, then the k-radial sum of K and L is a star body D whose radial function is equal to

$$\rho_D = \left(\rho_K^k + \rho_L^k\right)^{1/k}.$$

If K is a convex body, we write h_K for the support function of K:

$$h_K(x) = \max_{\xi \in K} \langle x, \xi \rangle \qquad \forall x \in \mathbb{R}^n.$$

We define the geometric distance between two bodies K_1 and K_2, $d_G(K_1, K_2)$ as the infimum of positive numbers r such that there exists some $a > 0$ so that

$$K_1 \subseteq aK_2 \subseteq raK_1.$$

The Banach-Mazur distance is defined as

$$d_{BM}(K_1, K_2) := \inf_{T \in GL_n} d_G(K_1, TK_2).$$

Let K be a star body in $\mathbb{R}^n$ of volume 1. We define the isotropic constant of K, L_K as

$$L_K^2 := \frac{1}{n} \inf\{\int_{TK} \|x\|_2^2 dx : T \in SL_n\}.$$

For two convex bodies K and L in $\mathbb{R}^n$, the covering number of K by L, denoted by $N(K, L)$, is defined as the minimal number of translates of L with their centers in K, needed to cover K.

The notation $a \simeq b$ means that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 b$. We refer to the books [12], [25], [41], [45], [47] for basic facts from the Brunn-Minkowski theory, the asymptotic theory of finite dimensional normed spaces and intersection bodies.

3. On the outer volume ratio of convex bodies with respect to the class $\mathcal{BP}_k^n$

The main idea of the proof of Theorem 1.1. is to first cover a given convex body with Euclidean balls and then show that, if the number of balls is not too big, one can approximate the union of the balls by a body in $\mathcal{BP}_k^n$, where k will be related to the covering number of K.

We will use the following theorem for covering numbers of Pisier (see [45]).

Theorem 3.1. *Let K a symmetric convex body in $\mathbb{R}^n$ and $\alpha \in (0, 2)$. Then there exist an $T \in SL_n$ such that if $K_1 = TK$ then*

(1) $|K_1| = |B_2^n|$,

(2) $\log N(K_1, tB_2^n) \leqslant \frac{cn}{t^a(2-\alpha)}$, $\forall t \geqslant 1$, *where* $c > 0$ *is a universal constant.*

It is known that the position of the body K, constructed in the previous theorem, satisfies the reverse Brunn-Minkowski inequality of V. Milman ([40]). For completeness we provide a proof.

Corollary 3.2. *Let K be a symmetric convex body with the properties of the body K_1 from the previous theorem. Then, for $t \geqslant 1$*

$$|K + tB_2^n|^{\frac{1}{n}} \leqslant ct|K|^{\frac{1}{n}}, \tag{3.1}$$

where $c > 0$ is an absolute constant.

Proof. Choose $\alpha = 1$. Then using Theorem 3.1, we have

$$\frac{|K + tB_2^n|^{\frac{1}{n}}}{t|K|^{\frac{1}{n}}} = \frac{|K + tB_2^n|^{\frac{1}{n}}}{t|B_2^n|^{\frac{1}{n}}} \leqslant N\left(K + tB_2^n, 2tB_2^n\right)^{\frac{1}{n}} \leqslant$$

$$N\left(K, tB_2^n\right)^{\frac{1}{n}} \leqslant c.$$

□

We will need the following elementary

Lemma 3.3. *Let $z \in \mathbb{R}^n$ and $t > 0$. Then there exists a centered ellipsoid $\mathcal{E}$ such that*

$$z + tB_2^n \subseteq \mathcal{E} \subseteq co\{2z + 2\sqrt{2}tB_2^n, -2z + 2\sqrt{2}tB_2^n\}. \tag{3.2}$$

Proof. We may assume that $z := z_1 e_1$. Let C be the "cylinder" defined as

$$C := \{(s, y) \in \mathbb{R}^n : |s| \leqslant z_1 + t,\ \|y\|_2 \leqslant t\}.$$

Then one can check that

$$z + tB_2^n \subseteq C \subseteq \text{co}\{z + \sqrt{2}tB_2^n, -z + \sqrt{2}tB_2^n\} =: K \tag{3.3}$$

Let $Q := \{(s, y) \in \mathbb{R}^n : |s| \leqslant 1,\ \|y\|_2 \leqslant 1\}$. Then $C := TQ$, where $T \in GL_n$. Define $\mathcal{E}_1 := TB_2^n$. Then

$$B_2^n \subseteq Q \subseteq \sqrt{2}B_2^n, \text{ or } \mathcal{E}_1 \subseteq C \subseteq \sqrt{2}\mathcal{E}_1. \tag{3.4}$$

Then, by (3.3) we get that

$$z + tB_2^n \subseteq C \subseteq \sqrt{2}\mathcal{E}_1 \subseteq 2C \subseteq 2K.$$

We set $\mathcal{E} := \sqrt{2}\mathcal{E}_1$ and the proof is complete. □

Note that the class $\mathcal{BP}_k^n$ is closed under k-radial sums and that ellipsoids belogn to this class for all $1 \leqslant k \leqslant n-1$.
Let V_1, V_2 two star bodies in $\mathbb{R}^n$. We define the distance between V_1 and V_2 as

$$d(V_1, V_2) := \sup_{x \in \mathbb{R}^n, x \neq 0} \Big\{ \frac{\rho_{V_1}(x)}{\rho_{V_2}(x)}, \frac{\rho_{V_2}(x)}{\rho_{V_1}(x)} \Big\}. \tag{3.5}$$

Observe that the definition implies that

$$\frac{1}{d} V_1 \subseteq V_2 \subseteq dV_1, \tag{3.6}$$

where $d := d(V_1, V_2)$.

Lemma 3.4. *Let $1 \leqslant k \leqslant n-1$ and $\log N \leqslant k$. Let $V_i, i \leqslant N$, be symmetric star bodies in $\mathcal{BP}_k^n$ and let $V = \cup_{i=1}^N V_i$. Then there exists $C \in \mathcal{BP}_k^n$ such that*

$$d(V, C) \leqslant e. \tag{3.7}$$

Proof. Let ρ_{V_i} be the corresponding radial functions of V_i. Then

$$\rho_V := \max_{i \leqslant N} \rho_{V_i}(x),\ x \in \mathbb{R}^n \setminus \{0\}. \tag{3.8}$$

For any $a \in \mathbb{R}^N$ we have that

$$\|a\|_\infty \leqslant \|a\|_p \leqslant N^{\frac{1}{k}} \|a\|_\infty \leqslant e\|a\|_\infty. \tag{3.9}$$

Let C be the star body defined by

$$\rho_C^k(x) := \sum_{i=1}^N \rho_{V_i}^k(x),\ 0 \neq x \in \mathbb{R}^n. \tag{3.10}$$

Note that $C \in \mathcal{BP}_k^n$ and by (3.9) we also have that $d(V, C) \leqslant e$. □

We can now give a proof of Theorem 1.1:
Let $\alpha := 2 - \frac{1}{\log e\frac{n}{k}}$. Note that o.v.r.$(TK, \mathcal{BP}_k^n) =$ o.v.r.$(K, \mathcal{BP}_k^n)$ for any $T \in GL_n$. So we may assume that K is in the position described in Theorem 3.1. We have that for every $t \geqslant 1$, there exists N, such that $\log N \leqslant \frac{cn}{t^a(2-a)}$, and $z_1, \ldots, z_N \in K$ such that

$$K \subseteq \cup_{i=1}^N z_i + tB_2^n.$$

Given any z_i, let $\mathcal{E}_i$ as in Lemma 3.3. Then we have that

$$K \subseteq \cup_{i=1}^N \mathcal{E}_i \subseteq 2\left(co\{z_1, \ldots, z_n\} + \sqrt{2}tB_2^n\right) \subseteq 2(K + \sqrt{2}tB_2^n).$$

Choose t such that $k = \frac{cn}{t^a(2-a)}$. Then $\log N \leqslant k$. So, by Lemma 3.4, there exist a $C \in \mathcal{BP}_k^n$ such that $d(C, V) \leqslant e$, where $V = \cup_{i=1}^N \mathcal{E}_i$. So,

$$\frac{1}{e}K \subseteq \frac{1}{e}V \subseteq C \subseteq eV \subseteq 2e\left(K + \sqrt{2}tB_2^n\right) \tag{3.11}$$

Recall that $t := \left(\frac{n}{k(2-a)}\right)^{\frac{1}{a}} \simeq \sqrt{\frac{n}{k}}\sqrt{\log e\frac{n}{k}}$. Then (3.11) becomes

$$\frac{1}{e}K \subseteq C \subseteq 2e\left(K + c\sqrt{\frac{n}{k}}\sqrt{\log e\frac{n}{k}}B_2^n\right) \tag{3.12}$$

Moreover, by Corollary 3.2, we have that

$$|C|^{\frac{1}{n}} \leqslant 2e|K + tB_2^n|^{\frac{1}{n}} \leqslant ct|K|^{\frac{1}{n}}.$$

Let $C_1 := eC$. We have that $C_1 \in \mathcal{BP}_k^n$, $K \subseteq C_1$ and

$$\frac{|C_1|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}} \leqslant c't \leqslant c''\sqrt{\frac{n}{k}}\sqrt{\log e\frac{n}{k}}. \tag{3.13}$$

This finishes the proof. □

4. Distances for k-intersection bodies

Let K a star body in $\mathbb{R}^n$. Recall that the k-intersection body of K (if it exists) is a star body that satisfies

$$|I_k(K) \cap F| = |K \cap F^{\perp}|, \ \forall F \in G_{n,k}. \tag{4.1}$$

Note that $I_k(K)$ (if it exists) is unique. Also if $t > 0$ and $T \in GL_n$, one has that

$$I_k(tK) = t^{\frac{n-k}{k}} I_k(K) \text{ and } I_k(TK) = |\det T| T^{-*}(I_k(K)). \tag{4.2}$$

The last equality follows from the Fourier characterization of the k-intersection bodies (see [29]).
Let K be a symmetric convex body of volume 1 and $p \geqslant 1$. We define the L_p-centroid body of K ([36],[37],[43]), as the symmetric convex body that has support function

$$h_{Z_p(K)}(\theta) := \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}}. \tag{4.3}$$

We will use the following L_q-version of Rogers-Shephard inequality ([43],[44]):

Proposition 4.1. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and $1 \leqslant k \leqslant n-1$. Then for every $F \in G_{n,k}$,*

$$|K \cap F^{\perp}|^{\frac{1}{k}} |P_F Z_k(K)|^{\frac{1}{k}} \simeq 1. \tag{4.4}$$

We will also use the Santaló and reverse Santaló inequality ([5]): If K is a symmetric convex body in $\mathbb{R}^n$, then

$$(|K||K^{\circ}|)^{\frac{1}{n}} \simeq \frac{1}{n}. \tag{4.5}$$

Proposition 4.2. *Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1 and $1 \leqslant k \leqslant n-1$. Assume that $I_k(K)$ exists. Then for $F \in G_{n,k}$,*

$$|I_k(K) \cap F|^{\frac{1}{k}} \simeq |\left(\frac{Z_k(K)}{k} \right)^{\circ} \cap F|^{\frac{1}{k}}. \tag{4.6}$$

Proof. Using the definition (4.1) and equations (4.4), (4.5), we get that

$$|I_k(K) \cap F|^{\frac{1}{k}} = |K \cap F^{\perp}|^{\frac{1}{k}} \simeq |P_F Z_k(K)|^{-\frac{1}{k}} \simeq$$

$$k|Z_k^{\circ}(K) \cap F|^{\frac{1}{k}} = |\left(\frac{Z_k(K)}{k} \right)^{\circ} \cap F|^{\frac{1}{k}}.$$

□

The following lemma is a well known application of the Brunn-Minkowski inequality:

Lemma 4.3. *Let K be a symmetric convex body of volume 1 in $\mathbb{R}^n$ and let $k \geqslant 2$. Then*

$$Z_2(K) \subseteq Z_k(K) \subseteq ckZ_2(K), \tag{4.7}$$

where $c > 0$ is an absolute constant.

Note that $Z_2(K)$ is an ellipsoid. Moreover, since $Z_p(TK) = TZ_p(K)$ for $T \in SL_n$, there exists $T \in SL_n$ such that $Z_2(TK) := L_K B_2^n$. In this case we say that K is isotropic (see [42], [14] for more information on isotropicity). Note that in the case

where K is convex the definition of the isotropic constant L_K that we give here is equivalent to the definition given in §2. (see e.g. [14]).
We have the following

Corollary 4.4. *Let K be an isotropic convex body in $\mathbb{R}^n$, $1 \leqslant k \leqslant n-1$ and assume that $I_k(K)$ exists. Then for all $F_1, F_2 \in G_{n,k}$,*

$$\frac{1}{(c_1 k)^k} \leqslant \frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} \leqslant (c_1 k)^k, \tag{4.8}$$

where $c_1 > 1$ is an absolute constant.

Proof. Using equations (4.6) and (4.7) we have that

$$\frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} \leqslant c^k \frac{|\left(\frac{Z_k(K)}{k}\right)^\circ \cap F_1|}{|\left(\frac{Z_k(K)}{k}\right)^\circ \cap F_2|}$$

$$\leqslant c^k \frac{|\left(\frac{Z_2(K)}{k}\right)^\circ \cap F_1|}{c_0^k |Z_2^\circ(K) \cap F_2|} \leqslant \frac{k^k c^k}{c_0^k} \frac{|Z_2^\circ(K) \cap F_1|}{|Z_2^\circ(K) \cap F_2|} \leqslant (c_1 k)^k,$$

since K is isotropic. We work similary for the left hand side inequality. □

We will also need the following

Lemma 4.5. *Let K be a symmetric convex body in $\mathbb{R}^{m+1}$. Let $r := r_K(e_{m+1})$ and $R := h_K(e_{m+1})$. Then*

$$\frac{2r}{m+1} |K \cap \mathbb{R}^m| \leqslant |K| \leqslant 2R |K \cap \mathbb{R}^m|. \tag{4.9}$$

Proof. To estimate the left-hand side observe that $re_{m+1} \in K$, so

$$|K| \geqslant |\mathrm{co}\{K \cap \mathbb{R}^m, re_{m+1}, -re_{m+1}\}| = 2\frac{r}{m+1} |K \cap \mathbb{R}^m|.$$

For the right-hand side, observe that the function $f(s) := |K \cap (se_{m+1} + e_{m+1}^\perp)|$ is even and *log*-concave by the Brunn-Minkowski inequality, therefore attains its maximum at 0. Then by Fubini's theorem,

$$|K| = 2\int_0^R f(s)ds \leqslant 2Rf(0) = 2R|K \cap \mathbb{R}^m|.$$

This finishes the proof. □

Proposition 4.6. *Let K be a symmetric convex body in $\mathbb{R}^n$, $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$. Assume that there exists $\delta \geqslant 1$ such that for every $F_1, F_2 \in G_{n,k}$,*

$$\frac{|K \cap F_1|}{|K \cap F_2|} \leqslant \delta^k.$$

Then

$$d_G(K, B_2^n) \leqslant k\delta^k. \tag{4.10}$$

Proof. Let $\theta_1 \in S^{n-1}$ be such that $R(K) := h_K(\theta_1) = \rho_K(\theta_1)$. Let $\theta_2 \in S^{n-1}$ be such that $r(K) := h_K(\theta_2) = \rho_K(\theta_2)$. It is enough to show that $\frac{R}{r} \leqslant k\delta^k$.

We may assume that $\theta_1 \neq \theta_2$, or else we have nothing to prove. Let $F_0 := \text{span}\{\theta_1, \theta_2\}$. Let $F \in G_{n,k-1}$, $F \perp F_0$, $F_1 := \text{span}\{\theta_1, F\}$ and $F_2 := \text{span}\{\theta_2, F\}$. Let $S := |K \cap F|$. Then by Lemma 4.5 we have that

$$\frac{2RS}{k} \leqslant |K \cap F_1| \leqslant 2RS \text{ and } \frac{2rS}{k} \leqslant |K \cap F_2| \leqslant 2rS \text{ or}$$

$$\frac{R}{kr} \leqslant \frac{|K \cap F_1|}{|K \cap F_2|} \leqslant \frac{kR}{r} \tag{4.11}$$

Then $\frac{R}{kr} \leqslant \delta^k$. □

Proof of Theorem 1.2: The case $k = 1$ is covered by the result of Hensley as noted in the Introduction. So we assume that $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$. Using (4.2) we may assume that K is isotropic. Then we want to show that $d_G(I_k(K), B_2^n) \leqslant k(ck)^k$, where $c > 0$ absolute constant. This follows from Corollary 4.4 and Proposition 4.6. □

The dependence on k in Theorem 1.2 is very bad, $c(k) \leqslant k(ck)^k$. This becomes meaningless for $k \geqslant \frac{\log n}{\log\log n}$. We can give a (slightly) better bound (but still exponential) using certain tools that were developed in order to attack the Hyperplane conjecture. For a proof of the following lemma see [42].

Lemma 4.7. *Let K be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. Then for any $F \in G_{n,k}$ there exists a symmetric convex body B in F such that, $B := B(F)$ is also isotropic and*

$$|K \cap F^\perp|^{\frac{1}{k}} \simeq \frac{L_B}{L_K}. \tag{4.12}$$

Second proof of Theorem 1.2: Again we may assume that K is isotropic. We will use the best known bound for the isotropic constant due to B. Klartag [23] (see also [24]): for every B in $\mathbb{R}^k$,

$$L_B \leqslant ck^{\frac{1}{4}}, \tag{4.13}$$

where $c > 0$ is a universal constant. Moreover, it is known (e.g.([42]) that $L_K \geqslant L_{B_2^n} \simeq 1$. So using (1.12) we get that for every $F_1, F_2 \in G_{n,k}$,

$$\frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} = \frac{|K \cap F_1^\perp|}{|K \cap F_2^\perp|} \leqslant \left(c\frac{L_{B(F_1)}}{L_{B(F_2)}}\right)^k \leqslant (c'k)^{\frac{k}{4}}. \tag{4.14}$$

So, using Proposition 4.6 again we get that $d_G(K, B_2^n) \leqslant k(ck)^{\frac{k}{4}}$. This finishes the proof. □

5. Volumetric estimates for k-intersection bodies

Let $p \neq 0$ and C be a symmetric star body in $\mathbb{R}^n$. We define

$$M_p(C) := \left(\int_{S^{n-1}} \|\theta\|_C^p d\sigma(\theta)\right)^{\frac{1}{p}}. \tag{5.1}$$

Moreover, if C is convex we write

$$W_p(C) := \left(\int_{S^{n-1}} h_C^p(\theta) d\sigma(\theta)\right)^{\frac{1}{p}}. \tag{5.2}$$

Let K be a compact set in $\mathbb{R}^n$ with $|K|=1$. Let $p>-n,\ p\neq 0$. We define

$$E_p(K):=\left(\int_K \|x\|_2^p dx\right)^{\frac{1}{p}}. \tag{5.3}$$

We have the following identity (see [44] for a generalization to the case of measures):

Lemma 5.1. *Let K be a symmetric star body in $\mathbb{R}^n$ of volume 1 and $1\leqslant k\leqslant n-1$. Then*

(5.4)

$$E_{-k}(K)\left(\int_{G_{n,k}}|K\cap F^{\perp}|d\mu(F)\right)^{\frac{1}{k}}=E_{-k}(D_n)\left(\int_{G_{n,k}}|D_n\cap F^{\perp}|d\mu(F)\right)^{\frac{1}{k}}.$$

Proof. Indeed, writing in polar coordinates, we get that

$$E_{-k}^{-k}(K)=\frac{n\omega_n}{n-k}\int_{G_{n,n-k}}\int_{S_E}\frac{d\sigma(\theta)}{\|\theta\|_K^{n-k}}d\mu(E)=$$

$$\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,n-k}}\omega_{n-k}\int_{S_E}\frac{d\sigma(\theta)}{\|\theta\|_K^{n-k}}d\mu(E)=\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,n-k}}|K\cap E|d\mu(E)=$$

$$\frac{n\omega_n}{(n-k)\omega_{n-k}}\int_{G_{n,k}}|K\cap F^{\perp}|d\mu(F).$$

Since the same holds also for D_n, equation (5.4) follows. □

We have the following application of the previous lemma and a definition of k-intersection bodies.

Lemma 5.2. *Let K be a symmetric star body in $\mathbb{R}^n$ of volume 1 and $1\leqslant k\leqslant n-1$. Assume that $I_k(K)$ exists. Then*

$$M_{-k}(I_k(K))\left(\int_{G_{n,k}}|K\cap F^{\perp}|d\mu(F)\right)^{\frac{1}{k}}=\omega_k^{\frac{1}{k}}. \tag{5.5}$$

Moreover,

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))}=\frac{E_{-k}(K)}{E_{-k}(D_n)}. \tag{5.6}$$

Proof. Integrating (4.1) over $G_{n,k}$ we have that

$$\int_{G_{n,k}}|I_k(K)\cap F|d\mu(F)=\int_{G_{n,k}}|K\cap F^{\perp}|d\mu(F). \tag{5.7}$$

Moreover,

$$\int_{G_{n,k}}|I_k(K)\cap F|d\mu(F)=\int_{G_{n,k}}\omega_k\int_{S_F}\frac{d\sigma_F(\theta)}{\|\theta\|_{I_k(K)}^k}d\mu(F)$$

$$=\omega_k\int_{S^{n-1}}\frac{d\sigma(\theta)}{\|\theta\|_{I_k(K)}^k}=\omega_k M_{-k}^{-k}(I_k(K)).$$

So, by (5.7) we get (5.5). (5.6) follows from (5.5) and (5.4). □

The following two propositions deal with the behavior of the ratio $\frac{E_p(K)}{E_p(D_n)}$.

It is not difficult to obtain a lower bound for the quantity $\frac{E_p(K)}{E_p(D_n)}$. The main tool is an argument of Milman and Pajor (see [42]).

Proposition 5.3. *Let K a compact set in $\mathbb{R}^n$ of volume 1 and let $p > -n,\ p \neq 0$. Then*

$$E_p(K) \geqslant E_p(D_n), \tag{5.8}$$

with equality if and only if $|K \cap D_n| = 1$.
In particular, if K is a star body, then we have equality if and only if $K = D_n$.

Proof. Note that $|K \setminus D_n| = |D_n \setminus K|$. Also if $x \in K \setminus D_n$ then $\|x\|_2 \geqslant |B_2^n|^{-\frac{1}{n}}$ while if $x \in D_n \setminus K$, $\|x\|_2 \leqslant |B_2^n|^{-\frac{1}{n}}$. So, if $p > 0$,

$$E_p^p(K) = \int_K \|x\|_2^p dx = \int_{K\setminus D_n} \|x\|_2^p dx + \int_{K\cap D_n} \|x\|_2^p dx \geqslant$$

$$\int_{D_n\setminus K} \|x\|_2^p dx + \int_{K\cap D_n} \|x\|_2^p dx = E_2^p(D_n).$$

It is clear that we have equality if and only if $|K \cap D_n| = 1$. We work similarly if $p < 0$. □

Proposition 5.4. *Let K a star body in $\mathbb{R}^n$ of volume 1, $-n < p < q \leqslant \infty$, $p, q \neq 0$. Then*

$$\frac{E_p(K)}{E_q(K)} \leqslant \frac{E_p(D_n)}{E_q(D_n)}, \tag{5.9}$$

with equality if and only if $K = D_n$.

Proof. We follow an argument from [3]. A simple computation shows that

$$\frac{E_q(D_n)}{E_p(D_n)} = \frac{(\frac{n}{n+q})^{1/q}}{(\frac{n}{n+p})^{1/p}}. \tag{5.10}$$

For every $q > -n$, $q \neq 0$, by integration in polar coordinates we have

$$E_q^q(K) = \omega_n \int_0^\infty r^{n+q-1} \sigma\left(\frac{1}{r}K\right) dr := \int_0^\infty r^{n+q-1} g(r), dr. \tag{5.11}$$

The function $g(r) := \omega_n \sigma\left(\frac{1}{r}K\right)$ is non-increasing on $(0, \infty)$ and it is supported in some $[r(K), R(K)]$. If we assume that K is not the Euclidean ball, then $[r(K), R(K)]$ is an interval and $g(r)$ can be assumed absolutely continuous. In this case we can write

$$g(r) = n \int_r^\infty \frac{\rho(s)}{s^n} ds, \quad (r > 0) \tag{5.12}$$

for some non-negative function ρ on $(0, \infty)$. Then, again by integration in polar coordinates,

$$1 = |K| = \int_0^\infty r^{n-1} g(r)\, dr = n \int_0^\infty \int_{0<r<s} r^{n-1} \frac{\rho(s)}{s^n} dr\, ds = \int_0^\infty \rho(s)\, ds.$$

Hence, ρ represents a probability density of a positive random variable, say, ξ. Then (5.11) becomes

$$E_q^q(K) = \int_0^\infty r^{q+n-1} g(r)\, dr = \frac{n}{n+q} \int_0^\infty s^q \rho(s)\, ds = \frac{n}{n+q} \mathbb{E}(\xi^q).$$

Applying Hölder's inequality for $-n < p < q \leqslant \infty$, we see that

$$(\mathbb{E}(\xi^q))^{1/q} > (\mathbb{E}(\xi^p))^{1/p}\ . \tag{5.13}$$

Note that, since ξ is a non-zero random variable with an absolute continuous density, there is no equality case in (5.13) (see [19, Th.188]). So,

$$\frac{E_q(K)}{E_p(K)} = \frac{\left(\frac{n}{n+q}\mathbb{E}(\xi^q)\right)^{1/q}}{\left(\frac{n}{n+p}\mathbb{E}(\xi^p)\right)^{1/p}} > \frac{\left(\frac{n}{n+q}\right)^{1/q}}{\left(\frac{n}{n+p}\right)^{1/p}} = \frac{E_q(D_n)}{E_p(D_n)},$$

as claimed. □

We will use the following immediate application of Hölder's inequality.

Lemma 5.5. *Let C be a star symmetric body in $\mathbb{R}^n$ and $p \leqslant q$, $p, q \neq 0$. Then*

$$M_p(C) \leqslant M_q(C), \tag{5.14}$$

with equality if and only if $C = aB_2^n$ for some $a > 0$.

Moreover writing the volume of C in polar coordinates we get that

$$M_{-n}(C) = \left(\frac{|B_2^n|}{|C|}\right)^{\frac{1}{n}}. \tag{5.15}$$

Results of Lewis [34], Figiel and Tomczak-Jaegermann [11], Pisier [45] establish the following "reverse Uryson" inequality:

Proposition 5.6. *Let C be a symmetric convex body in $\mathbb{R}^n$. Then there exists $T \in SL_n$ such that*

$$W(C_1) \leqslant c\sqrt{n}|C_1|^{\frac{1}{n}} \log\left(1 + d_{BM}(C, B_2^n)\right), \tag{5.16}$$

where $c > 0$ is an absolute constant and $C_1 = TC$.

Proof of Theorem 1.3: Using Lemma 5.5 and (5.15) we have that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} \geqslant \left(\frac{|I_k(D_n)|}{|I_k(K)|}\right)^{\frac{1}{n}}. \tag{5.17}$$

Using (4.2), we may assume that K is isotropic. Then $E_2(K) = \sqrt{n}L_K$. Then, by (5.6), and Proposition 5.4,

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} = \frac{E_{-k}(K)}{E_{-k}(D_n)} \leqslant \frac{E_2(K)}{E_2(D_n)} = \frac{L_K}{L_{D_n}}. \tag{5.18}$$

So, (5.17) and (5.18) imply equation (1.3).

We now assume that $I_k(K)$ is convex. We consider $I_k^\circ(K)$ to be in the position described in Proposition 5.6. (Again by using (4.2).) Then by Lemmas 5.5 and 5.6,

$$M_{-k}(I_k(K)) \leqslant M_1(I_k(K)) = W(I_k^\circ(K)) \leqslant$$

$$c\sqrt{n}||I_k^\circ(K)|^{\frac{1}{n}} \log\left(1 + d_{BM}(I_k(K), B_2^n)\right) \leqslant \frac{c' \log\left(1 + d_{BM}(I_k(K), B_2^n)\right)}{\sqrt{n}|I_k(K)|^{\frac{1}{n}}},$$

using also Santaló inequality. This implies that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} \leqslant c\left(\frac{|I_k(D_n)|}{|I_k(K)|}\right)^{\frac{1}{n}} \log\left(1 + d_{BM}(I_k(K), B_2^n)\right). \tag{5.19}$$

So again by (5.6) and Proposition 5.3, we have that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} = \frac{E_{-k}(K)}{E_{-k}(D_n)} \geqslant 1. \tag{5.20}$$

So, equations (5.19) and (5.20) imply (1.4). To conclude we use the fact that $d_{BM}(K, B_2^n) \leqslant \sqrt{n}$ and Theorem 1.2. □

6. Concluding remarks

Theorem 1.1 indicates that the classes $\mathcal{BP}_k^n$ increase in a "canonical" way. It is not clear to us if the reverse inequality (up to the logarithmic term) holds true even for the Banach-Mazur distance. We pose this as a

Question 1: Is it true that for every n and every k, $1 \leqslant k \leqslant n-1$, there exists a convex symmetric body K in $\mathbb{R}^n$ such that

$$d_{BM}(K, \mathcal{BP}_k^n) \geqslant c\sqrt{\frac{n}{k}},$$

where $c > 0$ is a universal constant?

We can show that this is true in the case $k = 1$:

Proposition 6.1. *There exists $c > 0$ such that for every $n \geqslant 1$,*

$$d_{BM}(B_\infty^n, \mathcal{I}_1^n) \geqslant c\sqrt{n}.$$

The proof of the latter Proposition depends on the following fact: all convex intersection bodies have bounded volume ratio. Let $\mathcal{VR}_n(a)$ be the class of symmetric convex bodies in $\mathbb{R}^n$ with v.r.$(K) := \inf\{\left(\frac{|TK|}{B_2^n}\right)^{\frac{1}{n}} : T \in GL_n\} \leqslant a$. In this notation we have the following

Proposition 6.2. *There exists $c > 0$ such that for every $n \geqslant 1$ every convex intersection body in $\mathbb{R}^n$ belongs to the class $\mathcal{VR}_n(c)$.*

Proof. The proof is simply a combination of certain known results. Let $K \in \mathcal{I}_1^n$ and convex, and let X the n-dimensional Banach space with norm $\|\cdot\| := \|\cdot\|_K$. Then it has been shown in [22] that X embeds isomorphically to $L_{1/2}$. It is known (e.g. [21]) that every Banach subspace of $L_{1/2}$ has (Rademacher) cotype 2. Next, by a result of Bourgain and Milman ([5]), every finite dimensional subspace with bounded cotype 2 constant, has the property that its unit ball has bounded volume ratio. This finishes the proof. □

Proof of Proposition 6.1: We will show that $d_{BM}(B_\infty^n, \mathcal{VR}_n(a)) \geqslant c(a)\sqrt{n}$. Then by Proposition 6.2 the proof would be complete. For simplicity we assume that $n = 4k$ for some $k \in \mathbb{N}$. The general case follows easily. Let $C \in \mathcal{VR}_n(a)$ such that $d_{BM}(B_\infty^n, \mathcal{VR}_n(a)) = d_G(B_\infty^n, C) =: d$. Then we have that $C \subseteq B_\infty^n \subseteq dC$. By a well known generalization of Kashin's theorem ([48]) there exists $F \in G_{n,\frac{n}{2}}$ and an ellipsoid $\mathcal{E}$ such that $c_1(a)\mathcal{E} \subseteq C \cap F \subseteq c_2(a)\mathcal{E}$. Moreover it is known (see e.g. [51]) that for any ellipsoid in $\mathbb{R}^m$ there exists $E \in G_{m,\frac{m}{2}}$ and some $r > 0$ such that $\mathcal{E} \cap E = rB_E$, where B_E is the Euclidean ball of E. So we get that there exists $E \in G_{n,\frac{n}{4}}$ and $r_1 := r_1(a) > 0$, $r_2 := r_2(a)$, such that

$$r_1 B_E \subseteq B_\infty^n \cap E \subseteq dr_2 B_E.$$

Hence, it is enough to show that $d_G(B_\infty^n \cap E, B_E) \geqslant c\sqrt{n}$, for every $E \in G_{n,\frac{n}{4}}$. Considering the polar body of $B_\infty^n \cap E$ it is enough to show that the convex hull of at most $8n$ points in $\mathbb{R}^n$ has geometric distance from the Euclidean ball at least $c\sqrt{n}$. Let $N \geqslant n+1$, $v_1, \cdots, v_N \in B_2^n$ and $\|v_1\|_2 = 1$. Let $K := \mathrm{co}\{v_1, \cdots, v_N\}$. Note that

$R(K) = 1$. Now it is enough to show that if $N = 8n$ then $r_K := \min_{\theta_{S^{n-1}}} \rho_K(\theta) \leqslant \frac{c}{\sqrt{n}}$. But (see e.g. [1], [2], [10], [15]) one has that $|K|^{\frac{1}{n}} \leqslant c' \frac{\sqrt{\log \frac{eN}{n}}}{n}$. Writing the volume of K in polar coordinates we get that there exists at least one $\theta \in S^{n-1}$ such that $\rho_K(\theta) \leqslant c'' \sqrt{\frac{\log \frac{eN}{n}}{n}}$. We complete the proof by choosing $N = 8n$. □

The estimate in Theorem 1.2 is exponential with respect to k. Even if we assume that the Hyperplane conjecture has a positive answer the existing proof would still give an estimate exponential with respect to k. We believe that a better estimate (polynomial) must be true. Having in mind equation (4.6), we pose the following question:

Question 2: Is it true that if K is symmetric and convex in $\mathbb{R}^n$, $1 \leqslant k \leqslant n$ and $I_k(K)$ exists and it is convex then $I_k(K)$ is isomorphic to $Z_k^\circ(K)$?

Note that a positive answer to the previous question would easily imply a linear in k estimate in Theorem 1.2.

The second conclusion of Theorem 1.3. can be viewed as a generalization of the classical Busemann inequality (see Introduction). We don't know if the assumption that $I_k(K)$ is convex is necessary in Theorem 1.3 and whether the estimate can be replaced by 1:

Question 3: Is it true that if K is a symmetric star body of volume 1, $1 \leqslant k \leqslant n-1$ and if $I_k(K)$ exists, then

$$|I_k(K)| \leqslant |I_k(D_n)| \ ?$$

Acknowledgments: The first name author wish to thank the US National Science Foundation for support through grants DMS-0652571 and DMS-1001234, and the Max Planck Institute for Mathematics for support and hospitality during his stay in Spring 2011. The second name author was partially supported by NSF 0906051. Part of this work was carried out when the third name author was visiting the Mathematics Department of Texas A& M which she thanks for its hospitality.

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