VANISHING CYCLES IN HOLOMORPHIC FOLIATIONS BY CURVES AND FOLIATED SHELLS

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Abstract. The purpose of this paper is the study of vanishing cycles in holomorphic foliations by complex curves on compact complex manifolds. The main result consists in showing that a vanishing cycle comes together with a much richer complex geometric object - we call this object a *foliated shell*.

Contents

1. Introduction.

1.1. Vanishing cycles, compact leaves and simultaneous uniformization. Let $\mathcal{L}$ be a holomorphic foliation by complex curves on a compact complex manifold X. For the sake of clarity and simplicity of exposition we describe our results in this Introduction assuming $\mathcal{L}$ to be smooth. In the main body of the paper this assumption will be removed (as well, the assumption of compactivity of X will be replaced by the *disc-convexity*).

Take a point $z \in X$ and denote by $\mathcal{L}_z$ the leaf of $\mathcal{L}$ passing through z. A cycle in $\mathcal{L}_z$ is, by definition, a closed path (a loop) $\gamma : [0,1] \to \mathcal{L}_z$. A cycle $\gamma \subset \mathcal{L}_z$ is called *a vanishing cycle* if the following two conditions hold:

- γ is not homotopic to zero in $\mathcal{L}_z$;
- there exist a sequence of points $z_n \to z$ and a sequence of loops $\gamma_n : [0,1] \to \mathcal{L}_{z_n}$ such that γ_n uniformly converge to γ and each γ_n is homotopic to zero in $\mathcal{L}_{z_n}$.

Classically vanishing cycles became the object of study in foliation theory since the seminal paper of Novikov [N], where he used them to produce a compact leaf in every smooth foliation by surfaces on $\mathbb{S}^3$, see also [H].

Apart of the question of existence of compact leaves vanishing cycles come into a play as obstructions to the simultaneous uniformization of leaves. Following Il'yashenko, see §2 in [Iy2], take a smooth complex hypersurface D in X transverse to the leaves of $\mathcal{L}$.

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Such D will be called simply *a transversal* in the sequel. Set $\mathcal{L}_D = \bigcup_{z\in D} \mathcal{L}_z$ and call this open subset of X the *cylinder* of $\mathcal{L}$ over the transversal D.

Let $\tilde{\mathcal{L}}_D = \bigcup_{z\in D} \tilde{\mathcal{L}}_z$ be the union of the universal coverings of the leaves $\mathcal{L}_z$ equipped with the natural topology, see Section 3. Let's call $\tilde{\mathcal{L}}_D$ the *universal covering cylinder* (or, simply the *covering cylinder* if no misunderstanding can occur) of $\mathcal{L}$ over D. It is clear (see Section 3 for more details) that a leaf $\mathcal{L}_z \subset \mathcal{L}_D$ containing a vanishing cycle exists if and only if the natural topology of $\tilde{\mathcal{L}}_D$ is not separable (*i.e.*, is not Hausdorff). Separability of $\tilde{\mathcal{L}}_D$ means that the leaves of $\mathcal{L}$ which cut D can be simultaneously uniformized. Therefore a vanishing cycle in some leaf $\mathcal{L}_z \subset \mathcal{L}_D$ is an obstruction to such simultaneous uniformization. $\mathcal{L}$ is called *uniformizable* if for any transversal D the cylinder $\mathcal{L}_D$ can be uniformized. Therefore $\mathcal{L}$ is uniformizable if and only if it doesn't contain a vanishing cycle in any of its leaves. This explains one more reason for the interest in studying of vanishing cycles.

1.2. Vanishing cycles and foliated shells. One of the main goals of this paper is to show that a vanishing cycle generates a very rich complex geometric object - a *foliated shell.*

Let $P = \{z = (z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} \leq 1\}$ be the unit bicylinder in $\mathbb{C}^2$ and $B = \{z = (z_1, z_2) \in \mathbb{C}^2 : \max\{|z_1|, |z_2|\} = 1\}$ its boundary. For some $0 < \varepsilon < 1$ let $B^\varepsilon = \{z \in \mathbb{C}^2 : 1-\varepsilon < \max\{|z_1|, |z_2|\} < 1+\varepsilon\}$ be a shell around B. Denote by $\pi : \mathbb{C}^2 \to \mathbb{C}$ the canonical projection $\pi(z) = z_1$ onto the first coordinate of $\mathbb{C}^2$. Note that B^ε is foliated by π over the disc $\Delta_{1+\varepsilon}$ of radius $1+\varepsilon$ (Δ_r denotes the disc of radius $r > 0$ in $\mathbb{C}$). Denote this foliation by $\mathcal{L}^{\mathrm{v}}$ and call it *a vertical foliation*. Its leaves $\mathcal{L}^{\mathrm{v}}_{z_1} := \pi^{-1}(z_1)$ are discs $\Delta_{1+\varepsilon}$ if $1-\varepsilon < |z_1| < 1+\varepsilon$ and are annuli $A_{1-\varepsilon,1+\varepsilon} := \Delta_{1+\varepsilon} \setminus \bar{\Delta}_{1-\varepsilon}$ if $|z_1| \leqslant 1-\varepsilon$.

Definition 1. *The pair $(B^\varepsilon, \mathcal{L}^{\mathrm{v}})$ will be called the standard foliated shell.*

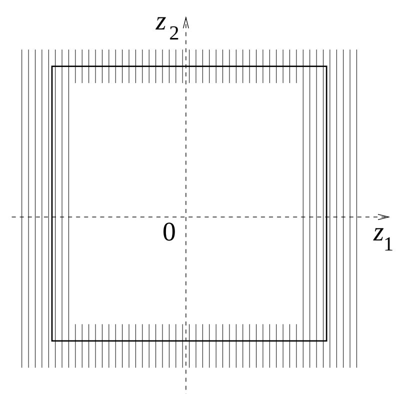

FIGURE 1. The *standard foliated shell* is foliated by discs and annuli over the disc $\Delta_{1+\varepsilon}$. In particular, $(B^\varepsilon, \mathcal{L}^{\mathrm{v}})$ is a foliated manifold.

By a foliated manifold in this paper we shall understand a pair $(X, \mathcal{L})$, where X is a complex manifold (separable and countable at infinity) and $\mathcal{L}$ is a holomorphic foliation by curves on X. Let $(X, \mathcal{L})$ be a foliated manifold and let $h : (B^\varepsilon, \mathcal{L}^{\mathrm{v}}) \to (X, \mathcal{L})$ be a foliated holomorphic immersion of the standard foliated shell into $(X, \mathcal{L})$ (an immersion between two foliated manifolds is called *foliated* if it sends leaves to leaves). Denote by Σ the image of the boundary B under h.

Definition 2. *The image $h(B^\varepsilon)$ is called* **a foliated shell** *in $(X,\mathcal{L})$ if:*

1) immersion h is a generic injection, i.e., is such that for all $z_1 \in \Delta_{1+\varepsilon}$ except of a finite set the restriction $h|_{\mathcal{L}^{\mathrm{v}}_{z_1}} : \{z_1\}\times A_{1-\varepsilon,1+\varepsilon} \to X$ is an imbedding;

2) Σ is not homologous to zero in X.

Roughly speaking the condition (1) means that h is (much) better then simply an immersion. The main point is of course the condition (2). It is very strong and our corollaries will demonstrate this.

Example 1. The reader should think about the Hopf surface $H^2 = \mathbb{C}^2\setminus\{0\}/z \sim 2z$. The same vertical foliation $\mathcal{L}^{\mathrm{v}}$ is invariant under the action $z\sim 2z$ and therefore projects to a foliation $\mathcal{L}$ on H^2. Let $h:\mathbb{C}^2\setminus\{0\}\to H^2$ be the canonical projection. It obviously induces a "foliated inclusion" $h:(B^\varepsilon,\mathcal{L}^{\mathrm{v}})\to(H^2,\mathcal{L})$. $\Sigma = h(B)$ is of course not homologous to zero in H^2.

Let ω be a $(1,1)$-form on X. ω is called *pluriclosed* if $dd^c\omega = 0$. Sometimes one calls such ω also dd^c-closed. Recall that $d^c := \frac{i}{4\pi}(\bar\partial - \partial)$ and therefore $dd^c = \frac{i}{2\pi}\partial\bar\partial$ (in particular $dd^c\ln|z|^2 = \delta_0$). We call a form ω *a taming form for $\mathcal{L}$* if $\omega|_{\mathcal{L}} > 0$. Foliations admitting a pluriclosed taming form we shall call *pluritamed*. Our first result is the following:

Theorem 1. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X which admits a pluriclosed taming form and let D be a transversal to $\mathcal{L}$ in X. Then the following statements are equivalent:*

i) Some leaf $\mathcal{L}_z \subset \mathcal{L}_D$ contains a vanishing cycle.

ii) The cylinder $\mathcal{L}_D$ contains a foliated shell.

Remark 1. (a) Statement (ii) means that the mapping $h: B^\varepsilon \to X$, which "supports" the foliated shell in X, actually takes values in the cylinder $\mathcal{L}_D$ (but $\Sigma = h(B)$ is not homologous to zero in the whole of X!).

(b) A transversal D is irrelevant in this theorem: if $\mathcal{L}_z$ contains a vanishing cycle then (*ii*) is true for *every* transversal $D\ni z$.

(c) Recall that a two-dimensional shell in a complex manifold X is a holomorphic image Σ of B such that Σ is not homologous to zero in X. Such shells can exist only in non-Kähler X by the Hartogs-type extension theorem for Kähler manifolds, see [Iv3] (and therefore foliations on Kähler manifolds don't have vanishing cycles). We want to stress here that X may contain a two-dimensional shell, but it may not be a *foliated* shell for the given foliation $\mathcal{L}$. A simple example is the elliptic fibration on the same Hopf surface H^2. This fibration doesn't admit a foliated shell, while H^2 itself does contain a two-dimensional shell.

(d) In fact in the process of the proof of Theorem 1 we establish the following useful characterization of shells:

Proposition 1. *Let w be a dd^c-closed taming form for $\mathcal{L}$. A holomorphic foliated immersion $h: B^\varepsilon \to X$ represents a foliated shell if and only if it is a generic injection and*

$$\int_B d^c(h^*\omega) \neq 0. \tag{1.1}$$

I.e. not only $h(B)$ is not homologous to zero in X but, moreover, the distinguished closed 3-form $d^c\omega$ doesn't vanish on $h(B)$. From Proposition 1 we immediately obtain the following:

Corollary 1. *If the taming form ω of the foliation $\mathcal{L}$ is d-closed then $\mathcal{L}$ has no vanishing cycles.*

The strategy of the proof follows that developed for the Kähler case in [Br3] with the replacement of the Thullen type extension theorem of Siu, [Si1], by Theorem 1.5 from [Iv6], and since all the results of this paper are valid for disc-convex manifolds it also includes the Stein case as found in [Iy2].

(e) The boundary B is topologically the three-dimensional sphere $\mathbb{S}^3$. It is not difficult to produce algebraic (and therefore Kähler) manifolds with nontrivial π_3, but none of them contains a shell. The reason is that a shell is a global *pseudoconvex* object in the complex manifold X and not simply an element of $\pi_3(X)$.

(f) The meaning of the Theorem 1 is that a topological property of $(X,\mathcal{L})$ to contain a vanishing cycle is equivalent to a complex geometric (even analytic) property to contain a foliated shell.

1.3. Imbedded cycles and imbedded shells. Note that our foliated shells are, after all, an *immersed* objects in X (even if they are "generic injections"). It would be definitely preferable to have really an *imbedded* ones. However, let us stress at this point that not all foliations with shells contain *imbedded* foliated shells, *i.e.,* such that $h: B^{\varepsilon} \to X$ is an imbedding. The reason is that the underlying manifold X may not contain an imbedded two-dimensional shell at all.

Example 2. Let, for example, $H^2/(z\sim -z)$ be the quotient of our Hopf surface by the antipodal involution. The vertical foliation $\mathcal{L}^{\mathrm{v}}$, described in the Example 1, is stable under this involution and we obtain a foliated manifold $(H^2/\mathbb{Z}_2, \mathcal{L}/\mathbb{Z}_2)$. The standard foliated shell immerses to $H^2/\mathbb{Z}_2$ and B maps onto the quotient $B/\mathbb{Z}_2$ which is topologically a lens space. I.e., we have here an immersed foliated shell. Due to a result of Kato, see [K1], would $H^2/(z\sim -z)$ contain an imbedded shell, it would be a deformation of a blown-up primary Hopf surface, *i.e.,* its fundamental group would be $\mathbb{Z}$. And this is not the case, because $\pi_1(H^2/\mathbb{Z}_2) = \mathbb{Z} \rtimes \mathbb{Z}_2$.

Nevertheless one can find an *imbedded* foliated shell provided that:

- the vanishing cycle γ is *imbedded* into its leaf $\mathcal{L}_z$;
- the shell itself is allowed to have somewhat more complicated topology.

Let us more carefully explain what does it mean that $\gamma \subset \mathcal{L}_z$ is imbedded. Let d be the order of the holonomy of $\mathcal{L}$ along the imbedded loop γ. It should be finite, otherwise γ cannot be approximated by the loops γ_n in the nearby leaves which are homotopic to zero. But then for a generic nearby leaf $\mathcal{L}_{z_n}$ the nearby loop $\gamma_n \subset \mathcal{L}_{z_n}$ will approximate $d\cdot\gamma$ (not just γ!) Therefore in the definition of an *imbedded vanishing cycle* one should specify that $\gamma_n \to d\cdot\gamma$ where $d \geqslant 1$ is the order of the holonomy of $\mathcal{L}$ along γ.

Now let us turn to the topology of shells. Recall that a cyclic surface quotient is a normal complex space $\mathcal{X}^{l,d}$ which is the quotient of $\mathbb{C}^2$ by the finite group $\Gamma_{l,d}$ of transformations given by $(z_1,z_2) \to (e^{\frac{2\pi i l}{d}}z_1, e^{\frac{2\pi i}{d}}z_2)$. Here $1 \leqslant l < d$ is relatively prime with d. This action preserves the vertical foliation on $\mathbb{C}^2$ and therefore $\mathcal{X}^{l,d}$ is equipped with the "vertical" foliation to, which we denote by $\mathcal{L}^{\mathrm{v}}$ again. Note that the standard "vertical" projection $\pi : \mathcal{X}^{l,d} \to \mathbb{C}/ < e^{\frac{2\pi i l}{d}} >= \mathbb{C}$ is well defined and its fibers are still the leaves of our vertical foliation. Take some smoothly bounded domain $G \Subset \Delta$ such that $\partial G \not\ni 0$ but $G \ni 0$ and consider the domain $P = \bigcup_{z\in G}\Delta_z \subset \mathcal{X}^{l,d}$ (here $\Delta_z := \{z\}\times\Delta$). Remark that the boundary B of P lies in the smooth part of $\mathcal{X}^{l,d}$. For some $\varepsilon > 0$ denote by B^{ε} the ε-neighborhood of B.

Definition 3. *A foliated cyclic shell in $(X,\mathcal{L})$ is a foliated holomorphic immersion $h : (B^{\varepsilon},\mathcal{L}^{\mathrm{v}}) \to (X,\mathcal{L})$ such that:*
1) h is a generic injection;
2) $\Sigma := h(B)$ is not homologous to zero in X.

With this notion at hand we can state the following:

Theorem 2. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X which admits a pluriclosed taming form and let D be a transversal to $\mathcal{L}$ in X. Then the following conditions are equivalent:*

i) Some leaf $\mathcal{L}_z \subset \mathcal{L}_D$ contains an imbedded vanishing cycle.

ii) The cylinder $\mathcal{L}_D$ contains an imbedded foliated cyclic shell.

We should point out that the topology of cyclic shell as we define it can be quite complicated. It is not just a lens space, *i.e.,* is not simply a quotient of $\mathbb{S}^3$ by a free action of a finite group.

Now we must to explain when the existence of a vanishing cycle in some leaf $\mathcal{L}_z$ of $(X,\mathcal{L})$ implies the existence of an *imbedded* one (in the same leaf). It occurs to depend on certain "almost Hartogs" property of the foliated pair $(X,\mathcal{L})$, see Definition 4.3. For the time being let us mention that if ω is actually a metric form on X then a foliated pair $(X,\mathcal{L})$ is almost Hartogs for every $\mathcal{L}$.

Theorem 3. *Let $(X,\mathcal{L},\omega)$ be a pluritamed compact foliated manifold. Suppose additionally that ω is a metric form. Then $(X,\mathcal{L})$ is almost Hartogs. In particular, if some leaf of $(X,\mathcal{L})$ contains a vanishing cycle then it contains also an imbedded vanishing cycle.*

Almost Hartogs are also all foliated pairs $(X,\mathcal{L})$, where the manifold X admits a rational or elliptic fibration, see Propositions 4.3 and 4.4. It is our understanding of the subject that if an immersed shell in a pluritamed foliated pair $(X,\mathcal{L})$ is found then the almost Hartogs property of $(X,\mathcal{L})$ is responsible for the presence also of an imbedded (but may be cyclic) shell.

Let's say a few more words about foliations with shells. First we remark that shells do come in families. Intuitively speaking we want to say that if our foliated manifold $(X,\mathcal{L})$ contains a foliated shell then it breaks into a complex $(\mathsf{dim}_{\mathbb{C}}X - 2)$ - parameter family of "foliated universes" each containing a foliated shell.

More precisely, the following is true:

Proposition 2. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact manifold X of complex dimension $n \geq 3$ which admits a pluriclosed taming form. Suppose that $(X,\mathcal{L})$ contains a foliated shell $h : (B^{\varepsilon},\mathcal{L}^{\mathrm{v}}) \to (X,\mathcal{L})$ (imbedded or immersed). Then there exists a smooth family $\{h_{\lambda}\}_{\lambda\in\Delta^{n-2}}$ of foliated shells containing h and transversal to $\mathcal{L}$ in the sense that:*

- *$h_0 = h$;*
- *$D_{\lambda}h_0(T_0\Delta^{n-2}) \cap D_z h_0(T_z B^{\varepsilon}) = \{0\}$ for every $z \in B^{\varepsilon}$.*

Such families of shells clearly come out in our proofs of Theorems 1 and 2. Remark also that due to the equivalence between shells and vanishing cycles the Proposition 2 reads

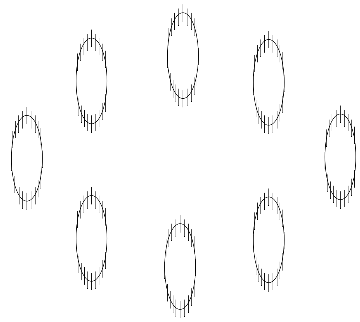

FIGURE 2. Shells persist transversely to $\mathcal{L}$: if there exists a foliated shell in $(X,\mathcal{L})$ then it is not disappearing, one can move it transversely to $\mathcal{L}$.

also as persistence of vanishing cycles in an obvious sense. If a two-dimensional "foliated universe" is, moreover, compact then it can be listed explicitly. Namely, the remarkable result of Kato in [K2] (but even more the "pseudoconvex surgery" invented there) allows us to describe all possible pairs $(X,\mathcal{L})$, where X is a compact complex surface, and $\mathcal{L}$ is a holomorphic foliation on X which contains a vanishing cycle:

Corollary 2. *Let X be a compact complex surface and $\mathcal{L}$ a (singular) holomorphic foliation by curves such that some leaf $\mathcal{L}_z$ of $\mathcal{L}$ contains a vanishing cycle γ. Then:*

i) either X is a modification of a Hopf surface and $\mathcal{L}_z$ is an elliptic curve;

ii) or, X is a modification of a Kato surface and the closure of $\mathcal{L}_z$ is a rational curve.

Remark 2. (a) In both cases (*i*) and (*ii*) of this Corollary the foliated shell in question is either spherical or a (holomorphic) quotient of the standard $\mathbb{S}^3$ by $\Gamma_{l,d}$ for some l,d. For the definition of a foliated *spherical* shell see Subsection 4.1 (in fact it means that as the boundary B one can take the standard sphere $\mathbb{S}^3 \subset \mathbb{C}^2$). This Corollary we state for singular foliations, the reason is that the case (*ii*) occurs only for a singular $\mathcal{L}$.

(b) Remark that in the case of surfaces we obtain a Novikov-type result, *i.e.,* the compactivity of the closure of the leaf supporting a vanishing cycle.

(c) The result, stated in this Corollary, is obtained also in [Br5].

1.4. Pluriexact foliations. Now let us clarify our assumption on a taming form ω to be pluriclosed. Let T be a $(1,1)$-current on X with measurable coefficients, *i.e.,* locally $T = T_{\alpha,\bar\beta}\frac{\partial}{\partial z^{\alpha}}\wedge\frac{\partial}{\overline{\partial z^{\beta}}}$, where $T_{\alpha,\bar\beta}$ are measures. Then there exists a $(1,1)$-vector field $\hat{T}$ and a complex Radon measure $\|T\|$ such that

$$< \varphi, T >= \int_X \varphi(\hat{T}_z)d\|T\|(z)$$

for every test $(1,1)$-form φ, see [HL].

Definition 4. *Following* [Su] *we shall use the following terminology throughout this paper:*

- *A $(1,1)$-current T is said to be **directed by $\mathcal{L}$** (or, **tangent to $\mathcal{L}$**) if for $\|T\|$ - a.a. $z\in X$ one has $\hat{T}_z = \frac{i}{2}\mathrm{v}\wedge\bar{\mathrm{v}}$, where $\mathrm{v}\in T_z\mathcal{L}$.*
- *A **foliated cycle** on $(X,\mathcal{L})$ is a **closed** $(1,1)$-current directed by $\mathcal{L}$.*

Remark that a smooth $(1,1)$-form ω on X is a *taming form* for $\mathcal{L}$ if and only if for every non-trivial positive $(1,1)$-current T directed by $\mathcal{L}$ one has $< \omega, T >> 0$. In the spirit of [HL] one can prove the following:

Proposition 3. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X. Then*

i) either $(X,\mathcal{L})$ admits a pluriclosed taming form,

ii) or, here exists a non-trivial, positive, dd^c-exact $(1,1)$-current on X directed by $\mathcal{L}$.

I.e., in the case (*ii*) $(X,\mathcal{L})$ carries a non-trivial, positive, dd^c*-exact foliated cycle*. A foliated manifold $(X,\mathcal{L},\omega)$ admitting a non-trivial, positive, dd^c-exact, bidimension $(1,1)$ current T tangent to (or, directed by) $\mathcal{L}$ we shall call *pluriexact*.

Via the aforementioned duality the characterization result of the Theorem 1 shows that the class of all holomorphic foliations by curves on compact complex manifolds splits naturally into the following three non-intersecting subclasses: the class $\mathcal{S}$ of *shelled foliations*, the class $\mathcal{U}$ of *uniformizable* foliations, and the class $\mathcal{E}$ of *pluriexact* foliations. A *shelled foliation* or a *foliation with shells* is a foliation on a compact manifold which contains foliated shells.

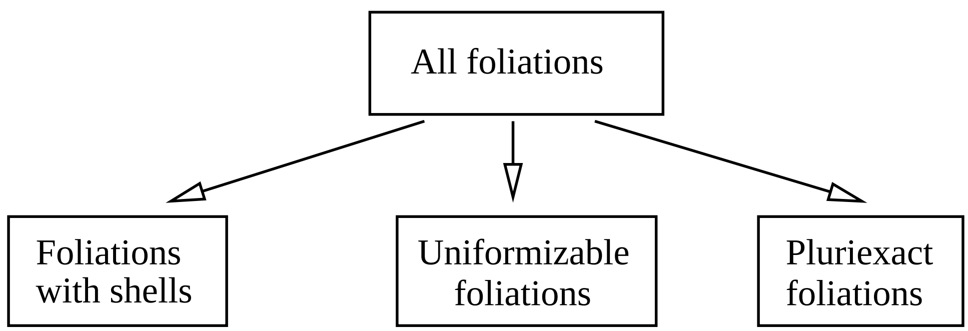

FIGURE 3

Note that in the definition of classes $\mathcal{S}$ and $\mathcal{U}$ we require both a pluriclosed taming form and a shell/or absence of shells. The point is that a foliated shell is of real importance only in the presence of such a taming form. In that case it turns out to be a dominating object in $(X,\mathcal{L})$. As well as the uniformizability condition on $\mathcal{L}$ implies more information about this foliation provided $\mathcal{L}$ admits a pluritaming form. While in the absence of such a form a dd^c-exact current tangent to $\mathcal{L}$ is (eventually) of much greater importance.

We see the future development of the subject as the study of each of these classes separately, eventually with the very different tools, and $\mathcal{E}$ to be certainly further subdivided. Let us outline one of the possible ways of doing that. For that remark that a foliated cycle T in a standard way, see [Go], defines a transverse invariant measure. Therefore:

Corollary 3. *A pluriexact holomorphic foliation by curves on a compact complex manifold admits a transversal invariant measure.*

Let us try now to say more about this measure. In order to do so let us define subclasses $\mathcal{E}_-,\mathcal{E}_+,\mathcal{E}_0$ such that $\mathcal{E}=\mathcal{E}_-\bigsqcup\mathcal{E}_0\bigsqcup\mathcal{E}_+$ and state our results for each of them.

Class $\mathcal{E}_-$. The first is the class $\mathcal{E}_-$ of pluriexact foliations carrying a plurinegative taming form, *i.e.*, $(X,\mathcal{L})\in\mathcal{E}_-$ if it is pluriexact and if there exists a $(1,1)$-form ω on X such that $\omega|_{\mathcal{L}}>0$ and $dd^c\omega\leqslant 0$. Such foliated manifolds we shall call *plurinegative*. In Section 5, Proposition 5.3 we shall see that in the presence of plurinegative taming form on $(X,\mathcal{L})$ every vanishing cycle $\gamma\subset\mathcal{L}_z$ results to a non-trivial exact foliated cycle, which will be denoted as T_z. Its support is contained in $\bar{\mathcal{L}}_z$.

Now let us describe the way a vanishing cycle appears in a foliated manifold of class $\mathcal{E}_-$. In the following theorem we suppose that the ambient manifold X carries a strictly positive dd^c-closed $(2,2)$-form (this is always the case if X is compact and $\mathsf{dim}_{\mathbb{C}}X = 3$ for example). Remark finally that $\mathcal{E}$ doesn't contain compact complex manifolds of dimension two. Therefore in the following theorem we deal with $\mathsf{dim}_{\mathbb{C}}X \geqslant 3$.

Theorem 4. *Let $(X,\mathcal{L},\omega_1)$ be a disc-convex foliated manifold of class $\mathcal{E}_-$ and suppose the manifold X itself admits a strictly positive dd^c-closed $(2,2)$-form ω_2. Let $\gamma \subset \mathcal{L}_z$ be a vanishing cycle in the leaf $\mathcal{L}_z$ of $\mathcal{L}$ and let T_z be the corresponding exact foliated cycle. Then:*

i) either $(X,\mathcal{L})$ admits a complex $(\mathsf{dim}_{\mathbb{C}}X-2)$-parameter family of distinct exact foliated cycles, which contains the cycle T_z;

ii) or $(X,\mathcal{L})$ contains a $(\mathsf{dim}_{\mathbb{C}}X - 3)$-parameter family of three-dimensional foliated shells.

The notion of three-dimensional foliation shell being clear let us explain the item (*ii*) of this Theorem by an example.

Example 3. *Take $H^3 = \mathbb{C}^3/z \sim 2z$ - the Hopf threefold. Let $\mathcal{L}^{\mathrm{v}}$ be again the vertical foliation $\mathcal{L}^{\mathrm{v}}_c = \{z_1 = c_1, z_2 = c_2\}$. $(H^3,\mathcal{L}^{\mathrm{v}})$ admits a plurinegative taming $(1,1)$-form but doesn't admit a pluriclosed one. It also contains a 3-dimensional foliated shell but not a 2-dimensional one.*

(i) Indeed, set $z' = (z_1,z_2)$ and consider the following $(1,1)$-form on H^3:

$$\theta = \frac{i}{2}\frac{(dz',dz')}{\|z'\|^2}. \tag{1.2}$$

θ is a well defined positive bidimension $(2,2)$-current on H^3. One easily checks that $dd^c\theta = -c_4[\mathcal{L}^{\mathrm{v}}_0]$, where $[\mathcal{L}^{\mathrm{v}}_0]$ is the current of integration over the central fiber $\mathcal{L}^{\mathrm{v}}_0$ of $\mathcal{L}^{\mathrm{v}}$ and c_4 is the volume of the unit ball in $\mathbb{C}^2$. θ is a clear obstruction to the existence of pluriclosed $\mathcal{L}^{\mathrm{v}}$-taming form. Indeed, would ω be such a form than one would have: $0 =< dd^c\omega,\theta >=< \omega, dd^c\theta >< 0$ - contradiction.

(ii) At the same time the $(1,1)$-form

$$\omega = \frac{i}{2}\frac{(dz,dz)}{\|z\|^2}, \tag{1.3}$$

where $z = (z_1,z_2,z_3)$, is strictly positive on H^3 (not only on $\mathcal{L}$, *i.e.*, is a metric form) and one easily checks that $dd^c\omega \leq 0$ (but $dd^c\omega \neq 0$ contrary to the two-dimensional case). I.e. ω serves as a plurinegative taming form for any foliation by curves on H^3. Our foliated Hopf manifold $(H^3,\mathcal{L}^{\mathrm{v}})$ contains an obvious foliated 3-dimensional shell, but doesn't contain *any* 2-dimensional shell because $H_3(H^3,\mathbb{Z}) = 0$. I.e., the case (*ii*) of Theorem 4 realizes here.

Remark 3. A behavior as in the case (*i*) of Theorem 4 appears in Example 3.6 of [Iv6], where also some other relevant examples can be found.

Class $\mathcal{E}_+$. A foliated manifold $(X,\mathcal{L})$ we call *pluripositive* if there exists a non-trivial $(1,1)$-current T tangent to $\mathcal{L}$ such that $T = dd^cR$ for some *positive* $(2,2)$-current R. We denote the class of pluripositive foliations as $\mathcal{E}_+$. Note that obviously $\mathcal{E}_- \cap \mathcal{E}_+ = \varnothing$. Our main result on class $\mathcal{E}_+$ is the following:

Theorem 5. *Let $(X,\mathcal{L})$ be a disc-convex foliated manifold which possesses a non-trivial, positive $(1,1)$-current T directed by $\mathcal{L}$ such that $T = dd^cS$ for some positive current S. Then:*

i) $\chi_{\mathcal{L}^s}T = 0$,

ii) the transversal measure μ induced by T on $X^0 := X\setminus\mathcal{L}^s$ has finite logarithmic potential.

Here $\mathcal{L}^s := \mathcal{L}^{\mathsf{sing}}$ is the singular locus of $\mathcal{L}$ (*i.e.*, it is admitted in this formulation that $\mathcal{L}$ can be singular). Remark also that for the Hopf foliated pair $(H^3,\mathcal{L}^{\mathrm{v}})$ of Example 3 the transverse measure is the delta function. This makes contrast to the statement of Theorem 5 for the pluripositive foliations.

Class $\mathcal{E}_0$. We define this class simply as $\mathcal{E}_0 := \mathcal{E}\setminus(\mathcal{E}_-\bigsqcup\mathcal{E}_+)$. I.e., $\mathcal{E}_0$ is the class of foliated manifolds which do not admit neither a plurinegative taming form no a positive dd^c-exact foliated cycle T. Surprisingly this can happen. Namely we shall see that:

Example 4. *There exists a compact Moishezon 3-fold X and a holomorphic foliation by curves $\mathcal{L}$ on X such that $(X,\mathcal{L})\in\mathcal{E}_0$.*

We also give a characterization of class $\mathcal{E}_0$ in terms of currents, see Propositions 5.1 and 5.2.

1.5. Uniformizable foliations. For uniformizable foliations tamed by a pluriclosed form we expect more or less the same results as for foliations on compact Kähler (or algebraic) manifolds. Let's give some typical statements.

Corollary 4. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X which admits a plurinegative taming form. Suppose that $\mathcal{L}$ contains a leaf whose universal cover is $\mathbb{CP}^1$. Then the universal cover of every leaf is $\mathbb{CP}^1$ and, moreover, $\mathcal{L}$ is a rational quasi-fibration.*

For foliations on Kähler manifolds this result is proved in [Br3]. In Section 6 we prove also the following version of the Reeb stability theorem:

Proposition 4. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X admitting a dd^c-negative taming form.*

i) If $\mathcal{L}$ has a compact leaf with finite holonomy then all leaves of $\mathcal{L}$ are compact with finite holonomy.

ii) If every leaf of $\mathcal{L}$ is compact then every leaf has finite holonomy. In that case there is an upper bound on volumes of leaves and the leaf space is Hausdorff.

For the foliations on Kähler manifolds this result is well known, see [Ga, P]. Both statements (and some others) follow from the compactness property of the corresponding cycle space in the presence of a plurinegative taming form, see Remark 6.1.

Take a cycle γ on some leaf $\mathcal{L}_z$. Following [LP], see also [Iy1], we define in Subsection 6.3 the domain of preservation of the homotopy class $[\gamma]$ and prove the following:

Proposition 5. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a compact complex manifold X admitting a pluriclosed taming form. Then:*

1) either the domain of preservation Ω_γ is Hausdorff (and therefore is a complex manifold) for every loop γ,

2) or $(X,\mathcal{L})$ contains a foliated shell.

1.6. The structure of the paper, notes. This paper is organized in the following way.

1. First, in Section 2 we develop the main technical tool - a meromorphic extension Theorem 2.3 from generalized Hartogs domains. This theorem should be viewed as a generalization of Theorem 2.2 from [Iv6] and Proposition 4.1 from [Br5] at a time. The new points here are replacement of *metric* forms by *taming* ones (this includes the replacement of the Siu's Thullen type extension theorem by Lemma 2.2 - a nonparametric version of Theorem 1.5 from [Iv6]) and detecting the obstructions to the extension as *foliated shells* and not simply as *shells* like in [Iv6] - the corresponding arguments are gathered in Lemmas 2.3 and 2.4.

2. We start the Section 3 with recalling the necessary definitions and notions around uniformization of foliations, vanishing ends and covering cylinders (tubes), which were developed by M. Brunella in [Br1]-[Br4]. The right notions were worked out in the sited papers in part after appearance of example in [CI], see discussion before Theorem 3.1 in [Br4]. This example is recalled and enhanced in subsection 2.3 of the present paper in the context of extension theorems. Its relevance to the vanishing ends is discussed in Remark 3.2. After that we prove a more precise version of Theorem 1, namely the Theorem 3.1, which includes the case of singular foliations, non-compact ambient manifolds and, more crucially, specifies the location of a foliated shell. A more precise version of Theorem 2 - the Theorem 3.2 about imbedded cycles and shells follows in Subsection 3.6.

3. Section 4 contains the proof of Theorem 3. More generally it describes the techniques to obtain an *imbedded* vanishing cycle from a non-imbedded one. The main achievement is Theorem 4.1, which relates an "almost Hartogs" property of a foliated pair $(X, \mathcal{L})$ with imbedded vanishing cycles. It contains also the description of examples of complex dimension two, *i.e.,* of complex surfaces. In particular, that means the Corollary 2.

The logic how to obtain from immersed cycles the imbedded ones is explained in [Br5], where the case of nonuniformizable foliations on complex surfaces is studied. Our paper follows this logic, but applies it in all dimensions for disc convex manifolds, and more crucially for taming (and not only metric forms). The advantage of this becomes clear via Proposition 3, *i.e.,* the complementary class $\mathcal{E}$ also is quite understandable.

4. Section 5 is devoted to pluriexact foliations and contains the proof of Theorems 4 and 5 as well as Example 4.

5. The last Section 6 contains multidimensional examples relevant to the subject of this paper, proofs of Corollary 4 and Propositions 4 and 5. Here we also formulate several open questions.

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3. The results of this paper had been reported at the conference "Recent Developments in non-Kähler Geometry" held on the honor of Masahide Kato at the University of Hokkaido, Sapporo on 5-7 March 2008, see [NK]. I would like to thank the organizers for the possibility to give this talk and for the useful post-talk discussions.
4. I'm also grateful to the Referee of this paper for the remarks and useful suggestions.

2. Pluriclosed Taming Forms and Foliated Immersions.

2.1. Generalized Hartogs figures. Let us start with some definitions. In the definition of a *foliated manifold* $(X,\mathcal{L})$ from now we aloud $\mathcal{L}$ to be a *singular* holomorphic foliation by curves on X. One of the ways to define such $\mathcal{L}$ is the following. Take a sufficiently fine open covering $\{\Omega_\alpha\}$ of X. Then $\mathcal{L}$ will be defined by the nonvanishing identically holomorphic vector fields $v_\alpha \in \mathcal{O}(\Omega_\alpha, TX)$ which are related on a non-empty intersections $\Omega_{\alpha,\beta} := \Omega_\alpha \cap \Omega_\beta$ as $v_\alpha = h_{\alpha,\beta} v_\beta$. Here $h_{\alpha,\beta} \in \mathcal{O}^*(\Omega_{\alpha,\beta})$. After contracting the common factors one immediately sees that the singular set $\mathcal{L}^{\mathsf{sing}}$ of $\mathcal{L}$, which is defined as $\mathcal{L}^{\mathsf{sing}} = \{z : v_\alpha(z) = 0\}$, is an analytic subset of X of codimension at least two. Set $X^0 := X \setminus \mathcal{L}^{\mathsf{sing}}$. The leaves of $\mathcal{L}$ are, in the first approximation, defined as the leaves of the smooth foliation $\mathcal{L}^0 := \mathcal{L}|_{X^0}$, *i.e.,* they are entirely off the singular set of $\mathcal{L}$. Then, depending on the someone goals, one adds to them some "ends". We shall do that in the following Section.

A particular class of foliated manifolds are fibrations by curves, *i.e.,* triples (W,π,V) where W is a complex manifold of dimension $\mathsf{dim}\, V + 1$ and $\pi : W \to V$ is a surjective holomorphic submersion with connected fibers. A holomorphic mapping $f : (X,\mathcal{L}) \to (X',\mathcal{L}')$ is said to be a foliated immersion if it is an immersion and sends leaves to leaves. In the case of fibrations, *i.e.,* if $f : (W,\pi,V) \to (W',\pi',V')$, one can be more precise: there exists a holomorphic map $f_{\mathrm{v}} : V \to V'$ such that for all $z \in V$ one has $f(W_z) \subset W'_{f_{\mathrm{v}}(z)}$. Dimension of W' might be bigger then that of W. If $V' = V$ one often supposes also that W_z goes to W'_z for all $z \in V$. This will be clear from the context.

Definition 2.1. *A generalized Hartogs figure is a quadruple (W,π,U,V), where W and V are complex manifolds, U an open subset of V and $\pi : W \to V$ is a holomorphic submersion such that:*

i) for all $z \in V \setminus U$ the fiber $W_z = \pi^{-1}(z)$ is diffeomorphic to an annulus;
ii) for $z \in U$ the fiber W_z is diffeomorphic to a disc.

Generalized Hartogs figures are foliated manifolds (even fibrations) of a special type: they are concave in the most naïve and clear sense. Manifold W has a distinguished part of the boundary formed by the outer boundaries $\partial_0 W_z$ of annuli W_z. We shall suppose that W is smooth up to this part of its boundary and denote it by $\partial_0 W$, *i.e.,* $\partial_0 W = \cup_{z\in V} \partial_0 W_z$. Projection π is also supposed to be smooth up to $\partial_0 W$ and therefore $\pi : \partial W_0 \to V$ is a circle fibration. For $z \in U$ the outer boundary $\partial_0 W_z$ is actually the boundary of the disc W_z.

Recall that the *standard Hartogs figure* is the open subset of $\mathbb{C}^{n+1}$ of the form

$$H_\varepsilon = \big(\Delta^n_{1+\varepsilon} \times A_{1-\varepsilon,1+\varepsilon}\big) \cup \big(\Delta^n_\varepsilon \times \Delta_{1+\varepsilon}\big) \tag{2.1}$$

for some $\varepsilon > 0$. H_ε likewise carries our "vertical foliation" $\mathcal{L}^{\mathrm{v}}$. This time the leaves $\mathcal{L}^{\mathrm{v}}_{z'}$ are discs $\Delta_{1+\varepsilon}$ if $||z'|| < \varepsilon$ and annuli for $\varepsilon \leqslant ||z'|| < 1+\varepsilon$. Here $z' = (z_1,...,z_n)$ and $||\cdot||$ is the polydisc-norm in $\mathbb{C}^n$. Remark now that $(H_\varepsilon, \mathcal{L}^{\mathrm{v}})$ fits, of course, into the Definition 2.1 with $V = \Delta^n_{1+\varepsilon}$, $U = \Delta^n_\varepsilon$ and π being the restriction of the canonical "vertical" projection $\mathbb{C}^{n+1} \to \mathbb{C}^n$ onto H_ε. Remark furthermore that the standard foliated shell is also a generalized Hartogs figure. Namely it can be viewed as $(B^\varepsilon, \pi, A_{1-\varepsilon,1+\varepsilon}, \Delta_{1+\varepsilon})$.

Definition 2.2. *If $U = \varnothing$ we call $(W,\pi,\varnothing,V)$ trivial, if $U = V$ we call (W,π,V,V) complete and in the latter case often denote it as (W,π,V).*

The standard Hartogs figure is newer trivial by definition, *i.e.*, it is commonly accepted that always $\varepsilon > 0$. Let D be a non-empty open subset of V. Set $W|_D = \pi^{-1}(D)$ and consider it also as a generalized Hartogs figure $(W|_D, \pi|_D, D\cap U, D)$ - a subfigure of (W,π,U,V). Moreover, if $S\subset V$ is a submanifold of V one can consider the restriction $(W|_S, \pi|_S, S\cap U, S)$ and it is again a generalized Hartogs figure. In the sequel we shall often avoid the word "generalized" and call our figures simply *Hartogs figures*, specifying over what V they are considered.

Remark 2.1. The necessity of considering generalized Hartogs figures in this paper comes from the simple observation that: *every vanishing cycle produces a natural generalized (or topological) Hartogs figure around it.* This will become clear in Section 3. The fact that the standard Hartogs figure is not sufficient for our considerations will be explained by an example in Subsection 2.3.

2.2. Extension after a reparametrization. The following notion comes back to [Ti], see also [Bl]. Let $f : A_{1-\varepsilon,1} \to X$ be a holomorphic immersion.

Definition 2.3. *We say that f extends to Δ after a reparametrization if for some $\delta > 0$ there exists an imbedding $h : A_{1-\delta,1} \to A_{1-\varepsilon,1}$ sending $\partial\Delta$ to $\partial\Delta$ and preserving the canonical orientation of $\partial\Delta$, such that $f\circ h$ holomorphically extends to Δ.*

It is clear that such h, if exists, should be holomorphic. We shall use also the following form of this notion. Let γ be a simple oriented loop on a bordered Riemann surface W. The latter should be viewed simply as a collar adjacent to γ. Let $f : W \to X$ be a holomorphic immersion. Suppose that there exist a Riemann surface $\widetilde{W}$ which is a bordered disc with boundary $\tilde{\gamma}$ (canonically oriented) and a biholomorphic mapping h from a collar adjacent to $\tilde{\gamma}$ to W (smooth up to the boundaries) and sending $\tilde{\gamma}$ onto γ, preserving orientations, such that the composition $f\circ h$ holomorphically extends onto the disc $\widetilde{W}$. Then we shall say that f extends onto the disc $\widetilde{W}$ after a reparametrization. If such $\widetilde{W}, \tilde{\gamma}$ and h do exist but are not specified we shall say simply that f holomorphically extends onto a disc after a reparametrization.

Remark 2.2. (a) There is one case when the extension of an immersion after a reparametrization may be not unique in the sense that there may not exist an automorphism φ of Δ such that one extension is equal to the another one composed with φ. For example, take a function $f(z) = 4z + \sqrt{z^2-1}$ and consider it as a holomorphic mapping from a thin annulus around $\partial\Delta(2)$ - the circle of radius 2, into $\mathbb{CP}^1$. Then f has two extensions after a reparametrization:

1) An injective one. Indeed, f is an imbedding of $\partial\Delta(2)$ into $\mathbb{CP}^1$ and therefore bounds a disc, say D. Let $r : \Delta(2) \to D$ be a Riemann mapping (it is biholomorphic in a neighborhoods of the closed discs). Set $h = f^{-1}\circ r$ - a reparametrization of $\partial\Delta(2)$. Then $f\circ h = r$ is the extension of f onto $\Delta(2)$ after a reparametrization.

2) A non-injective one. This is given by the formula defining f. It has two ramification points ± 1 and extends onto the union $\Delta(2)\cup\mathbb{CP}^1$ appropriately glued along the slit $[-1,1]$. The Riemann surface obtained is again a disc. This second extension is non-injective.

(b) At the same tame, if f was a *generic injection* (i.e. injective outside of a finite set) then its extension after reparametrization, which we also require to be a generic injection, is unique (if exists). Uniqueness means here up to a biholomorphic automorphism of the disc.

Now let's turn ourselves to the families of immersions.

Definition 2.4. *A holomorphic mapping $f:(W,\pi,V)\to X$ of a fibration (W,π,V) into a complex manifold X is called generically injective if for all $z\in V$ outside of a proper analytic subset $A\subset V$ the restriction $f_z:=f|_{W_z}$ is a generic injection.*

Note that we do not ask f to be "generically injective" itself but only its restrictions onto "generic" fibers. Actually f may not be even an immersion. However in most cases mappings appearing in this paper will be both immersions and generic injections. We shall also need a corresponding notion for the meromorphic case.

Definition 2.5. *A meromorphic mapping $f:W\to X$ between complex manifolds is a meromorphic immersion if it is an immersion outside of its indeterminacy set I_f. If, moreover, (W,π,V) is a holomorphic fibration then a meromorphic mapping f is called generically injective if $f|_{W_z}$ is a generic injection for z outside of a proper analytic subset of V.*

Here and always in this paper writing $f(C)$ for some meromorphic map and some complex curve C we mean that the restriction $f|_C$ of f onto C is well defined (this means that C in not contained in the indeterminacy set of f) and $f(C)$ is actually $f|_C(C)$. Again we will mostly work with meromorphic maps which are both meromorphic immersions and generic injections.

Let a holomorphic generic injection $f:(W,\pi,U,V)\to X$ of a generalized Hartogs figure into a complex manifold X be given and let $\hat U$ be some open subset of V containing U.

Definition 2.6. *We say that f meromorphically extends onto the Hartogs figure $(\widetilde{W},\pi,\hat U,V)$ over (the same!) V* after a reparametrization *if there exists a foliated biholomorphism of trivial figures $h:(\partial_0\widetilde{W},\pi,\varnothing,V)\to(\partial_0W,\pi,\varnothing,V)$ (i.e., h is defined in an one-sided neighborhood of $\partial_0\widetilde{W}$ and $h(z)$ tends to ∂_0W when z tends to $\partial_0\widetilde{W}$) such that $f\circ h$ extends to a generically injective meromorphic map $\tilde f:(\widetilde{W},\pi,\hat U,V)\to X$. A reparametrization map h is supposed to be constant over V, i.e., $h(\partial_0\widetilde{W}_{z_1})\subset W_{z_1}$ for all $z_1\in V$.*

Remark 2.3. (a) We say simply that f extends after a reparametrization if the data as in Definition 2.6 do exist (but, may be, are not specified). In that case we often omit tildes over the extended objects, such as W and f (π will newer come with a tilde in this context). I.e., we often say that f extends onto $(W,\pi,\hat U,V)$ after a reparametrization.
(b) Remark that if f extends as a meromorphic map being a generic injection on (W,π,U,V) with $U\neq\varnothing$ then its extension will be automatically a generic injection. However in the definition above we do not exclude the case when $U=\varnothing$.

Definition 2.7. *If for any point $z\in V$ there exists a neighborhood $V(z)$ such that the restriction $f|_{W|_{V(z)}}$ extends after a reparametrization onto a complete Hartogs figure $\big(\widetilde{W}|_{V(z)},\pi,V(z)\big)$ over $V(z)$ then we say that f locally extends after a reparametrization.*

Let us be very precise at this point: by saying that the "restriction $f|_{W|_{V(z)}}$ extends" we mean here that one is taking the restriction of f onto the Hartogs subfigure $(W|_{V(z)},\pi,V(z)\cap U,V(z))$ and this restriction extends onto the complete figure $(W|_{V(z)},\pi,V(z))$. I.e., a reparametrization is supposed to be made near $\partial_0W|_{V(z)}$ only.

If a generically injective (!) mapping f extends locally then it extends globally. Namely the following is true:

Lemma 2.1. *Let $V \supset U_1 \cup U_2$ with (may be empty) intersection $U_{12} := U_1 \cap U_2$. Let $(W, \pi, \varnothing, V)$ be a trivial generalized Hartogs figure over V and $f : W \to X$ be a generically injective holomorphic map into a complex manifold X such that f meromorphically extends onto a complete Hartogs figures $(W|_{U_k}, \pi, U_k)$ for $k = 1, 2$ after a reparametrization. Then f extends after a reparametrization onto a figure $(W, \pi, U_1 \cup U_2, V)$.*

Proof. *Step 1. Extending onto a complete figure* $(W|_{U_1\cup U_2}, \pi, U_1 \cup U_2)$. Denote by $h_k : (\partial_0 \widetilde{W}_k, \pi_k, \varnothing, U_k) \to (\partial_0 W|_{U_k}, \pi, \varnothing, U_k)$ the corresponding foliated biholomorphisms. The fact that f is a generic injection imply that $h_2^{-1} \circ h_1 = (f \circ h_2)^{-1} \circ (f \circ h_1) : \partial_0 \widetilde{W}_{1,z} \to \partial_0 \widetilde{W}_{2,z}$ extends for every $z \in U_{12}$ onto a corresponding disc and therefore extends to a foliated biholomorphism between complete figures $(\widetilde{W}_1|_{U_{12}}, \pi_1, U_{12})$ and $(\widetilde{W}_2|_{U_{12}}, \pi_2, U_{12})$. Therefore complete figures $(\widetilde{W}_1, \pi_1, U_1)$ and $(\tilde{W}_2, \pi_2, U_2)$ glue together to a complete figure $(\widetilde{W}_3, \pi, U_1 \cup U_2)$ and reparametrization maps h_k glue to a reparametrization map $h : (\partial_0 \widetilde{W}_3, \pi, \varnothing, U_1 \cup U_2) \to (\partial_0 W|_{U_1\cup U_2}, \pi, \varnothing, U_1 \cup U_2)$. Mapping f extends, after being reparameterized by h onto the complete figure $(W|_{U_1\cup U_2}, \pi, U_1 \cup U_2)$.

Step 2. Extending to $(W, \pi, U_1 \cup U_2, V)$. Reparametrization $h : \partial_0 \widetilde{W}_3 \to \partial_0 W|_{U_1\cup U_2}$ constructed in the first Step allows us to glue figures $(\widetilde{W}_3, \pi, U_1 \cup U_2)$ and $(W, \pi, \varnothing, V)$ together to a figure $(W, \pi, U_1 \cup U_2, V)$ and f extends onto it.

□

From this lemma we obtain the following

Corollary 2.1. *Let $f : W \to X$ be a generically injective holomorphic immersion of a generalized Hartogs figure (W, π, U, V) into a complex manifold X. Then there exists a maximal open $U \subset \hat{U} \subset V$ such that f meromorphically extends onto $(W, \pi, \hat{U}, V)$ after a reparametrization.*

2.3. Hartogs figures and reparametrizations. The necessity of considering generalized Hartogs figures in foliation theory will be very clearly seen along this paper. The fact that the work with them cannot be reduced to the standard Hartogs figures in $\mathbb{C}^2$ (or $\mathbb{C}^n$) comes from the example constructed in [CI]. This example is explained on the Figure 4. Namely the following can happen. There exists a generalized Hartogs figure W over a disc (*i.e.,* both $\varnothing \neq U \subset V$ are discs in $\mathbb{C}$) with the following property: whenever a holomorphic foliated imbedding $h : (z_1, z_2) \to (h_1(z_1), h_2(z_1, z_2))$ of H_ε into W is given such that $h_1(0) = z^0 \in U$ then necessarily $h_1(\Delta_{1+\varepsilon}) \subset U$ (whatever $\varepsilon > 0$ is).

This feature of the example in question is not explicitly stated in [CI] (a somewhat weaker property of it was sufficient there) and therefore we shall enhance (and simplify considerably) our example in this subsection. Let us briefly recall the construction from [CI].

Step 1. Construction. Consider the following domain

$$W = \Delta \times \Delta \setminus \{(z_1, z_2) : 1/3 \leqslant |z_1| \leqslant 2/3 \text{ and } z_2^2 = z_1 \lambda(|z_1|^2)\}.$$

Here $\lambda \in \mathcal{C}^\infty(\mathbb{R}), 0 \leqslant \lambda \leqslant 1, \lambda(t) \equiv 0$ for $t < 1/9$ and $\lambda(t) \equiv 1$ for $t > 4/9\}$.

In W consider the (almost) complex structure J defined by the basis of $(1,0)$-forms dz_1 and $dz_2 + a(z_1, z_2) d\bar{z}_1$ where

$$a(z_1, z_2) = \begin{cases} \frac{z_2 z_1^2 \lambda'(|z_1|^2)}{z_2^2 - z_1 \lambda(|z_1|^2)} \text{ if } 1/3 \leqslant |z_1| \leqslant 2/3, \\ 0 \text{ otherwise.} \end{cases} \tag{2.2}$$

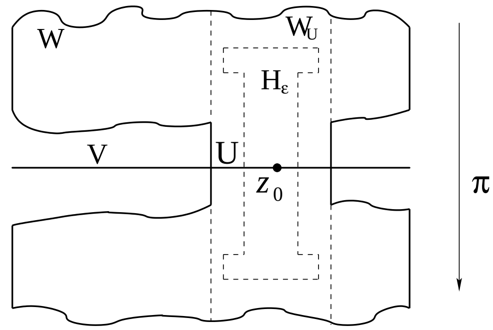

FIGURE 4. This is the standard Hartogs figure imbedded into a generalized Hartogs figure W constructed in [CI]. Every attempt to imbed H_ε into this W will look like on the picture: if the fiber over the origin in H_ε is mapped to a fiber over some point $z_0 \in U$ then the image of H_ε will newer leave $W|_U$.

In [CI], see Lemma 1, it was proved that:
i) J is integrable;
ii) $J = J_{\mathsf{st}}$ on $W \setminus \left(\bar{A}_{\frac{1}{3},\frac{2}{3}} \times \Delta\right)$ - this is obvious from the definition of J;
iii) functions $g(z) = z_1$ and $f(z) = z_2 + \frac{z_1}{z_2}\lambda(|z_1|^2)$ are J-holomorphic;
iv) $\mathsf{ind}_{|z_2|=1-\delta} f(z_1, z_2) = -1$ for every $|z_1| = 1$ and every $\delta > 0$ small enough ($0 < \delta < 1/6$ is sufficient).

Remark 2.4. (a) The construction of this example could be easily understood looking on the function f. It is designed to have properties incompatible with being "holomorphic". Then one computes its differential and gets

$$df(z_1, z_2) = \big(\frac{\lambda}{z_2} + \lambda' \frac{z_1}{z_2} \bar{z}_1\big) dz_1 + \big(1 - \frac{z_1}{z_2^2}\lambda\big)\big(dz_2 + \frac{z_2 z_1^2 \lambda'}{z_2^2 - z_1 \lambda} d\bar{z}_1\big).$$

Differential dz_1 should be of the type (1,0) in order to have a holomorphic fibration $(z_1, z_2) \to z_1$. All what is left is to take the structure such that df is a (1,0)-form. Now the choice for a made in (2.2) becomes clear.
(b) Item (*i)* follows immediately from the fact that J admits two holomorphic functions. Item (*iv)* will be not used here.

Remark that $(W, \pi, \Delta_{1/3}, \Delta_{2/3})$, equipped with the structure J, is a generalized Hartogs figure (π is the natural projection $(z_1, z_2) \to z_1$).
Step 2. Properties of (W, J). Let us list some important properties of this example.

i) Remark that W_{z_1} has the conformal structure of a (pluri-punctured) disc. This follows from the observation that $f(z_1, z_2)$ (z_1 is fixed) is holomorphic (meromorphic at zero) on z_2 with respect to both structures: J and J_{st}.

ii) The most important feature is the following

Proposition 2.1. *For any $\varepsilon > 0$ and any foliated holomorphic imbedding $h(z_1, z_2) = (h_1(z_1), h_2(z_1, z_2))$ of the standard Hartogs figure H_ε into W such that $h_1(0) \in \Delta_{1/3}$ one has $h_1(\Delta_{1+\varepsilon}) \subset \Delta_{1/3}$.*

Proof. Set $\beta(z_1) = \lambda(|h_1(z_1)|^2)$. A disc $\Delta_{z_1} := \{z_1\} \times \Delta_{1+\varepsilon}$ (if $|z_1| < \varepsilon$) or an annulus $A_{z_1} := \{z_1\} \times A_{1-\varepsilon, 1+\varepsilon}$ (if $|z_1| \geqslant \varepsilon$) is mapped by h to the fiber $W_{h_1(z_1)} := \pi^{-1}(h(z_1))$.

The holomorphic mapping $\Phi := (g,f) : W \to \mathbb{C}\times\mathbb{CP}^1$ realizes W as a Riemann domain over $\mathbb{C}\times\mathbb{CP}^1$. The composition $\Phi\circ h$ extends to a bidisc $\Delta^2_{1+\varepsilon}$ by the classical Hartogs-Levi theorem and this extension is a foliated meromorphic immersion. The last is because the zero divisor of the Jacobian (if non-empty) should have intersect H_ε, but there our map is locally biholomorphic.

This is a contradiction, because on each fiber Δ_{z_1}, $1/3 < |z_1| \leqslant 2/3$, the meromorphic function $(f\circ h)(z_1,z_2) = h_2(z_1,z_2) + \frac{h_1(z_1)}{h_2(z_1,z_2)}\lambda(|h(z_1)|^2)$ is a composition of a holomorphic function h_2 and Jukovskiy function $z+\frac{\text{const}}{z}$ and the last is not an immersion.

□

iii) Along the same lines as in the proof of this Proposition one can show that: for any $z_1^0 \in \partial\Delta_{1/3}$ and any $\varepsilon > 0$ there exists ***no imbedding*** of $W|_{\Delta(z_1^0,\varepsilon)}$ into $\mathbb{C}^2$, *i.e.,* there no holomorphic coordinates on $W|_{\Delta(z_1^0,\varepsilon)}$.

2.4. Key lemma. The following statement is the key lemma for this section. It replaces the Lemma 2.3 from [Iv6] for the "unparametric" case of the present paper. In fact it states even more general result that we need in this paper, see Remarks 2.5 and 2.7. But we are including it for the future references. We suppose that our complex manifold X is equipped with some Riemannian metric. The condition (*ii)* in the following lemma, where this metric is used, doesn't depend, in fact, on a particular choice of a metric.

Lemma 2.2. *Let $f : W \to X$ be a generically injective holomorphic map of a trivial Hartogs figure $(W,\pi,\varnothing,V)$ into a complex manifold X. Suppose that $\dim V = 1$ and that for some sequence $z_n \to z_0 \in V$ restrictions $f|_{W_{z_n}} : W_{z_n} \to X$ holomorphically extend as generic injections onto a discs $\widetilde{W}_{z_n}$ after a reparametrization. Suppose additionally that:*

i) $\tilde f|_{\widetilde{W}_{z_n}}(\widetilde{W}_{z_n})$ *stay in some compact of X;*

ii) $\mathrm{area}\left(\tilde f|_{\widetilde{W}_{z_n}}(\widetilde{W}_{z_n})\right)$ *are uniformly bounded.*

Then there exists a neighborhood $D \ni z_0$ such that f extends meromorphically onto a figure $(\widetilde{W},\pi,D,V)$ after a reparametrization. Moreover, the extension $\tilde f$ is a generically injective meromorphic map.

Proof. Writing $\tilde f|_{\widetilde{W}_{z_n}}$ in the statement of this lemma we mean that for every n a reparametrization map $h_{z_n} : \partial\widetilde{W}_{z_n} \to \partial W_{z_n}$ is given such that $\tilde f|_{\partial\widetilde{W}_{z_n}} := f|_{\partial W_{z_n}}\circ h_{z_n}$ extends generically injectively and holomorphically to the disc $\widetilde{W}_{z_n}$. The proof will use in a crucial way the description of convergence of analytic discs obtained in [IS3] and the structures of Banach neighborhoods of stable curves obtained in [IS1].

Set $f_n = f|_{\widetilde{W}_{z_n}}$ and consider them as complex discs over X, parameterized by a fixed disc Σ (see §3 from [IS1] or §2 from [IS3] for exact definitions). Applying Theorem 1 from [IS3] we can find a subsequence from $\{f_n\}$ that converge in the sense of Definition 2.5 from [IS3] to a stable curve f_0 over X, parameterized again by a disc. Be careful, this $f_0(\Sigma)$ may have compact components.

By Theorem 3.4 from [IS1] the space of discs over X which are close to f_0 is a Banach analytic set of finite codimension. Denote it by $\mathcal{C}$. By the Theorem of Ramis, see [Ra], $\mathcal{C}$ is the union of finitely many irreducible components $\mathcal{C}_j$ and each $\mathcal{C}_j$ is a finite ramified covering over a Banach ball. Take a component which contains infinitely many of f_n-s. In order not to complicate our notations we suppose that $\mathcal{C}$ is irreducible itself and contains all f_n.

For the sequel it is important to understand how $\mathcal{C}$ was constructed in [IS1]:

1) The parametrizing disc Σ is covered by finite number of discs, annuli and pants Σ_α (the boundary annulus is one of them, denote it as Σ_{α_0}). This covering has that property that each intersecting pair $\Sigma_\alpha, \Sigma_\beta$ intersect by an annulus denoted as $\Sigma_{\alpha,\beta}$.

2) For each α a Banach manifold H_α of holomorphic maps from Σ_α to X is considered. The same type manifolds $H_{\alpha,\beta}$ of holomorphic maps $\Sigma_{\alpha,\beta} \to X$ for intersecting Σ_α and Σ_β are considered.

3) For every pair of intersecting Σ_α and Σ_β a transition map $\psi_{\alpha,\beta} : H_\alpha \times H_\beta \to H_{\alpha,\beta}$ is defined.

Now $\mathcal{C}$ comes out as the zero set of some "gluing" holomorphic map $\Psi = \{\psi_{\alpha,\beta}\} : \bigcup_\alpha H_\alpha \to \bigcup_\alpha H_{\alpha,\beta}$. By construction $\mathcal{C}$ restricts as a Banach analytic subset to each of H_α.

All what is left to do is to replace H_{α_0} (the manifold of maps from the annulus adjacent to the boundary) by a 1-dimensional manifold $\mathcal{W} := \{f|_{\partial W_z} : z \text{ in a neighborhood of } z_0\}$ (∂W_z stays here for an annulus adjacent to $\partial_0 W_z$). The obtained Banach analytic set, we still denote it as $\mathcal{C}$, is of finite dimension (the proof goes along the same lines as the proof of Lemma 1.1 from [Iv6]). In fact it is clearly of dimension not more than one. But since it contains the sequence $\{f|_{\widetilde{W}_{z_n}}\}$ its dimension is actually one. Therefore $\mathcal{C}$ is a usual analytic set by Barlet-Mazet theorem, [M], *i.e.,* is a complex curve in our case. Restriction $\mathcal{C} \to \mathcal{W}$ is an analytic map and it is *proper* (!), because a nondegenerate analytic maps between complex curves are always proper. Therefore its image is the whole $\mathcal{W}$. We get an extension $\tilde{f}_z$ for all z close to z_0 as a family by a tautological map $\tilde{f} : \tilde{\mathcal{W}} \to X$. Here $\tilde{\mathcal{W}}$ is a tautological family of discs over $\mathcal{W}$.

□

Remark 2.5. (a) An analogous statement can be proved also in the case $\dim V \geq 2$, but then one should require the boundedness of rational cycle geometry of X as in [Iv6] (only cycles tangent to $\mathcal{L}$ are relevant). We shall do this later, see Lemma 2.5.

(b) One can seriously simplify the proof of this lemma if one imposes *ad hoc* the condition that (some subsequence of) the sequence $\{\tilde{f}|_{\widetilde{W}_{z_n}}(\widetilde{W_{z_n}})\}$ converges to f_0 without bubbles. The proof is then almost immediate, since then only one Banach manifold H_0 appears (no Banach analytic sets), that of deformations of f_0 and it has dimension at least one because it contains a converging sequence.

2.5. Two-dimensional case. Recall that a complex manifold X is called disc-convex if for any compact $K \subset X$ the exists a compact $\hat{K}$ in X such that for any holomorphic map $\varphi : \overline{\Delta} \to X$ such that $\varphi(\partial\Delta) \subset K$ one has $\varphi(\overline{\Delta}) \subset \hat{K}$. Let's adapt this notion to the foliation theory:

Definition 2.8. *A complex foliated manifold $(X, \mathcal{L})$ is called disc-convex if for any compact $K \subset X$ the exists a compact $\hat{K}_{\mathcal{L}}$ in X such that for any holomorphic map $\varphi : \overline{\Delta} \to X$ tangent to $\mathcal{L}$ and such that $\varphi(\partial\Delta) \subset K$ one has $\varphi(\overline{\Delta}) \subset \hat{K}_{\mathcal{L}}$.*

A holomorphic mapping $\varphi : \overline{\Delta} \to X$ is called tangent to $\mathcal{L}$ if it takes (almost all) its values in some leaf of $\mathcal{L}$. Note that for disc-convex $(X, \mathcal{L})$ and foliated mappings $f : (W, \pi, \varnothing, V) \to (X, \mathcal{L})$ the condition *(i)* in Lemma 2.2 is satisfied automatically.

Let ω be a $(1,1)$-form on X.

Definition 2.9. *We call ω plurinegative (dd^c-negative) if $dd^c\omega \leq 0$. We call ω pluriclosed (dd^c-closed) if $dd^c\omega = 0$.*

Denote by $\mathcal{E}^{p,p}_{\mathbb{R}}$ the sheaf of smooth real (p,p)-forms and by $\mathcal{E}^{p,g}$ the sheaf of smooth complex valued (p,q)-forms on X. Likewise by $\mathcal{E}_{p,q}$ we denote the dual to $\mathcal{E}^{p,g}$ space of currents of bidimension (p,q) and by $\mathcal{E}^{\mathbb{R}}_{p,p}$ the space of real currents of bidimension (p,p).

Fix some strictly positive $(1,1)$-form Ω on X. Given a holomorphic foliation by curves $\mathcal{L}$ on X define the following convex compact $K_{1,1}(\mathcal{L}) \subset \mathcal{E}^{\mathbb{R}}_{1,1}(X)$. For every point $z \in X^0$ take a $(1,1)$-vector $\frac{i}{2}\mathrm{v}\wedge\bar{\mathrm{v}}$ tangent to $\mathcal{L}^0_z$ such that $< \frac{i}{2}\mathrm{v}\wedge\bar{\mathrm{v}}, \Omega >= 1$.

For $z \in \mathcal{L}^{\mathsf{sing}}$ take any sequence $z_n \to z$, $z_n \in X^0$ and any $\frac{i}{2}\mathrm{v}_n \wedge \bar{\mathrm{v}}_n$ tangent to $\mathcal{L}^0_{z_n}$ such that $< \frac{i}{2}\mathrm{v}_n \wedge \bar{\mathrm{v}}_n, \Omega >= 1$ for all n. Subtract a converging subsequence from $\frac{i}{2}\mathrm{v}_n \wedge \bar{\mathrm{v}}_n$ and denote by $\frac{i}{2}\mathrm{v}_0\wedge\bar{\mathrm{v}}_0$ its limit. In this way we obtain all positive bidimension $(1,1)$ δ-currents tangent to $\mathcal{L}$. $K_{1,1}(\mathcal{L})$ is the closure of the convex hull of these δ-currents.

Definition 2.10. *A $(1,1)$-form ω we call a taming form for $\mathcal{L}$ if $< T,\omega >> 0$ for every $T \in K_{1,1}(\mathcal{L})$.*

The Theorem below plays the role of Lemmas 3.1 from [Iv4] and 2.4 from [Iv6] for the generalized Hartogs figures. The proof closely follows [Iv6].

Theorem 2.1. *Let $(X,\mathcal{L})$ be a disc-convex foliated complex manifold which admits a dd^c-negative taming form ω and let $f : (W,\pi,\varnothing,V) \to (X,\mathcal{L})$ be a generically injective foliated holomorphic mapping from a trivial two-dimensional Hartogs figure W over a disc V into X. Suppose that:*

i) for some sequence $z_n \to z_0 \in V$ restrictions $f_n := f|_{W_{z_n}}$ extend onto discs $\widetilde{W}_{z_n}$ after a reparametrization as generic injections;

ii) $\mathsf{area}\left(\tilde{f}|_{\widetilde{W}_{z_n}}(\widetilde{W}_{z_n})\right)$ are uniformly bounded.

Then mapping f extends after a reparametrization as a generically injective meromorphic map onto a complete Hartogs figure $(\widetilde{W},\pi,V)$ over V minus a closed "essential singularity" set S of the form $S = \bigcup_{z\in S_1} S_z$, where S_1 is a closed complete polar subset of V and S_z for every $z \in S_1$ is a compact in the disc $\widetilde{W}_z$.

Proof. Denote by $\hat{U}$ the maximal open subset of V such that f meromorphically extends onto a Hartogs figure $(W,\pi,\hat{U},V)$ after a reparametrization (tildes are skipped everywhere). By Corollary 2.1 we know that $\hat{U}$ exists and by Lemma 2.2 that $\hat{U} \neq \varnothing$. All we need to prove is that $\partial\hat{U} \cap V$ is a polar set. The question is local with respect to the base V and therefore we fix $z^0_1 \in \partial\hat{U} \cap \Delta$ and suppose that V is a disc around z^0_1. We denote by T the current $f^*\omega$. It is a smooth $(1,1)$-form outside of a discrete set A of points of indeterminacy of f which is dd^c-closed everywhere on W as a current.

Step 1. Laplacian of the area function. For points $z_1 \in \hat{U}$ the following area function is well defined:

$$a(z_1) = \mathsf{area}\left(f|_{W_{z_1}}(W_{z_1})\right) = \int\limits_{W_{z_1}} T, \tag{2.3}$$

here $W_{z_1} = \pi^{-1}(z_1)$. Let ρ be a bump function, equal to $\mathbf{1}$ ion a neighborhood of $\partial_0 W_V$. Fix a disc $D \subset \hat{U}$ and denote by $\pi : \bar{W}_D \to D$ the natural projection. Here witting $\bar{W}_D$ we mean $W_D \cup \partial_0 W_0$. The restriction $T|_{\bar{W}_D}$ we denote still as T. Write

$$dd^c a = dd^c[\pi_* \rho T + \pi_*(1-\rho)T] = dd^c a_\rho + dd^c[\pi_*(1-\rho)T], \tag{2.4}$$

where π_* stands for the push forward operator of the restriction $\pi|_{\bar{W}_D} : \bar{W}_D \to D$. a_ρ denotes here the area function, which corresponds to the form ρT. Remark that $\pi_*(\rho T)$ is well defined and smooth on the whole of V and therefore such is also $dd^c a_\rho$. What concerns the second term in (2.4) it is equal to $-\pi_*[dd^c\rho \wedge T + 2d\rho \wedge d^c T + (\rho - 1)dd^c T]$. As a result we see that

$$dd^c a = dd^c a_\rho - \pi_*[dd^c \rho \wedge T + 2d\rho \wedge d^c T + (\rho - 1)dd^c T]. \tag{2.5}$$

Since T is plurinegative we get that

$$dd^c a \leqslant dd^c a_\rho - \pi_*[dd^c \rho \wedge T + 2d\rho \wedge d^c T] \tag{2.6}$$

Remark that the right hand side of this expression is also well defined and smooth on the whole of V. (2.6) shows, in particular, that the Laplacian of a is bounded from above by a function, denote it as c, which smoothly extends from $\hat{U}$ to V.

Remark 2.6. (a) All computations on this stage were done on $\partial_0 W|_V$ minus a discrete set of points of indeterminacy of f. At points z_1 such that W_{z_1} cuts an indeterminacy point of f the area function a jumps. Usually this small "non-smoothness" of a plays no role in the forthcoming considerations. But, while working with pluriclosed forms it will be convenient to remark that a can be considered to be smooth everywhere due to the following observation. First: a is clearly smooth outside of a discrete set. Second: subtracting from it the Poisson integral of the right hand side of (2.6) (which becomes to be an equality if ω is pluriclosed), we get a bounded harmonic function which smoothly extends through these discrete set. Therefore we can "correct" a on a discrete set of points to make it genuinely smooth.

(b) Alternatively, following [Br5], one can exploit approach from [Iv6] and compute the Laplacian of a in such a way that the resulting expression involves only the boundary integrals.

Step 2. Polarity of $\partial\hat{U} \cap \Delta$. Area function a writes as $a(z_1) = b(z_1) + h(z_1)$, where b is the Poisson integral of c (and therefore is smooth on V), and $h := a - b$ has nonpositive Laplacian, *i.e.*, is superharmonic in $\hat{U}$. In addition h is bounded from below on $\hat{U}$. All what is left to do is to remark that $a(z_1)$ and therefore $h(z_1)$ tend to $+\infty$ when $z_1 \to \partial\hat{U} \cap V$ by Lemma 2.2. Therefore we are in the conditions of the proof of Lemma 3.1 from [Iv4] or Lemma 2.4 from [Iv6]. Indeed $\partial\hat{U} \cap V$ occurs to be a $+\infty$ set for a superharmonic in $\hat{U}$ function h. But then setting

$$h_n(z_1) = \min\{n, h(z_1)\}$$

we get an increasing sequence of superharmonic function *in the whole of* V (!), which therefore converges to a superharmonic function on V, equal to h on $\hat{U}$ and to $+\infty$ on $\partial\hat{U} \cap V$. One concludes that $\partial\hat{U} \cap V$ is complete polar in V (see Lemma 2.4 from [Iv6] for more details about this) and one sets $S_1 := \partial\hat{U} \cap V$.

□

Remark 2.7. One can modify the proof of Theorem 2.1 along the lines of [Iv3] and then the Remark 2.5 (b) would be sufficient. But anyway, this doesn't make the proof shorter and we shall not do that.

2.6. Condition $\int d^cT=0$ and foliated shells. In this subsection we suppose the timing form ω is pluriclosed and that f is already extended onto $W\setminus S$, see Theorem 2.1 (tildes are skipped everywhere). The polar set $\partial\hat{U}\cap V$ we had denote as S_1. Therefore the "essential singularity" set S of the extended map is actually $S=\bigcup_{z\in S_1}S_z$. Now we shall see how this leads to a foliated shell. We suppose therefore that S is not empty. Take a point $s_0\in S_1$ and let V be a small disc around s_0. Shrinking W_V if necessary we can suppose that W_V is a bidisc Δ^2 in $\mathbb{C}^2$ with coordinates z_1,z_2. Write $T=it^{\alpha\bar\beta}dz_\alpha dz_{\bar\beta}$ for $T:=f^*\omega$.

Lemma 2.3. *Suppose that the timing form ω is pluriclosed and that for a relatively compact disc $D\Subset V$ such that $\partial D\cap S_1=\varnothing$ one has*

$$\int\limits_{\partial W|_D} d^cT=0. \tag{2.7}$$

Then f meromorphically extends onto W_D.

Proof. We know already that dd^ca smoothly extends onto D (and a is positive!). Denote by $\widetilde{dd^ca}$ this extension. Again supposing that W_D is a bidisc we can compute the integral of d^cT over components ∂_0W_D and $W_{\partial D}$ of the boundary ∂W_D of W_D, and using the expression (2.2.5) from [Iv6] for the Laplacian dd^ca, get:

$$2\int\limits_{\partial_0W_D} d^cT=-\int\limits_D\widetilde{dd^ca}+\frac{1}{4\pi}\int\limits_{\partial_0W_D}\left(\frac{\partial t^{1,\bar1}}{\partial z_2}dz_2-\frac{\partial t^{1,\bar1}}{\partial\bar z_2}d\bar z_2\right)\wedge dz_1\wedge d\bar z_1 \tag{2.8}$$

and

$$\int\limits_{W_{\partial D}} d^cT=\int_{\partial D}d^ca+\frac{1}{4\pi}\int\limits_{\partial_0W_{\partial D}} t^{1\bar2}dz_1\wedge d\bar z_2-t^{2\bar1}dz_2\wedge d\bar z_1. \tag{2.9}$$

The second term in the right hand side of (2.9) is the integration over $\partial_0W_{\partial D}$ as the boundary of $W_{\partial D}$. Considering it as the boundary of ∂_0W_D (and thus changing its orientation) and applying Stokes we get after summing up (2.8) with (2.9) that

$$2\int\limits_{\partial_0W_D}d^cT+\int\limits_{W_{\partial D}}d^cT=\int_{\partial D}d^ca-\int\limits_D\widetilde{dd^ca}+\int\limits_{\partial_0W_D}d^cT, \tag{2.10}$$

i.e., that

$$\int\limits_{\partial W_D}d^cT=\int_{\partial D}d^ca-\int\limits_D\widetilde{dd^ca}. \tag{2.11}$$

Therefore for the negative measure $\mu_h:=dd^ch$ supported on S_1 we obtain (using smoothing by convolutions) the identity:

$$\mu_h(S_1\cap D)=\int\limits_D dd^ch=\int\limits_{\partial D}d^ca-\int\limits_D\widetilde{dd^ca}=\int\limits_{\partial W_D}d^cT. \tag{2.12}$$

The assumption of our Lemma mean now that $\mu_h(S_1\cap D)=0$, *i.e.*, that h is harmonic. Therefore a is smooth and Lemma 2.2 implies now the extendibility of f onto W_D.

□

We conclude with the following

Corollary 2.2. *If in the conditions of Theorem 2.1 the mapping f is additionally supposed to be an immersion, taming form ω to be pluriclosed, and the singularity set S of the extended mapping is non-empty then $(X,\mathcal{L})$ contains a foliated shell.*

Really, the extended mapping $\tilde f:\widetilde{W}\setminus S\to X$ might fail to be an immersion in this case only on a discrete subset of $\widetilde{W}\setminus (S\cup I_{\tilde f})$. We add this subset to S together with indeterminacy set $I_{\tilde f}$ of $\tilde f$ to get $\tilde S$. The projection $\tilde S_1=\pi(\tilde S)$ will stay complete polar. Let $0\in S$. Take a small disc Δ_r around 0 in such a way that $\partial\Delta_r\subset V\setminus \tilde S_1$ and such that $\widetilde{W}|_{\Delta_r}$ is a bidisc (after a slight shrinking of its outer boundary $\partial_0\widetilde{W}|_{\Delta_r}$), *i.e.*, $\widetilde{W}|_{\Delta_r}=\Delta_r\times\Delta$ as foliated manifolds. In $\widetilde{W}|_{\Delta_r}$ take a bidisc $P=\Delta_r\times\Delta_{1-\varepsilon}$ for $\varepsilon>0$ small enough to insure the immersivity of $\tilde f$ near the boundary B of P. Lemma 2.3 says now that $\int_{\tilde f(B)}\omega\neq 0$, *i.e.*, we got a foliated shell.

Remark that we also proved the Proposition 1 from the Introduction:

Corollary 2.3. *A generic holomorphic injection $h:(B^{\varepsilon},\mathcal{L}^{\mathrm{v}})\to (X^0,\mathcal{L}^0,\omega)$ into a disc-convex pluritamed foliated manifold defines a foliated shell if and only if it is an immersion and*

$$\int_{h(B)}d^c\omega\neq 0. \tag{2.13}$$

Finally let us proof that

Lemma 2.4. *If the timing form ω is pluriclosed then the set S_1 is at most countable.*

Proof. We use the notations of the proof of Lemma 2.3, *i.e.*, h denotes the superharmonic extension of $a-b$ to $\Delta=V$. Consider the following representation of $\pi_1(\Delta\setminus S_1)$ in $\mathbb{R}$:

$$R_1:\pi_1(\Delta\setminus S_1)\ni\gamma\to\int_{\gamma}d^ch. \tag{2.14}$$

We can always suppose that γ has only normal crossings and denote by Γ its interior, *i.e.*, a finite number of discs. By (2.12) we have that

$$\int_{\gamma}d^ch=\int_{\partial W_{\Gamma}}d^cT=\int_{f(\partial W_{\Gamma})}d^c\omega.$$

Therefore the image of the representation R_1 is contained in the image of the representation

$$R_2:H_3(X,\mathbb{Z})\ni M\to\int_{M}d^c\omega, \tag{2.15}$$

and the last is countable. Therefore $\mathsf{Im}\,R_1$ is at most countable. Would S_1 be uncountable one would be able to find a nontrivial $\gamma\in\pi_1(\Delta\setminus S_1)$ such that

$$\int_{\gamma}d^ch=0 \text{ and therefore } \int_{\Gamma}dd^ch=0.$$

But then for any component D of Γ we would have

$$\int_{\partial W_D} d^cT = \int_D dd^ch = 0,$$

and this by Lemma 2.3 implies that f extends to W_D, *i.e.*, that $S_1 \cap D = \varnothing$. Since this is true for every component of Γ we get that γ is trivial in $\pi_1(\Delta \setminus S_1)$. Contradiction.

□

2.7. Nonparametric extension in all dimensions. Let $n \geq 1$ be the dimension of the base V. In the following formulations tildes are skipped everywhere.

Theorem 2.2. *Let $(X,\mathcal{L})$ be a disc-convex foliated manifold admitting a plurinegative taming form ω and let $f : (W,\pi,U,V) \to (X,\mathcal{L})$ be a generically injective foliated holomorphic map from a non-trivial Hartogs figure (i.e., $U \neq \varnothing$) into X. Then f extends after a reparametrization to a foliated meromorphic map $\tilde{f} : (\widetilde{W},\pi,V) \setminus S \to X$ of a complete Hartogs figure minus a closed subset S of the form $S = \cup_{z\in S_1} S_z$, where:*

(a_1) S_1 is a complete $(n-1)$-polar subset of V of Hausdorff dimension $2n-2$.

(a_2) S_z is a compact in the disc $\widetilde{W}_z$ for every $z \in S_1$.

(a_3) If $\mathsf{dim}\, W = \mathsf{dim}\, X$ and f was an immersion then the extended map $\tilde{f}$ is a meromorphic immersion.

Complete $(n-1)$-polarity of S_1 means that every point $0 \in S_1$ admits a neighborhood $\Delta^n = \Delta^{n-1} \times \Delta$ with coordinates (λ, z_1) such that for every λ the disc $\Delta_\lambda := \{\lambda\} \times \Delta$ intersects S_1 by a complete polar compact set. Hausdorff zero-dimensionality of $\Delta_\lambda \cap S_1$ follows. For the purposes of this paper we will need to know more about the behavior of the extended map $\tilde{f}$ near the essential singularity set S. Supposing that S is nonempty take a point $s_0 \in S$ and find a coordinate n-disc $\Delta^n \ni 0 = \pi(s_0) \in S_1$ and a neighborhood P of s_0 in W biholomorphic to $\Delta^n \times \Delta$ such that $\pi|_P : P \to \Delta^n$ is the natural vertical projection $\Delta^n \times \Delta \to \Delta^n$. In what follows z_2 will denote the coordinate along the fiber of π.

Theorem 2.3. *Under the conditions of Theorem 2.2 suppose additionally that the taming form ω is pluriclosed and that the singular set S is nonempty. Then S has the following structure:*

(b_1) The coordinate polydisc Δ^n as above can be presented as $\Delta^n = \Delta^{n-1} \times \Delta$ with coordinates (λ, z_1) in such a way that the restriction to $S \cap \Delta^{n+1}$ of an another vertical projection $\pi_1 : \Delta^{n+1} \to \Delta^{n-1}$, i.e., of $(\lambda, z_1, z_2) \to \lambda$ is proper and surjective. In another words for every $\lambda \in \Delta^{n-1}$ the intersection $S_\lambda := \Delta^2_\lambda \cap S$ is nonempty.

(b_2) For every $\lambda \in \Delta^{n-1}$ let $B_\lambda = \partial\Delta^2_\lambda$. Then $\tilde{f}(\partial B_\lambda))$ is not homologous to zero in X, i.e., it is a foliated shell in $(X,\mathcal{L})$, provided that f was in addition an immersion.

(b_3) Denote by $\pi_2 : \Delta^n \to \Delta^{n-1}$ the natural projection $(\lambda, z_1) \to \lambda$. Then for every λ the set $S_{1,\lambda} := \pi_2^{-1}(\lambda) \cap S_1$ is a most countable.

In all applications/formulations S will be supposed to be minimal closed such that f extends onto $\widetilde{W} \setminus S$. The proof is not a direct generalization of the two-dimensional case. First of all we need to introduce one object relevant to a complex foliated manifold $(X,\mathcal{L})$. Denote by $\mathcal{R}_{\mathcal{L}}$ the analytic space of rational cycles on X tangent to $\mathcal{L}$. Recall that a rational cycle is a finite linear combination of rational curves with integer coefficients:

$C=\sum_j n_jC_j$. Here each C_j is a rational curve in X. We fix a Hermitian metric on X and denote by ω its associated (1,1)-form. The area of C is defined as

$$\mathrm{v}_\omega(\mathcal{C})=\sum_j n_j\int_{C_j}\omega. \tag{2.16}$$

Definition 2.11. *Let us say that $(X,\mathcal{L})$ has unbounded rational cycle geometry if there exists a path $\gamma:[0,1[\to\mathcal{R}_{\mathcal{L}}$ such that*

1) $C_{\gamma(t)}$ stays in some compact K of X for all $t\in[0,1[$;

2) $\mathrm{v}_\omega(C_{\gamma(t)})\to+\infty$ when $t\nearrow 1$.

Here $C_{\gamma(t)}$ is the rational cycle in X corresponding to the point $\gamma(t)\in\mathcal{R}_{\mathcal{L}}$. This notion doesn't depend on the particular choice of ω and represents from itself a pure complex-geometric property of $(X,\mathcal{L})$.

In Lemma 6.1 we shall prove that if $(X,\mathcal{L})$ admits a plurinegative taming form then the rational cycle geometry of $(X,\mathcal{L})$ is bounded. Recall finally, that a subset A of a complex manifold V is said to be *thick at* the point $z^0\in V$ if for any neighborhood U of z^0 $A\cap U$ is not contained in a proper analytic subset of U. Now we can state the needed lemma:

Lemma 2.5. *Let $f:W\to X$ be a generically injective foliated holomorphic mapping of a trivial Hartogs figure $(W,\pi,\varnothing,V)$ into a complex foliated manifold $(X,\mathcal{L})$. Suppose that $\mathsf{dim}\,V\geqslant 2$ and that for all z in some subset $A\subset V$ thick at z^0 all restrictions $f|_{W_z}:W_z\to X$ holomorphically extend onto discs $\widetilde{W}_z$ after a reparametrization. Suppose additionally that:*

i) $\mathsf{area}\left(\tilde f|_{\widetilde{W}_z}(\widetilde{W}_z)\right)$ is uniformly bounded for $z\in A$;

ii) $\tilde f|_{\widetilde{W}_z}(\widetilde{W}_z)$ stay in some compact K of X;

iii) $(X,\mathcal{L})$ has bounded rational cycle geometry.

Then there exists a neighborhood $D\ni z^0$ such that f extends meromorphically onto a complete Hartogs figure $(\widetilde{W},\pi,D)$ after a reparametrization.

Proof. We keep the notations used in the proof of Lemma 2.2. Only for the annulus Σ_{α_0} adjacent to the boundary of the disc Σ the manifold $\mathcal{W}:=\{f|_{\widetilde{W}_z}: z$ in a neighborhood of $z^0\}$ now has dimension $n=\mathsf{dim}\,V\geqslant 2$.

Let $\nu>0$ be the minimum of areas of rational curves tangent to $\mathcal{L}$ and contained in the compact K. We divide A into a finite union of increasing closed subsets: $A_1\subset\dots\subset A_k\subset\dots\subset A_K=A$ where $A_k=\{z\in A:\mathsf{area}_\omega\big(\tilde f|_{\widetilde{W}_z}\big)\leqslant k\frac{\nu}{2}\}$. For some k the set $A_k\setminus A_{k-1}$ is thick at origin. In the sequel we take it as A. As a result every converging sequence $\{\tilde f|_{\widetilde{W}_{z_n}}: z_n\in A, z_n\to z^0\}$ has the same limit. Really two different limits should differ by a rational cycle. Therefore their areas should differ at least by ν. Contradiction.

The Banach analytic set $\mathcal{C}$ obtained literally as in the proof of Lemma 2.2 is again finite dimensional. But the problem is that the restriction map $r:\mathcal{C}\to\mathcal{W}$ might be not proper. That mean that for some z close to z^0 the preimage $r^{-1}(\tilde f|_{\widetilde{W}_z})$ is not compact. But a cycle in this preimage is different from $\tilde f|_{\widetilde{W}_z}$ itself by a rational cycle tangent to $\mathcal{L}$. Therefore we got a contradiction with the boundedness of rational cycle geometry condition (iii) of this Lemma. So r is proper and by Remmert proper mapping theorem $r(\mathcal{C})$ is an analytic set in $\mathcal{W}$. Since it contains a thick subset it is the whole $\mathcal{W}$. We again get extension of all f_z for z close to z^0 as a family by a tautological map $\tilde f:\tilde{\mathcal{W}}\to X$.

□

Remark 2.8. Note that in this Lemma no taming conditions on $(X,\mathcal{L})$ need to be imposed (it is weakened, in fact, to a boundedness of the rational cycle geometry).

Proof of Theorems 2.2 and 2.3. Let again $\hat{U}$ be the maximal open subset of V such that f extends onto $(W,\pi,\hat{U})$ after a reparametrization.

Now we can proceed exactly as in the proof of Steps 1 and 2 on the page s 817-818 of [Iv6]. Our present situation is even somewhat simpler because f is holomorphic on the Hartogs figure. Reparametrizations do not cause any additional difficulties. In this way we get we get that $S_1 := \partial\hat{U}\cap V$ is of Hausdorff dimension $2n-2$. By the "rationalization trick" we extend our map f onto a complete Hartogs figure (W,π,V) over V minus the closed set of the form $S=\bigcup_{(\lambda,z_1)\in S_1}S_{\lambda,z_1}$ with S_{λ,z_1} being compact subsets of the discs Δ_{λ,z_1}. Due to our localization Lemma 2.1 we need to work in a neighborhood of a point $(\lambda,z_1)\in S_1$ only. Lemma 2.5 shows that for every natural N the set $\{z\in V : f|_{W_z}$ extends onto a disc $\widetilde{W}_z$ and $\mathsf{area}(\tilde{f}|_{\widetilde{W}_z})\leqslant N\}$ is thin in a neighborhood of S_1. The rest is obvious. In particular, one gets that for any two-dimensional submanifold $U\subset V$ the domain $U\setminus(S_1\cap U)$ is the maximal domain over which the restricted map $f|_{W|_U}$ extends after a reparametrization.

Starting from this point no further reparametrizations are needed. Therefore the proof of $(a_1),(a_2)$ and (b_1) is done.

(a_3) is clear, because $\tilde{f}$ could fail to be immersion only along a divisor, which should then intersect $\widetilde{W}_U$. But this is not the case.

(b_2) is exactly the Corollary 2.2 from the preceding subsection. Remark that this item easily implies the following:

Corollary 2.4. *In the conditions (and notations) of Theorem 2.3 denote by S_λ^0 the minimal closed subset of Δ_λ^2 such that the restriction $f_\lambda := f|_{\Delta_\lambda^2}$ extends onto $\Delta_\lambda^2\setminus S_\lambda^0$. Then $S_\lambda^0=S_\lambda$.*

Item (b_3) follows from Lemma 2.4. Theorems 2.2 and 2.3 are proved.

□

Let's repeat once more that S in Theorem 2.3 is always understood as being the minimal closed subset that f meromorphically extends to its complement.

Suppose now that a polydisc $P=\Delta^n\times\Delta_{1+\varepsilon}$ is fixed, a closed subset $S\subset P$ of the form $S=\bigcup_{z\in S_1}S_z$ in P is given, where S_z is a compact subset of the leaf $P_z:=\{z\}\times\Delta_{1+\varepsilon}$ for every $z\in S_1$. Suppose that $S_1\ni 0$ and that 0 is an accumulation point for $\Delta^n\setminus S_1$. Finally, let a meromorphic foliated generic injection $f:(P\setminus S,\pi)\to(X,\mathcal{L})$ into a disc-convex foliated manifold be given. We shall make use of the following:

Corollary 2.5. *For a fixed constant $C>0$ let A_C denote a set of z in a neighborhood of $0\in\Delta^n$ such that $(\lambda,z_1)\notin S_1$ and that*

$$\int_{\Delta_{P_z}} f^*\omega\leqslant C. \tag{2.17}$$

If $(X,\mathcal{L})$ has bounded rational cycle geometry (ex. admits a plurinegative taming form) then A_C is contained in a germ of a proper analytic subset of Δ^n at 0.

The proof follows immediately from Lemmas 2.5 and 6.1.

Remark 2.9. (a) The Hartogs type extension theorems for meromorphic maps with values in manifolds carrying pluriclosed metric forms were proved in [Iv4] and [Iv6]. The Kähler case was previously done in [Iv3].

(b) "Unparametric" versions of these results are due to Brunella, see [Br3] and [Br5].

(c) Here we prove such type of results for the forms positive only along a given foliation, which represents additional difficulties. One of them is that the singularity set S can be "massive" along the z_2-direction (we are using notations of Theorem 2.3). Another is the less obvious appearance of shells in Lemma 2.3.

(d) Example of Subsection 2.3 has all necessary features to appear in the proofs of this paper. It is a generalized Hartogs figure and, moreover, it admits a holomorphic foliated immersion, namely $(g,f):W\to\mathbb{C}\times\mathbb{CP}^1$, into an algebraic (!) surface. Therefore the Hartogs figure (W,π,U,V) of this example could be well an open subset of some holonomy covering cylinder $\hat{\mathcal{L}}^0_D$ for some foliation $\mathcal{L}$ and, say, $\gamma_0=\{\frac{2}{3}\}\times\partial\Delta$ could be a vanishing cycle.

(e) It is instructive to see how the mapping $(g,f):W\to\mathbb{C}\times\mathbb{CP}^1$ in this example extends to a complete Hartogs figure after a reparametrization. It is almost tautological: remark that $f(z_1,\cdot)$ is an imbedding into $\{z_1\}\times\mathbb{C}\subset\mathbb{C}\times\mathbb{CP}^1$ near $\{z_1\}\times\partial\Delta$ and therefore $\Phi=(g,f):\partial_0W\to\partial_0\widetilde{W}$ is a biholomorphism for an obvious $\widetilde{W}\subset\mathbb{C}\times\mathbb{C}$. Therefore Φ^{-1} will be a reparametrization.

3. Vanishing Cycles, Covering Cylinders and Foliated Shells.

3.1. Vanishing ends and holonomy covering cylinders. To the classical definitions of holonomy coverings, eg. [Iy1, Iy2, Sz], we will further employ a subtle extension of [Br3, Br4] to take account of "removable singularities". Let $(X,\mathcal{L})$ be a foliated manifold. We use the notations introduced at the beginning of the Section 2. Take a point $z^0\in X^0$ and denote by $\mathcal{L}^0_{z^0}$ the leaf of $\mathcal{L}^0$ passing through z^0. Recall that a parabolic end of $\mathcal{L}^0_{z^0}$ is a closed subset $E\subset\mathcal{L}^0_{z^0}$ which is biholomorphic to the closed punctured disc $\bar{\Delta}^*=\{\zeta\in\mathbb{C}:0<|\zeta|\leq 1\}$. By ∂E we shall denote the biholomorphic image of the circle $\{|\zeta|=1\}$ - the outer boundary of the end E. Foliation $\mathcal{L}$ may have a nontrivial holonomy along ∂E, which can be finite or infinite.

Consider the case when holonomy is finite. Recall what does that mean. Take a transversal D through z^0, *i.e.,* a complex hypersurface in X^0 which is everywhere transverse to the leaves of $\mathcal{L}$. Transversalis will be always taken small enough, in particular, we shall always suppose that $D\subset X^0$ and that D is transversal to $\mathcal{L}^0$ "up to its boundary ∂D". Take a path γ_{z^0} on $\mathcal{L}^0_{z^0}$ which goes from z^0 to some point $q\in\partial E$, then goes one time around ∂E and goes back to z^0. If one takes a point $z\in D$ close to z and travels on $\mathcal{L}^0_z$ along the path γ_z close to γ_{z^0} then one certainly hits D in a neighborhood of z^0 by a point $g(z)$. This defines a local biholomorphism $g:(D,z^0)\to(D,z^0)$ which generates a subgroup $<g>$ of the group $Bihol(D,z^0)$ of local biholomorphisms of D fixing z^0. We suppose that $<g>$ is finite, *i.e.,* $g^d=\mathsf{Id}$ for some $d\geq 1$ and this d is always taken to be the minimal satisfying this property. This d is called the order of the holonomy of $\mathcal{L}$ along ∂E.

Lemma 3.1. *Let $E\subset\mathcal{L}^0_{z^0}$ be a parabolic end. Then for a sufficiently small $\varepsilon>0$ there exists a foliated holomorphic immersion $f:\Delta^n\times A_{1-\varepsilon,1+\varepsilon}\to\mathcal{L}^0_D$ such that:*

i) $f(\{0\}\times A_{1-\varepsilon,1+\varepsilon})\subset \mathcal{L}^0_{z^0}$ and the restriction $f|_{\{0\}\times A_{1-\varepsilon,1+\varepsilon}} : \{0\}\times A_{1-\varepsilon,1+\varepsilon}\to \mathcal{L}^0_{z^0}$ is a regular covering of order d (i.e., covers d-times some imbedded annulus in $\mathcal{L}^0_{z^0}$ and $f(\{0\}\times\partial\Delta)=d\cdot\partial E$).

ii) For all $z\in\Delta^n$ outside of a proper analytic subset $A\subset\Delta^n$ the restriction $f|_{\{z\}\times A_{1-\varepsilon,1+\varepsilon}}$: $\{z\}\times A_{1-\varepsilon,1+\varepsilon}\to\mathcal{L}_z$ is an imbedding.

Proof. It will be convenient to move the point z^0 (and the transversal D) to ∂E. Take an annulus A_0 on $\mathcal{L}^0_{z^0}$ around ∂E. Let $g\in Bihol(z^0,D)$ generates the holonomy of $\mathcal{L}$ along ∂E as above. Denote by A the germ of a proper analytic subset of D at z^0 which consists from those $z\in D$ that the orbit of the corresponding holonomy has cardinality $l>1$ (l necessarily divides d). When one travels from $z\in D$ to z in the leaf $\mathcal{L}^0_z$ along a curve close to ∂E one cuts an imbedded annulus A_z on $\mathcal{L}^0_z$. For z in the exceptional set A one sweeps A_z d/l times (if the orbit of z has cardinality l), for z outside from A only once. The union $W=\bigcup_{z\in D}A_z$ has a natural structure of a complex manifold and possesses a natural foliated holomorphic immersion $f: D\times A_{1-\varepsilon,1+\varepsilon}\to W$ coming from the construction, which is a generic injection (all this provided D is shrinked to a small neighborhood of z^0). f sends each annulus $\{z\}\times A_{1-\varepsilon,1+\varepsilon}$ onto the corresponding A_z with corresponding multiplicity. For that one might need to shrink D and annuli A_z for $z\in D$ once more. Now we can suppose that D is biholomorphic to Δ^n. The rest is obvious.

□

As we remarked in the proof our f is a *generic injection* of the trivial Hartogs figure $\Delta^n\times A_{1-\varepsilon,1+\varepsilon}$ over a polydisc in the sense of Definition 2.4 and results of the previous section are applicable to such f.

Definition 3.1. *A parabolic end E is called a* vanishing end *of order d if:*

i) the holonomy of $\mathcal{L}$ along ∂E is finite of order $d\geq 1$;

ii) the generic injection $f:\Delta^n\times A_{1-\varepsilon,1+\varepsilon}\to\mathcal{L}^0_D$, constructed above, extends as a foliated meromorphic immersion $\tilde f:\widetilde W\to X$ from a complete Hartogs figure $(\widetilde W,\pi,\Delta^n)$ over Δ^n to X after a reparametrization;

iii) the intersection of $\widetilde W_0:=\pi^{-1}(0)$ with the set of points of indeterminacy $I_{\tilde f}$ of $\tilde f$ consists of a single point $a\in\widetilde W_0$.

The point $q=\tilde f|_{\widetilde W_0}(a)$ will be called the *endpoint* of the vanishing end E (or of the leaf $\mathcal{L}^0_z$). Following Brunella, see [Br3], we add all vanishing endpoints to the leaf $\mathcal{L}^0_{z^0}$ and call the curve obtained a *completed leaf* through z^0. Completed leaf will be denoted as $\mathcal{L}_{z^0}$.

Remark 3.1. Let us give two very simple examples explaining this notion.

1. Consider the radial foliation in $\mathbb{C}^2$, *i.e.,* $\mathcal{L}_c=\{z_2/z_1=c\}$ for $c\in\mathbb{CP}^1$. The origin of $\mathbb{C}^2$ is a parabolic end for every leaf $\mathcal{L}^0_c$. But it is never a vanishing end! Really, one cannot construct a foliated meromorphic immersion as in Definition 3.1 in this case. Any $\tilde f$ will contract some complex curve to a point.

2. Let $\mathcal{L}^v$ be the vertical foliation in $\mathbb{C}^2$, *i.e.,* $\mathcal{L}_c=\{z_1=c\}$ for $c\in\mathbb{C}$. Blow-up the origin $\pi:\hat{\mathbb{C}}^2\to\mathbb{C}^2$ and lift our foliation to $\hat{\mathbb{C}}^2$. The leaf $\mathcal{L}^0_0$ has now a parabolic end at its point of intersection with the exceptional divisor and this end is a vanishing end. The role of $f=\tilde f$ plays π^{-1}.

3. Let D be a transversal through z and let E be a vanishing end of $\mathcal{L}^0_z$. Remark that for points z' close to z on D only those ones which belong to some proper analytic subset could be such that $\mathcal{L}^0_{z'}$ has a vanishing end E' with $\partial E'$ close to E. Really, such z' should

lie in the projection $\tilde{\pi}:\widetilde{W}\to D$ of a point of indeterminacy of $\tilde{f}$. Therefore a generic leaf of $\mathcal{L}$ has no vanishing ends at all.

4. If the holonomy along ∂E is infinite then E is never a vanishing end by definition.

Remark 3.2. Example, which we discussed in the previous Section, showed the necessity of modifying the notion of a vanishing end, see discussion before the Theorem 3.1 in [Br4]. The necessary modification was undertaken in [Br3] and it is this notion which we use along this paper.

For each $z\in D$ take a holonomy cover $\hat{\mathcal{L}}_z^0$ of the leaf $\mathcal{L}_z^0$. Recall that a holonomy cover of $\mathcal{L}_z^0$ is a cover with respect to the holonomy subgroup $\mathsf{Hol}(z,\mathcal{L}_z^0)$ of the fundamental group $\pi(z,\mathcal{L}_z^0)$. That means that in the construction of $\hat{\mathcal{L}}_z^0$ two pathes γ_1,γ_2 from z to some $w\in\mathcal{L}_z^0$ define the same point of $\hat{\mathcal{L}}_z^0$ if and only if $\gamma_1\circ\gamma_2^{-1}\in\mathsf{Hol}(z,\mathcal{L}_z^0)$, *i.e.*, if the holonomy along $\gamma_1\circ\gamma_2^{-1}$ is trivial.

Set

$$\hat{\mathcal{L}}_D^0=\bigcup_{z\in D}\hat{\mathcal{L}}_z^0. \tag{3.1}$$

This set (introduced by Suzuki in [Sz] under the name of "tube normaux") has the natural structure of a complex manifold together with the natural projection $\pi:\hat{\mathcal{L}}_D^0\to D$ which sends $\hat{\mathcal{L}}_z^0$ to z. It admits also the natural locally biholomorphic foliated map $p:\hat{\mathcal{L}}_D^0\to\mathcal{L}_D^0\subset X^0$ which sends $\hat{\mathcal{L}}_z^0$ to $\mathcal{L}_z^0$ with $p|_{\hat{\mathcal{L}}_z^0}:\hat{\mathcal{L}}_z^0\to\mathcal{L}_z^0$ being the canonical holonomy covering map. Call $\hat{\mathcal{L}}_D^0$ the *holonomy covering cylinder* of $\mathcal{L}$ over D.

Vanishing ends of $\hat{\mathcal{L}}_z^0$ are defined similarly to that of $\mathcal{L}_z^0$. Let E be a parabolic end of $\hat{\mathcal{L}}_{z^0}^0$ Take $f:\Delta^n\times A_{1-\varepsilon,1+\varepsilon}\to\hat{\mathcal{L}}_D^0$ such that:

i) $f:\Delta^n\times A_{1-\varepsilon,1+\varepsilon}\to\hat{\mathcal{L}}_D^0$ is an imbedding;
ii) $f(\{0\}\times\partial\Delta)=\partial E$ (note that $d=1$ in this case).

The only difference that now f takes values in $\hat{\mathcal{L}}_D^0$ and f is an imbedding. The last is because the holonomy of the foliation $\hat{\mathcal{L}}^0$ on $\hat{\mathcal{L}}_D^0$ is trivial.

Definition 3.2. *E is called a vanishing end of $\hat{\mathcal{L}}_{z^0}^0$ if $h=p\circ f$ extends to a meromorphic foliated immersion $\tilde{h}:\widetilde{W}\to X$ after a reparametrization (not f itself as in Definition 3.1) and $\widetilde{W}_0$ intersects the indeterminacy set $I_{\tilde{h}}$ of $\tilde{h}$ by exactly one point.*

The union of $\hat{\mathcal{L}}_z^0$ with all its vanishing endpoints equipped with an obvious complex structure will be denoted as $\hat{\mathcal{L}}_z$. We shall call it also *a completed holonomy covering leaf* of the leaf $\mathcal{L}_z^0$. Set $\hat{\mathcal{L}}_D:=\bigcup_{z\in D}\hat{\mathcal{L}}_z$ and call it the *completed holonomy covering cylinder* over D. Now let us bring together the principal properties of $\hat{\mathcal{L}}_D$, which will be repeatedly used along this paper.

Lemma 3.2. *i) The completed holonomy covering cylinder possesses the natural structure of a foliated complex manifold with foliation given by the natural projection $\pi:\hat{\mathcal{L}}_D\to D$ defined as above by $\pi(\hat{\mathcal{L}}_z)=z$.*

ii) The natural foliated holomorphic immersion $p:\hat{\mathcal{L}}_D^0\to\mathcal{L}_D^0$ extends to a meromorphic foliated immersion $p:\hat{\mathcal{L}}_D\to X$ and its restrictions $p|_{\hat{\mathcal{L}}_z}:\hat{\mathcal{L}}_z\to\mathcal{L}_z$ are ramified at vanishing ends.

Proof. (*i*) Cylinder $\hat{\mathcal{L}}_D^0$ has a natural complex structure. Therefore we need to add vanishing ends to some leaves and extend this structure to a neighborhood of each added end. Take a vanishing endpoint $a \in \hat{\mathcal{L}}_{z^0}$. Let $f : \Delta^n \times A_{1-\varepsilon,1+\varepsilon} \to \hat{\mathcal{L}}_D^0$ be an imbedding from the Definition 3.2 with $h = p \circ f$ already extended to a meromorphic foliated immersion of $(\Delta^n \times \Delta_{1+\varepsilon}, \mathcal{L}^{\mathrm{v}})$ into $(X, \mathcal{L})$. Let I_h be the indeterminacy set of h. For $z \notin A := \pi(I_h)$ the restriction $h|_{\{z\}\times\Delta_{1+\varepsilon}} : \{z\} \times \Delta_{1+\varepsilon} \to X$ is an imbedding and therefore so is also the $f|_{\{z\}\times\Delta_{1+\varepsilon}} : \{z\} \times \Delta_{1+\varepsilon} \to \hat{\mathcal{L}}_D^0$. This implies that f is an imbedding on $\Delta^n \setminus (A \times \Delta)$. This immediately implies that f is an imbedding on $\Delta^n \setminus I_h$. Therefore we can complete $\hat{\mathcal{L}}_D^0$ by I_h over the image $(\pi \circ f)(\Delta^n) \subset D$. This defines the structure of a complex manifold on $\hat{\mathcal{L}}_D$. The rest is obvious.

(*ii*) This item follows readily from the construction above.

□

Remark 3.3. Let us make a remark which will be important for the future. The covering $p_{z^0} : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an orbifold covering. That means that its ramification index at point a depends only on $b := p_{z^0}(a)$. This is also an unbounded covering in the sense that for every a there exists a disc-neighborhood $V \ni b$ such that $p_{z^0}^{-1}(V)$ is a disjoint union of discs W_j with centers a_j - preimages of b, such that every restriction $p_{z^0}|_{W_j} : W_j \to V$ is a proper covering ramified over b.

3.2. Vanishing ends in dimension 3 vs dimension 2: an example. We want to exploit the example constructed in [Iv5]. This example of a compact (!) complex threefold X has the following very strange features:

i) First: for every domain $D \subset \mathbb{C}^2$ every meromorphic map $f : D \to X$ extends to a meromorphic map $\hat{f} : \hat{D} \to X$ of the envelope of holomorphy $\hat{D}$ of D into X. For example, if $D = H_\varepsilon$ is the standard Hartogs figure then every f extends from H_ε to the bidisc $\Delta^2_{1+\varepsilon}$. One can easily prove (following the lines in [Iv5] on the pp. 99-105) also a non-parametric version of this statement in the spirit of Theorem 2.1 of the present paper.

ii) Second: but there exists a meromorphic map $f : \Delta^3_* \to X$ of the punctured 3-disc into X which doesn't extend to the origin.

The construction goes (very roughly) as follows.

a) Take the standard three-ball $\mathbb{B}^3 \subset \mathbb{C}^3$ and blow-up the line $l_0 = \{z_2 = z_3 = 0\}$ in it. Denote by E_0 the exceptional divisor and by Y_0 the threefold obtained.

b) Denote by l_1 the intersection of the exceptional divisor with the proper transform of the plane $L_1 := \{z_1 = 0\}$. Proper transforms will be denoted with the same letters. Therefore $l_1 = L_1 \cap E_0$. This is a copy of a projective line $\mathbb{P}^1$. Now blow-up l_1. Denote by E_1 the exceptional divisor of this second blow-up.

c) Take some $\mathbb{P}^1$ on E_1 (for example the intersection of E_1 with the proper transform of the plane $L_2 := \{z_2 = 0\}$), denote is as l_2 and blow it up again. E_2 is again the exceptional divisor and by l_3 denote the projective line, which is the intersection of E_2 with (the proper transform of) E_0.

d) Fix a point 0_3 on l_3 and consider the biholomorphism g of $\mathbb{B}^3 \subset \mathbb{C}^3$ onto a neighborhood of 0_3 which sends the origin 0_0 of the space $\mathbb{C}^3$ to 0_3 and which, moreover, in the natural coordinates of $\mathbb{C}^3$ and of the resulting blowing-up is the identity. If one writes g in the natural coordinates (z_1, z_2, z_3) of ${}_c c^3$ only then it has the form

$$g : (z_1, z_2, z_3) \to (z_1, z_2, z_2 z_2^2 z_3, z_1, z_2, z_3),$$

see [Iv5] for much more details.

e) Blow-up l_3 and denote the resulting threefold as Y_3. g lifts to an imbedding of Y_0 into Y_3, denote this lift as $\hat{g}$. Remove $\hat{g}(Y_0)$ from Y_3, *i.e.*, consider the threefold $X_0 := Y_3 \setminus \hat{g}(Y_0)$. Identify the boundary components of X_0 by $\hat{g}$ and obtain a compact complex threefold X. This X is our example.

Universal covering $\tilde{X}$ of X can be obtained by gluing infinitely many copies $\{X_0^j\}_{j=-\infty}^{+\infty}$ of X_0 one to another by $\hat{g}$ (more precisely by corresponding powers of $\hat{g}$, *i.e.*, X_0^{-1} is attached to $X_0^0 := X_0$ by $\hat{g}$ and X_0^1 is attached to X_0^0 by $\hat{g}^{-1}$ and so on). Remark that $\tilde{X}$ can be naturally viewed as a blowed-up $\mathbb{C}^3 \setminus \{0\}$. For us the following feature of X will be sufficient:

iii) if one takes a (singular in general) complex surface Z in $\mathbb{B}^3$ and lifts $Z \setminus \{0\}$ naturally to $\tilde{X}$ then the closure $\tilde{Z}$ of this lift is contained in $\bigcup_{j=-N}^{N} X_0^j$ for some N, i.e., in some "finite" part of the universal covering $\tilde{X}$.

For example, it is not difficult to see that the closure of the lift of $(L_1 \cap \mathbb{B}^3) \setminus \{0\}$ is a three times blowed-up two-ball imbedded into $X_0^{-1} \cup X_0^0 \cup X_0^1$. In fact (*iii*) explains why (*ii*) is true.

Take a vertical foliation $\mathcal{L} := \{z_1 = \mathsf{const}, z_2 = \mathsf{const}\}$ on $\mathbb{B}^3$. It lifts to a foliation (denoted with the same letter) on $\tilde{X}$. Take the transversal $D := \{z_3 = 1/2, |z_1|^2 + |z_2|^2 < 1/2\}$. The leaf $\mathcal{L}_0$ has an obvious end - the image of the punctured disc $\{(0,0,z_3) : 0 < |z_3| < \frac{1}{2}\}$ in X. This end is ***not*** a vanishing end, because $\tilde{\mathcal{L}}_D$ has a puncture at this end. At the same time if one takes any one-disc $S \subset D$ then $\tilde{\mathcal{L}}_S$ lifts to some finite part $\bigcup_{j=-N}^{N} X_0^j$ of $\tilde{X}$ (N crucially depends on the choice of S and there is no any bound on it). Therefore our end will be a vanishing end for any $\tilde{\mathcal{L}}_S$ as above.

Remark 3.4. This example explains how subtle is the definitions of a vanishing end when working in the manifolds of higher dimension.

3.3. Vanishing cycles. Let now $\hat{\gamma} : [0,1] \to \hat{\mathcal{L}}_z^0$ be a loop in $\hat{\mathcal{L}}_z^0$ which is not homotopic to zero in $\hat{\mathcal{L}}_z^0$.

Definition 3.3. *We say that $\hat{\gamma}$ is a vanishing cycle if for some sequence $z_n \to z$ there exist loops $\hat{\gamma}_n$ in $\hat{\mathcal{L}}_{z_n}$ uniformly converging to $\hat{\gamma}$ which are homotopic to zero in the corresponding leaves $\hat{\mathcal{L}}_{z_n}$.*

(a) We say that $\hat{\gamma}$ is an algebraic vanishing cycle if γ is not homotopic to zero in $\hat{\mathcal{L}}_z^0$ but is homotopic to zero in the completed leaf $\hat{\mathcal{L}}_z$.

(b) If $\hat{\gamma}$ is not homotopic to zero also in the completed leaf $\hat{\mathcal{L}}_z$ we call it an essential vanishing cycle.

There is an analogy (rather deep in fact) between *algebraic*/*essential* vanishing cycles and poles/essential singularities of meromorphic functions. Really, pole of a meromorphic function f becomes a regular point if one completes $\mathbb{C}$ to $\mathbb{CP}^1$ and considers f as a holomorphic mapping into the latter manifold. However, an essential singular point stays to be a singularity of f also after this operation. The same with cycles. For the moment let us say that:

- If $\mathcal{L}^{\mathsf{sing}} = \varnothing$, *i.e.*, if $\mathcal{L}$ has no singularities, then every vanishing cycle is an essential vanishing cycle, more precisely projects to a vanishing cycle under the holonomy covering map $\hat{\mathcal{L}}_z \to \mathcal{L}_z$, see Remark 3.6.

- Algebraic vanishing cycles in the leaf $\hat{\mathcal{L}}_z^0$ can be removed (*i.e.,* one can make these cycles homotopic to zero) by adding to $\hat{\mathcal{L}}_z^0$ *vanishing ends*.
- It is known also (it follows from [Br3]) that if X is Kähler, then all vanishing cycles (of any $\mathcal{L}$) are algebraic.

In this paper we shall concentrate our attention on essential vanishing cycles only. In this subsection, following [Br1], we show that if $\hat{\mathcal{L}}_z$ contains an essential vanishing cycle then it contains an *imbedded* essential vanishing cycle. Take an immersed loop γ in a Riemann surface R which has only transversal self-intersections. Denote by N the closure of a sufficiently small tubular neighborhood of γ. Add to N all discs bounded by circles - components of ∂N, and denote the obtained compact as $\bar{N}$.

Lemma 3.3. *Imbedding $\overline{N} \subset R$ induces the natural injection $\pi_1(\bar{N}) \to \pi_1(R)$.*

Proof. Suppose that there exists a loop β in $\bar{N}$ not homotopic to zero in $\bar{N}$ which is homotopic to zero in R. Then the homotopy of β to zero is supported in a compact part of R and therefore we can suppose that R has finite topology, *i.e.,* finite number of handles and boundary circles. In the sequel the trivial case when $\bar{N}$ or R is a disc or an annulus will be omitted. Now we perform the following manipulations which obviously do not change the homotopy type of $\bar{N}$.

(a) Every connected component of $R' := R \setminus \bar{N}$ which is an annulus adjacent to ∂R we add to $\bar{N}$.

(b) If some component C of R' is an annulus with both boundary circles belonging to $\partial\bar{N}$ then we cut C on two annuli C_1 and C_2. Each of them we add to $\bar{N}$ and think about $\bar{N}$ as having ∂C_1 and $\partial C_2 = -\partial C_1$ as two boundary components.

Denote by g the Riemannian metric on R of curvature -1 having boundary circles as geodesics. Every loop γ in R is now homotopic to a unique geodesic $\tilde{\gamma}$ in metric g which is either not intersecting ∂R or is a component of ∂R, see for example [Bu] Theorem 1.6.6. We deform all boundary circles of $\bar{N}$ one by one to geodesics. If in the process of deformation a curve is touching β we move β appropriately enlarging (or contracting) $\bar{N}$ in a way to keep β inside.

We end up with having all boundary circles of $\bar{N}$ geodesics in g. Now we do the same with β getting from it a geodesic $\tilde{\beta}$ in $\bar{N}$. Note that it stays in $\bar{N}$ and do not intersect also $\partial C_2 = -\partial C_1$ from (b) (or coinciding with one of them). But this But $\tilde{\beta}$ stays to be geodesic in g on the whole of R and therefore is not homotopic to zero. Contradiction.

□

Now we are going to reduce the question of existence of essential vanishing cycles in $\hat{\mathcal{L}}_D$ to the existence of *imbedded* essential vanishing cycles in $\hat{\mathcal{L}}_D$. Namely, we shall prove that the following is true:

Lemma 3.4. *If there exists an essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ then there exists an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$.*

Proof. Let $\hat{\gamma}_0 : [0,1] \to \hat{\mathcal{L}}_{z^0}$ be our essential vanishing cycle. After perturbing it, if necessary, we can suppose that $\hat{\gamma}_0$ is an immersion with only transversal self-intersections. For every point $\hat{\gamma}_0(t)$ take an $(n-1)$-disc $Q_{\hat{\gamma}_0(t)}$ in $\hat{\mathcal{L}}_D$ transversal to the leaf $\hat{\mathcal{L}}_{z^0}$ and cutting it by the point $\hat{\gamma}_0(t)$. Make these discs depend smoothly on $\hat{\gamma}_0(t)$ in such a way that for $\hat{\gamma}_0(t_1) \neq \gamma_0(t_2)$ the corresponding $(n-1)$-discs do not intersect. Let's stress explicitly that $Q_{\hat\gamma_0(t)}$ depends only on the image point $\hat\gamma_0(t)$ on the curve and not on t. We have therefore a natural projection $\Pi : \bigcup_{\hat\gamma_0(t)} Q_{\hat\gamma_0(t)} \to \hat\gamma_0(t)$. Extend these data over a closure of a small tubular neighborhood N_0 of $\hat\gamma_0$. I.e., set $Q := \bigcup_{\tau\in N_0} Q_\tau$ and now Π maps this Q onto N_0.

For every z in our transversal D, which is close to z^0 each Q_τ cuts the leaf $\mathcal{L}^0_z$ exactly by one point and when τ runs over N_0 our discs Q_τ cuts a closure of a tubular neighborhood N_z of some closed curve γ_z which covers $\hat\gamma_0$ under the projection $\Pi|_{\gamma_z} : \gamma_z \to \hat\gamma_0$. Remark also that $\Pi|_{N_z} : N_z \to N_0$ is bijective. Denote by $\bar N_0$ the union of N_0 with all discs bounded by circles components of ∂N_0. Denote likewise by $\bar N_z$ the union of N_z with all discs bounded by circles components of ∂N_z.

Take some component $\hat\gamma_0'$ of ∂N_0 bounding a disc in $\hat{\mathcal{L}}_{z^0}$. Then the corresponding component γ_{z_n}' of ∂N_{z_n} bounds a disc in $\hat{\mathcal{L}}_{z_n}$, say D_0' and then $\Pi|_{N_{z_n}} : N_{z_n} \to N_0$ extend to a homeomorphism $\Pi|_{N_{z_n}\cup D'_{z_n}} : N_{z_n} \cup D'_{z_n} \to N_0 \cup D'_0$.

If γ_0' doesn't bound a disc in $\hat{\mathcal{L}}_{z^0}$ but γ_{z_n}' do bounds a disc in $\hat{\mathcal{L}}_z$ we get an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$.

So, unless an imbedded vanishing cycle was found in $\hat{\mathcal{L}}_{z^0}$ we end up with extending Π to a homeomorphism $\tilde\Pi : \bar N_{z_n} \to \bar N_0$.

Since γ_{z_n} is homotopic to zero in $\hat{\mathcal{L}}_{z_n}$ it will be homotopic to zero in $\bar N_{z_n}$ by Lemma 3.1. Therefore $\hat\gamma_0$ should be homotopic to zero in $\bar N_0$ and therefore in $\hat{\mathcal{L}}_{z^0}$. Contradiction. Therefore the only possibility left is that some component γ_0' of ∂N_0 doesn't bound a disc while γ_{z_n}' do bound disc, *i.e.,* γ_0' is an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$.

□

Remark 3.5. Remark that if $\hat\gamma_0$ is an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ then a sequence $z_n \to z$ such that there exists γ_{z_n} bounding a disc in $\hat{\mathcal{L}}_{z_n}$ and γ_{z_n} uniformly converging to γ_0 when $z_n \to z^0$ can be taken generic.

3.4. Universal covering cylinder. Further, for $z \in D$ denote by $\tilde{\mathcal{L}}_z$ the universal cover of the completed holonomy leaf $\hat{\mathcal{L}}_z$. I.e., we take the orbifold universal covering of $\mathcal{L}_z$, see Remark 3.3. On the union

$$\tilde{\mathcal{L}}_D = \bigcup_{z\in D} \tilde{\mathcal{L}}_z \tag{3.2}$$

one defines a natural topology in the following way. An element of $\tilde{\mathcal{L}}_D$ is a path γ in some leaf $\hat{\mathcal{L}}_z$ starting from z and ending at some point $w \in \hat{\mathcal{L}}_z$. γ and γ' define the same point if their ends coincide and they are homotopic (inside $\hat{\mathcal{L}}_z$) with ends fixed. A neighborhood of $\gamma \subset \hat{\mathcal{L}}_z$ in $\tilde{\mathcal{L}}_D$ is the set of pathes γ'-s in the leaves $\hat{\mathcal{L}}_{z'}$ with z' close to z which are themselves close to γ. γ' "close" to γ is understood here as closed in the topology of uniform convergence in the space $\mathcal{C}([0,1],X)$ of continuous mappings from $[0,1]$ to X.

Definition 3.4. *$\tilde{\mathcal{L}}_D$ with the topology just described is called the universal covering cylinder of $\mathcal{L}$ over D.*

The natural projection $\pi : \hat{\mathcal{L}}_D \to D$ lifts to $\pi : \tilde{\mathcal{L}}_D \to D$ (and will be denoted with the same letter). There is a distinguished section $\sigma : D \to \tilde{\mathcal{L}}_D$ sending z to z. The mapping $p : \hat{\mathcal{L}}_D \to X$ lifts to $\tilde{\mathcal{L}}_D$ and stays to be a *meromorphic foliated immersion* $\tilde p : \tilde{\mathcal{L}}_D \to X$ in the sense that it is a foliated immersion outside of its indeterminacy set.

Due to the eventual presence of essential vanishing cycles the natural topology on the covering cylinder might be not Hausdorff. Let us explain this in more details. Non-separability of the natural topology on $\tilde{\mathcal{L}}_D$ means that:

- there exist $z \in D$ and $w \in \hat{\mathcal{L}}_z$ and two pathes γ_1, γ_2 from z to w such that $\gamma_1 \circ \gamma_2^{-1}$ is not homotopic to zero in $\hat{\mathcal{L}}_z$;
- there exist some sequence $z_n \to z$ in D, some sequence $w_n \in \hat{\mathcal{L}}_{z_n}$ converging to w, some sequences of pathes γ_1^n and γ_2^n from z_n to w_n each converging uniformly to γ_1 and γ_2 such that $\gamma_1^n \circ (\gamma_2^n)^{-1}$ are homotopic to zero in $\mathcal{L}_{z_n}$.

And that exactly means that $\gamma_1 \circ \gamma_2^{-1}$ is an essential vanishing cycle. Vice verse, if $\gamma : [0,1] \to \hat{\mathcal{L}}_z$ is an essential vanishing cycle starting and ending at z, then γ and the trivial path $\beta \equiv z$ represent two non-separable points in $\tilde{\mathcal{L}}_D$.

3.5. Proof of Theorem 1. In the presence of a pluriclosed taming form the problem of the separability of the topology of $\tilde{\mathcal{L}}_D$ can be resolved by Theorem 1 from the Introduction. Now we shall state and prove somewhat more general and precise statement which contains the aforementioned result. To make the statement more precise let's turn to the definition of a *foliated shell*, *i.e.*, to the Definition 2 from the Introduction.

Definition 3.5. *In general, when foliation $\mathcal{L}$ is singular we require that the mapping $h : (B^{\varepsilon}, \mathcal{L}^{\mathrm{v}}) \to (X, \mathcal{L})$ which defines a foliated shell takes its values in X^0.*

By the Theorem 2.3 we know that mapping $h : (B^{\varepsilon}, \mathcal{L}^{\mathrm{v}}) \to (X^0, \mathcal{L})$, which defines a foliated shell in a pluritamed foliated manifold extends onto $P^{\varepsilon} \setminus \bigcup_{z_1 \in S_1} S_{z_1}$, where S_1 is at most countable compact in Δ. One more remark: for a transversal $D \subset X^0$ and an imbedded disc $\Delta \subset D$ the restriction $\mathcal{L}_{\Delta}^0 := \bigcup_{z \in \Delta} \mathcal{L}_z^0$ is well defined (we don't need to give this set more structure then this which it already has).

Theorem 3.1. *Let $(X, \mathcal{L})$ be a disc-convex foliated manifold which admits a dd^c-closed taming form and let $z^0 \in X^0$ be a point. Then the following statements are equivalent:*

i) The leaf $\hat{\mathcal{L}}_{z^0}$ contains an essential vanishing cycle.

ii) For every transversal $D \ni z^0$ there exists an imbedded disc $z^0 \in \Delta \subset D$ such that $\mathcal{L}_{\Delta}^0$ contains a foliated shell.

Remark 3.6. (a) Let us explain that the item (i) of this Theorem is equivalent to the item (i) of Theorem 1 from the Introduction in the case when $\mathcal{L}$ is smooth (*i.e.*, without singularities). In that case vanishing ends do not exist and, in particular, $p_{z^0} : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an unramified covering. Let $\gamma_0 \subset \mathcal{L}_{z^0}$ be a vanishing cycle and $\gamma_n \subset \mathcal{L}_{z_n}$ be cycles homotopic to zero and converging to γ_0. All γ_n lift to cycles $\hat{\gamma}_n \subset \hat{\mathcal{L}}_{z_n}$ converging to the lift $\hat{\gamma}_0 \subset \hat{\mathcal{L}}_{z^0}$ of γ_0. All $\hat{\gamma}_n$ are homotopic to zero. But $\hat{\gamma}_0$ cannot be homotopic to zero. Therefore we get a vanishing cycle $\hat{\gamma}_0$ in $\hat{\mathcal{L}}_{z^0}$. Vice verse, let $\hat{\gamma}_0$ and $\hat{\gamma}_n$ be as above in the holonomy covering leaves. Then $\hat{\gamma}_n$ project to cycles homotopic to zero in corresponding leaves. But $\hat{\gamma}_0$ project to some γ_0 which cannot be homotopic to zero because in the latter case its lift $\hat{\gamma}_0$ (as lift of any curve homotopic to zero) should be homotopic to zero itself. Therefore γ_0 is a vanishing cycle in $\mathcal{L}_{z^0}$.

(b) The item (ii) specifies that the "support" $\Sigma = h(B)$ of the foliated shell is in $\mathcal{L}_{\Delta}^0$ (but it is not homologous to zero in the whole of X!) Remark also that the existence of an essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$ is unrelated to the choice of a transversal $D \ni z^0$ (and also on the imbedded disc $z^0 \in \Delta \subset D$). Therefore if for some transversal and disc in it $D \supset \Delta \ni z^0$ there is a shell in $\mathcal{L}_{\Delta}^0$ then it persists in all others.

Proof. *(i)* $\Rightarrow$ *(ii)* By Lemma 3.4 we can suppose that our vanishing cycle $\hat{\gamma}_0$ is imbedded into $\hat{\mathcal{L}}_{z^0} \subset \hat{\mathcal{L}}_D$. Deforming it, if necessary, we suppose that $\hat{\gamma}_0$ is contained in $\hat{\mathcal{L}}^0_{z^0}$. Therefore we can suppose that for an imbedded loop $\hat{\gamma}_0 \subset \hat{\mathcal{L}}^0_{z^0}$ started at z^0 the following holds:

- $\hat{\gamma}_0$ does not bound a disc in $\hat{\mathcal{L}}_{z^0}$;
- but for a generic sequence $z_n \to z^0$ and a sequence of imbedded loops $\hat{\gamma}_n \subset \hat{\mathcal{L}}_{z_n}$ uniformly converging to $\hat{\gamma}_0$ every $\hat{\gamma}_n$ bounds a disc Δ_n in $\hat{\mathcal{L}}_{z_n}$.

Take a neighborhood U of some z_N such that for every $z \in U$ there is an imbedded loop $\hat{\gamma}_z$ close to $\hat{\gamma}_N$ bounding a disc Δ_z in $\hat{\mathcal{L}}_z$. We can suppose that $\hat{\gamma}_z$ smoothly depend on $z \in U$. Take some open cell $V \subset D$ containing U and z^0 and extend our family $\Gamma := \{\hat{\gamma}_z\}$ smoothly over $z \in V$ (after shrinking it over U, if necessary) in such a way that $\hat{\gamma}_{z^0}$ coincides with $\hat{\gamma}_0$. Perturbing the family Γ, if necessary, we can suppose that some neighborhood W of $\Gamma \cup \Delta_{z_N}$ in $\hat{\mathcal{L}}_D$ forms a generalized Hartogs figure (W, π, U, V). Projection $\pi : W \to V$ here is the restriction to W of the natural projection $\pi : \hat{\mathcal{L}}_D \to D$.

Mapping $p : \hat{\mathcal{L}}_D \to X$ restricted to W will be likewise denoted as $p : W \to X^0 \subset X$ and it is a holomorphic foliated immersion, because the construction can be obviously fulfilled in such a way that $W \subset \hat{\mathcal{L}}^0_D$. Note also that p is a generic injection because for generic z_N our $p|_{\gamma_N}$ is an imbedding. But $p|_{\hat{\gamma}_0}$ might be only an immersion in general. By Theorems 2.2 and 2.3 p extends after a reparametrization onto $\widetilde{W} \setminus S$, where $\widetilde{W}$ is a complete Hartogs figure over V and S is of the form $S = \bigcup_{z \in S_1} S_z$ with S_1 being a complete $(n-1)$-polar subset of V and every S_z is a compact subdisc of the corresponding disc $\widetilde{W}_z$. This extension $\tilde{p}$ is a foliated meromorphic immersion, *i.e.*, it is an immersion outside of its indeterminacy set $I_{\tilde{p}}$ and takes values in X (not more in X^0). The family, which corresponds in $\widetilde{W}$ to our family Γ will be denoted still by Γ and no new notation for the loops $\hat{\gamma}_z$ will be introduced.

Observe that $z^0 \in S_1$. Otherwise take an n-disc Δ^n around z^0 in D such that $\Delta^n \cap S_1 = \varnothing$ and such that:

- $\pi^{-1}(\Delta^n)$ is biholomorphic to $\Delta^n \times \Delta_{1+\varepsilon}$ with π being the vertical projection $\Delta^n \times \Delta \to \Delta^n$ (one might need to shrink Δ^n and $\widetilde{W}$ to achieve this).
- For $z \in \Delta^n$ circles $\hat{\gamma}_z = \partial \Delta_z$ belong to our family Γ (for this one might need to perturb Γ).

Our $\tilde{p}$ now is meromorphically extended to $\Delta^n \times \Delta_{1+\varepsilon}$. But that means (by the very definition of vanishing ends) that $p^{-1} \circ \tilde{p}$ lifts to a holomorphic map $\tilde{f} : \Delta^n \times \Delta_{1+\varepsilon} \to \hat{\mathcal{L}}_D$. Therefore $\tilde{f}|_{\{z^0\} \times \bar{\Delta}}$ realizes the homotopy of $\hat{\gamma}_0 = \tilde{f}(\{z^0\} \times \partial \Delta)$ to zero. Contradiction.

Denote by A the proper analytic subset in a neighborhood of z^0 on D which consists from points z such that $\tilde{p}|_{\partial_0 \widetilde{W}_z}$ is not a generic injection. Again we locally represent $\widetilde{W}$ as a product $\widetilde{W} = \Delta^n \times \Delta_{1+\varepsilon}$ with π being the vertical projection $\pi : \Delta^n \times \Delta_{1+\varepsilon} \to \Delta^n$ and with z^0 being the origin in these coordinates. Decompose $\Delta^n = \Delta^{n-1} \times \Delta$ in such a way that $(\{0\} \times \Delta) \cap A = \{$ a finite set $\}$. Then by Theorem 2.3 for $\lambda \in \Delta^{n-1}$ close to 0 (if $n \geqslant 2$), or equal to 0 (if $n = 1$) we have $\tilde{p}(\partial \Delta^2_\lambda) \not\sim 0$ (not homologous to zero in X). Moreover, for every $z = (\lambda, z_1)$ in Δ^n such that $(\lambda, z_1) \notin A$ we have that $\tilde{p}|_{\{z\} \times \partial \Delta}$ is an imbedding.

Therefore we get a foliated shell in $\mathcal{L}^0_\Delta$, where $\Delta = \{0\}\times\Delta$ if $n=1$, or a family of foliated shells $\tilde p(\partial\Delta^2_\lambda)$ if $n\geqslant 2$.

(ii) $\Rightarrow$ *(i)* Suppose now that $\mathcal{L}^0_D$ contains a foliated shell $h:(B^\varepsilon,\mathcal{L}^{\rm v})\to(X^0,\mathcal{L}^0)$. By the Theorem 2.3 h can be extended to a foliated meromorphic map (in particular this extension stays to be a generic injection) $h:(P^\varepsilon\setminus S,\mathcal{L}^{\rm v})\to(X,\mathcal{L})$, where P^ε is the ε-neighborhood of the polydisc P and $S=S_1\times\bar\Delta_{1-\varepsilon}$ for some non-empty (!) at most countable compact $S_1\subset\Delta_{1-\varepsilon}$. Note that we don't need to make any reparametrizations here. Without loss of generality we suppose that $S_1\ni\{0\}$. Denote by z^0 the image under h of the point $q=(0,1)\in P^\varepsilon$ - the future reference point for the leaf $\mathcal{L}_{z^0}$ which contains $h(\{0\}\times\partial\Delta)$. Since $h|_{\{z_1\}\times\partial\Delta}$ is not an imbedding only for finite set of z_1-s we can shrink Δ and suppose that for all $z_1\neq 0$ the restriction $h|_{\{z_1\}\times\partial\Delta}$ is an imbedding. In fact it will be an imbedding on some annulus $A_{1-\varepsilon,1+\varepsilon}$ for some $\varepsilon>0$ - the same for all $z_1\neq 0$ - and therefore it will be also an imbedding on the disc $\{z_1\}\times\Delta_{1+\varepsilon}$ provided $z_1\notin S_1$.

Now we can lift $h|_{\Delta_\varepsilon\times A_{1-\varepsilon,1+\varepsilon}}$ for some $\varepsilon>0$, small enough, to an imbedding $f:=p^{-1}\circ h:\Delta_\varepsilon\times A_{1-\varepsilon,1+\varepsilon}\to\hat{\mathcal{L}}^0_D$. This should be explained in more details. Consider $p^{-1}\circ h$ along $\partial\Delta_0$. It cannot be multivalued because for $z_1\sim 0, z_1\notin S_1$ this map is defined and singlevalued on the disc Δ_{z_1}. Moreover $(p^{-1}\circ h)|_{\partial\Delta_0}$ is also univalent. This follows from the same property of $(p^{-1}\circ h)_{\Delta_{z_1}}$, Rouche's theorem and absence of the holonomy in $\hat{\mathcal{L}}^0_D$. The rest is clear.

If $n\geqslant 2$ we can extend this lifting to a holomorphic foliated imbedding $f:\Delta^n_\varepsilon\times A_{1-\varepsilon,1+\varepsilon}\to\hat{\mathcal{L}}^0_D$ (taking a smaller $\varepsilon>0$ if necessary). This follows from the fact that $\Delta_\varepsilon\times A_{1-\varepsilon,1+\varepsilon}$ is Stein, so $f(\Delta_\varepsilon\times A_{1-\varepsilon,1+\varepsilon})$ has a Stein neighborhood (after shrinking ε, see [Si2]) and from the absence of holonomy on $\hat{\mathcal{L}}_D$. At this moment we fix an *imbedded* transversal $h(\Delta^n_\varepsilon\times\{q\})$ and name it as D. From that moment our mappings f (respectively h) and their future reparametrizations are mappings over Δ^n_ε (in the regions where this makes sense), *i.e.*, fibers Δ_z are mapped into fibers $\hat{\mathcal{L}}_z$ (or $\mathcal{L}_z$ respectively). That means that by z we denote further both a point in Δ^n_ε and its image $h(z,q)$ in $D\subset X^0$.

We know already that for $z=(0,...,0,z_1)\in\Delta^{n-1}_\varepsilon\times\Delta_\varepsilon, z_1\notin S_1$ $h|_{\{z\}\times\partial\Delta}$ extends to an imbedding of $\{z\}\times\Delta_{1+\varepsilon}$ to $\mathcal{L}^0_D$. Therefore it extends after a reparametrization onto $\{z\}\times\Delta_{1+\varepsilon}$ for z-s in an open non-empty subset of Δ^n_ε (a neighborhood of any such $(0,...,0,z_1)$). The same is true therefore for $f=p^{-1}\circ h$. Theorem 2.3 gives us an extension $\tilde h$ of h after a reparametrization onto $\widetilde W\setminus\tilde S$ and this extension is a foliated meromorphic immersion which is generically injective. Therefore the same is true for f, *i.e.*, f extends after a reparametrization to $\tilde f:\widetilde W\setminus\tilde S\to\hat{\mathcal{L}}_D$. Remark that $\tilde S$ is not empty and up to introducing new coordinates (locally near the fiber $\widetilde{W_0}$) we can suppose that $\widetilde W=\Delta^n\times\Delta_{1+\varepsilon}$, $\tilde S=\tilde S_1\times\Delta_{1-\varepsilon}$ where $\tilde S_1$ as in Theorem 2.3 and $\tilde S_1\ni 0$. The diagram on the Figure 5 could be useful here.

We claim that $\hat\gamma_0:=\tilde f|_{\Delta_0}(\partial\Delta)$ is a vanishing cycle in $\hat{\mathcal{L}}_{z^0}$. Since for all $z_1\notin S_1$ the restriction $\tilde f|_{\{(0,z_1)\}\times\partial\Delta_{1+\varepsilon}}$ is an imbedding, we get that $\tilde f|_{\{(0,z_1)\}\times\Delta_{1+\varepsilon}}$ is an imbedding to and therefore $\hat\gamma_{z_1}:=\tilde f|_{\{(0,z_1)\}\times\partial\Delta}$ is homotopic to zero in the corresponding leaf. All is left to prove is that $\hat\gamma_0$ doesn't bound a disc in $\hat{\mathcal{L}}_{z^0}$. But would $\hat\gamma_0$ bound a disc Δ^0 in $\hat{\mathcal{L}}_{z^0}$ our foliation on $\hat{\mathcal{L}}_D$ in a neighborhood of Δ^0 would be biholomorphic to the product $\Delta^n\times\Delta$ with $\Delta^0:=\{0\}\times\Delta$ and $\Delta^z:=\{z\}\times\Delta$ being the leaves of $\hat{\mathcal{L}}_D$. For all $(\lambda,z_1)\in\Delta^n_\varepsilon$ (with

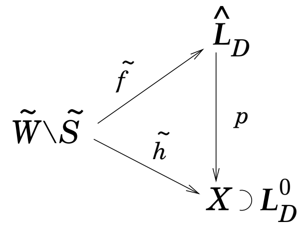

FIGURE 5. Diagram relating $\tilde{f}$, p and $\tilde{h}$: $\tilde{f}$ is a foliated imbedding (*i.e.*, it is holomorphic), while both $\tilde{h}$ and p are, in general, meromorphic immersions.

$\varepsilon > 0$ again to be taken small enough) $\tilde{f}$ sends $\partial\Delta_{\lambda,z_1}$ to some imbedded loop in some $\Delta_{\varphi(\lambda,z_1)}$, where $\varphi : \Delta^n_\varepsilon \to \Delta^n$ is some holomorphic map sending 0 to 0.

Now observe that $\mathsf{area}\left(p(\Delta_{\varphi(\lambda,z_1)})\right)$ stays bounded when $\varphi(\lambda,z_1) \to 0$. This follows from [Ba], see Corollary 2.4.2 in [Iv5] for more details. All is left is to remark that $p\left(\Delta_{\varphi(\lambda,z_1)}\right) = \tilde{h}(\Delta_{(\lambda,z_1)})$ for $(\lambda,z_1) \notin \tilde{S}_1$. Therefore this implies that $\mathsf{area}(\tilde{h}(\Delta_{(\lambda,z_1)}))$ stays bounded as $(\lambda,z_1) \to 0$ and $(\lambda,z_1) \notin \tilde{S}_1$. But this contradicts to (2.17) and to the fact that $0 \in \tilde{S}_1$ is an essential singular point of $\tilde{h}$.

□

In the process of proof of Theorem 3.1 we saw that vanishing cycles appear exactly in the fibers $\hat{\mathcal{L}}_z$ for z belonging to the closed $(n-1)$ - polar set S_1 of Hausdorff dimension $2n-2$. Therefore we obtain the following:

Corollary 3.1. *Let $\mathcal{L}$ be a holomorphic foliation by curves on a disc-convex $(n+1)$-dimensional complex manifold X which admits a pluriclosed taming form and let D be a transversal. Then the subset $S_1 \subset D$ of points s such that the completed holonomy leaf $\hat{\mathcal{L}}_s$ contains an essential vanishing cycle is complete $(n-1)$-polar of Hausdorff dimension $2n-2$.*

3.6. Imbedded vanishing cycles and proof of Theorem 2. First of all let us make precise what we mean by an imbedded essential vanishing cycle in the case of a *singular* foliation. Let $\gamma_0 \subset \mathcal{L}^0_{z^0}$ be an imbedded loop and let $d \geqslant 1$ be the order of the holonomy of $\mathcal{L}$ along γ_0. Denote by $\hat{\gamma}_0 \subset \hat{\mathcal{L}}^0_{z^0}$ the lift of γ_0. Then $p|_{\hat{\gamma}_0} : \hat{\gamma}_0 \to \gamma_0$ is a regular covering of order d.

Definition 3.6. *An imbedded essential vanishing cycle in $\mathcal{L}_{z^0}$ is a loop $\gamma_0 \subset \mathcal{L}_{z^0}$ for which the following items are satisfied:*

- *γ_0 is imbedded in $\mathcal{L}^0_{z^0}$, it admits a lift $\hat{\gamma}_0$ which is imbedded in $\hat{\mathcal{L}}^0_{z^0}$ and regularly covers γ_0 with degree d.*
- *$\hat{\gamma}_0$ doesn't bound a disc on $\hat{\mathcal{L}}_{z^0}$.*
- *For some (and therefore for a generic) sequence $\{z_n\} \subset D$ converging to z^0 there are imbedded loops $\hat{\gamma}_n$ in $\hat{\mathcal{L}}_{z_n}$ uniformly converging to $\hat{\gamma}_0$, each bounding a disc D_{z_n} in $\hat{\mathcal{L}}_{z_n}$.*

Remark 3.7. The condition on γ_0 to be in $\mathcal{L}^0_{z^0}$ and not just in $\mathcal{L}_{z^0}$ is not innocent at all. One may not be able to perturb an imbedded $\gamma_0 \subset \mathcal{L}_{z^0}$ (which admits a lift) in the way that this perturbation still admits a lift to $\hat{\mathcal{L}}_{z^0}$. And this will be needed in the proof (and it is actually an important issue).

Now we state the precise version of the Theorem 2 from the Introduction.

Theorem 3.2. *Let $(X,\mathcal{L})$ be a disc-convex foliated manifold which admits a dd^c-closed taming form and let $D \subset X^0$ be a transversal. Then the following statements are equivalent:*

i) Some leaf $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ contains an imbedded essential vanishing cycle.
ii) $\mathcal{L}_D$ contains an imbedded foliated cyclic shell.

Proof. *(i)* $\Rightarrow$ *(ii)* For a given transversal $D \ni z^0$ we need to produce from an *imbedded essential vanishing cycle* in $\mathcal{L}_{z^0} \subset \mathcal{L}_D$ an *imbedded foliated cyclic shell* in $\mathcal{L}_D^0$.

Take open cells $U \ni z_n$, $V \ni z^0$, $U \subset V \subset D$ such that for an appropriate Hartogs figure $(W,\pi,U,V) \subset \hat{\mathcal{L}}_V$ mapping $p : \hat{\mathcal{L}}_V \to X$ restricted to W is a foliated holomorphic immersion, which extends (after a reparametrization) to a foliated meromorphic immersion $p: W\setminus S \to X$ as in Theorem 2.3 (we drop tildes for the simplicity of our notations).

Note that d is the maximal cardinality of the holonomy along loops $\gamma_z := p(\partial\Delta_z) \subset \mathcal{L}_z$ close to $\gamma_{z^0} = \gamma_0$ for z in a neighborhood of z^0. Find a coordinate system $\Delta^{n-1}\times\Delta^2$ in a neighborhood of W_{z^0} in W as in Theorem 2.3, actually we shrink W to have $W = \Delta^{n-1}\times\Delta^2$ in the sequel. We keep noting coordinates in $\Delta^{n-1}\times\Delta^2$ as (λ, z_1, z_2). Note that (λ, z_1) are coordinates in a neighborhood of z^0 on D. Coordinates are chosen in such a way that z^0 correspond to $(\lambda = 0, z_1 = 0)$.

Due to Theorem 2.3 the restriction to S of the natural projection $\pi_2 : \Delta^{n-1}\times\Delta^2 \to \Delta^{n-1}$ is proper and surjective. Of course, for that to be true one should remark here that $S \neq \varnothing$ and, moreover, $\pi(S) = S_1 \ni z^0$, because otherwise γ_{z^0} would not be an essential vanishing cycle. Here, as usual, $S_1 = \pi(S)$ is the image of the singularity set S under the natural projection $\pi : \Delta^{n+1} \to \Delta^n$. By our assumption the restriction $p|_{W_{z^0}} : W_{z^0}\setminus S_{z^0} \to \mathcal{L}_{z^0} \subset X^0$ is a regular covering of order $d \geq 1$ between an appropriate annuli in the source and target complex curves. For $1 \leq l \leq d$ denote by A_l the analytic set in Δ^n which consists from points q such that the cardinality of the holonomy along γ_q is at least l. Remark that $z^0 = 0 \in A_d$, $A_1 = D$ and we set by definition $A_{d+1} = \varnothing$. Take a minimal l such that $S_1 \cap (A_l \setminus A_{l+1}) \neq \varnothing$. Call it l_0.

Case 1. $l_0 = 1$.

Take a point $s_1 \in S_1 \cap (A_1 \setminus A_2)$ and shrink our transversal D once more to a polydisc $D = \Delta^n$ - a neighborhood of s_1. We can suppose that $s_1 = 0$ in these coordinates. If this neighborhood was taken small enough our foliation has no holonomy along γ_z for $z \in D$. Therefore $p: \partial_0 W|_D \to X^0$ is an imbedding. Consider the disc $\Delta_0 := \{0\}\times\Delta \subset \Delta^{n-1}\times\Delta$ and consider the restriction $W|_{\Delta_0} = \Delta_0\times\Delta$ and the restriction of p to $W|_{\Delta_0}$. Recall that $\Delta_0 \cap S_1$ is at most countable compact subset of Δ_0.

Lemma 3.5. *There exists a finite union of imbedded loops $\beta \subset \Delta_0$ which bound a relatively compact domain $G \subset \Delta_0$ such that:*

a) $G\cap S_1 \neq \varnothing$ and $\partial G \cap S_1 = \varnothing$.

b) $p|_{\cup_{z\in\beta} W_z}$ is injective.

c) Moreover, $p\left(\bigcup_{z\in\beta} W_z \cup \partial_0 W|_G\right)$ is an imbedding.

Proof. As in Section 2 consider the area function $a(z_1) = \int_{W_{z_1}} p^*\omega$ for $z_1 \in \Delta_0 \setminus S_1$. Function a is positive, smooth (see Remark 2.6) and tends to infinity when $z_1 \to S_1$, see Theorem 2.17 (by S_1 here we understand now $S_1 \cap \Delta_0$ - but we not introduce any new notations). By Sard's lemma for a generic positive c the level set $\beta_c = \{z_1 : a(z_1) = c\}$ is a union of smooth curves in Δ_0. In the sequel c will be always taken bigger then $\inf\{a(z_1) : z_1 \in \partial\Delta_0\}$, *i.e.*, our curves will be all closed and situated away from $\partial\Delta_0$.

Claim 1. β_c has finite number of irreducible components. Suppose not and denote by β_c^i the sequence of irreducible components of β_c. Let q be an accumulation point of β_c^i. q belongs to S_1, because $\bigcup_i \beta_c^i$ is a smooth manifold. But this contradicts Lemma 2.2. Really, $\bigcup_i \beta_c^i$ is thick at q and therefore p should extend to a neighborhood of W_q. This contradicts to the fact that $q \in S_1$.

Remark that we are working here with $p|_{W_{\Delta_0}}$ and use the fact that W_q contains a singular point of this restriction. This follows from the homological characterization (b_3) of essential singularities of p in Theorem 2.3.

Claim 2. p is injective on $W|_{\beta_c}$. First of all p is injective on each W_{z_1}, $z \in \Delta \setminus S_1$ because it is injective on ∂W_{z_1}. Suppose that for some $z_1, z_2 \in \beta_c$, $z_1 \neq z_2$ one has $p(W_{z_1}) \cap p(W_{z_2}) \neq \varnothing$. Since $p(\partial W_{z_1}) \cap p(\partial W_{z_2}) = \varnothing$ we have that $p(W_{z_1}) \subset p(W_{z_2})$ (or vice verse). But this contradicts to the fact that $\text{area}\,(p(W_{z_1})) = \text{area}\,(p(W_{z_2})) = c$.

For every i denote by D^i the compact component of $\Delta_0 \setminus \beta_c^i$. Fix some point $s_1 \in S_1$. Take one of D^i-s, namely such that $D^i \ni s_1$. Denote it as D^1 and its boundary curve as β^1. If p is not injective on $\partial_0 W|_{D^1} \cup W|_{\beta^1}$ then there exists $z_1 \in D^1$ such that $p(W_{z_1}) \cap p(W_{z_2}) \neq \varnothing$ for some $z_2 \in \beta^1$. Since $p(\partial W_{z_1}) \cap p(\partial W_{z_2}) = \varnothing$ we have two possibilities. First: $p(W_{z_1}) \supset p(W_{z_2})$ but this simply doesn't imply that p is not injective on $\partial_0 W|_{D^1} \cup W|_{\beta^1}$. Therefore we are left with the second one: $p(W_{z_1}) \subset p(W_{z_2})$.

Claim 3. If $p(W_{z_1}) \subset p(W_{z_2})$ then there exists $\beta_c^j \Subset D^1$. This is obvious, take a path from z_1 to S_1 inside D^1. Then it will contain a point z with $a(z) = c$.

If this β_c^j surrounds our point s_1 call it β^2 and the compact component of $\Delta_0 \setminus \beta^2$ call D^2. If this is not the case call $\beta^1 \cup \beta_c^j$ as β^2 and the region bounded by them as D^2. Note that in both cases D^2 contains s_1.

The process $D^1 \supset D^2 \supset \ldots$ is finite because the number of β_c^i-s is finite. Therefore after a finite number of steps we will get $D^N =: G$ and $\beta^N =: \beta = \partial G$ such that p injective on $\partial_0 W|_G \cup W|_\beta$ and G has the required properties.

□

Since (taking initially Δ_0 small enough) we can suppose that $W|_\Delta$ is biholomorphic to $\Delta \times \Delta$ we get a pseudoconvex domain $G \times \Delta \subset W|_\Delta$ such that p has an essential singularity inside of this domain. By Theorem 2.3 this means that $p(\partial(G \times \Delta))$ is not homologous to zero in X. Set $B = \partial(G \times \Delta)$, then $p(G)$ is an imbedded foliated shell in $(X, \mathcal{L})$.

Remark 3.8. (a) Note that cyclic quotients didn't appear at this case, but the topology of the shell became complicated.

(b) Note also that G is found such that it contains an ad hoc taken point $s_1 \in S_1 \cap (A_1 \setminus A_2)$, *i.e.*, the constructed shell is centered at this s_1. This will be used in the sequel, see Remark 3.10.

Case 2. $l_0 > 1$.

Set $A = \bigcup_{l\geq 2} A_l$. This is a proper analytic subset of D. Changing the slope of z_1-coordinate and shrinking a neighborhood of z^0, if necessary, we can suppose that the projection $\pi_1|_A : A \to \Delta^{n-1}$ is proper. Here $\pi_1 : \Delta^n \to \Delta^{n-1}$ is the natural projection $(\lambda, z_1) \to \lambda$. $A_{l_0} \setminus A_{l_0+1}$ contains a point $s_1 \in S_1$. Shrinking D, if necessary, we can suppose that $A_{l_0+1} \cap D = \varnothing$ and $D \ni s_1$. From now on intersect $A_{l_0} = A_{l_0} \cap D$.

Claim 4. There exists an irreducible component A' of A_{l_0} of pure dimension $n-1$ which is entirely contained in S_1.

Choose coordinates (λ, z_1) in a neighborhood of $s_1 = (0,0)$ in $D = \Delta^{n-1} \times \Delta$ in such a way that $\pi_1|_{A_{l_0}}$ is proper. If $\mathsf{dim}\, A_{l_0} < n-1$ then $\mathsf{dim}\, \pi_1(A_{l_0}) < n-1$. But we know that for every $\lambda \in \Delta^{n-1} \setminus \pi_1(A_{l_0})$ there exists at least one z_1 such that $(\lambda, z_1) \in S_1$. Remark also that the holonomy along γ_{λ, z_1} for such $z = (\lambda, z_1)$ is less then l_0. Contradiction to the definition of l_0.

Therefore $\mathsf{dim}\, A_{l_0} = n-1$. Note that $S_1 \subset A_{l^0}$ by the definition of l_0. If there exists a point $q \in A_{l^0} \setminus S_1$ then from homological characterization (b_3) in Theorem 2.3 it follows that no point of A_{l^0} in a neighborhood of q belongs to S_1. Therefore all irreducible components of A_{l_0} intersecting this neighborhood do not belong to S_1. In this way we find an irreducible component A' of A_{l^0} which is entirely contained in S_1.

From now on we can suppose that $A' = S_1$ is smooth and is given by the equation $z_1 = 0$ in D. Let $g : D \to D$ be a local biholomorphism generating the holonomy along γ_0. Remark that $g|_{A'} \equiv \mathsf{Id}$ and $g^{l_0} \equiv \mathsf{Id}$. γ_0 here is the boundary $\partial\Delta_0$ and $s_1 = 0$.

Claim 5. In an appropriate coordinates with center at s_1 the automorphism g has the form $g(\lambda, z_1) = (\lambda, e^{\frac{2\pi i l}{l_0}} z_1)$ for some $l \in \{1, ..., l_0\}$ relatively prime with l_0.

This is a nearly standard fact which easily follows from the famous Bochner's linearization theorem, see [Bo]. Really, literally repeating the proof of Theorem 1 from [Bo] one can find coordinates in which g is linear and still preserving $A' = \{z_1 = 0\}$. Therefore in an appropriate coordinates g has the form $g(\lambda, z_1) = (\lambda, e^{\frac{2\pi i l}{l_0}} z_1)$ for some $l \in \{1, ..., l_0\}$ relatively prime with l_0.

Factorize $\Delta^{n-1} \times \Delta \times \Delta$ by the action $(\lambda, z_1, z_2) \to (\lambda, e^{\frac{2\pi i l}{l_0}} z_1, e^{\frac{2\pi i}{l_0}} z_2)$ to get $\Delta^{n-1} \times \mathcal{X}^{l,l_0}$, where $\mathcal{X}^{l,l_0}$ is a surface with cyclic quotient singularity. We get a holomorphic foliated immersion $p : \Delta^{n-1} \times \mathcal{X}^{l,l_0} \setminus S^{l,l_0} \to X$, where S^{l,l_0} - image of S under the factorization.

Remark that $p|_{\partial(\{0\} \times \mathcal{X}^{l,l_0})}$ is now an imbedding. Therefore we can repeat arguments of Lemma 3.5 and prove that $p|_{\partial(\{0\} \times \mathcal{X}^{l,l_0})}$ is injective in an neighborhood of the boundary B of the domain $W_{l,l_0} = \bigcup_{z\in G} W_z$ for some $G \Subset \Delta$. Would $p(B)$ be homologous to zero in X then by (b_3) of the Theorem 2.3 would imply the extensibility of p onto our domain W_{l,l_0} and this is not the case. I.e. we got an *imbedded* foliated cyclic shell.

Remark 3.9. We silently used here a version of Theorem 2.3 in the spaces with cyclic singularities. One can either prove such version directly, or "lift" the problem to the covering of $\mathcal{X}$ (which is a bicylinder), apply extension there and push the extended map down. This is possible, because the extended map will be also invariant under the action of the cyclic group by the uniqueness theorem for holomorphic functions.

(ii) $\Rightarrow$ *(i)* Let $h : (B^{\varepsilon}, \mathcal{L}^{\mathrm{v}}) \to (X^0, \mathcal{L}^0)$ be an imbedded foliated cyclic shell. We can proceed literally as in the proof of *(ii)* $\Rightarrow$ *(i)* of Theorem 3.1. All we need to do is to see that the cycle $h|_{\{0\}\times\partial\Delta} : \{0\} \times \partial\Delta \to \mathcal{L}^0_{z^0}$ - proved to be a vanishing one - was imbedded from the every beginning. Further details will be omitted. Theorem is proved.

□

Remark 3.10. Note, that in the Theorem 3.2 the place for the shell is less precise then in Theorem 3.1. But let us still make a precision here. Let $D = \Delta^{n-1}\times\Delta$ in a neighborhood of $z^0 = (0,0)$ as above. Then we proved, in fact, that we can find $\lambda \in \Delta^{n-1}$ arbitrarily close to 0 such that $\mathcal{L}^0_{\Delta^2_\lambda}$ will contain an imbedded foliated cyclic shell centered at given $s_1 \in \Delta_\lambda$. If $n = 1$ then this λ is 0. Remember that we were able to center our shell in a generic point s_1 on S_1 near z^0, see Remark 3.6.

Remark 3.11. With the definition of a foliated shell of this paper, analogues of implications *(i)* $\Rightarrow$ *(ii)* of Theorems 3.1 and 3.2 can be found for surfaces in [Br5].

4. Pluriclosed Metric Forms and Foliated Spherical Shells.

4.1. Pluriclosed metric forms and foliated spherical shells. Up to now our immersed shells were boundaries of the bicylinder (or pseudoconvex hypersurfaces close to it). One might ask if the CR-geometry is relevant here? The test question would be: can one take as shells the images of the standard spheres (with the standard vertical foliation) and not such a Levi-flat objects as boundaries of bicylinders? In the context of this paper this issue goes together in one line with reducing of the size of the essential singularity set S that is "virtually present" in the heart of all our proofs. And this task is of capital importance. It appears to be crucial for getting from vanishing cycles the *imbedded* ones.

At present we are able to reduce the size of S (equivalently to pass to spheres as shells) only in the case when our dd^c-closed taming form is actually a metric form on X, *i.e.,* ω should be not just a dd^c-closed form positive in the directions tangent to $\mathcal{L}$ but in all directions in TX.

Let $B = \mathbb{S}^3 = \{z = (z_1,z_2) \in \mathbb{C}^2 : ||z|| = 1\}$ denote the unit sphere in $\mathbb{C}^2$, $P = \{z \in \mathbb{C}^2 : ||z|| < 1\}$ - the unit ball. For some $0 < \varepsilon < 1$ set $B^\varepsilon = \{z \in \mathbb{C}^2 : 1-\varepsilon < ||z|| < 1+\varepsilon\}$ - a shell around $\mathbb{S}^3$. As before, denote by $\pi : \mathbb{C}^2 \to \mathbb{C}$ the canonical projection $\pi(z) = z_1$ onto the first coordinate of $\mathbb{C}^2$. Note that B^ε is foliated by π over the disc $\Delta_{1+\varepsilon}$ of radius $1+\varepsilon$. Denote this foliation again as $\mathcal{L}^{\mathrm{v}}$. Its leaves $\mathcal{L}_{z_1} := \pi^{-1}(z_1)$ are discs if $1-\varepsilon < |z_1| < 1+\varepsilon$ and are annuli if $|z_1| < 1-\varepsilon$.

Definition 4.1. The pair $(B^\varepsilon, \mathcal{L}^{\mathrm{v}})$ we shall call the *standard foliated spherical shell.*

Let $h : (B^\varepsilon, \mathcal{L}^{\mathrm{v}}) \to (X^0, \mathcal{L}^0)$ be some generically injective foliated holomorphic immersion of the standard foliated spherical shell into $(X^0, \mathcal{L}^0)$. Denote by Σ the image of the unit sphere $\mathbb{S}^3$ under h.

Definition 4.2. $h(B^\varepsilon)$ is called *a foliated spherical shell* in $(X, \mathcal{L})$ if Σ is not homologous to zero in X.

Remark 4.1. (a) Let us recall the Main Theorem from [Iv6], where we worked with pluriclosed *metric* forms. We proved there that the singularity set S from the Theorem 2.3 is "small" in the sense that for every $\lambda \in \Delta^{n-1}$ (see notations in Theorem 2.3) the set $S_\lambda := S \cap \Delta^2_\lambda$ is a complete pluripolar compact of Δ^2_λ of Hausdorff dimension zero.

(b) The arguments of Lemma 2.4 can be repeated here and give that S_λ are, in fact, at most countable. The crucial issue, however, here is not a countability (null-polarity is perfectly sufficient) but the size of the sets S_{λ,z_1} for $(\lambda, z_1) \in S_1$. If the taming form is actually a metric form then these last sets are also small, *i.e.,* at most countable. It is

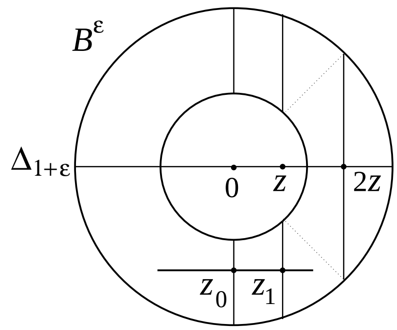

FIGURE 6. The "vertical foliation" on the Hopf surface is again the simplest example. The leaf $\mathcal{L}_0$ is a torus, $\mathcal{L}_z = \mathcal{L}_{2z}$ is $\mathbb{C}$ for $z \neq 0$. The cycle $\gamma = \{(0,z) : |z| = 1\}$ is a vanishing cycle. Image of the ε- neighborhood of $\mathbb{S}^3$ together with the "vertical foliation" under the natural projection $\mathbb{C}^2 \setminus \{0\} \to H$ is a foliated spherical shell in $(H^2, \mathcal{L})$.

this fact which leads to the foliated *spherical* shells in the Proposition below and finally allows to produce imbedded vanishing cycles and shells.

Proposition 4.1. *Let $(X, \mathcal{L}, \omega)$ be a disc-convex pluritamed holomorphic foliation by curves. Suppose that ω is actually a metric form and that a foliated manifold $(X, \mathcal{L})$ contains a foliated shell $h : (B^{\varepsilon}, \mathcal{L}^{\mathrm{v}}) \to (X^0, \mathcal{L}^0)$. Then:*

i) h extends to a foliated meromorphic immersion of $P^{\varepsilon} \setminus S$ where S is at most countable compact subset of P.

ii) $(X, \mathcal{L})$ contains a foliated spherical shell.

Proof. The proof is immediate because as a shell we can take a standard 3-sphere $\mathbb{S}^3_r(s)$ around any point s of S. A radius r should be chosen in such a way that $\mathbb{S}^3_r(s) \cap S = \varnothing$. And this is possible due to the null-dimensionality of S.

Countability of S (absent in [Iv6]) can be now achieved due to the Part (b_3) of Theorem 2.3.

□

Remark 4.2. (a) In this proof we didn't use the condition on h to be a foliated map, because any holomorphic mapping $h : B^{\varepsilon} \to X$, where X admits a pluriclosed metric form, extends to a meromorphic map from $P^{\varepsilon} \setminus S$ to X with S being at most countable compact subset of P.

(b) The conclusion of this Proposition remains valid (with countability replaced by null-polarity) if ω is supposed to be a pluriclosed taming form for $\mathcal{L}$ and there exists some other plurinegative metric form ω_1 on X (irrelevant to $\mathcal{L}$). Really, all we need is to reduce the size of the essential singularity set S along "z_2-direction" and this can be done with the help of ω_1.

4.2. Almost Hartogs property of foliated pairs. It occurs that the eduction of the size of S already made is exactly what one needs in order to produce imbedded vanishing cycles. Let us formalize this by giving the following (we use notations from Theorems 2.2 and 2.3):

Definition 4.3. *A foliated manifold $(X,\mathcal{L})$ of dimension $n+1 \geqslant 2$ is called* almost Hartogs *if the following is satisfied:*

i) Every foliated holomorphic immersion $h:(W,\pi,U,D)\to(X,\mathcal{L})$ of a non-trivial generalized Hartogs figure of dimension $\dim X$ extends to a foliated meromorphic immersion of $(W\setminus S,\pi,D)$ into $(X,\mathcal{L})$ after a reparametrization, where S is a closed subset of W of zero Hausdorff $(2n-2)$-dimensional measure.

*ii) Moreover, the essential singularity set S (*i.e.*, the minimal set with property as in (i)) has the following structure:*

a) for every point $s^0\in S$ there exists a neighborhood of it biholomorphic to Δ^{n+1} such that the restriction of π to this neighborhood is the natural vertical projection $\pi:\Delta^{n+1}\to\Delta^n$;

b) the restriction to $S\cap\Delta^{n+1}$ of the natural projection $\pi_1:\Delta^{n+1}\to\Delta^{n-1}$ is proper and has at most countable fibers.

As usual "meromorphic foliated immersion" means here that the extended h is a foliated immersion outside of its indeterminacy set. However, one should remark that the only point here is to extend h: if a meromorphic extension of h onto $\Delta^{n+1}_{1+\varepsilon}\setminus S$ is possible then it will be automatically a foliated immersion outside of its indeterminacy set. If S happens to be empty for every such mapping into $(X,\mathcal{L})$ then the latter is called simply "Hartogs". Needless to say that the set S appearing here is always closed. In the case of the presence of a plurinegative taming form on $(X,\mathcal{L})$ the item *(i)* is automatic by Theorem 2.2 and only the item *(ii)* represents itself a condition.

Our goal in this subsection is to reduce the problem of finding imbedded vanishing cycles in a shelled foliations to the proof of the almost Hartogs extension property of $(X,\mathcal{L})$. And the latter can be proved in many interesting cases, see [Iv1, Iv2, Iv3, Iv4]. In particular, the Theorem 3.3 from [Iv6] (Proposition 4.1 of the present paper) can be restated in the following form:

Proposition 4.2. *Suppose that a foliated manifold $(X,\mathcal{L})$ admits a pluriclosed taming form ω, such that ω is actually a metric form. Then $(X,\mathcal{L})$ is almost Hartogs.*

One more example is a result from [Iv4] (it doesn't require any special metric form on the total space X):

Proposition 4.3. *Suppose that the manifold X is an elliptic fibration (with possibly singular fibers) over a disc-convex Kähler manifold Y. Then every holomorphic foliation by curves on X is almost Hartogs.*

Really, let $f:(W,\pi,U,V)\to X$ be a holomorphic map. If $p:X\to Y$ is the holomorphic mapping defining the elliptic fibration then the composition $p\circ f$ extends onto W after a reparametrization by [Iv3] and [Br3]. Following the arguments in [Iv4] one gets an extension of f onto $W\setminus S$ where S is the indeterminacy set of $p\circ f$ (reparametrizations do not cause any problems here). One also obviously has he following:

Proposition 4.4. *Suppose that the manifold X is a rational fibration (with possibly singular fibers) over a disc-convex Kähler manifold Y. Then every holomorphic foliation by curves on X is almost Hartogs.*

4.3. Imbedded vanishing cycles. Recall that the classical result of Ohtsuka states the following: if $h:\Delta^*\to P$ is a holomorphic map of a punctured disc to a hyperbolic Riemann surface, then h extends to zero as a holomorphic mapping from Δ either to P, or to a *completed* by one point surface $R := P \cup \{b_0\}$ with $b_0 := h(0)$. One can express this by saying that in the situation described above h cannot have an essential singularity at zero. See [Oh1, Oh2] and for a much simpler proofs see [Re, Ro].

We shall need an orbifold version of Ohtsuka's theorem in this paper. The reason is the following: according to Remark 3.3 our holonomy covering map $h := p_{z_0} : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an *orbifold covering* map (this follows from the very definition of the vanishing end) and we like to prove that in the case when $\hat{\mathcal{L}}_{z^0}$ is hyperbolic this map behaves near an eventual puncture as in the Ohtsuka's theorem, *i.e.*, extends to a puncture after a one point completion of $\mathcal{L}_{z^0}$. A proof using the metric structure of orbifolds may be found in [Br5], §3.3, while we present here a group theoretic alternative.

For all notions and facts about orbifold Riemann surfaces that we are going to use below we refer to the book of Milnor [Mi1] and references there. For the rudiments on Fuchsian groups see [Be]. Recall that for a Riemann surface orbifold (R,ν) the Euler characteristic is defined as

$$\chi(R,\nu) = \chi(R) + \sum_j \left(\frac{1}{\nu(z_j)} - 1 \right), \tag{4.1}$$

where $\chi(R)$ is the Euler characteristic of the underlying Riemann surface R and $\nu(z_j)$ is the value of ramification function ν at ramification point z_j. Riemann surface orbifold (R,ν) is called hyperbolic if its orbifold universal covering $\tilde{S}_\nu$ is the unit disc and parabolic in the opposite case. According to Lemma E.4 from [Mi1] the Riemann surface orbifold (S,ν) is hyperbolic if and only if $\chi(S,\nu) < 0$. A *regular* holomorphic map $h : S \to (R,\nu)$ from a Riemann surface S to a Riemann surface orbifold is by definition a holomorphic map $h : S \to R$ such that for every $z \in S$ the branch index of h at z is equal to $\nu(h(z))$ (ex., no branching whenever $\nu(h(z)) = 1$).

Lemma 4.1. *Let $h : \Delta^* \to (P,\nu)$ be a regular holomorphic map from the punctured disc to a hyperbolic Riemann surface orbifold. Then h cannot have an essential singularity at the origin. More precisely:*

i) either h holomorphically extends to zero as a mapping with values in P,

ii) or, there exists a Riemann surface $R \supset P$ such that $R \setminus P = \{b_0\}$ and h holomorphically extends to zero as a mapping to R with $h(0) = b_0$.

Proof. We are following [Ro]. Denote by Γ the Fuchsian group of deck transformations of the universal covering $H \to P = H/\Gamma$ (H stands for the upper half plane). Wright $\Delta^* = H/G$, where G is generated by the translation $T_1(z) = z+1$. Mapping h induces a homomorphism $h_* : G \to \Gamma$. As in [Ro] we get that if for a circle $h_*(T_1) = 0$ then h lifts to $\tilde{h} : \Delta^* \to H$. In that case the Riemann extension theorem applies and gives the extension of $\tilde{h}$ (and therefore of h) to the origin.

Suppose now that $h_*(T_1) \neq 0$ be not homotopic to zero in P. Write $h_* T_1 = T^n$, where T is primitive. Then h lifts to a mapping $\tilde{h} : \Delta^* \to H/\Gamma_T$ where Γ_T is the cyclic subgroup of Γ generated by T. From here and by Lemma 3 from [Ro] we get that T is parabolic and therefore $H/\Gamma_T = \Delta^*$. If we prove that the natural mapping $\psi : \Delta^* = H/\Gamma_T \to H/\Gamma = P$ extends to the puncture (after, may be, completing P) our lemma will be proved, because then $h = \psi \circ \tilde{h}$ will extend to.

It will be convenient to break the proof into two cases.

Case 1. The orders of elliptic elements of Γ *are uniformly bounded.* In that case we can apply the result of Purzitsky, see [Pr]: Γ contains a torsion free subgroup Γ_1 of finite index. Since $H \to H/\Gamma_1$ is an unbranched covering the quotient H/Γ_1 is hyperbolic. Therefore the theorem of Ohtsuka applies: for $r > 0$ small enough Δ_r^* projects properly onto some puncture $\Delta_\rho^* \subset H/\Gamma_1$. But H/Γ_1 is a *finite* branched covering of $H/\Gamma = P$. Therefore under the resulting covering ψ our puncture Δ_r^* is mapped onto a neighborhood of a puncture in H/Γ and we are done.

Case 2. The orders of elliptic elements of Γ *are not bounded.* Let $f := \{f_1, f_2, ...\} \subset P$ be the images of centers of all elliptic elements of Γ. This sequence is infinite by assumption. Denote by F the set of all elliptic elements of Γ with fixed points which project to $\{f_6, f_7, ...\}$. Let Γ_F be the subgroup of Γ normally generated by T and F. Remark that $\psi : \Delta^* \to P$ lifts to some $q : \Delta^* \to H/\Gamma_F$, *i.e.,* is a composition of q with the projection $\psi_1 : H/\Gamma_F \to P$. But now H/Γ_F is hyperbolic by formula (4.1). Indeed, $H/\Gamma_F \to P$ is ramified over $f_1, ..., f_5$ and therefore the orbifold Euler characteristic of H/Γ_F is negative whatever $\chi(P)$ is. Therefore q extends to a puncture by Ohtsuka's theorem (after, may be, completing H/Γ_F). At the same time remark that the group Γ/Γ_F of the deck transformations of the cover $\psi_1 : H/\Gamma_F \to P$ has torsions only over the centers $f_1, ..., f_5$, *i.e.,* their orders are uniformly bounded. That means that we are now under the Case 1 and this finishes the proof.

□

Now we are prepared to state the main result of this Section, which is a precise version of Theorem 2 from the Introduction:

Theorem 4.1. *Let* $(X, \mathcal{L})$ *be a disc-convex foliated manifold and let* $f : (H_\varepsilon, \mathcal{L}^{\mathrm{v}}) \to \hat{\mathcal{L}}_D$ *be a foliated holomorphic imbedding of the standard Hartogs figure into the holonomy covering cylinder* $\hat{\mathcal{L}}_D$ *for some transversal* $D \subset X^0$*. Suppose that:*

1) $h := p \circ f$ *extends as a foliated meromorphic immersion to a complement of a closed subset* $S \subset \Delta_{1+\varepsilon}^{n+1}$ *of the form* $S = \cup_{z \in S_1} S_z$*, where* S_1 *is* $(n-1)$*-pluripolar in* $\Delta_{1+\varepsilon}^n$ *and all* S_z *are at most countable.*

2) For some $z^0 \in \Delta_{1+\varepsilon}^n$ *and the disc* $\Delta_{z^0} := \{z^0\} \times \Delta_{1+\varepsilon}$ *the cycle* $\hat{\gamma}_0 := f|_{\Delta_{z^0}}(\partial \Delta_{z^0}) \subset \hat{\mathcal{L}}_{z^0}$ *is an imbedded essential vanishing cycle in the holonomy covering leaf* $\hat{\mathcal{L}}_{z^0}$*.*

Then the leaf $\mathcal{L}_{z^0}$ *itself contains an imbedded essential vanishing cycle* $\gamma_0 \subset \mathcal{L}_{z^0}^0$*.*

Proof. Note that from (1) we get that f itself extends as a foliated imbedding of $\Delta_{1+\varepsilon}^{n+1} \setminus S$ into $\hat{\mathcal{L}}_D$. The condition that every point $s \in S$ is an essential singularity of h (and therefore also of f) means that there exists no neighborhood $V \ni s$ such that h (and f) meromorphically (holomorphically) extends to V. Note also that Δ_{z^0} intersects S, otherwise $f|_{\Delta_{z^0}}(\partial \Delta)$ cannot be a vanishing cycle.

We shall work locally around point $z^0 \in \Delta_{1+\varepsilon}^n$ and therefore we shall take coordinates in which this point is the origin 0. $h_0 := h|_{\Delta_0} : \Delta_0 \setminus S_0 \to \mathcal{L}_{z^0}$ is a holomorphic mapping of a pluri-punctured disc $\Delta_0 \setminus S_0$ to the Riemann surface $\mathcal{L}_{z^0}$ which factors as $h_0 = p_0 \circ f_0$ through the holomorphic imbedding $f_0 : \Delta_0 \setminus S_0 \to \hat{\mathcal{L}}_{z^0}$. Here $S_0 := S \cap \Delta_0$ is at most countable compact in Δ_0. Take some isolated point in S_0, suppose it is the origin and remark that for a boundary of a small disc around the origin its image by f is an imbedded essential vanishing cycle in $\hat{\mathcal{L}}_{z^0}$. Therefore we shrink our polydisc to Δ^{n+1} to be as small as necessary to have that 0 is the only intersection point of Δ_0 with S, *i.e.,* $\{0\} = S_0 = S \cap \Delta_0$.

Remark that $f_0 : \Delta_0 \setminus \{0\} \to \hat{\mathcal{L}}_{z^0}$ extends to a holomorphic imbedding of the disc Δ_0 into a Riemann surface $\hat{R}$ which is obtained from $\hat{\mathcal{L}}_{z^0}$ by adding to it a point, *i.e.*, $\hat{R} \setminus \hat{\mathcal{L}}_{z^0} = \{a_0\}$ and a_0 is the image of 0 under the extended map (which we still denote as $f_0 : \Delta_0 \to \hat{R}$). This follows easily from the fact that $f_0 : \Delta_0 \setminus \{0\} \to \hat{\mathcal{L}}_{z^0}$ is an imbedding.

If $\mathcal{L}_{z^0}$ is hyperbolic the mapping p restricted to $\hat{\mathcal{L}}_{z^0}$ extends to a holomorphic map $p_0 : \hat{R} \to R$ which is a ramified covering between neighborhoods of a_0 and b_0. Here $R := \mathcal{L}_{z^0} \cup \{b_0\}$ with $b_0 := h_0(0) = f(a_0)$. This results to an imbedded essential vanishing cycle in $\mathcal{L}_{z^0}$.

If $\mathcal{L}_{z^0}$ is orbifold-hyperbolic, *i.e.*, hyperbolic is $\hat{\mathcal{L}}_{z^0}$, then Lemma 4.1 still does the job as above.

In all other cases $\mathcal{L}_{z^0}$ will be parabolic, *i.e.*, torus, sphere, plane or punctured plane, as well as $\hat{\mathcal{L}}_{z^0}$. These few cases can be listed explicitly with the help of [Mi1, Mi2]. Note that in all these cases we have both $\chi(\mathcal{L}_{z^0}) \geqslant 0$ and $\chi(\hat{\mathcal{L}}_{z^0}) \geqslant 0$. If h_0 extends to zero as a mapping from Δ_0 to $R = \{b_0\} \cup \mathcal{L}_{z^0}$ then everything goes as above. Therefore below we shall be concerned with h_0 not extending to the origin in the sense just described, *i.e.*, with p_0 having an essential singularity at "added" point a_0.

Case 1. $\chi(\hat{\mathcal{L}}_{z^0}) > 0$ and $\mathcal{L}_{z^0}$ is compact. In that case it should be a sphere as well as $\hat{\mathcal{L}}_{z^0}$, and the covering $p : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ should be finite, see Remark E.5 in [Mi1]. This case is trivial, *i.e.*, a vanishing cycle doesn't occur ($\hat{\mathcal{L}}_{z^0}$ cannot contain a curve, which is nonhomotopic to zero).

Remark 4.3. By Lemma 6.2, which will be proved in the last Section in this case $(X, \mathcal{L})$ is a rational quasifibration.

Case 2. $\mathcal{L}_{z^0}$ is non-compact. I.e., is $\mathbb{C}$ or $\mathbb{C}^*$. Then the formula (4.1) tells us that $\mathcal{L}_{z^0}$ can be either $\mathbb{C}$ with one ramification point, or $\mathbb{C}$ with two of index two, or $\mathbb{C}^*$ with no ramifications. All these cases are trivial, *i.e.*, we always get an imbedded vanishing cycle.

Now we consider the cases when $\mathcal{L}_{z^0}$ is compact and $\chi(\hat{\mathcal{L}}_{z^0}) = 0$.

Case 3. $\mathcal{L}_{z^0}$ is a torus. In that case formula (4.1) tells us that $p : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is an unramified covering. Now every loop in $\mathbb{T}^2$ is homotopic to a multiply covered imbedded one and therefore $h_0(\partial\Delta_0)$ is homotopic to a multiply covered imbedded loop γ_0 and this homotopy lifts again to $\hat{\mathcal{L}}_{z^0}$. This again produces an imbedded essential vanishing cycle.

In the last two cases $\mathcal{L}_{z^0}$ is a sphere. Then the formula (4.1) tells that $p : \hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is a ramified covering with either three or four ramification points $\{z_j\}$ with multiplicity function ν satisfying

$$\sum_j \left(1 - \frac{1}{\nu(z_j)}\right) = 2. \tag{4.2}$$

There are only four integer solutions of (4.2), see [Mi1] Remark E.6 and [Mi2] Corollary 4.5 for more details. Here we only list them together with the needed facts.

- The (orbifold) universal covering $\tilde{\mathcal{L}}_{z^0}$ of $\mathcal{L}_{z^0}$ (*i.e.*, the usual universal covering of $\hat{\mathcal{L}}_{z^0}$) is $\mathbb{C}$ in all these cases and the group of deck transformations of the covering $\tilde{p}_{z^0} : \tilde{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is the extension of $\mathbb{Z}^2$ by a finite group $\mathbb{Z}_n$ of n-roots of 1 for $n = 2, 3, 4, 6$ (*i.e.*, one has four options). In another words the group in question is $\mathbb{Z}^2 \rtimes \mathbb{Z}_n$ - the semidirect product of $\mathbb{Z}^2$ with $\mathbb{Z}_n$.

- $\mathbb{Z}^2$ acts on $\mathbb{C}$ by translations along some lattice Λ and $\mathbb{Z}_n$ by rotations onto the angle $e^{\frac{2\pi i}{n}}$.
- In the case $(2,2,2,2)$ the lattice Λ is generated by 1 and τ where τ is an arbitrary complex number which belongs to the Siegel region $\mathcal{S} := \{\tau : |\tau| \geqslant 1, |\mathsf{Re}(\tau)| \leqslant 1/2, \mathsf{Im}(\tau) > 0$ and $\mathsf{Re}(\tau) \geqslant 0$ if $|\tau| = 1$ or $|\mathsf{Re}(\tau)| = 1/2\}$ - the fundamental domain of $SL(2,\mathbb{Z})$. The finite group is $\mathbb{Z}_2 = \{\pm 1\}$ in this case.
- In all other cases the lattice is rigid, *i.e.,* unique, and is determined by the condition to be invariant under the rotations from $\mathbb{Z}_n$ for $n = 3,4,6$.

With this information at hand one should distinguish here two cases.

Case 4. $\mathcal{L}_{z^0}$ is a sphere and the ramification function is one of $(2,4,4)$, $(2,3,6)$, $(3,3,3)$.

Recall the following well known fact:

The group $G_n := \mathbb{Z}^2 \rtimes \mathbb{Z}_n$ for $n = 3,4,6$ has no nontrivial normal subgroups of infinite index.

Really, let $N \triangleleft G_n$ be a nontrivial normal subgroup. We see it as acting on $\mathbb{C}$ as described. Suppose N contains a rotation ρ. Take any translation $t \in G_n$. Then the commutator $t_1 := [\rho, t]$ is (obviously) a translation and it belongs to N because $[\rho, t] = \rho(t\rho^{-1}t^{-1})$ and N is normal. But $t_2 := \rho t \rho^{-1}$ is a translation transversal to t_1 (if $n \neq 2$) and therefore $N \supset \mathbb{Z}\cdot t_1 \times \mathbb{Z}\cdot t_2$ and we are done.

Remark now that the group N of deck transformations of the covering $\tilde{\mathcal{L}}_{z^0} \to \hat{\mathcal{L}}_{z^0}$ should be a normal subgroup of the group G_n of the deck transformations of the covering $\tilde{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$. By the fact, just mentioned, N is either trivial or of finite index in G_n. In both cases there cannot be any vanishing cycles in $\hat{\mathcal{L}}_{z^0}$.

Case 5. $\mathcal{L}_{z^0}$ is a sphere and the ramification function is $(2,2,2,2)$.

This case is not rigid in the sense that there is one conformal parameter, namely the cross-ratio of four (ramification) points on $\mathbb{CP}^1$. But this doesn't matter. Again, if $N \triangleleft G_2$ is a nontrivial normal subgroup then it contains a translation t_1 as it was explained above. So $N \supset \mathbb{Z}\cdot t_1$. If N contains also a rotation then it contains also an another translation $t_2 = [\rho, t]$ transversal to t_1 if t was taken transversal to t_1, the proof goes exactly as above.

Therefore we are left with the case $N = \mathbb{Z}\cdot t$, *i.e.,* the group $N \triangleleft G_2$ of the deck transformations of the cover $\tilde{\mathcal{L}}_{z^0} \to \hat{\mathcal{L}}_{z^0}$ can be only $\mathbb{Z}\cdot t$ in this case (other cases are trivial). Take $t = k$ for simplicity (after and appropriate choice of a basis for Λ, *i.e.,* $1, \tau$ as above). Then $\hat{\mathcal{L}}_{z^0}$ is a cylinder $\mathbb{C}/\mathbb{Z}\cdot t$, *i.e.,* $\hat{\mathcal{L}}_{z^0} = \mathbb{C}/k\mathbb{Z} = [0,k] + \mathbb{R}\tau$ with left and right boundary lines identified by $z \to z + k$. Every imbedded loop $\hat{\gamma}_0$ in this cylinder is homotopic to the interval $[0,k]$. Covering $\hat{\mathcal{L}}_{z^0} \to \mathcal{L}_{z^0}$ is a composition of a unramified k-sheeted covering $p_1 : \mathbb{C}/\mathbb{Z}\cdot k \to \mathbb{C}/\mathbb{Z}$ and a ramified one $p_2 : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}^2 \times \mathbb{Z}_2$. Under the first mapping $\hat{\gamma}_0$ maps to a k-times taken imbedded loop $[0,1]$. This loop is homotopic to (k-times taken) $[0,1] + \frac{i}{4}$ and the last lies entirely in the fundamental domain of G_2 (only the ends are identified). Therefore it projects to an imbedded loop γ_0 in the factor $\mathcal{L}_{z^0}$. As a result we got an imbedded vanishing cycle. Figure 5 from [Mi2] might be helpful for better understanding the last few lines above. Theorem is proved.

□

Theorem 3 from the Introduction follows now immediately from Theorem 4.1 and Proposition 4.2. More precisely, we obtain the following result.

Corollary 4.1. *Let $(X,\mathcal{L},\omega)$ be a disc-convex, singular holomorphic foliation by curves such that the pluritaming form ω is actually a metric form. If some leaf $\mathcal{L}_{z^0}$ of $\mathcal{L}$ contains an essential vanishing cycle then it contains also an imbedded essential vanishing cycle.*

Remark 4.4. The same is true for disc-convex foliated manifolds $(X,\mathcal{L})$ provided X is a total space of an elliptic fibration (with possibly singular fibers) over a disc-convex Kähler manifold (apply Proposition 4.3).

4.4. Imbedded shells in dimension two. It would be instructive to understand something to the very end. Also it is a time to get more examples and see how restrictive is the presence of a foliated shell in $(X,\mathcal{L})$. That's why let us look closely to foliations on compact complex surfaces. X in this subsection will denote a compact complex surface, *i.e.*, a complex manifold of dimension two. $\mathcal{L}$ will be a singular holomorphic foliation by curves on X. We will work only with $(X,\mathcal{L})\in\mathcal{S}$ in this subsection.

As we know on a compact complex surface there always exists a dd^c-closed metric form. This was for the first time observed by Gauduchon in [Ga]. Moreover all compact complex surfaces are almost Hartogs, this is explained in [Iv1, Iv4]. Really, the Kähler ones are simply Hartogs, elliptic ones are served by Proposition 4.3 and that of class VII by the Proposition 4.2. Therefore results of this paper are applicable to compact complex surface in their full scale. Our task here is simple: to get consequences from the presence of shells. This can be done using the following beautiful and extremely powerful idea (I call it a "pseudoconvex surgery") due to Kato, we shall step by step use his results from [K1, K2] adapting them to our "foliated" case.

Pseudoconvex surgery. Let $h:(P^{\varepsilon}\setminus\{0\},\mathcal{L}^{\mathrm{v}})\to(X,\mathcal{L})$ be an imbedded foliated shell. We keep the notations of the Introduction and of the proof of Theorem 3.2. Recall that $P=\cup_{z\in G}\Delta_z$ for a domain $G\ni 0$. In an ε- neighborhood B^{ε} of the boundary $B=\partial P$ the mapping h is a foliated imbedding (but it is only immersion on the whole of $P\setminus\{0\}$). The origin $\{0\}$ is the only essential singular point of h. $\gamma_0:=h(\partial\Delta_0)$ is an essential vanishing cycle. Set $\Sigma:=h(B)$. Denote by $B^{\varepsilon}_{\pm}$ one sided neighborhoods of B. Set $\Sigma^{\varepsilon}_{\pm}=h(B^{\varepsilon}_{\pm})$ as on the Figure 7. Cut X along Σ to get a connected open set $E:=X\setminus\Sigma$.

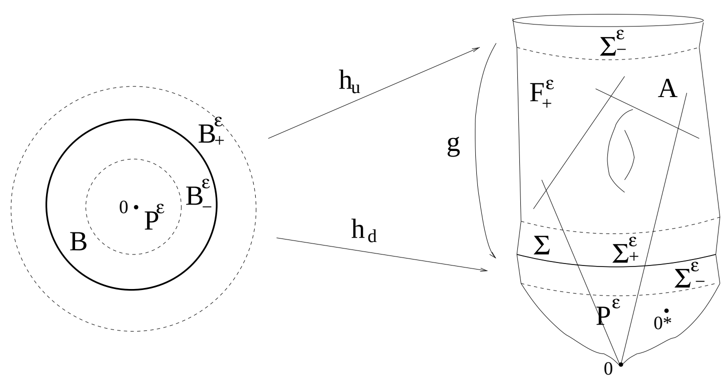

FIGURE 7. Pseudoconvex surgery.

Construct a pseudoconvex manifold F^{ε}_{+} by gluing to E the domain P^{ε} by the biholomorphism $h_d:B^{\varepsilon}_{+}\to\Sigma^{\varepsilon}_{+}$ - a copy of h (in fact it may have one cyclic singularity). Note that F^{ε}_{+} inherits the foliation $\mathcal{L}$. Moreover, F^{ε}_{+} contains two copies of Σ^{ε}_{-}, one near its boundary - second in the interior (see our Figure). There is a natural map g between these two copies of Σ^{ε}_{-}, we refer to [K1], §1 for the construction of g. For us it will be important that g is a foliated biholomorphism in its domain of definition. Really, g comes in [K1] and [K2] as a part of a deck transformation $\tilde{g}$ of a certain unramified covering $\tilde{X}$. The latter inherits a foliation $\tilde{\mathcal{L}}$ which, of course, must be preserved by $\tilde{g}$ and therefore is preserved by g. In fact, one can see g in our Figure: in coordinates on both copies of Σ_-^ε in F_+^ε given by h_u and h_d the mapping g is the identity. h_u is an "upper" copy of h. Anyway, by the Hartogs extension theorem for holomorphic functions g extends onto the whole F_+^ε as a foliated holomorphic map $g:(F_+^\varepsilon,\mathcal{L})\to(F_+^\varepsilon,\mathcal{L})$.

F_+^ε also contains a point - the "origin" - it comes from the origin 0 in P^ε when attaching it to E. We keep noting this point as 0.

Claim 1. (M. Kato, Lemma 1 in [K1]*, Lemma 2 in* [K2]*.) There exists a point* $0^*\in F_+^\varepsilon$ *such that*

$$\bigcap_{n\geqslant 1} g^n(F_+^\varepsilon)=\{0^*\}. \tag{4.3}$$

Let A be the maximal compact subvariety of F_+^ε. Note that A is contracted by g to points. The set A on our Figure 7 is drawn as a chain of four segments. We need to distinguish two cases.

Case 1. $0^*\notin A$.

0^* may coincide with 0 or not, we treat both cases simultaneously. Remark that due to the fact that g is foliated and $0^*\notin A$ it is a biholomorphism in a neighborhood of 0^*. Take a cyclic quotient $\mathbb{B}_{l,d}$ of the standard ball $\mathbb{B}=\{z\in\mathbb{C}^2:\|z\|<1\}$ centered at 0^* and contained in some $g^{n_0}(F_+^\varepsilon)$ if $0^*=0$ and our shell was (l,d)-cyclic shell. If $0^*\neq 0$ or $d=1$ then it is just the ball.

Now as in [K2], Lemma 5 one proves that, if $d=1$, then:

i) g is a contracting biholomorphism in a neighborhood of 0^*.

ii) Moreover there exists a strongly plurisubharmonic function φ near 0^* such that for every $c>0$ small enough $P_c:=\{z:\varphi(z)<c\}$ is biholomorphic to $\mathbb{B}$ and g contracts each P_c, i.e., $g(P_c)\Subset P_c$.

The proof if entirely local. Therefore if $d>1$ one lifts g from $\mathbb{B}_{l,d}$ to $\mathbb{B}$ and has the same properties for the lifted local biholomorphism. In the first case one gets a primary Hopf surface in the second - non-primary. Namely, one masters from the shell between ∂P_c and $g^n(\partial P_c)$ (for appropriate n and $c>0$) a Hopf surface and proves that our surface X blows down to this one, call it Y and the foliation obtained in Y denote by $\mathcal{F}$. Bimeromorphic transformations/unramified coverings do not effect essential (!) vanishing cycles. In particular our leaf $\mathcal{L}_0$ with a vanishing cycle γ_0 descends to the same in Y with foliation $\mathcal{F}$. Now let us see what happens in Y. Our foliation is vertical in an appropriate coordinates near 0^*. Therefore, after appropriate change in z_1-coordinate can write the contracting map g (or its lift) in $\mathbb{B}$ in the following form:

$$g(z_1,z_2)=(\alpha_1 z_1,\alpha_2 z_2+z_1 g_1(z_1,z_2)), \tag{4.4}$$

where $0<|\alpha_1|,|\alpha_2|<1$. Now it is obvious that it is the central fiber $\mathcal{F}_0$ of $\mathcal{F}$ which carries an essential vanishing cycle and this fiber is a torus.

Case 2. $0^*\in A$.

Take a connected component of A containing 0^* and from this moment denote it as A. Let $\lambda:\tilde{F}_+^\varepsilon\to F_+^\varepsilon$ be the minimal resolution of singularities of F_+^ε and let B be the proper preimage of 0. Remark that in the case of a cyclic quotient singularity all components of B are rational curves, see pp. [BHPV] 107-110. Consider an, a priori meromorphic mapping $\tilde g := \lambda^{-1}\circ g\circ\lambda : \tilde{F}_+^\varepsilon \to \tilde{F}_+^\varepsilon$. Kato proved in [K2] that:

i) $\tilde g$ is holomorphic (and foliated in our case) and there exists n such that $\tilde g^n(A\cup B) = \{\text{point}\}$.

ii) There exists $0^{**}\in B$ such that $\bigcap_{n\geqslant 1}\tilde g^n(\tilde{F}_+^\varepsilon) = \{0^{**}\}$.

iii) There exists a strongly plurisubharmonic function φ near 0^{**} such that for every c small enough $P_c := \{z : \varphi(z) < c\}$ is biholomorphic to $\mathbb{B}$ and $\tilde g$ contracts each P_c, *i.e.,* $\tilde g(P_c) \Subset P_c$.

iv) $\tilde g : \tilde g^{-1}(P_c\setminus\{0^{**}\} \to P_c\setminus 0^{**}$ is a biholomorphism.

See again Lemma 5 in [K2]. Kato then masters from these data (in a clear way) a surface Y with Global Spherical Shell in the terminology of Kato, or a Kato surface (his shell is clearly foliated in our sense) and proves that our X blows down to Y (as well as foliation $\mathcal{L}$ goes down to some $\mathcal{F}$). On Y one gets a divisor C as factor of $(A\cup B)\setminus\tilde g^n(F_+^\varepsilon)$ by $\tilde g^n$ for an appropriate n. C is proved to be a chain (or two chains) of rational curves. Again the foliation in a neighborhood of 0^{**} is vertical in an appropriate coordinates. The image of the leaf $\mathcal{L}_0$ which supports a vanishing cycle under $\tilde g^n$ cannot miss the set $\tilde g^n(F_+^\varepsilon)\cap A\cup B$, otherwise the corresponding $\mathcal{F}_0$ would not contain a vanishing cycle - this was already once explained. Therefore $\mathcal{L}_0\subset C$. I.e. it is contained in a rational curve and we are done.

Remark 4.5. Corollary 2 from the Introduction is proved.

It would be instructive to see clearly an example of the Case 2. Let's take the simplest one.

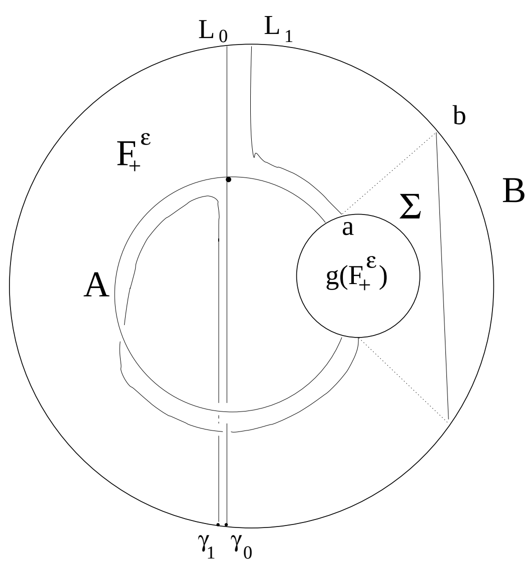

Figure 8. Example to the case $0^*\in A$. B is the standard sphere. F_+^ε is the one time blown up unit ball. $\tilde g$ is given by $(z_1,z_2)\to(\frac12 z_1,\frac12 z_2)$ and Σ is the image of B under $\tilde g$. The image $\tilde g(F_+^\varepsilon)$ of the blown up ball is removed and X is obtained by identifying B with Σ. $\mathcal{L}_0$ lands to A which is a rational curve with one point of selfintersection. γ_0 is a circle (a point on this Figure) and γ_1 on a nearby leaf $\mathcal{L}_1$ (complicated curve on the Figure) bounds a disc. To understand this note that circles a and b should be identified.

5. Pluriexact Manifolds and Foliations

5.1. Characterization of pluriexact foliations. This Section is entirely devoted to pluriexact foliations. We start with the following:

Lemma 5.1. *If X is a compact complex manifold then $dd^c : \mathcal{E}^{\mathbb{R}}_{2,2} \to \mathcal{E}^{\mathbb{R}}_{1,1}$ has closed range.*

Proof. Observe the following resolution of the sheaf $\mathcal{H}_{\mathbb{C}}$ of complex valued pluriharmonic functions

$$0 \longrightarrow \mathcal{H}_{\mathbb{C}} \xrightarrow{(-\partial,\mathsf{Id})} \Omega^1 \oplus [\mathcal{H}_{\mathbb{R}} + i\mathcal{E}_{\mathbb{R}}] \xrightarrow{(\mathsf{Id}\oplus\partial)} \mathcal{E}^{1,0} \xrightarrow{(\partial\oplus\bar\partial)} \mathcal{E}^{1,1}_{\mathbb{R}} \xrightarrow{dd^c} \mathcal{E}^{2,2}_{\mathbb{R}} \xrightarrow{d} \dots \tag{5.1}$$

Here $\mathcal{H}_{\mathbb{R}}$ is the sheaf of real valued pluriharmonic functions, Ω^1 the sheaf of holomorphic 1-forms, $\mathcal{E}_{\mathbb{R}}$ the sheaf of smooth real valued functions. This resolution tells that

$$\mathsf{Ker}\{d : \mathcal{E}^{2,2}_{\mathbb{R}} \to \mathcal{E}^{3}_{\mathbb{R}}\}/\mathsf{Im}\{dd^c : \mathcal{E}^{1,1}_{\mathbb{R}} \to \mathcal{E}^{2,2}_{\mathbb{R}}\} \equiv H^4(X, \mathcal{H}_{\mathbb{C}}). \tag{5.2}$$

Therefore $dd^c : \mathcal{E}^{1,1}_{\mathbb{R}} \to \mathcal{E}^{2,2}_{\mathbb{R}}$ has closed range (in fact of finite codimension). By duality $dd^c : \mathcal{E}^{\mathbb{R}}_{2,2} \to \mathcal{E}^{\mathbb{R}}_{1,1}$ has also closed range.

□

Fix some strictly positive $(1,1)$-form Ω on X. Let $\mathcal{L}$ be a holomorphic foliation by curves on X. Denote by $K_{1,1}(\mathcal{L})$ the compact set in $\mathcal{E}^{\mathbb{R}}_{1,1}$ which consists from positive $(1,1)$-currents T tangent to $\mathcal{L}$ such that $(\Omega, T) = 1$, *i.e.,* the compact base of currents directed by $\mathcal{L}$. Let us prove now the Proposition 3 from the Introduction, it is analogous to Theorem 3.18 from [Go], a non-foliated version for dd^c-closed metric forms was given in [Iv1].

Proof of Proposition 3. Let ω be a pluriclosed taming form for $\mathcal{L}$. If $dd^cS = T \in K_{1,1}(\mathcal{L})$ for some $S \in \mathcal{E}^{\mathbb{R}}_{2,2}$ then $0 < (\omega, T) = (\omega, dd^cS) = (dd^c\omega, S) = 0$ - a contradiction. Vice verse, if $K_{1,1}(\mathcal{L}) \cap dd^c\mathcal{E}^{\mathbb{R}}_{2,2} = \varnothing$ then, since $dd^c\mathcal{E}^{\mathbb{R}}_{2,2}$ is closed, by Hanh-Banach theorem there exists ω such that $\omega|_{K_{1,1}(\mathcal{L})} > 0$ and $\omega|_{dd^c\mathcal{E}^{\mathbb{R}}_{2,2}} = 0$.

□

5.2. Plurinegative metric and taming forms. A form $\omega \in \mathcal{E}^{p,p}$ is positive if its restriction onto any germ of p-dimensional complex submanifold is positive (meaning $\geqslant 0$). This is equivalent to the positivity of (n,n)-forms

$$\omega \wedge i\theta_1 \wedge \bar\theta_1 \wedge \dots \wedge i\theta_{n-p} \wedge \bar\theta_{n-p}$$

for all $(1,0)$-forms $\theta_1, \dots, \theta_{n-p}$. A current $T \in \mathcal{E}^{\mathbb{R}}_{p,p}$ is positive if $\langle T, \omega \rangle \geqslant 0$ for every positive $w \in \mathcal{E}^{p,p}$. Denote by $K_{p,p}$ the compact set in $\mathcal{E}^{\mathbb{R}}_{p,p}$ which consists from such strictly positive (p,p)-currents T on X that $< T, \Omega^p >= 1$.

Proposition 5.1. *Let X be a compact complex manifold. Then the following alternative holds true:*

i) either X admits a plurinegative metric form,

ii) or, there exists a sequence $S_n \in K_{2,2}$ and increasing sequence of real numbers t_n such that $t_n dd^c S_n$ converges to some $T \in K_{1,1}$.

Proof. *(i)* $\Rightarrow$ *(ii)* Let ω be a plurinegative metric form. Then we need to see that such sequence cannot exist. Suppose it does. Then

$$0 \geqslant < t_nS_n, dd^c\omega > = < t_n dd^cS_n, \omega > \to < T, \omega > > 0.$$

Therefore $< T,\omega >= 0$, contradiction.

(ii) $\Rightarrow$ *(i)* Set $B = dd^c(K_{2,2}$. This is a convex compact in $\mathcal{E}^{\mathbb{R}}_{1,1}(X)$. Let K_B be the cone generated by B. Nonexistence of a sequence as in *(ii)* means exactly that $K_B \cap K_{1,1} = \varnothing$. Hanh-Banach theorem gives us a continuous linear form ω on $\mathcal{E}^{\mathbb{R}}_{1,1}(X)$ such that $\omega|_{K_{1,1}} > 0$, *i.e.,* ω is a metric form, and such that $\omega|_B \leqslant 0$, *i.e.,* ω is plurinegative.

□

Let $\mathcal{L}$ be a holomorphic foliation by curves on X. Denote by $K_{1,1}(\mathcal{L})$ the compact set in $\mathcal{E}^{\mathbb{R}}_{1,1}$ which consists from positive $(1,1)$-currents T tangent to $\mathcal{L}$ such that $(\Omega, T) = 1$, *i.e.,* the compact base of currents directed by $\mathcal{L}$. The following Proposition is analogous to the previous one and the proof is identic to that already given.

Proposition 5.2. *Let $(X,\mathcal{L})$ be a compact foliated pair. Then the following alternative holds true:*

i) either $(X,\mathcal{L})$ admits a plurinegative taming form,

ii) or, there exists a sequence $S_n \in K_{2,2}$ and increasing sequence of real numbers t_n such that $t_n dd^c S_n$ converges to some $T \in K_{1,1}(\mathcal{L})$.

5.3. Subdivision of pluriexact manifolds and foliations. Based on what was said in this section we divide the class $\mathcal{E}$ of compact pluriexact (foliated) manifolds into three classes:

- Class $\mathcal{E}_-$ of (foliated) manifolds from $\mathcal{E}$ admitting a plurinegative (taming) form.
- Class $\mathcal{E}_+$ of (foliated) manifolds from $\mathcal{E}$ admitting a (non-trivial) positive (directed) $(1,1)$-current T such that $T = dd^c S$ for some positive $(2,2)$-current S.
- Class $\mathcal{E}_0 := \mathcal{E} \setminus \big(\mathcal{E}_- \sqcup \mathcal{E}_+\big)$.

Let us give an example of a foliated manifold from class $\mathcal{E}_0$. The example is the classical one due to Hironaka. In nonhomogeneous coordinates (x,y,z) in $\mathbb{CP}^3$ consider the rational curve C with exactly one transverse self-intersection, which is defined by the equation

$$y^2 = x^2 + x^3, \qquad z = 0. \tag{5.3}$$

Manifold X of the example is a proper modification of $\mathbb{CP}^3$ along C. In an neighborhood of the origin one blows-up first one local irreducible branch of C and then an another. Let $p : \mathbb{CP}^1 \to C \subset \mathbb{CP}^3$ be a holomorphic parameterizing map. Let D_∞ be a disc in $\mathbb{CP}^1$ near ∞ and let $p(D_\infty)$ be its image. We perform a local blow-up of $\mathbb{CP}^3$ with the center $p(D_\infty)$ and denote by C'_∞ the strict transform of the origin. Then we blow-up again along $p(D_0)$, where D_0 is a disc around 0 n $\mathbb{CP}^1$. By C_∞ we denote the strict transform of C'_∞ under this second blow-up, by C_0 the strict transform of the point of intersection of $p(D_0)$ with the exceptional divisor of the first blow-up. We perform the blow-up along the remaining part of C and denote by E the exceptional divisor of the resulting blow-down map $\pi : X \to \mathbb{CP}^3$. As it is well known (and obvious), see [S] for example, $[C_0]$ is homologous to $[C_0] + [C_\infty]$ and therefore C_∞ is homologous to zero.

Note. In fact $[C_\infty]$ is dd^c-exact as a current. To see this take as $S = \frac{1}{\pi}\pi^* \ln |t|^2 \cdot [E]$ where t is a parameter on the parameterizing curve $\mathbb{CP}^1$. This means that we consider $\frac{1}{\pi}\pi^* \ln |t|^2$ as a function on E and the action of S on $(2,2)$-form φ is given by

$$< S, \varphi >= -\frac{1}{\pi} \int_E \pi^* \ln |t|^2 \varphi. \tag{5.4}$$

Since $dd^c \ln|t|^2 = \pi\delta_{\{0\}}$ we get that for any $(1,1)$-form ψ

$$< dd^c S, \psi > = < S, dd^c\psi > = -\frac{1}{\pi}\int_E \pi^* \ln|t|^2 dd^c\psi = -\frac{1}{\pi}\int_E dd^c\big(\pi^*\ln|t|^2\big)\psi =$$

$$= -\frac{1}{\pi}\int_E \pi^* dd^c\big(\ln|t|^2\big)\psi = \int_{C_\infty}\psi = < [C_\infty], \psi > . \tag{5.5}$$

Let us see that $[C_\infty]$ is not a dd^c of a **positive** $(2,2)$-current.

Lemma 5.2. *There exists no* **positive** $(2,2)$*-current* S *such that* dd^cS *is positive and non-zero.*

Proof. Suppose the opposite, *i.e.*, that there exists positive $(2,2)$-current S on X such that $T := dd^cS$ is a non-zero positive $(1,1)$-current. Then for positive currents $\tilde{S} := \pi_*S$ and $\tilde{T} := \pi_*T$ one has $dd^c\tilde{S} = \tilde{T}$. Since $\mathbb{CP}^3$ is Kähler this implies that $\tilde{T} = 0$. Since $\pi : X\setminus E \to \mathbb{CP}^3\setminus C$ is a biholomorphism the current T is supported on E.

By the Cut-off Theorem of Bassanelli $\chi_E\cdot S$ is a dd^c-positive current supported on E, *i.e.*, $\chi_E\cdot S = h[E]$ where h is a plurisubharmonic function on E. But then h is constant and $dd^c\chi_E\cdot S = 0$. Now we can write

$$T = dd^cS = dd^c(\chi_E\cdot S) + dd^c(\chi_{X\setminus E}\cdot S) = dd^c(\chi_{X\setminus E}\cdot S). \tag{5.6}$$

Let us see that $\chi_E\cdot dd^c(\chi_{X\setminus E}\cdot S) = 0$. Since $\chi_E\cdot T = T$, from this will follow that $T = 0$, which is a contradiction. By one more theorem of Bassanelli, see Theorem 3.5 in [Bs] $dd^c(\chi_{X\setminus E}\cdot S) - \chi_{X\setminus E}dd^cS$ is a negative current supported on E. Therefore $\chi_E dd^c(\chi_{X\setminus E}\cdot S)$ is negative and supported on E. But from (5.6) we see that this current is positive. So it is zero.

□

Lemma 5.3. *There exists a sequence of positive bidimension* $(2,2)$ *currents* $\{S_n\}$ *of mass one and an increasing sequence of positive real numbers* $t_n \nearrow +\infty$ *such that*

$$[C_\infty] = \lim_{n\to\infty} t_n dd^c S_n.$$

Proof. For $p > 0$ consider the following function on the Riemann sphere $\mathbb{CP}^1$

$$\varphi_p(z) = \begin{cases} |z|^{2p} & \text{if } |z|\leqslant 1 \\ 2 - \frac{1}{|z|^{2p}} & \text{if } |z|\geqslant 1. \end{cases} \tag{5.7}$$

We have that

$$dd^c\varphi_p = \frac{i}{2}\partial\bar\partial\varphi_p = \frac{i}{2}\partial(p|z|^{2p-2}zd\bar z) = p^2|z|^{2p-2}\frac{i}{2}dz\wedge d\bar z \quad \text{for } |z|\leqslant 1. \tag{5.8}$$

Analogously in coordinate $w := 1/z$ we have

$$dd^c\varphi_p = dd^c(2 - |w|^{2p}) = -p^2|w|^{2p-2}\frac{i}{2}dw\wedge d\bar w \quad \text{for } |w|\leqslant 1. \tag{5.9}$$

For $p > 0$ consider $(2,2)$-currents S_p which are define as

$$< S_p, \psi > = \frac{1}{\pi}\int_E \pi^*\varphi_p\cdot\psi|_E \tag{5.10}$$

for a $(2,2)$-form ψ on X. Note that S_p are positive. Set $t_p = \frac{1}{\pi p}$. We want to prove that $t_p dd^c S_p \to [C_\infty]$ as $p \searrow 0$.

First remark that $\langle t_p dd^c S_p, \psi\rangle = \langle t_p dd^c \varphi_p, \psi\rangle$ for a test $(1,1)$-form ψ. Here $dd^c\varphi_p$ should be understood in the sense of distributions. From (5.8) we see that

$$t_p \int\limits_{\Delta} dd^c\varphi_p = 1, \quad dd^c\varphi_p \geqslant 0 \text{ and } t_p dd^c\varphi_p = \frac{p}{\pi}|z|^{2p-2}\frac{i}{2} \to 0$$

as $p \searrow 0$ on compacts of $\Delta^* := \Delta \setminus \{0\}$. This implies that $t_p dd^c\varphi_p \to \delta_{\{0\}}$ in Δ. In the same manner from (5.9) we get that $t_p dd^c\varphi_p \to -\delta_{\{\infty\}}$ in $\mathbb{CP}^1 \setminus \bar{\Delta}$.

Claim. $t_p dd^c\varphi_p \to 0$ *on the annulus* $A := \Delta(2)\setminus\bar{\Delta}(1/2)$.

Take a test function ψ with support in A, set $\tilde{\psi}(r) = \int_0^{2\pi}\psi(r,\theta)d\theta$ and write

$$4t_p \int\limits_{A} dd^c\varphi_p \cdot \psi := t_p \int_{1/2}^{2}\int_0^{2\pi}\varphi_p(r)\Delta\psi r d\theta dr = t_p\int_{1/2}^{2}\int_0^{2\pi}\varphi_p(r)\big(r\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial\psi}{\partial r}\big)d\theta dr =$$

$$= t_p\int_{1/2}^{2}\varphi_p(r)\big(r\tilde{\psi}'' + \tilde{\psi}'\big)dr = t_p\Big[\int_{1/2}^{1} r^{2p+1}\tilde{\psi}''dr + \int_1^2\big(2 - \frac{1}{r^{2p}}\big)r\tilde{\psi}''dr\Big] +$$

$$t_p\Big[\int_{1/2}^{1} r^{2p}\tilde{\psi}'dr + \int_1^2\big(2-\frac{1}{r^{2p}}\big)\tilde{\psi}'dr\Big] =: I_p'' + I_p'.$$

Next we compute these integrals separately.

$$I_p' = t_p\Big[\int_{1/2}^{1}\big(r^{2p}\tilde{\psi}\big)'dr + \int_1^2\big(\big(2-\frac{1}{r^{2p}}\big)\tilde{\psi}\big)'dr\Big] - \frac{2p}{\pi p}\int_{1/2}^{1}r^{2p-1}\tilde{\psi}dr - \frac{2p}{\pi p}\int_1^2 r^{-2p-1}\tilde{\psi}dr =$$

$$= \frac{\tilde{\psi}(1)}{\pi p} - \frac{\tilde{\psi}(1)}{\pi p} - \frac{2}{\pi}\Big(\int_{1/2}^{1}r^{2p-1}\tilde{\psi}dr + \int_1^2 r^{-2p-1}\tilde{\psi}dr\Big) \to -\frac{2}{\pi}\int_{1/2}^{2}\frac{\tilde{\psi}}{r}dr \tag{5.11}$$

as $p \searrow 0$. At the same time

$$I_p'' = t_p\int_{1/2}^{1}\big(r^{2p+1}\tilde{\psi}'\big)'dr + t_p\int_1^2\big(\big(2-\frac{1}{r^{2p}}\big)r\tilde{\psi}'\big)'dr - \frac{2p+1}{\pi p}\int_{1/2}^{1}r^{2p}\tilde{\psi}'dr -$$

$$-\frac{1}{\pi p}\int_1^2\big(2+\frac{2p-1}{r^{2p}}\big)\tilde{\psi}'dr = \frac{\tilde{\psi}'(1)}{\pi p} - \frac{\tilde{\psi}'(1)}{\pi p} - \frac{2p+1}{\pi p}\int_{1/2}^{1}\big(r^{2p}\tilde{\psi}\big)'dr - \frac{1}{\pi p}\int_1^2\big(\big(2+\frac{2p-1}{r^{2p}}\big)\tilde{\psi}\big)'dr$$

$$+\frac{2(2p+1)}{\pi}\int_{1/2}^{1}r^{2p-1}\tilde{\psi}dr - \frac{2p(2p-1)}{\pi p}\int_1^2\frac{\tilde{\psi}}{r^{2p+1}}dr \to -\frac{2p+1}{\pi p}\tilde{\psi}(1) + \frac{2p+1}{\pi p}\tilde{\psi}(1) +$$

$$+\frac{2}{\pi}\Big(\int_{1/2}^{1}\frac{\tilde{\psi}}{r}dr + \int_1^2\frac{\tilde{\psi}}{r}dr\Big) = \frac{2}{\pi}\int_{1/2}^{2}\frac{\tilde{\psi}}{r}dr. \tag{5.12}$$

(5.11) and (5.12) cancel each other and the Claim is proved. We conclude that

$$t_p dd^c\varphi_p \to \delta_{\{0\}} - \delta_{\{\infty\}} \tag{5.13}$$

on $\mathbb{CP}^1$. From (5.10) we conclude immediately that $t_p dd^c S_p \to [C_\infty]$.

□

Remark 5.1. Lemmas 5.2 and 5.3 show that the manifold in the example of Hironaka belongs to class $\mathcal{E}_0$, *i.e.,* that $\mathcal{E}_0 \neq \varnothing$. In order to get a holomorphic foliation by curves $\mathcal{L}$ on X such that the foliated manifold $(X, \mathcal{L})$ belongs to $\mathcal{E}_0$ it is sufficient to take $\mathcal{L}$ in such a way that the current $T := [C_\infty]$ is directed by $\mathcal{L}$. The rational fibration $\{x = y = \mathsf{const}\}$ will do the job.

5.4. Foliations with Plurinegative Taming Forms. In this section we consider the class $\mathcal{E}_-$ of foliated manifolds $(X,\mathcal{L})$ which possess a plurinegative taming form ω. Our goal is to understand the vanishing cycles in $\mathcal{L}$. We start with the following:

Proposition 5.3. *If a leaf $\mathcal{L}_z$ of a disc-convex foliated manifold $(X,\mathcal{L},\omega)\in\mathcal{E}_-$ contains an essential vanishing cycle then there exists a nontrivial positive d-exact $(1,1)$-current T tangent to $\mathcal{L}$. The support of T is contained in $\overline{\mathcal{L}_z}$.*

Proof. Let $\gamma\subset\mathcal{L}_z^0$ be (an essential) vanishing cycle. From the proofs of Theorems 2.2 and 3.1 we see that there exists a foliated meromorphic immersion $h:(\Delta^{n+1},\mathcal{L}^{\mathrm{v}})\setminus S\to(X,\mathcal{L})$ such that $h(\partial\Delta_0)=\gamma$ and the essential singularity set S of h intersects Δ_0. From (2.17) we see that there exist $q_n\to 0$ in Δ^n such that $h|_{\partial\Delta_{q^n}}$ converges but $\mathsf{area}(h(\Delta_{q^n}))$ diverges to infinity. Therefore by standard (and obvious) reasoning currents

$$T_n=\frac{[h(\Delta_{q^n})]}{\mathsf{area}(h(\Delta_{q^n}))} \tag{5.14}$$

converge to a closed, positive current T of mass one tangent to $\mathcal{L}$.

To prove that T is, in fact, exact, observe that by Lemma 2.2 from [Iv3] one have that $H^2_{DR}(\Delta^2\setminus S)=0$, where H_{DR} denotes the de Rham cohomology. Let φ be a d-closed 2-form on X. Then $h^*\varphi=d\psi$ for some 1-form in $\Delta^2\setminus S$. Therefore

$$<T,\varphi>=\lim_{n\to\infty}<T_n,\varphi>=\lim_{n\to\infty}\frac{1}{\mathsf{area}(h(\Delta_{q^n}))}\int\limits_{\Delta_{q^n}}h^*\varphi=\lim_{n\to\infty}\frac{1}{\mathsf{area}(h(\Delta_{q^n}))}\int\limits_{\partial\Delta_{q^n}}h^*\psi=$$

$$=\int\limits_{\partial\Delta_0}h^*\psi\lim_{n\to\infty}\frac{1}{\mathsf{area}(h(\Delta_{q^n}))}=0.$$

□

Now we turn to the

Proof of Theorem 4. Let $z_0\in X^0$ be a point such that the leaf $\mathcal{L}^0_{z^0}$ of the foliation $\mathcal{L}$ through z^0 contains a vanishing cycle γ_0. According to subsection 3.2 the situation can be reduced to the following. Take a transverse n-disc D through z^0, $n+1$ being the complex dimension of X, in such a way that after identification of D with Δ^n the point z^0 corresponds to 0. Let $\hat{\mathcal{L}}_D$ be the completed holonomy covering cylinder of $\mathcal{L}$ over D. Then there exists an imbedded vanishing cycle $\hat\gamma_0\subset\hat{\mathcal{L}}_{z^0}$ which projects to (some another) vanishing cycle $\gamma_0\subset\mathcal{L}_{z^0}$ under the natural meromorphic projection $p:\hat{\mathcal{L}}_D\to X$. This γ_0 may not be an imbedded loop and may be different from that one which was taken at the beginning.

Shrinking D, if necessary, we can construct a generalized Hartogs figure (W,π,U,D) over D, where W is an open subset of $\hat{\mathcal{L}}_D$, U is a neighborhood of some point $z\in D$, $\pi:W\to D$ is the restriction of the natural projection $\pi:\hat{\mathcal{L}}_D\to D$ to W.

According to Theorem 2.2 we get that our meromorphic foliated immersion $p:(W,\pi,U,D)\to(X,\mathcal{L},\omega)$ extends, after a reparametrization, to a meromorphic foliated immersion $\tilde p:\widetilde{W}\setminus S\to X$, where::

- $(\widetilde{W},\pi,D)$ is a *complete* Hartogs figure over D. That means the complex manifold $\widetilde{W}$ is holomorphically foliated by discs by the holomorphic submersion $\pi:\widetilde{W}\to D$.

- S is a closed subset of $\widetilde{W}$ which has the following structure: there exists a closed, complete $(n-1)$-polar subset $S_1 \subset \Delta^n$ such that $S = \bigcup_{s\in S_z} S_z$ where S_z is a compact subset of $\widetilde{W}_z := \pi^{-1}(z)$.
- $(n-1)$-polarity of S_1 means that Δ^n in its turn (after shrinking, if necessary) can be decomposed as $\Delta^{n-1}\times\Delta$ in such a way that for every $\lambda\in\Delta^{n-1}$ the intersection $S_{1,\lambda} := \big(\{\lambda\}\times\Delta\big)\cap S_1$ is a complete polar (and compact in our case) subset of Δ.

Now we have the following two possible cases.

Case 1. For every $\lambda\in\Delta^{n-1}$ the set $S_{1,\lambda}$ is not empty.

Take a point $s_\lambda \in S_{1,\lambda}$ for every $\lambda\in\Delta^{n-2}$. From Theorem 2.1 it readily follows that for every λ there exists a sequence $(\lambda_n, z_{1,n}) \to s_{1,\lambda}$ such that

$$\mathsf{area}\big[\tilde{p}(\Delta_{\lambda_n,z_{1,n}})\big] \to \infty. \tag{5.15}$$

As it was explained in Proposition 5.3 such sequence accumulates to an exact, positive $(1,1)$-current T_λ directed by $\mathcal{L}$. Remark that the support of each T_λ belongs to $\mathcal{L}_{\Delta^2_\lambda}$ and therefore they are distinct for different λ-s.

Case 2. For some $\lambda_0\in\Delta^{n-1}$ the set S_{1,λ_0} is empty.

Then $S_{1,\lambda}=\varnothing$ for λ in a neighborhood of λ_0. We can assume that λ_0 is as close to 0 in Δ^{n-1} as we wish. Otherwise for some neighborhood of 0 the Case 1 will occur. Restricting $\widetilde{W}$ to a smaller polydisc, if needed, we find ourselves in the conditions where $(\widetilde{W},\pi,D)$ is isomorphic to $(\Delta^{n+1},\pi,\Delta^n)$ where $\pi:\Delta^{n+1}\to\Delta^n$ is the canonical projection. We find ourselves in the assumptions of [IS]. More precisely, we have a meromorphic mapping $\tilde{p}: H^n_2(\varepsilon)\to X$ where:

i) $H^n_2(\varepsilon) = \big[\Delta^{n-1}(\lambda_0,\varepsilon)\times\Delta^2\big]\cup\big[\Delta^{n-1}\times A^2(1-\varepsilon,1)\big]$ - the Hartogs figure of bidimension $(n-1,2)$;

ii) $\tilde{p}$ is holomorphic on $\Delta^{n-1}\times A^2(1-\varepsilon,1)$;

iii) the image manifold X admits a dd^c-closed positive $(2,2)$-form ω_2.

By the result of [IS] the mapping $\tilde{p}$ meromorphically extends to $\Delta^n\setminus R$, where the singularity set R is either empty, or has the following structure:

i) $R=\cup_{\lambda\in R_1}R_\lambda$, where R_1 is complete $(n-3)$-polar closed subset of Δ^{n-2} and each R_λ is complete polar compact of Δ^2_λ of Hausdorff dimension zero.

ii) Again, the $(n-3)$-polarity of R_1 means that in a neighborhood of zero we can decompose $\Delta^{n-1}=\Delta^{n-2}\times\Delta$ in such a way that for every $\lambda'=(\lambda_1,\dots,\lambda_{n-2})$ the set $R_{\lambda'} := R\cap\Delta^3_{\lambda'}$ is a compact, pluripolar subset of Hausdorff dimension zero.

iii) Moreover $\tilde{p}(\partial\Delta^3_{\lambda'})$ is not homologous to zero in X.

The last item means that $\tilde{p}(\partial\Delta^3_{\lambda'})$ is a foliated three-dimensional shell in X for every $\lambda'\in\Delta^{n-2}$.

Theorem 4 is proved.

5.5. Foliations with Pluriexact Directed Currents. This subsection is devoted to the class $\mathcal{E}_+$. By $\mathcal{L}^s := \mathcal{L}^{\mathsf{sing}}$ we denote the singular set of $\mathcal{L}$. $\mathcal{L}^s$ has complex codimension at least two.

Proof of Theorem 5. Let R be a positive $(2,2)$-current such that $T := dd^cR$ is non-trivial, positive and directed by $\mathcal{L}$. Exactly as in the proof of Lemma 5.2 (taking $\mathcal{L}^s$ instead of E and R as S in the notations of the proof) we get that $\chi_{\mathcal{L}^s}T = \chi_{\mathcal{L}^s}dd^c(\chi_{X\setminus\mathcal{L}^s}R)$ is a negative current supported on $\mathcal{L}^s$. But T is positive and therefore $\chi_{\mathcal{L}^s}T = 0$. This proves the part *(i)* of our theorem.

Now we shall prove the part *(ii)* . We fix a strictly positive $(1,1)$-form Ω on X. Masses of positive (p,p)-currents R on X of order zero will be measured as $\|R\| := \langle R, \Omega^p \rangle$. First remark that for every $(1,1)$-form ω on X such that $dd^c\omega + \Omega^2 \geqslant 0$ one has $\langle T, \omega \rangle \geqslant -\|R\|$. This is immediate, just write

$$\langle T, \omega \rangle = \langle dd^c R, \omega \rangle = \langle R, dd^c \omega \rangle \geqslant \langle R, -\Omega^2 \rangle = -\|R\|. \tag{5.16}$$

Second, for $n \geqslant 3$ consider the following $(n-2, n-2)$-form in $\mathbb{C}^n$

$$\omega_0 = -\frac{i}{2} \frac{(dz, dz)}{\|z\|^2} \wedge \left(dd^c \|z\|^2 \right)^{n-3}. \tag{5.17}$$

One checks easily that:

i) $dd^c\omega_0$ is positive;

ii) $dd^c\omega_0 \in L^p_{loc}$ for every $p < \frac{n}{2}$.

Both properties follow immediately from the following expression:

$$dd^c \frac{i}{2} \frac{(dz, dz)}{\| z \|^2} = \frac{i^2}{4\pi} \frac{(dz, dz)}{\| z \|^6} \wedge (2(dz, z) \wedge (z, dz) - \| z \|^2 (dz, dz)),$$

where the form on the right hand side is non-positively definite. Indeed, for a vector $v = (v_1, ..., v_n) \in \mathbb{C}^n$ one has

$$\begin{aligned}(\| z \|^2 (dz, dz) - 2(dz, z) \wedge (z, dz)) \wedge v \wedge \bar{v} &= \| z \|^2 \| v \|^2 - 2(v, z)(z, v) \\ &= \| z \|^2 \| v \|^2 - 2|(v, z)|^2 \geq 0\end{aligned}$$

by Cauchy-Schwarz inequality.

Taking $\rho\omega$ for an appropriate cut-off function ρ with support in the unit polydisc we can extend ω_0 to $\mathbb{C}^n$ by zero and with $dd^c\omega_0 \geqslant -\eta$ for $\eta > 0$ as small as we need. Smoothing by convolution we get ω_ε converging to ω_0 with support in the unit polydisc and having $dd^c\omega_\varepsilon \geqslant -2\eta$. Now we can push $\omega_\varepsilon(z - \zeta)$ to X verifying $dd^c\omega_\varepsilon(z - \zeta) + \Omega^2 \geqslant 0$ for all ζ in a neighborhood of a fixed center p_0 of some foliated coordinate chart.

Let $B = \Delta^n \times \Delta$ be a foliated chart for $\mathcal{L}$ and μ the induced by T Radon measure on Δ^n. Coordinates in B denote as $(z', z_{n+1}) = (z_1, ..., z_n, z_{n+1})$. Write

$$\langle T, \omega_\varepsilon \rangle \sim -\frac{i}{2} \int\limits_{\Delta^n} \int\limits_{\Delta} \frac{dz_{n+1} \wedge d\bar{z}_{n+1}}{\|z' - \zeta'\|^2 + |z_{n+1}|^2} d\mu(z') = -2\pi \int\limits_{\Delta^{n-1}} d\mu(z') \int_0^1 \frac{r dr}{r^2 + \|z' - \zeta'\|^2} \sim$$

$$\sim \pi \int\limits_{\Delta^{n-1}} \ln \|z' - \zeta'\|^2 d\mu(z') \geqslant -\|S\|$$

by (5.16) and independently of $\zeta' \in \Delta^n$. Theorem is proved.

□

Remark 5.2. The singularity set S repeatedly comes out in the proofs of this paper. Let us make a few remarks about this issue.

(a) An appearance of S is a highly *non-algebraic* phenomenon. It is sufficient to say that due to the Hartogs type extension theorem of [Iv3] the set S is always empty if X is Kähler (or if $(X, \mathcal{L})$ admits a d-closed taming form).

(b) In the case of a pluriclosed taming form S is *proper* over a subspace of complex codimension two, this is due to the homological nature of shells, see Lemma 2.3 and Theorem 2.3.

(c) If the taming form is only plurinegative the size of S can fall down, and this phenomena is responsible for the appearance of three-dimensional shells, see Theorem 4 and Example 3.

6. Other Results, Examples and Open Questions

We still owe the proofs of some statements used in the text of this paper and of some propositions from the Introduction. Moreover it is the time to give more interesting examples (from the point of view of this text) then just foliations on complex surfaces or on Kähler manifolds. Looking on each example in this Section we shall be rather attentive to its Hartogs properties because, as it should be clear from the proofs of this paper, the failure of a foliated manifold $(X,\mathcal{L})$ to be Hartogs is "almost equivalent" to the presence of essential vanishing cycles/foliated shells in $(X,\mathcal{L})$.

6.1. Hartogs foliation on a compact non-Hartogs threefold. The following example is due to Nakamura, see [Na]. We only interpret it according to our needs adding a foliation to it.

Example 6.1. Take any matrix $A\in SL(2,\mathbb{Z})$ with real eigenvalues $\alpha<1$ and $1/\alpha$. For example the following one:

$$A=\begin{pmatrix} 1 & 1\\ 1 & 2\end{pmatrix}. \tag{6.1}$$

Here $\alpha=3/2-\sqrt{5}/2$. Consider the standard integer lattice $\Lambda_0:=\mathbb{Z}^4$ in $\mathbb{C}^2$. A preserves Λ_0 and therefore defines a holomorphic automorphism A of the torus $\mathbb{T}_0:=\mathbb{C}^2/\Lambda_0$. Therefore we can construct a compact complex threefold $X_0:=\mathbb{C}^*\times\mathbb{T}_0/<g>$ where $g(z,Z)=(\alpha z,AZ)$. X_0 is a complex 2-torus bundle over a complex 1-torus $\mathbb{C}^*/<\alpha>$. We fix the coordinate z for $\mathbb{C}^*$.

Let v be the eigenvector of A with eigenvalue α and w be that with $1/\alpha$. It will be appropriate for the forthcoming construction to take v, w as the basis in $\mathbb{C}^2$, where A acts, and to introduce coordinates $Z=(z_1,z_2)$ in this basis, *i.e.*, now we have: $\mathrm{v}=(1,0)$ and $\mathrm{w}=(0,1)$. In these coordinates A acts as $AZ=(\alpha z_1, 1/\alpha z_2)$. Observe that our lattice Λ_0 is irrational in these coordinates. A foliation on X_0 we construct as follows. Take first the "vertical" foliation $\{z_1=\mathsf{const}\}$ in $\mathbb{C}^2$, factor it by Λ_0. Due to the irrationality of Λ in the new basis it will have dense leaves. Now we observe that this foliation is obviously invariant under the action of A, which is simply multiplication by $1/\alpha$ on the leaves. Therefore the "vertical" foliation $\mathcal{L}^{\mathrm{v}}=\{z=\mathsf{const}, z_1=\mathsf{const}_1\}$ descends from $\mathbb{C}^*\times\mathbb{C}^2$ to X_0 and we denote it as $\mathcal{L}_0$.

Now, following [Na], we shall deform $(X_0,\mathcal{L}_0)$. In the subspace $\mathbb{C}^2_{z,z_1}:=\mathbb{C}_z\times\mathbb{C}_{z_1}$ of our coordinate space $\mathbb{C}^3_{z,Z}:=\mathbb{C}_z\times\mathbb{C}^2_{z_1,z_2}$ we take a real subspace $\mathbb{R}^2_\tau$ - a deformation of $\{0\}_z\times\mathbb{C}_{z_1}$. Parameter τ here runs in $Gr_{\mathbb{R}}(2,4)$. This subspace $\mathbb{R}^2_\tau\subset\mathbb{C}^2_{z,z_1}$ we see as the graph of the uniquely defined $\mathbb{R}$-linear map $L_\tau:\mathbb{C}^2_{z_1}\to\mathbb{C}_z$ and therefore the subspace $\mathbb{R}^4_\tau:=\mathbb{R}^2_\tau\times\mathbb{C}_{z_2}$ is a graph of $(L_\tau,\mathsf{Id}):\mathbb{C}^2_{z_1,z_2}\to\mathbb{C}_z$. By Λ_τ we denote the image of the lattice Λ_0 under (L_τ,Id) - a deformation of Λ_0. Denote by $\mathbb{T}_\tau$ the torus $\mathbb{R}^4_\tau/\Lambda_\tau$. Remark that A still preserves Λ_τ and therefore $\mathbb{C}^3_{z,Z}\setminus\{0\}_z\times\mathbb{R}^4_\tau$ factors first by Λ_τ and then by (α,A) to a compact complex threefold X_τ which is a real 4-torus bundle over a complex

1-torus $\mathbb{C}^*/<\alpha>$. Our "vertical" foliation $\mathcal{L}^{\mathrm{v}}$ descends again to X_τ and we denote the result as $\mathcal{L}_\tau$. The construction of $(X_\tau, \mathcal{L}_\tau)$ is finished.

In the following Proposition $\mathcal{V}$ denotes a sufficiently small neighborhood of $\{0\}\times\mathbb{C}_{z_1}$ in $Gr_{\mathbb{R}}(2,\mathbb{C}^2_{z,z_1})$.

Proposition 6.1. *The family of foliated* 3*-folds* $\{(X_\tau,\mathcal{L}_\tau) : \tau\in\mathcal{V}\}$*, constructed above, possesses the following properties:*

i) Manifolds X_τ *do not admit a* dd^c*-closed (even* dd^c*-negative) metric form for all* $\tau\in\mathcal{V}\setminus\mathbb{CP}^1$ *and* X_τ *is not even almost Hartogs.*

ii) At the same time all $(X_\tau,\mathcal{L}_\tau)$ *are Hartogs.*

Proof. (*i*) The fact that X_τ are not Kähler is explained in [BK], see pp. 82-84. For $\tau\in V\setminus\mathbb{CP}^1$ our X_τ has $\mathbb{C}^3\setminus\mathbb{R}^4_\tau$ as an unramified covering. For this reason it is also not almost Hartogs. Really, the covering map is singular along $\mathbb{R}^4_\tau$ which is much more massive then just a countable union of complex curves. But it is also to massive as a singularity set for the covering map in the event that X_τ would admit a plurinegative metric form, see the Main Theorem from [Iv6].

(*ii*) Let $h:(W,\pi,U,V)\to(X_\tau,\mathcal{L}_\tau)$ be a holomorphic foliated generic injection of a three dimensional generalized Hartogs figure into the foliated manifold $(X_\tau,\mathcal{L}_\tau)$. Without a loss of generality assume that $U\subset V$ are bidiscs, so W is simply connected. Lift h to a foliated generic injection $\tilde h$ of (W,π,U,V) into $(\mathbb{C}^3\setminus\mathbb{R}^4_\tau,\mathcal{L}^{\mathrm{v}})$. Then it extends after a reparametrization as a map with values in $(\mathbb{C}^3,\mathcal{L}^{\mathrm{v}})$. But the fiber of $\mathcal{L}^{\mathrm{v}}$ which touches $\mathbb{R}^4_\tau$ is entirely contained in $\mathbb{R}^4_\tau$ and therefore the extended map never hits $\mathbb{R}^4_\tau$. After that we can descend the extended map back to $(X_\tau,\mathcal{L}_\tau)$.

□

6.2. Rationality. First of all we shall prove the following

Lemma 6.1. *Let* ω *be a plurinegative taming form for the vertical foliation on the product* $\Delta\times\mathbb{CP}^1$*. Then the volume function*

$$\mathrm{v}_\omega(z_1)=\int\limits_{\{z_1\}\times\mathbb{CP}^1}\omega$$

is superharmonic. If, moreover, ω *is pluriclosed then* v_ω *is harmonic.*

Proof. For any test function ψ in Δ we have

$$<\psi,\Delta\mathrm{v}_\omega>=\frac{i}{2}\int_\Delta\Delta\psi\left(\int_{\{z_1\}\times\mathbb{CP}^1}\omega\right)d\zeta\wedge d\bar\zeta=\int_{\Delta\times\mathbb{CP}^1}dd^c(\pi^*\psi)\wedge\omega=$$

$$=\int_{\Delta\times\mathbb{CP}^1}\pi^*\psi\wedge dd^c\omega\le 0, \tag{6.2}$$

i.e., v_ω is superharmonic. If ω was pluriclosed then (6.2) becomes an equality and consequently in this case v_ω is harmonic.

□

Recall that by $\mathcal{R}_{\mathcal{L}}$ we denoted the analytic space of rational cycles on X tangent to $\mathcal{L}$. Fix a plurinegative taming form ω and consider the area function $\mathrm{v}_\omega:\mathcal{R}_{\mathcal{L}}\to\mathbb{R}^+$ defined by (2.16).

Corollary 6.1. *Suppose that $\mathcal{L}$ is tamed by a plurinegative form ω. Then every irreducible component of $\mathcal{R}_{\mathcal{L}}$ is compact and every connected component consists of finitely many irreducible ones. The volume function v_{ω} is constant on every connected component of $\mathcal{R}_{\mathcal{L}}$.*

Proof. Let first that $\mathcal{K}$ is an irreducible component of $\mathcal{R}_{\mathcal{L}}$. Denote by $\mathcal{C}_{\mathcal{K}}$ the universal family over $\mathcal{K}$. $\mathcal{C}_{\mathcal{K}}$ comes with two natural mappings: projection $\pi : \mathcal{C}_{\mathcal{K}} \to \mathcal{K}$ and inclusion $p : \mathcal{C}_{\mathcal{K}} \to X$. Take an analytic disc $\varphi : \Delta \to \mathcal{K}$ and the restriction $\mathcal{C}_{\Delta}$. Lemma 6.1 shows that $\mathrm{v}_{\omega}|_{\varphi(\Delta)}$ is superharmonic. Therefore v_{ω} is plurisuperharmonic on $\mathcal{K}$. Suppose $\mathcal{K}$ is not compact. Take a divergent sequence of points $\{k_n\} \subset \mathcal{C}$. Now two cases could occur:

Case 1. $\mathrm{v}_{\omega}(k_n)$ *stays bounded (may be on some subsequence).*

In that case we can subtract a converging subsequence of rational cycles C_{k_n}. The limit is again a rational cycle C_0 which obviously should belong to our irreducible component $\mathcal{K}$. Contradiction.

Case 2. $\mathrm{v}_{\omega}(k_n) \to \infty$. So $v(k)$ increases when k goes to infinity in $\mathcal{K}$, *i.e.,* leaves every compact. But this contradicts to the minimum principle for (pluri)-harmonic functions.

Therefore $\mathcal{K}$ is compact and v_{ω} is constant on $\mathcal{K}$. This implies that v is constant on every connected component of $\mathcal{R}_{\mathcal{L}}$. Suppose there exist a sequence $\mathcal{K}_n$ of irreducible components of some connected component $\mathcal{N}$. Take $k_n \in \mathcal{K}_n$. Then $\mathrm{v}_{\omega}(C_{k_n})$ is constant and therefore some subsequence C_{k_n} converges to some rational cycle C_0 which corresponds to a point k_0 in $\mathcal{R}_{\mathcal{L}}$. But in this case $\mathcal{R}_{\mathcal{L}}$ contains a sequence of compact irreducible components having an accumulation point k_0. This contradicts to the fact that $\mathcal{R}_{\mathcal{L}}$ is a complex space.

Therefore each connected component of $\mathcal{R}_{\mathcal{L}}$ consists from a finite number of compact irreducible ones.

□

The following is immediate:

Corollary 6.2. *A foliated manifold $(X, \mathcal{L})$ which admits a plurinegative taming form has bounded rational cycle geometry.*

Let us turn to the proof Corollary 4 from the Introduction. It is sufficient to establish the following:

Lemma 6.2. *Let $(\tilde{\mathcal{L}}_D, \pi)$ be a covering cylinder of holomorphic foliation by curves $\mathcal{L}$ on a compact complex manifold X which admits dd^c-negative metric form. Suppose that D is biholomorphic to the polydisc and that there exists $z \in D$ such that the fiber $\tilde{\mathcal{L}}_z = \pi^{-1}(z)$ is isomorphic to $\mathbb{CP}^1$. Then $\pi^{-1}(D) \sim D \times \mathbb{CP}^1$.*

Proof. The set U of $z \in D$ such that $\tilde{\mathcal{L}}_z \sim \mathbb{CP}^1$ is clearly open. Each connected component U' of U naturally is included in some irreducible component $\mathcal{K}$ of $\mathcal{R}_{\mathcal{L}}$. Therefore the area function $\mathrm{v}_{\omega}(z) = \mathsf{area}_{\omega}(\tilde{\mathcal{L}}_z)$ is constant on U'. But this implies that for any boundary point $z^0 \in \partial U' \cap D$ the fiber $\tilde{\mathcal{L}}_{z^0}$ is again rational. Therefore $U' = D$ and $\tilde{\mathcal{L}}_D = D \times \mathbb{CP}^1$.

□

Remark 6.1. Precisely the same argument as in (6.2) gives that every irreducible component of a (not necessarily rational) cycle space of compact curves tangent to $(X, \mathcal{L}) \in \mathcal{E}_{-}$ is compact and the area is constant. This implies, for example, that if a holonomy covering of some leaf is compact then the same is true for all generic leaves.

Example 6.2. *As it is shown in* [K3] *there exists a compact complex manifold X of dimension* 5 *and a smooth holomorphic foliation by curves $\mathcal{L}$ on X such that there exists a non-empty domain $W \subset X$ with $X \setminus \bar{W} \neq \varnothing$ having the following properties:*

i) If $z^0 \in W$ then $\mathcal{L}_{z^0} \subset W$ and $\mathcal{L}_{z^0} \equiv \mathbb{CP}^1$.

ii) There exists thin subset S of $X \setminus \bar{W}$ such all compact leaves in $X \setminus \bar{W}$ are contained in S.

Lemma 6.2 implies now that this $(X, \mathcal{L})$ doesn't admit a plurinegative taming form.

6.3. Preservation of cycles. Let $\mathcal{L}$ be a foliation by curves on a disc-convex complex manifold X and D be a transversal smooth hypersurface. We shall work on the holonomy covering cylinder $\hat{\mathcal{L}}_D$. If $\mathcal{L}$ is smooth the same works also for $\mathcal{L}_D$. Take a point $z \in D$ and a loop $\gamma \in \pi_1(\hat{\mathcal{L}}_z)$. Reference point for $\pi_1(\hat{\mathcal{L}}_z)$ will be always z.

Definition 6.1. *The domain of preservation of the homotopy class $[\gamma]$ is a topological space $\Omega_{\gamma,D}$ defined as follows:*

1) the points of $\Omega_{\gamma,D}$ are homotopy classes $[\gamma^{'}] \in \pi_1(\hat{\mathcal{L}}_{z'})$ (where $z^{'}$ is any point of D) such that some representative $\gamma^{'}$ of $[\gamma^{'}]$ can be joined by a homotopy γ_t of loops in $\hat{\mathcal{L}}_{z(t)}$ with some representative γ of $[\gamma]$. Here $z(t)$ is a path in D from $z^{'}$ to z.

2) the topology on $\Omega_{\gamma,D}$ is defined in a natural way saying that $[\gamma_n]$ converge to $[\gamma]$ if some representatives converge uniformly.

Let Ω_γ be the domain of preservation of the (homotopy class $[\gamma]$ in fact) of our loop γ. There is a natural projection $p : \Omega_\gamma \to D$ sending $[\gamma^{'}] \in \pi_1(z^{'})$ to $z^{'}$.

Proof of Proposition 5. Suppose that for some loop $\gamma \subset \hat{\mathcal{L}}_z$ the space $\Omega_{\gamma,D}$ is not Hausdorff. That means that there exists $z^0 \in D$, two loops $\gamma, \beta \subset \hat{\mathcal{L}}_{z^0}$ representing different homotopy classes in $\pi_1(\hat{\mathcal{L}}_{z^0})$ and two sequences of loops $\gamma_n, \beta_n \subset \hat{\mathcal{L}}_{z_n}$, homotopic to each other in $\hat{\mathcal{L}}_{z_n}$, converging to γ and β respectively.

Taking $\alpha_n := \gamma_n \circ \beta_n^{-1}$ we obtain a sequence of loops, homotopic to zero and converging to a loop $\alpha \subset \hat{\mathcal{L}}_{z^0}$ which is not homotopic to zero.

We are exactly in the situation of the proof of the Theorem 1 and therefore deduce the existence of a foliated shell in $\mathcal{L}$.

The local biholomorphicity of the projection p is obvious.

□

The phenomena of preservation of cycles to our knowledge was first studied by Landis-Petrovsky in [LP], see also [Iy1].

6.4. Foliations with compact fibers. Let $\mathcal{L}$ be a smooth holomorphic foliation of dimension one on an n-dimensional complex manifold X. We suppose that all leaves of $\mathcal{L}$ are compact.

Proof of Proposition 4. (i) We denote by ω an adapted to $\mathcal{L}$ plurinegative (1,1)-form. Let $\mathcal{L}_z$ be a leaf of $\mathcal{L}$ through the point $z \in X$. If $\mathcal{L}_z$ is compact with finite holonomy we denote by $n(\mathcal{L}_z)$ the cardinality of the holonomy group of $\mathcal{L}_z$ and set

$$\mathrm{v}(z) = \mathsf{Vol}(\mathcal{L}_z) = n(\mathcal{L}_z)\int_{\mathcal{L}_z} \omega. \tag{6.3}$$

Denote by Ω the connected component of the set of $z \in X$ such that the leaf $\mathcal{L}_z$ of $\mathcal{L}$ through z is compact and has finite holonomy which contains our compact leaf. By the Reeb local stability theorem Ω is an open set in X.

Case 1. There exists $z^0 \in \partial\Omega$ which is a limit of $z_n \in \Omega$ with $\mathrm{v}(z_n)$ uniformly bounded.

For any transversal D to $\mathcal{L}_{z^0}$ the intersection $D \cap \Omega$ is open in D and every $\mathcal{L}_{z_n}$ cuts D by a bounded number of points, say N. This readily follows from the boundedness of volumes of $\mathcal{L}_{z_n}$. Therefore for every $h \in \mathsf{Hol}(\mathcal{L}_{z^0})$ its order is at most $N!$, *i.e.*, $h^{N1} = \mathsf{Id}$. Therefore the holonomy group $\mathsf{Hol}(\mathcal{L}_{z^0})$ has finite exponent and therefore it is finite itself, see Lemma 2 from [P]. Therefore z^0 is an interior point of Ω. Contradiction.

We are left with the following possibility:

Case 2. $\mathrm{v}(z) \to \infty$ *when* $z \to \partial\Omega$.

This case is excluded by Remark 6.1. All is left to remark that if $\partial\Omega \neq \varnothing$ we obtain a contradiction with the minimum principle for plurisuperharmonic functions.

Therefore $\Omega = X$ and (i) is proved.

(ii) By the standard observation in foliation theory, see ex. [Go] the set of leaves without holonomy is not thin in X. Therefore we are done by (i).

□

Remark 6.2. Without any changes this proof applies to smooth holomorphic q-dimensional foliations on compact complex manifolds admitting dd^c-negative taming (q,q)-forms.

6.5. Holomorphicity of Complex Fibrations. Let us give one more corollary of the compactness of the spaces of cycles tangent to a foliation admitting a plurinegative adapted form. Denote by $\mathcal{B}_{\mathcal{L}}$ the space of q-cycles tangent to $\mathcal{L}$. By $\mathcal{Z}$ we denote the corresponding universal family and let $\mathsf{ev} : \mathcal{Z} \to X$ be the evaluation map.

Theorem 6.1. *Let $\mathcal{L}$ be a smooth real foliation on compact complex manifold X with all leaves being compact complex manifolds of complex dimension q. Suppose that $\mathcal{L}$ admits a plurinegative adapted (q,q)-form ω and that $q \geqslant \frac{1}{2}\mathsf{dim}_{\mathbb{C}} X$. Then $\mathcal{L}$ is holomorphic.*

Proof. It is easy to see that $\mathcal{B}_{\mathcal{L}}$ has at most countably many irreducible components, they are all compact and the volume function is constant on each of them. The argument is similar to that of Lemma 6.1. That immediately gives that some irreducible component $\mathcal{K}$ of $\mathcal{B}_{\mathcal{L}}$ such that $\mathsf{ev}(\mathcal{Z}_{\mathcal{K}})$ has positive measure and therefore $\mathsf{ev}(\mathcal{Z}_{\mathcal{K}}) = X$.

Let us prove that $\mathcal{K}$ contains all leaves of $\mathcal{L}$. Suppose that there is a leaf $\mathcal{L}_z$ which is not in $\mathcal{K}$. Find $k \in \mathcal{K}$ such that $\mathcal{Z}_k \cap \mathcal{L}_z \neq \varnothing$. This intersection represents a nontrivial element in corresponding homology group. And this homological intersection doesn't depend on the choice of $k\mathcal{K}$ and $\mathcal{L}_z$. This is a contradiction, because for some k the cycle $\mathcal{Z}_k$ coincides with a leaf of $\mathcal{L}$.

Therefore all leaves of $\mathcal{L}$ belong to $\mathcal{K}$. In the same manner one establishes that all cycles $\mathcal{Z}_k$ from $\mathcal{K}$ are leaves of $\mathcal{L}$.

□

Remark 6.3. In dimension two, *i.e.*, when X is a compact complex surface ,this result is due to J. Winkelmann, see [W].

6.6. Foliations on Iwasawa manifold. Example 6.2 of Kato already provided us a foliation without a plurinegative taming form. However it is very inexplicit. Let us give a very simple one.

Example 6.3. Let $H(3)$ be the group of matrices of the form

$$A = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.4}$$

with complex z_1, z_2, z_3. Denote by $\mathbb{Z}(3)$ the subgroup of $H(3)$ which consists from $z_1, z_2, z_3 \in \mathbb{Z} + i\mathbb{Z}$. The quotient $H(3)/\mathbb{Z}(3)$ is a compact, complex three-dimensional manifold $\mathcal{I}$ - Iwasawa manifold. The holomorphic forms $\omega_1 = dz_1$, $\omega_2 = dz_2$ and $\omega_3 = dz_3 - z_1 dz_2$ are left invariant with respect to the action of $\mathbb{Z}(3)$ and therefore project to holomorphic forms on $\mathcal{I}$. Define a holomorphic foliation by curves $\mathcal{L}_1$ on $\mathcal{I}$ by $\omega_1 = \omega_2 = 0$.

Proposition 6.2. *Foliated manifold $(\mathcal{I}, \mathcal{L}_1)$ possesses the following properties:*
i) It is Hartogs.
ii) It doesn't admit a plurinegative taming form.

Proof. *(i)* Hartogs property is invariant with respect to unramified coverings. Since the universal covering of $\mathcal{I}$ is $H(3) \equiv \mathbb{C}^3$ we are done.
(ii) Consider $S := \frac{i}{2}\omega_3 \wedge \bar{\omega}_3$ as a positive $(2,2)$-current on $\mathcal{I}$. A simple calculation

$$dd^c S = i\partial\bar{\partial} S = \partial\omega_3 \wedge \bar{\partial}\omega_3 = \frac{i^2}{2}\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2 =: T.$$

And T is a positive current directed by $\mathcal{L}_1$. A positive current S such that $dd^c S$ is also positive and directed by $\mathcal{L}$ is a clear obstruction to the existence of a plurinegative taming form for $\mathcal{L}_1$.

□

Remark 6.4. Iwasawa manifold carries also other foliations. For example $\mathcal{L}_2 := \{\omega_1 = \omega_3 = 0\}$. This $\mathcal{L}_2$ is tamed by the closed form $\frac{i}{2}\omega_2 \wedge \bar{\omega}_2$ and therefore the Proposition 1 from the Introduction applies to $\mathcal{L}_2$ - it belongs to $\mathcal{U}$.

6.7. Open questions. In this subsection we shall formulate some open questions which are important to complete our knowledge about holomorphic foliation by curves on compact complex manifolds and which naturally come out from the discussions in this paper.

Class $\mathcal{S}$.

Recall that foliations of class $\mathcal{S}$ are pluritamed foliations containing foliated shells.

Question 1. *In the conditions of the Theorem 3.1 let $\mathcal{L}_z$ be the leaf which contains an essential vanishing cycle. Is it true that its closure $\bar{\mathcal{L}}_z$ is a compact complex curve?*

Recall that this is like the proof of Novikov's theorem works, see [Go] for example. If $\dim X = 2$ we proved it in Corollary 2.

Question 2. *Prove that a foliation with shells is parabolic.*

Question 3. *Let $(X, \mathcal{L}) \in \mathcal{S}$ be compact and $\dim X \geq 3$. Does $(X, \mathcal{L})$ contains a total space of a deformation $(\mathcal{X}, \pi, Y)$ of foliated compact Hopf or Kato surfaces $\mathcal{X}_\lambda, \lambda \in Y$ with compact Y?*

Most probably the answer to this question is “no” if stated in its full generality. But even a partial positive result (or a counterexample) would be important. In this concern let us give an example shoving that the total space of deformation may not sweep the whole of X, *i.e.,* that a foliated shell may "disappear in the limit".

Example 6.4. Let E' be a holomorphic rank two bundle over a Hopf surface $H^2 = \mathbb{C}\setminus\{0\}/z\sim 2z$ which admits a holomorphic section σ vanishing exactly at one point $z_0\in H^2$ with multiplicity one, see [GH], p.726. Denote by E the bundle dual to E'. Let τ_0 be the zero section of E. The quotient of $E\setminus\tau_0$ by the action $(z,\mathrm{v})\to(z,\frac{1}{2}\mathrm{v})$ is a compact complex 4-manifold which we denote as X. It is fibered over H^2 and the fiber over $z\in H^2$ we denote as X_z.

$E\setminus\tau_0$ carries a singular holomorphic foliation by curves defined as follows: its leaves in each fiber $E_z\setminus\{0\}, z\neq z_0$ are $\{x\in E_z : \sigma_z(x) = const\}$. Actually on each $E_z\setminus\{0\}$ it is again our "vertical" foliation. It factors under the chosen action to a foliation $\mathcal{L}$ on X. The singularity set of $\mathcal{L}$ is E_{z_0}. $(X,\mathcal{L})$ carries an obvious family of foliated shells over $H^2\setminus\{z_0\}$, and this family extends over z_0 (!) as a family of shells. But $\mathcal{L}$ itself is singular over z_0 and therefore the shell in X_{z_0} is not a foliated one.

Question 4. *Is it true that immersed foliated shells could be always made spherical? The same question about imbedded ones. In that case one expects them to be holomorphic foliated images of quotients of the standard sphere in $\mathbb{C}^2$ with the standard vertical foliation.*

The problem here lies in reducing of the size of the singularity set S, see Subsection 4.1 for a more detailed discussion.

Question 5. *Let $(X,\mathcal{L})$ be pluritamed by a dd^c-closed* metric *form ω. Is it true that the singularity set S of meromorphic foliated immersions appearing in Theorem 2.3 is at most countable union of analytic subsets of pure codimension two?*

Class $\mathcal{U}$.

Question 6. *Let D be a transversal polydisc. Suppose that the skew cylinder $\tilde{\mathcal{L}}_D$ exists (and $\mathcal{L}$ admits a plurinegative adapted form). Prove that $\tilde{\mathcal{L}}_D$ is disc-convex.*

This is known for Stein X, [Iy1], in that case $\tilde{\mathcal{L}}_D$ is Stein. It is also known for algebraic X, [Br3].

Question 7. *Prove that the set of $z\in D$ such that $\tilde{\mathcal{L}}_z = \mathbb{C}$ is pluripolar in D or is the full D.*

Algebraic case is treated in [Br3].

Question 8. *Suppose that the domain Ω_γ of preservation of cycle γ (as in Definition 1.4) exists. Prove that Ω_γ is good in the sense of Landis-Petrovsky, i.e., that for a natural projection $p:\Omega_\gamma\to D$ the set $\Sigma := D\setminus p(\Omega_\gamma)$ doesn't separate D.*

This question for holomorphic foliations by curves on arbitrary compact complex manifolds Il'yashenko calls the generalized Landis-Petrovsky conjecture. The answer is positive for algebraic foliations, but it is wrong for holomorphic ones on Stein manifolds, see [Iy2]. Example of Kato in [K3] leaves little hope for the positive answer in general, but it is not a direct counterexample. In Question 7 We propose to solve the Landis-Petrovsky conjecture when $\mathcal{L}$ admits an adapted pluriclosed (or plurinegative) taming form.

6.8. Class $\mathcal{E}$.

Question 9. *Suppose T is a nontrivial dd^c-exact foliated cycle for $\mathcal{L}$ such that $T = dd^c S$ where S is also positive. Can one provide more restrictions on the support of T then it is done in Theorem 5? Can one say something about the structure os S?*

Not that S has a well defined Lelong numbers, see [Sk].

Question 10. *Is it true that every pluriexact foliation contains a compact curve tangent to the leaves?*

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