LOCAL INDEX THEOREM FOR PROJECTIVE FAMILIES

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Abstract. We give a superconnection proof of the Mathai-Melrose-Singer index theorem for the family of twisted Dirac operators [28, 29].

Contents

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1. Introduction

Let $\pi : M \to B$ be a smooth fibration. Given a class $\delta \in H^3(B, \mathbb{Z})$ Mathai, Melrose and Singer defined an algebra of twisted by this class (also called projective) families of pseudodifferential operators. In order to give this definition in [28] it is assumed that δ is a torsion class. In [29] the corresponding assumption is that $\delta = \alpha \cup \beta$, $\alpha \in H^1(B, \mathbb{Z})$, $\beta \in H^2(B, \mathbb{Z})$ and $\pi^*\beta = 0$.

There is a notion of ellipticity for such a twisted pseudodifferential family and for such an elliptic family one can then define its index as an element of the K-theory of the algebra of smoothing operators. Mathai, Melrose and Singer than prove in [28, 29] an index theorem for such elliptic families thus giving an extension of the Atiyah-Singer family index theorem [1] to the twisted case. general theory of connections In this paper we give a superconnection proof of the cohomological version of this index theorem for a projective family of Dirac operators. We assume that we are given a gerbe $\mathcal{L}$ on B with a class $\delta = [\mathcal{L}] \in H^3(B, \mathbb{Z})$. Our conditions on this class are somewhat weaker than in [28, 29]; namely we assume that $\pi^*\delta$ is a torsion element in $H^3(M, \mathbb{Z})$. Assume that we are given a $\mathbb{Z}_2$-graded $\mathcal{L}$ -twisted Clifford module $\mathcal{E}$ on M as defined in the Section 4.1. We note that such Clifford modules always exist under our assumptions on the class of the gerbe $\mathcal{L}$. We define the algebra $\Psi_{\mathcal{L}}(M|B; \mathcal{E})$ of projective families of pseudodifferential operators on $\mathcal{E}$. One can then define the index of (the positive part of) the twisted Dirac operator $\operatorname{ind} D^+ \in K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B; \mathcal{E}))$. Here by $\Psi^{-\infty}_{\mathcal{L}}(M|B; \mathcal{E})$ we denote the algebra of smoothing $\mathcal{L}$-twisted pseudodifferential operators on $\mathcal{E}$.

In order to define the Chern character of the index we use cyclic homology and the map constructed in [30]. This Chern character takes values in the twisted cohomology of B defined as follows. Let $\mathit{\Omega} \in \Omega^3(B)$ be a form representing the class $[\mathcal{L}]$ and let u be a formal variable of degree -2. Twisted cohomology $H^\bullet_{\mathcal{L}}(B)$ is then the homology of the complex $\Omega^*(B)[u]$ with the differential $d_{\mathit{\Omega}} = ud + u^2 \mathit{\Omega} \wedge \cdot$. Note that the form $\mathit{\Omega}$ and thus the complex depend on the choice of connection on $\mathcal{L}$. Nevertheless the homologies of all the complexes thus obtained are canonically isomorphic. Following Mathai and Stevenson [30] one defines a morphism of complexes $\Phi_{\nabla^{\mathcal{H}}} \colon CC^-_\bullet(\Psi^{-\infty}_{\mathcal{L}}(M|B; \mathcal{E})) \to (\Omega^*(B)[u], d_{\mathit{\Omega}})$. Here $CC^-_\bullet$ denotes the negative cyclic complex. By composing $\Phi_{\nabla^{\mathcal{H}}}$ with the Chern character $\operatorname{ch} \colon K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B; \mathcal{E})) \to HC^-_0(\Psi^{-\infty}_{\mathcal{L}}(M|B; \mathcal{E}))$ we obtain the class $[\Phi_{\nabla^{\mathcal{H}}}(\operatorname{ch}(\operatorname{ind} D^+)) \in H^\bullet_{\mathcal{L}}(B)]$. The main result of the paper is the proof of the following theorem expressing the Chern character of the index in terms of characteristic classes:

Theorem 1.1. *Let D be a projective family of Dirac operators on a horizontally $\mathcal{L}$-twisted Clifford module $\mathcal{E}$ on a fibration $\pi : M \to B$. Then the following equality holds in $H^\bullet_{\mathcal{L}}(B)$:*

$$\left[\Phi_{\nabla^{\mathcal{H}}}(\operatorname{ch}(\operatorname{ind} D^+))\right] = \left[u^{-\frac{k}{2}} \int_{M|B} \widehat{A}\left(\frac{u}{2\pi i} R^{M|B}\right) \operatorname{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})\right],$$

where $k = \dim M - \dim B$ is the dimension of the fibers.

Here, as usual, $\widehat{A}$ is the power series defined by $\widehat{A}(x) = \det{}^{1/2}\left(\frac{x/2}{\sinh x/2}\right)$ and $\operatorname{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})$ is the twisted relative Chern character form of $\mathcal{E}$, see Proposition 2.13 and equation (4.1.1).

The characteristic classes appearing in the right hand side are defined in the Section 4.1.

Our proof uses the Bismut superconnection formalism [5]. We extend the notion of superconnection to the twisted context and show in the Theorem 4.6 that the Chern character of superconnection adapted to D computes the Chern character of $\operatorname{ind} D^+$. Then in the Theorem 4.8 using the results of [5] and [6] we compute the limit of the Chern character of the rescaled superconnection obtaining the expression in the right hand side of the index formula. Together these results imply the Theorem 1.1.

The paper is organized as follows. In the Section 2 we give a brief review of cyclic homology, gerbes and connections on them. We also discuss twisted bundles and their characteristic forms. In the Section 3 we give the definitions of the algebra of twisted families of pseudodifferential operators. Finally in the section 4 we define the projective family of Dirac operators on a twisted Clifford module and give the proofs of the main theorems of the paper.

Some work in related direction recently appeared in [13, 12, 14]. Related questions of deformation theory are considered in [9, 10]. This paper is a byproduct of a joint project with E. Leichtnam. The authors would like to thank him and C. Blanchet, M. Karoubi, M. Lesch, V. Mathai, R. Nest and B. Wang for helpful discussions. The authors worked on this paper while visiting CIRM, Luminy as well as Hausdorff institute and Max Planck Institute for Mathematics in Bonn. Part of this work was done while the second author was visiting the Laboratoire de Mathématiques et Applications of Metz and he is grateful for the hospitality.

2. Preliminaries

2.1. Cyclic homology. The general reference for this material is the book [27].

Let A be a complex unital algebra. Set $C_k(A) = A \otimes (A/\mathbb{C}1)^{\otimes k}$. Let u be a formal variable of degree -2. The space of negative cyclic chains of degree $l \in \mathbb{Z}$ is defined by

$$CC_l^-(A) = (C_\bullet(A)[[u]])_l = \prod_{-2n+k=l,\ n\geq 0} u^n C_k(A).$$

The boundary is given by $b+uB$ where b and B are the Hochschild and Connes boundaries of the cyclic complex. The homology of this complex is denoted $HC_\bullet^-(A)$. When the algebra A is $\mathbb{Z}_2$ graded they incorporate the relevant signs. If A is not necessarily unital denote by A^+ its unitalisation and set $CC_l^-(A) = CC_l^-(A^+)/CC_l^-(\mathbb{C})$. If I is an ideal in a unital algebra A the relative cyclic complex is defined by $CC_\bullet^-(A, I) = \operatorname{Ker}(CC_\bullet^-(A) \to CC_\bullet^-(A/I))$. One has a natural morphism of complexes $\iota\colon CC_\bullet^-(I) \to CC_\bullet^-(A, I)$ induced by the homomorphism $I^+ \to A$.

Recall that for an algebra A we have Chern character in cyclic homology $\operatorname{ch}\colon K_0(A) \to HC_0^-(A)$. It is defined by the following formula. Let $P, Q \in M_n(A^+)$ be two idempotents in $n \times n$ matrices of the algebra A^+, representing a class $[P-Q] \in K_0(A)$. Then

(2.1.1)
$$\operatorname{Ch}([P-Q]) = \operatorname{tr}(P-Q) + \sum_{n=1}^{\infty}(-u)^n \frac{(2n)!}{n!} \operatorname{tr}\left(\left(P - \frac{1}{2}\right) \otimes P^{\otimes(2n)} - \left(Q - \frac{1}{2}\right) \otimes Q^{\otimes(2n)}\right)$$

We will use the notation $\operatorname{Ch}([P-Q])$ for the cyclic cycle defined above and $\operatorname{ch}([P-Q])$ for its class in cyclic homology $HC_0^-(A)$.

We will also need to use the entire cyclic complex. For our purpose the algebraic version from [17], IV.7.α Remark 7 b. will be sufficient. First recall that one has the periodic cyclic complex $(CC^{per}_\bullet(A), b+uB)$ where $CC^{per}_\bullet(A) = C_\bullet(A)[u^{-1}, u]]$. Assume we are given a periodic chain $\alpha = \sum_{k\geq 0} \alpha_k u^k \in CC^{per}_m(A)$, $\alpha_k \in C_{2k+m}(A)$. Then α is called entire if there exist a finite dimensional subspace $V \subset A$, $1 \in V$ and $C > 0$ (depending on α) such that $\alpha_k \in V \otimes (V/\mathbb{C}1)^{\otimes k}$ and $\|\alpha_k\| \leq C^k k!$. Here the norms on $V \otimes (V/\mathbb{C}1)^{\otimes k}$ are induced by an arbitrary norm on V. We denote the entire cyclic complex of A by $CC^{entire}_\bullet(A)$.

Note that the chain $\operatorname{Ch}([P-Q])$ defined in (2.1.1) is an element in $CC^{entire}_0(A)$.

2.2. **Notion of a gerbe.** We give here only the general overview, refering the reader to [11] and [24] for the details. The differential geometry of not necessarily abelian gerbes is described in [8]. We will describe the gerbes in terms of their descent data.

Let M be a smooth manifold. Given an open cover $(U_\alpha)_{\alpha\in\Lambda}$ of M, we set as usual

$$U_{\alpha_1\cdots\alpha_k} = \bigcap_{1\leq j\leq k} U_{\alpha_j}.$$

Definition 2.1. A descent datum for a gerbe $\mathcal{L}$ on M is the collection $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ where $(U_\alpha)_{\alpha\in\Lambda}$ is an open cover of M, $(\mathcal{L}_{\alpha\beta} \to U_{\alpha\beta})_{\alpha,\beta\in\Lambda}$ is a collection of line bundles and $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \to \mathcal{L}_{\alpha\gamma}$ are line bundle isomorphisms over each triple intersection $U_{\alpha\beta\gamma}$ such that over each quadruple intersection $U_{\alpha\beta\gamma\delta}$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{L}_{\alpha\beta}\otimes\mathcal{L}_{\beta\gamma}\otimes\mathcal{L}_{\gamma\delta} & \xrightarrow{\mu_{\alpha\beta\gamma}\otimes \mathrm{id}} & \mathcal{L}_{\alpha\gamma}\otimes\mathcal{L}_{\gamma\delta} \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\mu_{\beta\gamma\delta}} & & \Big\downarrow{\scriptstyle \mu_{\alpha\gamma\delta}} \\
\mathcal{L}_{\alpha\beta}\otimes\mathcal{L}_{\beta\delta} & \xrightarrow{\mu_{\alpha\beta\delta}} & \mathcal{L}_{\alpha\delta}
\end{array}$$

Notice that we don't assume in this definition that the open sets $(U_\alpha)_{\alpha\in\Lambda}$ are contractible.

Given two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ on the same open cover $\{U_\alpha\}$ an isomorphism between them is given by line bundles S_α on U_α and isomorphisms of line bundles $\lambda_{\alpha\beta}\colon S_\alpha^{-1}\otimes\mathcal{L}_{\alpha\beta}\otimes S_\beta \to \mathcal{L}'_{\alpha\beta}$ over $U_{\alpha\beta}$ so that the diagram

$$\begin{array}{ccc}
S_\alpha^{-1}\otimes\mathcal{L}_{\alpha\beta}\otimes S_\beta\otimes S_\beta^{-1}\otimes\mathcal{L}_{\beta\gamma}\otimes S_\gamma & \xrightarrow{\mathrm{id}\otimes\mu_{\alpha\beta\gamma}\otimes\mathrm{id}} & S_\alpha^{-1}\otimes\mathcal{L}_{\alpha\gamma}\otimes S_\gamma \\
\Big\downarrow{\scriptstyle \lambda_{\alpha\beta}\otimes\lambda_{\beta\gamma}} & & \Big\downarrow{\scriptstyle \lambda_{\alpha\gamma}} \\
\mathcal{L}'_{\alpha\beta}\otimes\mathcal{L}'_{\beta\gamma} & \xrightarrow{\mu'_{\alpha\beta\gamma}} & \mathcal{L}'_{\alpha\gamma}
\end{array}$$

commutes.

Given two isomorphisms $(S_\alpha, \lambda_{\alpha\beta})$ and $(S'_\alpha, \lambda'_{\alpha\beta})$ between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$, a two-morphism between them is a collection of line bundle isomorphisms $\nu_\alpha\colon S_\alpha \to S'_\alpha$ such that

$$\lambda'_{\alpha\beta}\circ(\nu_\alpha^{-1}\otimes\mathrm{id}\otimes\nu_\beta) = \lambda_{\alpha\beta}$$

where we denote by ν_α^{-1} the isomorphism $S_\alpha^{-1} \to (S'_\alpha)^{-1}$ induced by ν_α.

Let $(V_i, \varrho)_{i\in I}$ be a refinement of the open cover of $(U_\alpha)_{\alpha\in\Lambda}$ of M. So $(V_i)_{i\in I}$ is an open cover of M and

$$\varrho : I \to \Lambda \quad \text{such that} \quad V_i \subset U_{\varrho(i)}.$$

Then restriction to the refinement of $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ is the descent datum $\mathcal{L}' = (V_i, \mathcal{L}'_{ij}, \mu'_{ijk})$ given by:

$$\mathcal{L}'_{ij} := \mathcal{L}_{\varrho(i)\varrho(j)}|_{V_{ij}} \text{ and } \mu'_{ijk} := \mu_{\varrho(i)\varrho(j)\varrho(k)}|_{V_{ijk}}.$$

Similarly one defines restriction of the isomorphisms and 2-morphisms to a refinement. We do not distinguish between a descent datum, isomorphisms of descent data, etc., and their restrictions to a refinement. Thus for instance, the isomorphism between two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ is an isomorphism between their restrictions to some common refinement of $\{U_\alpha\}$ and $\{U'_\alpha\}$.

An equivalence class of Dixmier-Douady gerbes on M is an equivalence class of the descent data on M. More precisely a gerbe is a maximal collection of descent data $\mathcal{D}_i$, $i \in A$ together with the isomorphisms $s_{ij} \colon \mathcal{D}_j \to \mathcal{D}_i$ for each $i, j \in A$ and 2-morphisms $\nu_{ijk} \colon s_{ij}s_{jk} \to s_{ik}$ satisfying the natural associativity condition. We refer the reader to the book [11] for the details.

If the cover U_α is good [7] all the bundles $\mathcal{L}_{\alpha\beta}$ are trivializable. After choice of such trivialization the collection $(\mu_{\alpha\beta\gamma})$ can be viewed as a Čech 2-cochain with coefficents in the sheaf $\underline{\mathbb{C}^*}$ of smooth functions with values in the nonzero complex numbers $\mathbb{C}^*$. The compatibility condition over $U_{\alpha\beta\gamma\delta}$ tells us that μ is a 2-cocycle and hence defines a cohomology class $[\mu] \in H^2(M; \underline{\mathbb{C}^*}) \cong H^3(M, \mathbb{Z})$. This class is a well defined invariant of the gerbe called the Dixmier-Douady class. We denote this class by $[\mathcal{L}]$. Every class in $H^3(M, \mathbb{Z})$ is a class of a gerbe defined by this class uniquely up to an isomorphism (see [11]).

Given a smooth map $f : M' \to M$ between smooth manifolds M' and M, we can pull-back any decent datum for a gerbe on M to a descent datum on M'. The pull backs of isomorphic descent data are isomorphic and thus we obtain a well-defined pull-back of a gerbe. Clearly the Dixmier-Douady class of the pull-back is the pull-back of the Dixmier-Douady class.

An unitary descent datum is $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ together with a choice of metric on each $\mathcal{L}_{\alpha\beta}$ such that each $\mu_{\alpha\beta\gamma}$ is an isometry. A notion of unitary equivalence of two unitary descent data on the same open cover U_α is obtained from the notion of equivalence above by requiring that each line bundle S_α is Hermitian and each $\lambda_{\alpha\beta}$ is an isometry. The definition of 2-morphisms is modified by requiring each ν_α to be an isometry. It is clear that the restriction of a unitary descent datum to a refinement is again unitary. Then a unitary gerbe is an equivalence class of unitary descent data in the sense described above.

2.3. **Connections on gerbes.**

Lemma 2.2. *Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ be a descent datum on M. There exists a collection $(\nabla_{\alpha\beta})$ of connections on $(\mathcal{L}_{\alpha\beta})$ such that for any $(\alpha, \beta, \gamma) \in \Lambda^3$ with $U_{\alpha\beta\gamma} \neq \emptyset$:*

$$\mu^*_{\alpha\beta\gamma}\nabla_{\alpha\gamma} = \nabla_{\alpha\beta} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\beta\gamma}.$$

If the descent datum is unitary each $\nabla_{\alpha\beta}$ can be chosen Hermitian.

Proof. Fix for any α, β with $U_{\alpha\beta} \neq \emptyset$ a connection $\nabla'_{\alpha\beta}$ on $\mathcal{L}_{\alpha\beta}$. We set for $U_{\alpha\beta\gamma} \neq \emptyset$,

$$A_{\alpha\beta\gamma} := \mu^*_{\alpha\beta\gamma}\nabla_{\alpha\gamma} - \nabla_{\alpha\beta} \otimes \mathrm{id} - \mathrm{id} \otimes \nabla_{\beta\gamma}.$$

Then using the identification

$$\mathrm{End}(\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma}) \simeq U_{\alpha\beta\gamma} \times \mathbb{C};$$

we see that $A_{\alpha\beta\gamma}$ is identified with a differential 1-form on the open set $U_{\alpha\beta\gamma}$. We also have for $U_{\alpha\beta\gamma\delta} \neq \emptyset$

$$A_{\beta\gamma\delta} + A_{\alpha\beta\delta} = A_{\alpha\gamma\delta} + A_{\alpha\beta\gamma}.$$

Therefore, there exists $A' = (A'_{\alpha\beta})$ such that

$$A_{\alpha\beta\gamma} = A'_{\beta\gamma} - A'_{\alpha\gamma} + A'_{\alpha\beta}.$$

The collection of connections $(\nabla_{\alpha\beta} = \nabla'_{\alpha\beta} + A'_{\alpha\beta})$ is then a connection on the gerbe $\mathcal{L}$. □

Lemma 2.3. *Let $(\nabla_{\alpha\beta})$ be as above, and denote by $\omega_{\alpha\beta} = \nabla^2_{\alpha\beta}$ the curvatures of the connections $\nabla_{\alpha\beta}$. Then there exists a collection of differential 2-forms $\omega_\alpha \in \Omega^2(U_\alpha)$ such that*

$$\omega_{\alpha\beta} = \omega_\alpha - \omega_\beta, \quad \text{for } U_{\alpha\beta} \neq \emptyset.$$

Proof. The collection $\omega_{\alpha\beta}$ satisfies

$$\omega_{\alpha\gamma} = \omega_{\alpha\beta} + \omega_{\beta\gamma}.$$

This shows that $(\omega_{\alpha\beta})$ is a Čech cocycle and hence by the acyclicity of the Čech complex of forms, there exist (ω_α) such that

$$\omega_{\alpha\beta} = \omega_\alpha - \omega_\beta.$$

□

We will say that the collection $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta})$.

Connection on descent datum yields a connection on its restriction to a refinement in an obvious manner; we will identify connection with its restriction. An isomorphism between descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta})$ with connection $(\nabla_{\alpha\beta}, \omega_\alpha)$ and descent datum $(U_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta})$ with connection $(\nabla'_{\alpha\beta}, \omega'_\alpha)$ is given by $(S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha)$ where $(S_\alpha, \lambda_{\alpha\beta})$ is an isomorphsm betweeween the descent data (without connections) and each ∇_α is a connection on S_α satsfying the following conditions. Let $\pi_\alpha = \nabla^2_\alpha$ be the curvatures of these connections and let ∇^{-1}_α be the dual connection on S^{-1}_α. Then we require the following dentities (using the isomorphism $\mathcal{L}_{\alpha\beta} = S_\alpha \otimes (S^{-1}_\alpha \otimes \mathcal{L}_{\alpha\beta} \otimes S_\beta) \otimes S^{-1}_\beta$):

$$\nabla_{\alpha\beta} = (\lambda_{\alpha\beta})^*(\nabla_\alpha \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \nabla'_{\alpha\beta} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \nabla^{-1}_\beta) \tag{2.3.1}$$

and

$$\omega_\alpha = \omega'_\alpha + \pi_\alpha. \tag{2.3.2}$$

If $s = (S_\alpha, \lambda_{\alpha\beta}, \nabla_\alpha)$ and $s' = (S'_\alpha, \lambda'_{\alpha\beta}, \nabla'_\alpha)$ are two isomorphisms between the descent data with connections as above, the 2-morphisms between s and s' are the same as the 2 morphisms between $(S_\alpha, \lambda_{\alpha\beta})$ and $(S'_\alpha, \lambda'_{\alpha\beta})$.

Assume $\mathcal{L}$ is a gerbe given by a collection of descent data $\mathcal{D}_i$, $i \in A$, isomorphisms $s_{ij} \colon \mathcal{D}_j \to \mathcal{D}_i$ for each $i, j \in A$ and 2-morphisms $\nu_{ijk} \colon s_{ij}s_{jk} \to s_{ik}$. A connection on a gerbe $\mathcal{L}$ is a lift of each $\mathcal{D}_i$ to a descent datum with connection $\widetilde{\mathcal{D}_i}$ and each s_{ij} to an isomorphsm $\widetilde{s_{ij}} \colon \widetilde{\mathcal{D}_j} \to \widetilde{\mathcal{D}_i}$. An easy argument using the Lemma 2.2 shows that on a given gerbe always exists a connection.

Lemma 2.4. *Let $\mathcal{L}$ be a gerbe represented by a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$. Choose a connection on $\mathcal{L}$ represented by a connection $(\nabla_{\alpha\beta}, \omega_\alpha)$ on the descent datum.*

1. *There exists a well-defined closed form $\Omega \in \Omega^3(M)$ – curvature 3-form of the connection – such that $\Omega|_{U_\alpha} = \frac{d\omega_\alpha}{2\pi i}$.*
2. *Let $(\nabla'_{\alpha\beta}, \omega'_\alpha)$ be another connection on $\mathcal{L}$ and let Ω' be the corresponding curvature 3-form. Then there exists a canonical $\eta \in \Omega^2(M)/d\Omega^1(M)$ such that $\Omega' = \Omega + d\eta$.*
3. *Let $(\nabla_{\alpha\beta}, \omega_\alpha)$ $(\nabla'_{\alpha\beta}, \omega'_\alpha)$, $(\nabla''_{\alpha\beta}, \omega''_\alpha)$ be 3 connections on $\mathcal{L}$ with the corresponding curvature 3-forms Ω, Ω', Ω''. Let $\eta, \eta', \eta'' \in \Omega^2(M)/d\Omega^1(M)$ be the canonical elements constructed above such that $\Omega' - \Omega = d\eta$, $\Omega'' - \Omega' = d\eta'$, $\Omega'' - \Omega = d\eta''$. Then $\eta'' = \eta + \eta'$.*

The 3-form Ω is a de Rham representative of the Dixmier-Douady class of the gerbe.

Proof. We use notations of the previous Lemma. For the first part notice that since $d\omega_{\alpha\beta} = 0$ for any α, β we see that

$$d\omega_\alpha|_{U_{\alpha\beta}} = d\omega_\beta|_{U_{\alpha\beta}}$$

which shows the existence of Ω; it is clearly closed. Ω does not change if one restricts the descent datum to a refinement. The equation (2.3.2) implies that if one uses a different descent datum for the same gerbe one obtains the same 3-curvature form.

We proceed to the proof of the second part. Let $(\nabla_{\alpha\beta}, \omega_\alpha)$ and $(\nabla'_{\alpha\beta}, \omega'_\alpha)$ be two connections on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$. Set $\delta_{\alpha\beta} = \nabla'_{\alpha\beta} - \nabla_{\alpha\beta} \in \Omega^1(U_{\alpha\beta})$. Choose $\delta_\alpha \in \Omega^1(U_\alpha)$ such that $\delta_{\alpha\beta} = \delta_\alpha - \delta_\beta$ on $U_{\alpha\beta}$. Then $\omega'_{\alpha\beta} - \omega_{\alpha\beta} = d\delta_{\alpha\beta} = d\delta_\alpha - d\delta_\beta$. Hence $\omega'_\alpha - \omega_\alpha - d\delta_\alpha = \omega'_\beta - \omega_\beta - d\delta_\beta$ on $U_{\alpha\beta}$. Therefore there exists $\eta \in \Omega^2(M)$ such that $\eta|_{U_\alpha} = \frac{1}{2\pi i}(\omega'_\alpha - \omega_\alpha - d\delta_\alpha)$. Then $\Omega' - \Omega = d\eta$. The formula for η given above depends on the choice of δ_α. If $\bar{\delta}_\alpha$ is a different such choice, then $\bar{\delta}_\alpha - \delta_\alpha = \epsilon|_{U_\alpha}$ for some $\epsilon \in \Omega^1(M)$. Then $\bar{\eta} = \eta - d\epsilon$. Hence the class of $\eta \in \Omega^2(M)/d\Omega^1(M)$ does not depend on the choice made. It is easy to see that η does not depend on the choice of the particular descent datum used.

The verification of the third statement is straightforward and is left to the reader. □

2.4. **Twisted cohomology.** For a smooth manifold M let $\Omega \in \Omega^3(M)$ be a closed 3-form. Denote by u a formal variable of degree -2. The twisted de Rham complex is defined as the complex $\Omega^*(M)[u]$ with the differential $d_\Omega = ud + u^2\Omega \wedge \cdot$. Note that if $\Omega' = \Omega + d\eta$ is cohomologous to Ω then the complexes $(\Omega^*(M)[u], d_\Omega)$ and $(\Omega^*(M)[u], d_{\Omega'})$ are isomorphic via the isomorphism

$$I_\eta \colon \xi \mapsto e^{-u\eta} \wedge \xi \tag{2.4.1}$$

Lemma 2.5. *The map induced by I_η on cohomology depends only on the class of η in $\Omega^2(M)/\left(d\Omega^1(M)\right)$.*

Proof. Indeed, for $\epsilon \in \Omega^1(M)$ define $h_\epsilon \colon (\Omega^*(M)[u])_\bullet \to (\Omega^*(M)[u])_{\bullet-1}$ by

$$h_\epsilon = u\epsilon \left(\sum_k \frac{(u\eta)^k}{(k+1)!} \right) \wedge \cdot .$$

Then

$$I_{\eta+d\epsilon} - I_\eta = h_\epsilon \circ d_\Omega + d_{\Omega'} \circ h_\epsilon$$

and therefore the maps I_η and $I_{\eta+d\epsilon}$ are chain homotopic. □

The identity $d_{\Omega+\Omega'}(\xi \wedge \eta) = d_\Omega \xi \wedge \eta + (-1)^{|\xi|}\xi \wedge d_{\Omega'}\eta$ implies that the product of forms induces the product $H^\bullet_\Omega(M) \otimes H^\bullet_{\Omega'}(M) \to H^\bullet_{\Omega+\Omega'}(M)$ and in particular endows $H^\bullet_\Omega(M)$ with the structure of $H^\bullet(M)[u]$-module.

We will also need to consider the following situation. Let $\pi\colon M \to B$ be an oriented fibration with compact fibers. Then we have an integration along the fibers map $\int_{M|B}\colon \Omega^*(M) \to \Omega^{*-k}(B)$, $k = \dim M - \dim B$. Let $\Omega \in \Omega^3(B)$ be a closed form. Then $\int_{M|B} d\eta = d\int_{M|B}\eta$, $\int_{M|B}\pi^*\Omega \wedge \eta = \Omega \wedge \int_{M|B}\eta$ and therefore

$$\int_{M|B} d_{\pi^*\Omega}\eta = d_\Omega \int_{M|B}\eta.$$

Hence we obtain a chain map $\int_{M|B}\colon\ (\Omega^*(M), d_{\pi^*\Omega})_\bullet \to (\Omega^*(B), d_\Omega)_{\bullet-k}$.

Let $\mathcal{L}$ be a gerbe. The choice of connection defines a closed 3-form Ω. We can therefore consider the complex $(\Omega^*(M)[u], d_\Omega)$. Different choice of connection leads to a different complex $(\Omega^*(M)[u], d_{\Omega'})$. Lemma 2.4 however implies that there exists however a canonical isomorphism I of the homologies of these complexes. We denote this homology $H^\bullet_{\mathcal{L}}(B)$

2.5. **Twisted bundles.**

Definition 2.6. A descent datum for a twisted vector bundle $\mathcal{E}$ consists of a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for a gerbe $\mathcal{L}$ together with a collection $(\mathcal{E}_\alpha \to U_\alpha)_{\alpha\in\Lambda}$ of vector bundles and a collection of vector bundle isomorphisms $\varphi_{\alpha\beta} : \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \cong \mathcal{E}_\beta$ such that for every α, β, γ the following diagram commutes

$$\begin{array}{ccc}
\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\mathrm{id}\otimes\mu_{\alpha\beta\gamma}} & \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\gamma} \\
{\scriptstyle \varphi_{\alpha\beta}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi_{\alpha\gamma}} \\
\mathcal{E}_\beta \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\varphi_{\beta\gamma}} & \mathcal{E}_\gamma
\end{array}$$

Restriction of the descent datum for $\mathcal{E}$ to a refinement is given by the restriction of the descent datum for $\mathcal{L}$ together with restriction of the vector bundles $\mathcal{E}_\alpha$.

Definition 2.7. An (iso)morphism between two descent data $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$ is given by the collection $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ where $(S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ and $\rho_\alpha\colon \mathcal{E}_\alpha \otimes S_\alpha \to \mathcal{E}'_\alpha$ is a collection of (iso)morphisms such that the diagram

$$\begin{array}{ccc}
\mathcal{E}_\alpha \otimes S_\alpha \otimes S_\alpha^{-1} \otimes \mathcal{L}_{\alpha\beta} \otimes S_\beta & \xrightarrow{\varphi_{\alpha\beta}\otimes\mathrm{id}} & \mathcal{E}_\beta \otimes S_\beta \\
{\scriptstyle \rho_\alpha\otimes\lambda_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \rho_\beta} \\
\mathcal{E}'_\alpha \otimes \mathcal{L}'_{\alpha\beta} & \xrightarrow{\varphi'_{\alpha\beta\gamma}} & \mathcal{E}'_\beta
\end{array}$$

commutes.

An isomorphism between two descent data on two different covers is defined as an isomorphism between their restriction on a common refinement. A 2-morphism between two isomorphisms is the 2-morphisms between the corresponding isomorphisms of the gerbe descent data.

A twisted bundle is then defined as an equivalence class of descent data of twisted vector bundles. "Forgetting" the bundle data we obtain from the descent datum for a twisted vector bundle a descent datum for a gerbe, and the same applies to morphisms and 2-morphisms. We say that $\mathcal{E}$ is an $\mathcal{L}$-twisted vector bundle if "forgetting" the bundle data one obtains the equivalence class of the gerbe descent data defining $\mathcal{L}$.

Assume now that the gerbe $\mathcal{L}$ is unitary. An Hermitian descent datum for $\mathcal{E}$ consists of a unitary descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for $\mathcal{L}$ and a collection h_α of metrics on $\mathcal{E}_\alpha$ such that the maps $\varphi_{\alpha\beta}$ are isometries. One obtains a notion of isomorphism of Hermitian descent data by requiring ρ_α to be isometries. An Hermitian twisted bundle is then an equivalence class of Hermitian descent data.

Given a gerbe $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ on M, it is well known that (finite dimensional) twisted vector bundles exist if and only if the gerbe is torsion (see e.g. [19, 25] and references therein).

Lemma 2.8. *Let $\mathcal{L}$ be a gerbe on M and $\mathcal{E}$ an $\mathcal{L}$-twisted bundle. Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ be a descent datum for $\mathcal{L}$. Then there exists a descent datum for $\mathcal{E}$ isomorphic to the one of the form $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$.*

Proof. Let $(V_i, L_{ij}, m_{ijk}, E_i, r_{ij}), i, j, k \in I$ be a descent datum for $\mathcal{E}$. Without a loss of generality we may assume that $\{V_i\}$ is refinement of $\{U_\alpha\}$ given by the map $\varrho : I \to \Lambda$ and that there exists an isomorphism (S_i, λ_{ij}) between (V_i, L_{ij}, m_{ijk}) and restriction of $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ to $\{V_i\}$.

We define the vector bundle $\mathcal{E}_\alpha$ as follows. On every non empty open set $U_\alpha \cap V_i$, we set

$$\mathcal{E}_\alpha^{(i)} := E_i \otimes S_i \otimes \mathcal{L}_{\varrho(i)\alpha}.$$

Notice that $V_i \cap U_\alpha \subset U_{\varrho(i)\alpha}$ so that this definition makes sense as a vector bundle over $V_i \cap U_\alpha$. Moreover, if $(i,j) \in I^2$ is such that $V_{ij} \cap U_\alpha \neq \emptyset$, then we have a bundle isomorphism

$$\psi_\alpha^{(ij)} \colon \mathcal{E}_\alpha^{(j)}|_{V_{ij} \cap U_\alpha} \longrightarrow \mathcal{E}_\alpha^{(i)}|_{V_{ij} \cap U_\alpha}.$$

defined by the composition

$$E_j \otimes S_j \otimes \mathcal{L}_{\varrho(j)\alpha} \overset{\varphi_{ij}^{-1} \otimes \mathrm{id}}{\longrightarrow} E_i \otimes L_{ij} \otimes S_j \otimes \mathcal{L}_{\varrho(j)\alpha} \longrightarrow E_i \otimes S_i \otimes S_i^{-1} \otimes L_{ij} \otimes S_j \otimes \mathcal{L}_{\varrho(j)\alpha}$$
$$\overset{\mathrm{id} \otimes \lambda_{ij} \otimes \mathrm{id}}{\longrightarrow} E_i \otimes S_i \otimes \mathcal{L}_{\varrho(i)\varrho(j)} \otimes \mathcal{L}_{\varrho(j)\alpha} \overset{\mathrm{id} \otimes \mu_{\varrho(i)\varrho(j)\alpha}}{\longrightarrow} E_i \otimes S_i \otimes \mathcal{L}_{\varrho(i)\alpha}$$

It is easy to see that

$$\psi_\alpha^{(ik)} = \psi_\alpha^{(ij)} \circ \psi_\alpha^{(jk)}.$$

So, we can glue the bundles $(\mathcal{E}_\alpha^{(i)})_i$ for $V_i \cap U_\alpha \neq \emptyset$ together and form a vector bundle $\mathcal{E}_\alpha$ over U_α. More precisely

$$\mathcal{E}_\alpha := \left(\amalg_{V_i \cap U_\alpha \neq \emptyset} \mathcal{E}_\alpha^{(i)} \right) / \{\psi_\alpha^{(ij)}\}.$$

Next we introduce for $i \in I$ with $V_i \cap U_{\alpha\beta} \neq \emptyset$,

$$\varphi^{(i)}_{\alpha\beta} := \mathrm{id} \otimes \mu_{\varrho(i)\alpha\beta} \colon \mathcal{E}^{(i)}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{E}^{(i)}_\beta.$$

When $V_{ij} \cap U_{\alpha\beta} \neq \emptyset$, a straightforward inspection shows again that the following diagram commutes

$$\begin{array}{ccc} \mathcal{E}^{(i)}_\alpha \otimes \mathcal{L}_{\alpha\beta} & \xrightarrow{\varphi^{(i)}_{\alpha\beta}} & \mathcal{E}^{(i)}_\beta \\ {\scriptstyle \psi^{(ji)}_\alpha \otimes \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \psi^{(ji)}_\beta} \\ \mathcal{E}^{(j)}_\alpha \otimes \mathcal{L}_{\alpha\beta} & \xrightarrow{\varphi^{(j)}_{\alpha\beta}} & \mathcal{E}^{(j)}_\beta \end{array}$$

Therefore, the isomorphisms $\varphi^{(i)}_{\alpha\beta}$ induce an isomorphism

$$\varphi_{\alpha\beta} : \mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{E}_\beta.$$

We leave it to the reader to show that $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a descent datum for a twisted vector bundle isomorphic to $(V_i, L_{ij}, m_{ijk}, E_i, r_{ij})$.

□

We now fix a gerbe $\mathcal{L}$ with a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ on the smooth manifold M together with a twisted vector bundle $\mathcal{E}$ represented by $(\mathcal{E}_\alpha, \varphi_{\alpha\beta})$. We denote by $\mathcal{A}_\alpha$ the collection of bundles of algebras

$$\mathcal{A}_\alpha := \mathrm{End}(\mathcal{E}_\alpha), \quad \alpha \in \Lambda.$$

For any $U_{\alpha\beta} \neq \emptyset$, we have a canonical vector bundle isomorphism over $U_{\alpha\beta}$

$$\rho_{\alpha\beta} : \mathrm{End}(\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \longrightarrow \mathcal{A}_\alpha,$$

extending the canonical isomorphism $\mathrm{End}(\mathcal{L}_{\alpha\beta}) \simeq U_{\alpha\beta} \times \mathbb{C}$. Therefore, the bundle isomorphism $\varphi_{\alpha\beta}$ together with the identification $\rho_{\alpha\beta}$ induce the isomorphism of algebra bundles over $U_{\alpha\beta}$ given by

$$\mathcal{A}_\beta \xrightarrow{\varphi^*_{\alpha\beta}} \mathrm{End}(\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \xrightarrow{\rho^*_{\alpha\beta}} \mathcal{A}_\alpha.$$

We denote this isomorphism by $\varphi^*_{\alpha\beta}$. It is then easy to check that

$$\varphi^*_{\alpha\beta} \circ \varphi^*_{\beta\gamma} = \varphi^*_{\alpha\gamma}, \qquad \text{over } U_{\alpha\beta\gamma}.$$

Therefore, the collection $(\mathcal{A}_\alpha)$ defines a bundle $\mathcal{A}$ of algebras over M, which we denote $\mathrm{End}(\mathcal{E})$. We leave to the reader an easy check that the isomorphism class of $\mathrm{End}(\mathcal{E})$ depends only on the isomorphism class of $\mathcal{E}$.

Note that for every α we have the trace $\mathrm{tr}_\alpha \colon \mathrm{End}(\mathcal{E}_\alpha) \to C^\infty(U_\alpha)$. For $a \in \Gamma(U_{\alpha\beta}; \mathcal{A}_\beta)$ $\mathrm{tr}_\alpha(\varphi^*_{\alpha\beta}(a)) = \mathrm{tr}_\beta(a)$. Therefore we obtain the trace $\mathrm{tr} \colon \mathrm{End}(\mathcal{E}) \to C^\infty(M)$ defined by

$$\mathrm{tr}(a)|_{U_\alpha} = \mathrm{tr}_\alpha(a|_{U_\alpha}) \text{ for } a \in \mathrm{End}(\mathcal{E})$$

If the bundle $\mathcal{E}$ is $\mathbb{Z}_2$-graded then $\mathrm{End}(\mathcal{E})$ is a bundle of $\mathbb{Z}_2$-graded algebras with a supertrace $\mathrm{str} \colon \mathrm{End}(\mathcal{E}) \to C^\infty(M)$.

Definition 2.9. The bundle $\mathcal{A} = \mathrm{End}(\mathcal{E})$ is called the Azumaya bundle associated with the $\mathcal{L}$-twisted bundle $\mathcal{E}$.

If $\mathcal{E}$ is an Hermitian twisted bundle, each of the bundles $\mathrm{End}(\mathcal{E}_\alpha)$ is a bundle of *-algebras, with the *-operation given by taking the adjoint endomorphism. This induces a structure of a bundle of *-algebras on $\mathrm{End}(\mathcal{E})$.

Notice that, more generally, if $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{L}$-twisted bundles, we have a well defined bundle $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$, defined similarly by $\mathrm{Hom}(\mathcal{E},\mathcal{E}')|_{U_\alpha} = \mathrm{Hom}(\mathcal{E}|_{U_\alpha},\mathcal{E}'|_{U_\alpha})$.

Let $\mathcal{L}$ be a gerbe on M and $\mathcal{E}$ an $\mathcal{L}$-twisted bundle on M. Let $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma},\mathcal{E}_\alpha,\varphi_{\alpha\beta})$ be a descent datum for $\mathcal{E}$.

Definition 2.10. A connection on $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma},\mathcal{E}_\alpha,\varphi_{\alpha\beta})$ is a collection $(\nabla_\alpha,\nabla_{\alpha\beta},\omega_\alpha)$ where $(\nabla_{\alpha\beta},\omega_\alpha)$ is a connection on $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma})$ and each ∇_α is a connection on $\mathcal{E}_\alpha$ such that the identities

$$\varphi^*_{\alpha\beta}\nabla_\beta = \nabla_\alpha\otimes \mathrm{id} + \mathrm{id}\otimes\nabla_{\alpha\beta} \tag{2.5.1}$$

hold for $U_{\alpha\beta}\neq\emptyset$.

Lemma 2.11. *Let $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma},\mathcal{E}_\alpha,\varphi_{\alpha\beta})$ be a descent datum for an $\mathcal{L}$-twisted bundle $\mathcal{E}$. Then every connection $(\nabla_{\alpha\beta},\omega_\alpha)$ on the descent datum $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma})$ for $\mathcal{L}$ can be extended to a connection for the descent datum for $\mathcal{E}$.*

Proof. We start with a collection (∇'_α) of connections on $(\mathcal{E}_\alpha)$. Then we set for $U_{\alpha\beta}\neq\emptyset$:

$$A_{\alpha\beta} = \varphi^*_{\alpha\beta}\nabla'_\beta - \nabla'_\alpha\otimes\mathrm{id} - \mathrm{id}\otimes\nabla_{\alpha\beta}$$

So, $A_{\alpha\beta}$ is differential 1-form on $U_{\alpha\beta}$ with coefficients in the bundle $\mathrm{End}(\mathcal{E}_\alpha\otimes\mathcal{L}_{\alpha\beta})$. This latter being canonically isomorphic to $\mathrm{End}(\mathcal{E}_\alpha)$, we see that

$$A_{\alpha\beta}\in\Omega^1(U_{\alpha\beta},\mathrm{End}(\mathcal{E}_\alpha)).$$

Using the functoriality conditions on the isomorphisms $(\varphi_{\alpha\beta})$, it is then easy to check that

$$A_{\alpha\gamma} = A_{\alpha\beta} + \varphi^*_{\alpha\beta}A_{\beta\gamma},$$

where $\varphi_{\alpha\beta}$ is viewed here as the isomorphism over $U_{\alpha\beta}$ between $\mathrm{End}(\mathcal{E}_\beta)$ and $\mathrm{End}(\mathcal{E}_\alpha)$. Since the sheaf of sections of the bundle $\mathrm{End}(\mathcal{E})$ is soft, there exists a collection $A_\alpha\in\Omega^1(U_\alpha;\mathrm{End}(\mathcal{E}_\alpha))$ such that

$$A_{\alpha\beta} = A_\alpha - \varphi^*_{\alpha\beta}A_\beta.$$

The collection $\nabla_\alpha = \nabla'_\alpha + A_\alpha$ is then a connection on the $\mathcal{L}$-twisted vector bundle $\mathcal{E}$ compatible with the curving $\nabla_{\alpha\beta}$. □

An isomorphism between two descent data $(U_\alpha,\mathcal{L}_{\alpha\beta},\mu_{\alpha\beta\gamma},\mathcal{E}_\alpha,\varphi_{\alpha\beta})$ and $(U'_\alpha,\mathcal{L}'_{\alpha\beta},\mu'_{\alpha\beta\gamma},\mathcal{E}'_\alpha,\varphi'_{\alpha\beta})$ for $\mathcal{E}$ with connections $(\nabla_\alpha,\nabla_{\alpha\beta},\omega_\alpha)$ and $(\nabla'_\alpha,\nabla'_{\alpha\beta},\omega'_\alpha)$ respectively is given by the collections $s=(\rho_\alpha,S_\alpha,\lambda_{\alpha\beta},)$ where $(\rho_\alpha,S_\alpha,\lambda_{\alpha\beta})$ is a morphism between the descent data without connections, ∇^S_α are connections on S_α such that $\left(S_\alpha,\lambda_{\alpha\beta},\nabla^S_\alpha\right)$ is a morphism of the corresponding gerbe descent data with connectictions (i.e. the equatons (2.3.1), (2.3.2) are satisfied) and the equality

$$\rho^*_\alpha\nabla'_\alpha = \nabla_\alpha\otimes\mathrm{id} + \mathrm{id}\otimes\nabla^S_\alpha$$

holds (in the notations of (2.3.1) and (2.3.2)).

A connection on a twisted bundle is then a choice of connections on each descent datum of this twisted bundle and lifting of isomorphisms of descent data to isomorphism of descent data with connections. Every connection on a gerbe $\mathcal{L}$ can be extended to a connection on any $\mathcal{L}$-twisted bundle.

Proposition 2.12. *Let $\mathcal{E}$ be an $\mathcal{L}$-twisted bundle with connection. Choose a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ with a connection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ representing $\mathcal{E}$. Then the collection $(\theta_\alpha + \omega_\alpha)$, where $\theta_\alpha = \nabla_\alpha^2$ is the curvature of ∇_α, defines a global differential 2-form θ on M with coefficients in the Azumaya bundle $\mathcal{A} = \operatorname{End}(\mathcal{E})$. This form is independent of the choice of the representing descent datum.*

Proof. We have for any $\alpha \in \Lambda$, $\theta_\alpha \in \Omega^2(U_\alpha, \mathcal{A}_\alpha)$ where $\mathcal{A}_\alpha = \operatorname{End}(\mathcal{E}_\alpha))$ is the Azumaya bundle associated with the twisted bundle $\mathcal{E}$. The equation (2.5.1) implies that

$$\varphi^*_{\alpha\beta}\theta_\beta = \theta_\alpha + \omega_{\alpha\beta} \quad \in \quad \Omega^2(U_{\alpha\beta}, \operatorname{End}(\mathcal{E}_\alpha)).$$

Therefore, the collection $(\theta_\alpha + \omega_\alpha)$ of elements of $\Omega^2(U_\alpha, \mathcal{A}_\alpha)$ satisfies the relations

$$\varphi^*_{\alpha\beta}(\theta_\beta + \omega_\beta) = \theta_\alpha + \omega_\alpha.$$

It is easy to see that the form θ is independent of the equivalence class of the connection and is functorial with respect to the isomorphism of descent data. □

In the notations above let $\nabla\colon \mathcal{A} \to \Omega^1(M, \mathcal{A})$ be connection defined for a fixed descent datum by

$$(\nabla\xi)|_{U_\alpha} = [\nabla_\alpha, \xi]. \tag{2.5.2}$$

It is easy to see that ∇ is well defined and by derivation with respect to the product on $\mathcal{A}$.

Note that

$$\nabla^2 = [\theta, \cdot] \text{ and } \nabla\theta = 2\pi i\Omega \tag{2.5.3}$$

where Ω is the 3-curvature form of the connection on $\mathcal{L}$, see Lemma 2.4.

Proposition 2.13. (1) *Let $\mathcal{E}$ be an $\mathcal{L}$-twisted bundle and ∇ a connection on $\mathcal{E}$. Set $\operatorname{Ch}_{\mathcal{L}}(\nabla) = \operatorname{tr} e^{-\frac{u\theta}{2\pi i}} \in \Omega^*(M)[u]$. Then $d_\Omega \operatorname{Ch}_{\mathcal{L}}(\nabla) = 0$*

(2) *The class of $\operatorname{Ch}_{\mathcal{L}}(\nabla)$ in $H_\Omega(M)$ is independent of choice of connection. Namely assume we are given a different connection ∇' on $\mathcal{E}$ (and therefore on $\mathcal{L}$) and let Ω' be the associated 3-curvature form. Then $I([\operatorname{Ch}_{\mathcal{L}}(\nabla)]) = [\operatorname{Ch}_{\mathcal{L}}(\nabla')]$, where I is the canonical isomorphism of cohomology of $(\Omega^*(M)[u], d_\Omega)$ with cohomology of $(\Omega^*(M)[u], d_{\Omega'})$.*

We denote the class of $\operatorname{Ch}_{\mathcal{L}}(\nabla)$ by $\operatorname{Ch}_{\mathcal{L}}(\mathcal{E})$.

Proof. Since Ω is central, $\nabla(e^{-\frac{u\theta}{2\pi i}}) = -u\Omega e^{-\frac{u\theta}{2\pi i}}$. Hence

$$d \operatorname{tr} e^{-\frac{u\theta}{2\pi i}} = \operatorname{tr} \nabla(e^{-\frac{u\theta}{2\pi i}}) = -u\Omega \wedge \operatorname{Ch}_{\mathcal{L}}(\nabla)$$

and $d_\Omega \operatorname{Ch}_{\mathcal{L}}(\nabla) = 0$, which proves the first statement.

Fix now a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for $\mathcal{E}$ together with a connection $\nabla = (\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ and let $\nabla' = (\nabla'_\alpha, \nabla'_{\alpha\beta}, \omega'_\alpha)$ be another connection on $\mathcal{E}$. Set $\delta_{\alpha\beta} = \nabla'_{\alpha\beta} - \nabla_{\alpha\beta}$

and let δ_α be such that $\delta_\alpha - \delta_\beta = \delta_{\alpha\beta}$. Recall that the isomorphism I is induced by the map of complexes

$$\xi \longmapsto e^{-u\eta} \wedge \xi$$

where $\eta \in \Omega^2(M)$ is defined by $\eta|_{U_\alpha} = \frac{1}{2\pi i}(\omega'_\alpha - \omega_\alpha - d\delta_\alpha)$.

Consider now the manifold $\widetilde{M} = \mathbb{R} \times M$. Denote by $\pi\colon \widetilde{M} \to M$ the projection on the second factor and by $t\colon \widetilde{M} \to \mathbb{R}$ the projection on the first factor. Consider on $\widetilde{M}$ the gerbe $\widetilde{\mathcal{L}} = \pi^*\mathcal{L}$ given by the descent datum $(\pi^{-1}U_\alpha, \pi^*\mathcal{L}_{\alpha\beta}, \pi^*\mu_{\alpha\beta\gamma})$. Then $\pi^*\mathcal{E}$, $\pi^*\phi_{\alpha\beta}$ describe a $\widetilde{\mathcal{L}}$ twisted bundle $\widetilde{\mathcal{E}}$. Moreover,

$$\widetilde{\nabla}_{\alpha\beta} := (1-t)\pi^*\nabla_{\alpha\beta} + t\pi^*\nabla'_{\alpha\beta} \text{ and } \widetilde{\omega}_\alpha = (1-t)\pi^*\omega_\alpha + t\pi^*\omega'_\alpha + dt \wedge \delta_\alpha$$

define respectively, connective structure and curving of $\widetilde{L}$. Similarly, $\widetilde{\nabla}_\alpha = (1-t)\pi^*\nabla_\alpha + t\pi^*\nabla'_\alpha$ defines a compatible connection on $\widetilde{\mathcal{E}}$. The corresponding 3-form is given by

$$\widetilde{\Omega} = (1-t)\pi^*\Omega + t\pi^*\Omega' + \frac{1}{2\pi i} dt \wedge (\omega'_\alpha - \omega_\alpha - d\delta_\alpha) = (1-t)\pi^*\Omega + t\pi^*\Omega' + dt \wedge \pi^*\eta.$$

Now, the map $\xi \mapsto e^{ut\eta} \wedge \xi$ is an isomorphism of complexes $\left(\Omega^*(\widetilde{M})[u], d_{\widetilde{\Omega}}\right)$ and $\left(\Omega^*(\widetilde{M})[u], d_{\pi^*\Omega}\right)$. Set $\widetilde{\mathcal{A}} = \pi^*\mathcal{A}$ and let $\widetilde{\theta} \in \Omega^2(\widetilde{M}, \widetilde{\mathcal{A}})$ be the form defined by $\pi^*\theta_\alpha + \widetilde{\omega}_\alpha$. By the result of the first part of the proposition the differential form $\operatorname{tr} e^{-\frac{u\widetilde{\theta}}{2\pi i}}$ is a cocycle in the complex $\left(\Omega^*(\widetilde{M})[u], d_{\widetilde{\Omega}}\right)$. Hence $\operatorname{tr} e^{ut\eta} \wedge e^{-\frac{u\widetilde{\theta}}{2\pi i}}$ is a cocycle in the complex $\left(\Omega^*(\widetilde{M})[u], d_{\pi^*\Omega}\right)$. This implies the relation

$$\operatorname{tr} e^{u\eta} \wedge e^{-\frac{u\theta'}{2\pi i}} - \operatorname{tr} e^{-\frac{u\theta}{2\pi i}} = d_\Omega \int_0^1 \iota_{\frac{\partial}{\partial t}} \operatorname{tr} e^{ut\eta} \wedge e^{-\frac{u\widetilde{\theta}}{2\pi i}} dt.$$

Therefore, we finally deduce that the differential forms $\operatorname{tr} e^{u\eta} \wedge e^{-\frac{u\theta'}{2\pi i}}$ and $\operatorname{tr} e^{-\frac{u\theta}{2\pi i}}$ are cohomologous in $(\Omega^*(M)[u], d_\Omega)$, and hence that the differential forms $\operatorname{tr} e^{-\frac{u\theta'}{2\pi i}}$ and $\operatorname{tr} e^{-u\eta} \wedge e^{-\frac{u\theta}{2\pi i}}$ are cohomologous in $(\Omega^*(M)[u], d_{\Omega'})$, which finishes the proof. □

We will need also a notion of superconnection on the twisted bundle. We now briefly indicate the modifications which need to be made to the notion of connection to obtain that of superconnection. Assume that we are given a gerbe $\mathcal{L}$ and a $\mathbb{Z}_2$-graded $\mathcal{L}$-twisted vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Let $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ be a descent datum for $\mathcal{E}$.

Definition 2.14. A superconnection $\mathbb{A}$ on the descent datum is $\mathbb{A} = (\mathbb{A}_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ where $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum for $\mathcal{L}$ and each $\mathbb{A}_\alpha$ is a superconnection on $\mathcal{E}_\alpha$ satisfying the relations

$$\varphi^*_{\alpha\beta}\mathbb{A}_\beta = \mathbb{A}_\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\alpha\beta}, \qquad \text{for } U_{\alpha\beta} \neq \emptyset.$$

Each superconnection $\mathbb{A}_\alpha$ can be written as $\mathbb{A}_\alpha = \sum_{k\geq 0} \mathbb{A}_\alpha^{[k]}$ where $\mathbb{A}_\alpha^{[k]} \in \Omega^k(U_\alpha; \mathrm{End}(\mathcal{E}_\alpha)^-)$ for k-even, $\mathbb{A}_\alpha^{[k]} \in \Omega^k(U_\alpha; \mathrm{End}(\mathcal{E}_\alpha)^+)$ for k-odd, $k \neq 1$, and $\mathbb{A}_\alpha^{[1]}$ is a grading preserving connection on $\mathcal{E}_\alpha$. It is easy to see that for each $k \neq 1$ there exists a form $\mathbb{A}^{[k]} \in \Omega^k(M; \mathrm{End}(\mathcal{E}))$ such that $\mathbb{A}^{[k]}|U_\alpha = \mathbb{A}_\alpha^{[k]}$. For $k = 1$ $(\mathbb{A}_\alpha^{[1]}, \nabla_{\alpha\beta}, \omega_\alpha)$ defines a connection on the (descent datum of) $\mathcal{E}$.

Let now $u^{1/2}$ be a formal variable of degree -1 such that $(u^{1/2})^2 = u$. Define the rescaled superconnection

$$\mathbb{A}_{u^{-1}} := \sum u^{(k-1)/2}\mathbb{A}^{[k]}.$$

Let Ω be as before the curvature 3-form of the connection on $\mathcal{L}$. Define the curvature of the rescaled superconnection $\theta^{\mathbb{A}_{u^{-1}}}$ by

$$\theta^{\mathbb{A}_{u^{-1}}}|_{U_\alpha} = (\mathbb{A}_\alpha)^2_{u^{-1}} + \omega_\alpha.$$

Then $u\theta^{\mathbb{A}_{u^{-1}}} \in \Omega^{even}(M, \mathrm{End}(\mathcal{E})^+)[u] + u^{1/2}\Omega^{odd}(M, \mathrm{End}(\mathcal{E})^-)[u]$. We therefore have the differential form $\exp\left(-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}\right)$ which belongs to $\Omega^{even}(M, \mathrm{End}(\mathcal{E})^+)[u]+u^{1/2}\Omega^{odd}(M, \mathrm{End}(\mathcal{E})^-)[u]$ and the differential form $\mathrm{str}\exp\left(-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}\right)$ which belongs to $\Omega^{even}(M)[u]$. The following is an analogue of the Proposition 2.13 for the superconnections with the essentially identical proof.

Proposition 2.15.

(1) *Set* $\mathrm{Ch}_{\mathcal{L}}(\mathbb{A}) = \mathrm{str}\exp\left(-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}\right)$. *Then* $d_\Omega \mathrm{Ch}_{\mathcal{L}}(\mathbb{A}) = 0$

(2) *The class of* $\mathrm{Ch}_{\mathcal{L}}(\mathbb{A})$ *in* $H_\Omega(M)$ *is independent of choice of superconnection. Specifically, assume we are given a different superconnection* $\mathbb{A}'$ *on* $\mathcal{E}$ *(and therefore a different connection on* $\mathcal{L}$*) and let* Ω' *be the associated* 3*-curvature form. Then* $I([\mathrm{Ch}_{\mathcal{L}}(\mathbb{A})]) = [\mathrm{Ch}_{\mathcal{L}}(\mathbb{A}')]$, *where* I *is the canonical isomorphism of cohomology of* $(\Omega^*(M)[u], d_\Omega)$ *with cohomology of* $(\Omega^*(M)[u], d_{\Omega'})$.

2.6. **Horizontally twisted bundles.** Let $\pi : M \to B$ be a smooth fibration. Let $\mathcal{L}$ be a gerbe on B.

Definition 2.16. A descent datum for a horizontally $\mathcal{L}$-twisted bundle $\mathcal{E}$ on M consists of the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ for $\mathcal{L}$ together with a collection $(\mathcal{E}_\alpha \to \pi^{-1}U_\alpha)_{\alpha\in\Lambda}$ of vector bundles and a collection of vector bundle isomorphisms $\varphi_{\alpha\beta}\colon \mathcal{E}_\alpha \otimes \pi^*\mathcal{L}_{\alpha\beta} \cong \mathcal{E}_\beta$ so that

$$(\pi^{-1}U_\alpha, \pi^*\mathcal{L}_{\alpha\beta}, \pi^*\mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$$

is a descent datum for a twisted vector bundle on M.

An (iso)morphism between two such descent data

$$(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta}) \text{ and } (U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$$

is given by the collection $(\rho_\alpha, S_\alpha, \lambda_{\alpha\beta})$ where $(S_\alpha, \lambda_{\alpha\beta})$ is an isomorphism between $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma})$ and $(U'_\alpha, \mathcal{L}'_{\alpha\beta}, \mu'_{\alpha\beta\gamma})$ and $\rho_\alpha\colon \mathcal{E}_\alpha\otimes\pi^*S_\alpha \to \mathcal{E}'_\alpha$ is such that $(\rho_\alpha, \pi^*S_\alpha, \pi^*\lambda_{\alpha\beta})$ is an (iso)morphism between $(\pi^{-1}U_\alpha, \pi^*\mathcal{L}_{\alpha\beta}, \pi^*\mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ and $(\pi^{-1}U'_\alpha, \pi^*\mathcal{L}'_{\alpha\beta}, \pi^*\mu'_{\alpha\beta\gamma}, \mathcal{E}'_\alpha, \varphi'_{\alpha\beta})$.

With these definitions one can now define a horizontally twisted bundle as an equivalence class of descent data. Let $Tw(\pi^*\mathcal{L})$ denote the set of isomorphism classes of all $\pi^*\mathcal{L}$-twisted bundles on M and $Tw_h(\mathcal{L})$ denote the set of isomorphism classes of all horizontally $\mathcal{L}$-twisted bundles. Then we have an obvious map $Tw_h(\mathcal{L}) \to Tw(\pi^*\mathcal{L})$. According to Lemma 2.8 this map is surjective. In particular $Tw_h(\mathcal{L}) \neq \emptyset$ if and only if $\pi^*[\mathcal{L}]$ is torsion in $H^3(M, \mathbb{Z})$. It is however not injective. Indeed, if $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a descent datum for a horizontally twisted bundle and S is a line bundle on M then $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha \otimes S|_{\pi^{-1}U_\alpha}, \varphi_{\alpha\beta} \otimes \mathrm{id})$ is another such descent datum. These data define the same element of $Tw(\pi^*\mathcal{L})$ but, unless S is a pull-back of a line bundle from B, different elements of $Tw_h(\mathcal{L})$.

A connection on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a collection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ where $(\nabla_{\alpha\beta}, \omega_\alpha)$ is a connection on the descent datum for $\mathcal{L}$ and ∇_α is a connection on $\mathcal{E}_\alpha$ such that $(\nabla_\alpha, \pi^*\nabla_{\alpha\beta}, \pi^*\omega_\alpha)$ is a connection on the descent datum $(\pi^{-1}U_\alpha, \pi^*\mathcal{L}_{\alpha\beta}, \pi^*\mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$. With these definitions one can now define a notion of connection on the horizontally $\mathcal{L}$-twisted bundle in complete analogy with the defnitions for the twisted bundles. If ∇ is such a connection and $\mathit{\Omega}$ is the curvature 3-form of the gerbe $\mathcal{L}$, one defines $\mathrm{Ch}_{\mathcal{L}}(\nabla)$ – a closed form in $(\Omega^*(M), d_{\pi^*\mathit{\Omega}})_\bullet$. The analogues of Propositions 2.12 and 2.13 hold in this context with the same proofs.

3. Projective families and the analytic index

3.1. **Families of pseudodifferential operators.** Here we collect several facts about the (untwisted) families of pseudodifferential operators.

Let $\pi\colon X \to Y$ be a smooth fibration and E a vector bundle on X.

We denote by $\Psi^m_{\mathcal{L}}(X|Y; E)$ the space of classical fiberwise pseudodifferential operators of order $\leq m$ on π, acting on the sections of the vector bundle E. As usual, we set

$$\Psi(X|Y;E) := \bigcup_{m\in\mathbb{Z}} \Psi^m(X|Y;E)$$

and

$$\Psi^{-\infty}(X|Y;E) := \bigcap_{m\in\mathbb{Z}} \Psi^m(X|Y;E).$$

Recall that composition endows each $\Psi(X|Y;E)$ with the structure of a filtered algebra and that $\Psi^{-\infty}(X|Y;E)$ is an ideal in this algebra. $\Psi(X|Y;E)$ is also a module over $C^\infty(Y)$ and the composition is $C^\infty(Y)$-linear.

We have the following elementary general result:

Lemma 3.1. *Assume $\pi\colon X \to Y$ is a smooth fibration, E a vector bundle on X, L a line bundle on Y. Define a map $\chi_L\colon \Psi(X|Y;E) \to \Psi(X|Y; E\otimes \pi^*L)$ by*

$$\chi_L(D)(e\otimes\pi^*(l)) = D(e)\otimes \pi^* l$$

for $D \in \Psi_{\mathcal{L}}(X|Y;\mathcal{E})$, $e \in \Gamma_c(E)$, $l \in \Gamma(L)$. Then χ_L is a well-defined isomorphism of algebras and $C^\infty(Y)$-modules.

$$\chi_{L_1\otimes L_2} = \chi_{L_2}\circ\chi_{L_1}$$

If we have two vector bundles E, E' on X we denote by $\Psi(X|Y;E,E')$ the set of fiberwise pseudodifferential operators $\Gamma_c(E) \to \Gamma(E')$. For L – line bundle on Y we again have the isomorphism of $C^\infty(Y)$-modules $\chi_L\colon \Psi(X|Y;E,E') \to \Psi(X|Y;E\otimes\pi^*L, E'\otimes\pi^*L)$ defined by the same formula.

We have the vertical cotangent bundle $T^*(X|Y) = T^*X/(Ker\pi_*)^\perp$. $\mathring{T}^*(X|Y)$ denotes (the total space of) this bundle with the zero section removed, and $p\colon \mathring{T}^*(X|Y) \to X$ is the natural projection. Recall that for $P \in \Psi^m(X|Y;E,E')$ the principal symbol $\sigma_m(P)$ is an m-homogeneous smooth section over $\mathring{T}^*(X|Y)$ of vector bundle $p^*\operatorname{Hom}(E,E')$. Then identifying canonically isomorphic bundles $\mathrm{Hom}(E, E')$ and $\mathrm{Hom}(E\otimes\pi^*L, E'\otimes\pi^*L)$ we have $\sigma_m(P) = \sigma_m(\chi_L(P))$.

3.2. **Projective families.** Let $\pi : M \to B$ be a smooth fibration with compact fibers. Let $\mathcal{L}$ be a gerbe on B such that $\pi^*[\mathcal{L}]$ is a torsion class in $H^3(M,\mathbb{Z})$. Let $\mathcal{E}$ be a horizontally $\mathcal{L}$-twisted bundle on M, cf. Section 2.6. We fix a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for $\mathcal{E}$. For any $(\alpha, \beta) \in \Lambda^2$ with $U_{\alpha\beta} \neq \emptyset$ we have an isomorphism of filtered algebras, respecting the $C^\infty(U_{\alpha\beta})$-module structure:

$$\phi_{\alpha\beta} \colon \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta) \to \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\alpha), \tag{3.2.1}$$

It is defined as the composition

$$\Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta) \overset{\psi_{\alpha\beta}}{\to} \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\beta \otimes \pi^*\mathcal{L}_{\alpha\beta}) \overset{\varphi_{\alpha\beta}}{\to} \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}; \mathcal{E}_\alpha)$$

where $\psi_{\alpha\beta} = \chi^{-1}_{\mathcal{L}_{\alpha\beta}} = \chi_{\mathcal{L}_{\beta\alpha}}$.

Recall (cf. [4]) that for every $\alpha \in \Lambda$ we have an infinite dimensional bundle $\pi_*\mathcal{E}_\alpha$ on U_α defined by $\Gamma(V, \pi_*\mathcal{E}_\alpha) = \Gamma(\pi^{-1}V, \mathcal{E}_\alpha)$, $V \subset U_\alpha$. Over $U_{\alpha\beta}$ we have isomorphisms $\pi_*\varphi_{\alpha\beta} \colon \pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \pi_*\mathcal{E}_\beta$ defined by

$$\pi_*\varphi_{\alpha\beta}(\xi \otimes l) = \varphi(\xi \otimes \pi^*(l)).$$

Here $\xi \in \Gamma(U_{\alpha\beta}, \pi_*\mathcal{E}_\alpha) = \Gamma(\pi^{-1}U_{\alpha\beta}, \mathcal{E}_\alpha)$, $l \in \Gamma(U_{\alpha\beta}, \mathcal{L}_{\alpha\beta})$.

Note that the isomorphisms $\pi_*\varphi_{\alpha\beta} \colon \pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \pi_*\mathcal{E}_\beta$ induce the isomorphisms

$$(\pi_*\varphi_{\alpha\beta})^* \colon \mathrm{End}(\pi_*\mathcal{E}_\beta) \to \mathrm{End}(\pi_*\mathcal{E}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \cong \mathrm{End}(\pi_*\mathcal{E}_\alpha)$$

over $U_{\alpha\beta}$. The restriction of this isomorphism to $\Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\beta) \subset \mathrm{End}(\pi_*\mathcal{E}_\beta)$ coincides with the isomorphism $\phi_{\alpha\beta} \colon \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\beta) \to \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\alpha)$. Since the isomorphisms $(\pi_*\varphi_{\alpha\beta})^*$ satisfy the natural cocycle identity we have the following:

Lemma 3.2. *The isomorphisms $\phi_{\alpha\beta}$ satisfy*

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$$

whenever $U_{\alpha\beta\gamma} \neq \emptyset$.

Recall that the isomorphisms $\varphi_{\alpha\beta}$ induce the natural isomorphisms $\varphi^*_{\alpha\beta} \colon \mathrm{End}(\mathcal{E}_\beta) \to \mathrm{End}(\mathcal{E}_\alpha)$. Then

$$\sigma_m \circ \phi_{\alpha\beta} = p^*\left(\varphi^*_{\alpha\beta}\right) \circ \sigma_m \tag{3.2.2}$$

Definition 3.3. A fiberwise pseudodifferential operator P of order $\leq m$ with coefficients in the horizontally $\mathcal{L}$-twisted vector bundle $\mathcal{E}$ is a collection $\{P_\alpha\}_{\alpha\in\Lambda}$, $P_\alpha \in \Psi^m(\pi^{-1}U_\alpha|U_\alpha; \mathcal{E}_\alpha)$ such that

$$P_\alpha = \phi_{\alpha\beta}(P_\beta).$$

where $\phi_{\alpha\beta}$ is defined in Equation 3.2.1. The space of fiberwise pseudodifferential operators of order $\leq m$, with coefficients in the $\pi^*\mathcal{L}$-twisted vector bundle $\mathcal{E}$, is denoted by $\Psi^m_{\mathcal{L}}(M|B; \mathcal{E})$.

Note that the equation (3.2.2) implies that if $P = \{P_\alpha\} \in \Psi^m_{\mathcal{L}}(M|B; \mathcal{E})$ then the collection $\sigma_m(P_\alpha)$ defines a section of the (untwisted) bundle $p^*\,\mathrm{End}(\mathcal{E})$. We will call this section the principal symbol of $P = \{P_\alpha\}$

Remark 3.4. We define in the same way the space $\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ of fiberwise pseudodifferential operators of order $\leq m$, from the horizontally $\mathcal{L}$-twisted vector bundle $\mathcal{E}$ to the horizontally $\mathcal{L}$-twisted vector bundle $\mathcal{E}'$. In particular $\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}) = \Psi^m_{\mathcal{L}}(M|B;\mathcal{E})$. We also have a principal symbol map $\sigma_m\colon \Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}) \to p^*\operatorname{Hom}(\mathcal{E},\mathcal{E}')$.

We set

$$\Psi_{\mathcal{L}}(M|B;\mathcal{E}) := \bigcup_{m\in\mathbb{Z}} \Psi^m_{\mathcal{L}}(M|B;\mathcal{E}) \text{ and } \Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}) := \bigcap_{m\in\mathbb{Z}} \Psi^m_{\mathcal{L}}(M|B;\mathcal{E}).$$

Introduce now a composition in $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$ by

$$\{P_\alpha\}\circ\{Q_\alpha\} = \{P_\alpha Q_\alpha\}$$

Since $\phi_{\alpha\beta}$ are algebra isomorphisms the right hand side of this equality defines an element in $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$.

Proposition 3.5. *The composition of operators is $C^\infty(B)$-linear and endows $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$ with the structure of associative algebra; $\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E})$ is an ideal in $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$.*

We can now define the algebra of forms on B with values in $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$ by

$$\Omega^*\left(B,\Psi_{\mathcal{L}}(M|B;\mathcal{E})\right) = \Omega^*(B)\otimes_{C^\infty(B)}\Psi_{\mathcal{L}}(M|B;\mathcal{E}).$$

Recall that for every α and $V\subset U_\alpha$ we have a fiberwise trace $\operatorname{Tr}_\alpha\colon \Psi^{-\infty}(\pi^{-1}V|V;\mathcal{E}_\alpha)\to C^\infty(V)$. It is easy to see that for $P\in\Psi^{-\infty}(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta};\mathcal{E}_\beta)$

$$\operatorname{Tr}_\alpha\phi_{\alpha\beta}(P) = \operatorname{Tr}_\beta(P).$$

We therefore obtain a well defined map $\operatorname{Tr}\colon \Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E})\to C^\infty(B)$ by setting $\operatorname{Tr}\{P_\alpha\}|_{U_\alpha} = \operatorname{Tr}_\alpha(P_\alpha)$. This trace is a $C^\infty(B)$-module map satisfying $\operatorname{Tr}[A,B]=0$. It extends naturally to define a map $\operatorname{Tr}\colon \Omega^*\left(B,\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E})\right)\to\Omega^*(B)$.

If the bundle $\mathcal{E}$ is $\mathbb{Z}_2$ graded we have a similarly defined supertrace

$$\operatorname{Str}\colon \Omega^*\left(B,\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E})\right)\to\Omega^*(B).$$

Note that the definition of the $\Psi_{\mathcal{L}}(M|B;\mathcal{E})$ depends on the descent datum for $\mathcal{E}$. It is straightforward however to see that an isomorphism of descent data defines canonically an isomorphism of the corresponding bundles of algebras.

3.3. **Analytic index.**

Definition 3.6. Let $m\geq 0$ be fixed. A fiberwise pseudodifferential operator $P\in\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ is fiberwise elliptic if the principal symbol $\sigma_m(P)\in\Gamma(p^*\operatorname{Hom}(\mathcal{E},\mathcal{E}'))$is an isomorphism.

We say that $Q\in\Psi^{-m}_{\mathcal{L}}(M|B;\mathcal{E}',\mathcal{E})$ is a parametrix of $P\in\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ if $PQ-1\in\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}',\mathcal{E}')$ and $QP-1\in\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E})$.

Lemma 3.7. *Every elliptic $P\in\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ has a parametrix.*

Proof. Construct first a parametrix R_α for P_α. Let ρ_α be a partition of unity subordinate to $\{U_\alpha\}$. Then define $R'_\alpha = \sum_\beta \rho_\beta\phi_{\alpha\beta}(R_\beta)$. Then each R'_α is a parametrix for P_α and $\phi_{\alpha\beta}(R'_\beta) = R'_\alpha$. □

Let $D \in \Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ be elliptic. Let $F \in \Psi^0_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ be such that $\sigma_0(F)|_{S^*(M|B)} = \sigma_m(D)|_{S^*(M|B)}$. Choose a parametrix R for F. Let $U_D \in \Psi^0_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')$ be an invertible operator such that $U_D - \begin{bmatrix} 0 & -R \\ F & 0 \end{bmatrix} \in \Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')$. An explicit construction of an example of such an operator is as follows. Let $S_0 = 1 - RF$, $S_1 = 1 - FR$. Then set $U_D = \begin{bmatrix} S_0 & -(1+S_0)R \\ F & S_1 \end{bmatrix}$. With such a choice the inverse is given by an explicit formula $U_D^{-1} = \begin{bmatrix} S_0 & (1+S_0)R \\ -F & S_1 \end{bmatrix}$.

Definition 3.8. The index of D is the K-theory class of the algebra $\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')$ defined by

$$\operatorname{ind}(D) = [P_D - Q] \in K_0(\Psi^0_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}'), \Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}')) \cong K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}')),$$

where P_D and Q are the idempotents given by $P_D = U_D \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_D^{-1}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We leave to the reader the standard K-theoretic proof that the index is well defined and is stable under the homotopies of D in the class of elliptic operators in $\Psi^m_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$. Assume that the horizontally $\mathcal{L}$-twisted bundles $\mathcal{E}$ and $\mathcal{E}'$ are hermitian (and in particular $\mathcal{L}$ is unitary) and that fibers of π are equipped with smoothly varying volume forms. In this situation for a projective family D we can define an formally adjoint projective family D^* by forming the formal adjoints for each family D_α.

Lemma 3.9. *Then, identifying $K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}'))$ with $K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}'\oplus\mathcal{E}))$ we have*

$$\operatorname{ind} D^* = -\operatorname{ind} D.$$

Proof. We have

$$\operatorname{ind} D^* = \left[U_{D^*} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U_{D^*}^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] \in K_0(\Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}')).$$

By deforming D we may assume that $\sigma_m(D)|_{S^*(M|B)}$ is an isometry. In this case we may choose $U_{D^*} = U_D^{-1}$, and the statement follows. □

3.4. **Chern character of the index.** We continue in the notations of the previous section. We assume that we are given a horizontally $\mathcal{L}$-twisted bundle $\mathcal{E}$ with a connection represented by a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ with connection $(\nabla_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ see the Section 2.6. Following Mathai and Stevenson [30], we describe in this paragraph a morphism of complexes $CC^-_\bullet(\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E})) \to (\Omega^*(M)[u], d_\Omega)_\bullet$.

Recall the bundles $\pi_*\mathcal{E}_\alpha$ and isomorphisms $\pi_*\varphi_{\alpha\beta}$ defined in the Section 3.2. It is easy to see that $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \pi_*\mathcal{E}_\alpha, \pi_*\varphi_{\alpha\beta})$ is a descent datum for an infinite dimensional twisted bundle. We now proceed to define a connection on this descent datum.

Choose horizontal distribution i.e. a subbundle $\mathcal{H} \subset TM$ such that $TM = \mathcal{H} \oplus T(M|B)$. This choice together with connections $\nabla^{\mathcal{E}}_\alpha$ defines for each α a connection $\nabla^{\mathcal{H}}_\alpha$ as follows:

$$(\nabla^{\mathcal{H}}_\alpha)_X \xi = (\nabla^{\mathcal{E}}_\alpha)_{X^{\mathcal{H}}} \xi$$

where $X^{\mathcal{H}}$ is the horizontal lift of $X \in \Gamma(B, TB)$.

Lemma 3.10. $(\pi_*\varphi_{\alpha\beta})^*\nabla^{\mathcal{H}}_\beta = \nabla^{\mathcal{H}}_\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\alpha\beta}$

The curvature of the connection $\nabla^{\mathcal{H}}_\alpha$ is a 2-form $\theta^{\mathcal{H}}_\alpha$ on U_α with values in fiberwise differential operators given by

$$\theta^{\mathcal{H}}_\alpha(X,Y) = \theta^{\mathcal{E}}_\alpha(X^{\mathcal{H}}, Y^{\mathcal{H}}) + (\nabla^{\mathcal{E}}_\alpha)_{T(X,Y)}.$$

where

$$T^{\mathcal{H}}(X,Y) = [X^{\mathcal{H}}, Y^{\mathcal{H}}] - [X,Y]^{\mathcal{H}},\ \ X, Y \in \Gamma(B, TB). \tag{3.4.1}$$

Each $\nabla^{\mathcal{H}}_\alpha$ defines a filtration-preserving derivation $\partial^{\mathcal{H}}_\alpha$ of the algebra of fiberwise pseudodifferential operators

$$\partial^{\mathcal{H}}_\alpha \colon \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha) \to \Omega^1(U_\alpha, \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha)) \text{ defined by } \partial^{\mathcal{H}}_\alpha(D) = [\nabla^{\mathcal{H}}_\alpha, D].$$

If $D \in \Psi(\pi^{-1}U_{\alpha\beta}|U_{\alpha\beta}, \mathcal{E}_\beta)$ then the result of Lemma 3.10 implies that

$$\partial^{\mathcal{H}}_\alpha(\phi_{\alpha\beta}(D)) = \phi_{\alpha\beta}(\partial^{\mathcal{H}}_\beta(D)).$$

Therefore if $\{D_\alpha\}$, $D_\alpha \in \Psi_{\mathcal{L}}(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha)$, defines an element in $\Psi^m_{\mathcal{L}}(M|B,\mathcal{E})$ then $\{\partial^{\mathcal{H}}_\alpha(D_\alpha)\} \in \Omega^1(B, \Psi^m_{\mathcal{L}}(M|B,\mathcal{E}))$. We therefore obtain a derivation

$$\partial^{\mathcal{H}} \colon \Psi_{\mathcal{L}}(M|B,\mathcal{E}) \to \Omega^1(B, \Psi_{\mathcal{L}}(M|B,\mathcal{E})),$$

which extends to a derivation of the algebra $\Omega^*(B, \Psi_{\mathcal{L}}(M|B,\mathcal{E}))$.

Lemma 3.11. *There exists* $\theta^{\mathcal{H}} \in \Omega^2(B, \Psi^1_{\mathcal{L}}(M|B,\mathcal{E}))$ *such that*

$$\theta^{\mathcal{H}}|_{U_\alpha} = \theta^{\mathcal{H}}_\alpha + \pi^*\omega_\alpha$$

Proof. By Lemma 3.10, $\phi^*_{\alpha\beta}\theta^{\mathcal{H}}_\beta = \theta^{\mathcal{H}}_\alpha + \pi^*(\omega_\alpha - \omega_\beta)$, and the statement follows as in Proposition 2.12. □

We have

$$(\partial^{\mathcal{H}})^2(D) = [\theta^{\mathcal{H}}, D] \text{ and } \partial^{\mathcal{H}}(\theta^{\mathcal{H}}) = 2\pi i(\pi^*\Omega).$$

where Ω is the 3-curvature form of the connection on $\mathcal{L}$.

Following Mathai and Stevenson [30] one can construct the morphism of complexes

$$\Phi_{\nabla^{\mathcal{H}}} \colon CC^-_\bullet(\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')) \to (\Omega^*(B)[u], d_\Omega)_\bullet$$

as follows. (Note that in the nontwisted case similar morphism was constructed in [21, 33, 34].) Denote by

$$\Delta^k := \{(t_0, \cdots, t_k) \in \mathbb{R}^k \mid 0 \le t_i, \sum_{i=0}^k t_i = 1\}.$$

the standard k-simplex.

Define the maps $\Phi^k_{\nabla^{\mathcal{H}}} \colon C_k\left(\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}')\right) \to \Omega^*(M)[u]$ by

$$\Phi^k_{\nabla^{\mathcal{H}}}(A_0, \cdots, A_k) := \int_{\Delta^k} \mathrm{Tr}\left(A_0 e^{-ut_0\frac{\theta^{\mathcal{H}}}{2\pi i}} \partial^{\mathcal{H}}(A_1) e^{-ut_1\frac{\theta^{\mathcal{H}}}{2\pi i}} \cdots e^{-ut_{k-1}\frac{\theta^{\mathcal{H}}}{2\pi i}} \partial^{\mathcal{H}}(A_k) e^{-ut_k\frac{\theta^{\mathcal{H}}}{2\pi i}}\right) dt_1 \ldots dt_k,$$

for $A_0 \otimes \ldots \otimes A_k \in C_k\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')\right)$. Then let $\Phi_{\nabla^{\mathcal{H}}} = \sum_{k=0}^\infty \Phi^k_{\nabla^{\mathcal{H}}}$.

Theorem 3.12. [30] *The map* $\Phi_{\nabla^{\mathcal{H}}} \colon (CC^-_\bullet(\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E})), b + uB) \to (\Omega^*(B)[u], d_\Omega)_\bullet$ *is a morphism of complexes.*

This morphism depends on the choice of horizontal distribution $\mathcal{H}$. However the results of [30] show that a different choice of $\mathcal{H}$ leads to a chain homotopic morphism.

Assume now that $D \in \Psi_{\mathcal{L}}(M|B;\mathcal{E},\mathcal{E}')$ is an twisted elliptic family. In the Definition 3.8 we defined $\operatorname{ind}(D) = [P_D - Q] \in K_0(\overline{\Psi_{\mathcal{L}}^{-\infty}}(M|B;\mathcal{E},\mathcal{E}'))$. Here the idempotents P_D and Q belong to the algebra $\overline{\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))}$ – the unitalization of $\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}')$ with the multiplier $Q \in \Psi^0_{\mathcal{L}}(M|B;\mathcal{E}\oplus\mathcal{E}')$ adjoined. It follows that if we directly apply the formula (2.1.1) to $[P_D - Q]$ we obtain a 0-chain $\operatorname{Ch}[P_D - Q]$ in the relative cyclic complex $CC^-_\bullet\left(\overline{\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))}, \Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))\right)$. We have the natural morphism $\iota\colon CC^-_\bullet\left(\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))\right) \to CC^-_\bullet\left(\overline{\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))}, \Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))\right)$, and the following equality of homology classes: $\iota(\operatorname{ch}(\operatorname{ind} D)) = [\operatorname{Ch}[P_D - Q]]$. It is straightforward to see that the map $\Phi_{\nabla^{\mathcal{H}}}$ extends to a morphism

$$\Phi_{\nabla^{\mathcal{H}}}\colon CC^-_\bullet\left(\overline{\Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))}, \Psi_{\mathcal{L}}^{-\infty}(M|B;\mathcal{E}\oplus\mathcal{E}'))\right) \to (\Omega^*(B)[u], d_\Omega)_\bullet$$

defined by the same formula, and that we have an equality $\Phi_{\nabla^{\mathcal{H}}} \circ \iota = \Phi_{\nabla^{\mathcal{H}}}$. It follows that

$$\Phi_{\nabla^{\mathcal{H}}}(\operatorname{ch}(\operatorname{ind} D)) = [\Phi_{\nabla^{\mathcal{H}}}(\operatorname{Ch}[P_D - Q])]\,.$$

4. Dirac operators and superconnections

4.1. **Dirac operators.** The goal of this section is to give a superconnection proof of the family index theorem for projective families of Dirac operators. We assume that the fibers of the smooth fibration $\pi : M \to B$ are even dimensional compact Riemannian manifolds.

We begin with the definition of a horizontally twisted Clifford module. Denote by $C(M|B)$ the Clifford algebra of the fiberwise cotangent bundle $T^*(M|B) = T^*M/(\ker \pi_*)^\perp$. Let $\mathcal{L}$ be a unitary gerbe on B.

Definition 4.1.

- A twisted Clifford module is a horizontally $\mathcal{L}$-twisted Hermitian $\mathbb{Z}_2$-graded vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ on M together with the homomorphism $c\colon C(M|B) \to \operatorname{End}(\mathcal{E})$ of bundles of unital $\mathbb{Z}_2$-graded $*$-algebras.
- A Clifford connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is an Hermitian connection such that $\nabla^{\mathcal{E}}(c(a)) = c(\nabla^{M|B}(a))$.

Clifford connections on Clifford modules always exist; the proof is analogous to the proof of existence of twisted connections in Lemma 2.11.

Choose horizontal distribution i.e. a subbundle $\mathcal{H} \subset TM$ such that $TM = \mathcal{H} \oplus T(M|B)$. This choice together with the Riemannian metric on the fibers of π allows one to define a connection $\nabla^{M|B}$ on the fiberwise tangent bundle $T(M|B)$, see [4] Section 10.1. We denote by $R^{M|B}$ the curvature of this connection.

Set $\operatorname{End}_{C(M|B)}(\mathcal{E}) = \{A \in \operatorname{End}(\mathcal{E}) \mid [A, c(a)] = 0 \text{ for every } a \in C(M|B)\}$. Let $\Gamma \in C(M|B)$ be the chirality operator defined locally by $\Gamma = i^{k/2}e^1 \dots e^k$ where $k = \dim M - \dim B$ and $e^1, \dots, e^k$ is the local orthonormal basis of $T^*(M|B)$. Define then the relative supertrace

$$\operatorname{str}_{\mathcal{E}/\mathcal{S}}\colon \operatorname{End}_{C(M|B)}(\mathcal{E}) \to C^\infty(M) \text{ by } \operatorname{str}_{\mathcal{E}/\mathcal{S}}(A) = 2^{-k/2} \operatorname{str} c(\Gamma)A.$$

We fix from now on a Clifford connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ and a descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ for the horizontally $\mathcal{L}$-twisted Clifford module $\mathcal{E}$. The connection $\nabla^{\mathcal{E}}$ defines a connection $(\nabla^{\mathcal{E}}_\alpha, \nabla_{\alpha\beta}, \omega_\alpha)$ (defined up to equivalence) on this descent datum. Each $\mathcal{E}_\alpha$ is then a Clifford module on the fibration $\pi^{-1}U_\alpha \to U_\alpha$, and each connection $\nabla^{\mathcal{E}}_\alpha$ is a Clifford connection.

Recall (see Proposition 2.12) that one defines $\theta^{\mathcal{E}} \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ by setting $\theta^{\mathcal{E}}|_{\pi^{-1}U_\alpha} = \theta^{\mathcal{E}}_\alpha + \pi^*\omega_\alpha$. Denote by $c(R^{M|B})$ the action of the 2 -form with values in the Clifford algebra obtained from $R^{M|B}$ via the Lie algebra isomorphism $\mathfrak{so}(T(M|B)) \to C^2(M|B)$. Here $C^2(M|B) \subset C(M|B)$ is a subspace consisting of elements $\sum u_i v_i$, $u_i, v_i \in T^*(M|B)$ with $\sum \langle u_i, v_i\rangle = 0$. Define $\theta^{\mathcal{E}/\mathcal{S}} = \theta^{\mathcal{E}} - c(R^{M|B})$.

The argument in [4], Proposition 3.43, shows that $\theta^{\mathcal{E}/\mathcal{S}} \in \Omega^2(M, \mathrm{End}_{C(M|B)}(\mathcal{E}))$. Over an open set $\pi^{-1}U_\alpha$ we have $\theta^{\mathcal{E}/\mathcal{S}}|_{U_\alpha} = \theta^{\mathcal{E}/\mathcal{S}}_\alpha + \pi^*\omega_\alpha$ where $\theta^{\mathcal{E}/\mathcal{S}}_\alpha \in \mathrm{End}_{C(M|B)}(\mathcal{E}_\alpha)$ is defined via the equality $\theta^{\mathcal{E}}_\alpha = \theta^{\mathcal{E}/\mathcal{S}}_\alpha + c(R^{M|B})$. We can then define a differential form $\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})$ by

$$\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S}) = \mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{u\theta^{\mathcal{E}/\mathcal{S}}}{2\pi i}} \in \Omega^*(M)[u]. \tag{4.1.1}$$

The proof of the following result is standard and analogous to the proof of Proposition 2.13.

Lemma 4.2. *We have $d_{\pi^*\Omega}\,\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S}) = 0$ and the corresponding class is $H^*_{\pi^*\Omega}(M)$ is independent of the choice of Clifford connection $\nabla^{\mathcal{E}}$.*

We also introduce the fiberwise $\widehat{A}$-genus by $\widehat{A}(TM|B) = \widehat{A}\left(\frac{u}{2\pi i} R^{M|B}\right) \in \Omega^*(M)[u]$, where $\widehat{A}(x)$ is the power series defined by

$$\widehat{A}(x) = \det{}^{1/2}\left(\frac{x/2}{\sinh x/2}\right).$$

Using the above data, we can define on each fibration $\pi^{-1}U_\alpha \to U_\alpha$ a family of Dirac operators D_α acting on the sections of the bundle $\mathcal{E}_\alpha$. Locally $D_\alpha = \sum_i c(e^i)\left(\nabla^{\mathcal{E}}_\alpha\right)_{e_i}$ where $\{e_i\}$, $\{e^i\}$ are dual bases of $T(M|B)$ and $T^*(M|B)$ respectively. We leave it to the reader to check the following.

Lemma 4.3. *The collection $D = \{D_\alpha\}$ defines an element in $\Psi^1_{\mathcal{L}}(M|B; \mathcal{E})$.*

With respect to the decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ we have the decomposition $\Psi_{\mathcal{L}}(M|B;\mathcal{E}) = \Psi_{\mathcal{L}}(M|B;\mathcal{E}^+,\mathcal{E}^+) \oplus \Psi_{\mathcal{L}}(M|B;\mathcal{E}^+,\mathcal{E}^-) \oplus \Psi_{\mathcal{L}}(M|B;\mathcal{E}^-,\mathcal{E}^+) \oplus \Psi_{\mathcal{L}}(M|B;\mathcal{E}^-,\mathcal{E}^-)$. The Dirac operator then decomposes as $D = D^+ \oplus D^-$ where $D^+ \in \Psi^1_{\mathcal{L}}(M|B;\mathcal{E}^+,\mathcal{E}^-)$, $D^- \in \Psi^1_{\mathcal{L}}(M|B;\mathcal{E}^-,\mathcal{E}^+)$

Classical arguments show that D^+ is fiberwise elliptic and hence the analytical index $\mathrm{ind}(D^+)$ of D^+ is well defined in $K(\Psi^{-\infty}_{\mathcal{L}}(M|B;\mathcal{E}))$.

We then have the following result

Theorem 4.4. *The following formula holds in $H^\bullet_{\mathcal{L}}(B)$:*

$$\Phi_{\nabla^{\mathcal{H}}}(\mathrm{ch}(\mathrm{ind}\, D^+))] = u^{-\frac{\dim M - \dim B}{2}}\left[\int_{M/B} \widehat{A}\left(\frac{u}{2\pi i} R^{M|B}\right) \wedge \mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})\right].$$

This index theorem is established in [28, 29] for general projective families however with the more restrictive conditions on the class $[\mathcal{L}]$ of the gerbe $\mathcal{L}$. The superconnection proof of this result occupies the rest of the paper.

4.2. **Superconnections and index.** We continue in the notations of the previous section.

A twisted superconnection $\mathbb{A}$ on the descent datum $(U_\alpha, \mathcal{L}_{\alpha\beta}, \mu_{\alpha\beta\gamma}, \mathcal{E}_\alpha, \varphi_{\alpha\beta})$ is a collection $(\mathbb{A}_\alpha)_{\alpha\in\Lambda}$ of superconnections on the vector bundles $\pi_*\mathcal{E}_\alpha$ over the open sets U_α such that when $U_{\alpha\beta} \neq \emptyset$,

$$(\pi_*\varphi_{\alpha\beta})^*\mathbb{A}_\beta = \mathbb{A}_\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{\alpha\beta}. \tag{4.2.1}$$

We say that $\mathbb{A}$ is a Bismut superconnection if each $\mathbb{A}_\alpha$ is. Specifically we have

$$\mathbb{A}_\alpha = D_\alpha + \nabla^{\mathcal{H}}_\alpha - \frac{1}{4}c(T^{\mathcal{H}})$$

where $T^{\mathcal{H}}$ is defined in (3.4.1).

The fact that $\nabla^{\mathcal{H}}$ is a connection on the descent datum, and $D \in \Psi^1(M|B, \mathcal{E})$, $c(T^{\mathcal{H}}) \in \Omega^2(B, \Psi^0(M|B, \mathcal{E}))$ implies that the conditions (4.2.1) are satisfied. From now on $\mathbb{A}$ will denote the Bismut superconnection.

We will also consider the rescaled Bismut superconnection $\mathbb{A}_s$ defined by

$$(\mathbb{A}_s)_\alpha = \mathbb{A}_{\alpha,s} := s^{1/2}D_\alpha + \nabla^{\mathcal{H}}_\alpha - \frac{1}{4}s^{-1/2}c(T^{\mathcal{H}})$$

where s is either a positive number or a multiple of the formal variable u. Denote by $\theta^{\mathbb{A}_s}_\alpha$ the curvature of the rescaled superconnection $(\mathbb{A}_s)^2_\alpha$. In particular we have forms $u\theta^{\mathbb{A}_{\alpha,u^{-1}}} \in \Omega^*(U_\alpha, \Psi(\pi^{-1}U_\alpha|U_\alpha, \mathcal{E}_\alpha))[u^{1/2}]$.

Proposition 4.5. *There exists a form* $u\theta^{\mathbb{A}_{u^{-1}}} \in \Omega^*(B, \Psi(M|B;\mathcal{E}))[u^{1/2}]$ *such that*

$$u\theta^{\mathbb{A}_{u^{-1}}}\big|_{U_\alpha} = u(\theta^{\mathbb{A}_{\alpha,u^{-1}}} + \pi^*\omega_\alpha).$$

Proof. Recall that the curvature of $\nabla_{\alpha\beta}$ is equal to $\omega_\alpha - \omega_\beta$. Therefore from the equation (4.2.1) we obtain

$$\phi_{\alpha\beta}\left(u\theta^{\mathbb{A}_{\beta,u^{-1}}}\right) = u(\theta^{\mathbb{A}_{\alpha,u^{-1}}} + \pi^*\omega_\alpha - \pi^*\omega_\beta).$$

The statement of the Proposition follows. □

Notice that $u\theta^{\mathbb{A}_{u^{-1}}} = D^2+$ forms of degree > 0. Therefore we can define $e^{-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}} \in \Omega^*(B, \Psi^{-\infty}(M|B;\mathcal{E}))[u^{1/2}]$ by the usual Duhamel's formula.

Note that the parity considerations as in finite dimensional case show that the coefficients for the nonintegral powers of u are odd with respect to the grading and hence have a vanishing supertrace.

Given a superconnection $\mathbb{A} = (\mathbb{A}_\alpha)_{\alpha\in\Lambda}$ on the $\mathbb{Z}_2$-graded horizontally $\mathcal{L}$-twisted Clifford module $\mathcal{E}$ by the infinite-dimensional version of the Proposition 2.15 the the differential form

$$\mathrm{Str}(e^{-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}}) \in \Omega^*(B)[u],$$

is closed with respect to the twisted de Rham differential d_Ω. The proof is identical to the proof of the first part of the Proposition 2.13. The proof of the following Theorem is adapted from [22] and it uses ideas from [23]. The use of cyclic theory is inspired by [35].

Theorem 4.6. *The following equality holds in the* $H^\bullet_{\mathcal{L}}(B)$*:*

$$\Phi_{\nabla^{\mathcal{H}}}(\mathrm{ch}(\mathrm{ind}\, D^+)) = \left[\mathrm{Str}(e^{-\frac{u\theta^{\mathbb{A}_{u^{-1}}}}{2\pi i}})\right]$$

Proof. Consider the twisted bundle $\widetilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{E}$ with the grading given by $\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$. The algebra $\Psi_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})$ of operators on $\widetilde{\mathcal{E}}$ is naturally $\mathbb{Z}_2$ graded. When discussing cyclic complexes of this algebra and its subalgebras we always consider it as $\mathbb{Z}_2$-graded algebra. For an operator or (super) connection K on $\mathcal{E}$ set $\widetilde{K} = K \oplus K$; so for example $\widetilde{\mathbb{A}} = \mathbb{A} \oplus \mathbb{A}$, $\widetilde{\nabla}^{\mathcal{H}} = \nabla^{\mathcal{H}} \oplus \nabla^{\mathcal{H}}$, etc.

Let $F = D(1+D^2)^{-1/2}$. Then it is immediate that $F \in \Psi^0_{\mathcal{L}}(M|B;\mathcal{E})$ is odd with respect to the grading Γ and fiberwise elliptic. Construct the invertible operator $U_D \in \Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})$ by the same formula as before. Namely, choose a parametrix R for F. Let $S_0 = 1 - RF$, $S_1 = 1 - FR$. Then set $U_D = \begin{bmatrix} S_0 & -(1+S_0)R \\ F & S_1 \end{bmatrix}$. With such a choice the inverse is given by an explicit formula $U_D^{-1} = \begin{bmatrix} S_0 & (1+S_0)R \\ -F & S_1 \end{bmatrix}$. Set $P_D = U_D^{-1} \begin{bmatrix} 1_{\mathcal{E}} & 0 \\ 0 & 0 \end{bmatrix} U_D$.

The choices in the constructions can be made so that we have $(P_D)^{\pm} = P_{D^{\pm}}$, see Definition 3.8.

Define the map $\Phi_{\widetilde{\nabla}^{\mathcal{H}}} \colon CC^{-}_{\bullet}\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})\right) \to (\Omega^*(M)[u], d_\Omega)$ by

$$\Phi^k_{\widetilde{\nabla}^{\mathcal{H}}}(A_0, \cdots, A_k) := \int_{\Delta^k} \operatorname{Str}\left(A_0 e^{-ut_0\frac{\widetilde{\theta}^{\mathcal{H}}}{2\pi i}} \widetilde{\partial}^{\mathcal{H}}(A_1) e^{-ut_1\frac{\widetilde{\theta}^{\mathcal{H}}}{2\pi i}} \cdots e^{-ut_{k-1}\frac{\widetilde{\theta}^{\mathcal{H}}}{2\pi i}} \widetilde{\partial}^{\mathcal{H}}(A_k) e^{-ut_k\frac{\widetilde{\theta}^{\mathcal{H}}}{2\pi i}}\right) dt_1 \ldots dt_k$$

for $A_0 \otimes \ldots \otimes A_k \in C_k\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})\right)$ and set again $\Phi_{\widetilde{\nabla}^{\mathcal{H}}} = \sum_{k=0}^{\infty} \Phi^k_{\widetilde{\nabla}^{\mathcal{H}}}$. Notice that

$$\left[\Phi_{\widetilde{\nabla}^{\mathcal{H}}}\left(\operatorname{Ch}\left(P_D - \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{E}} \end{bmatrix}\right)\right)\right] = \Phi_{\nabla^{\mathcal{H}}}\left(\operatorname{ch}\left(\operatorname{ind} D^+\right)\right) - \Phi_{\nabla^{\mathcal{H}}}\left(\operatorname{ch}\left(\operatorname{ind} D^-\right)\right) = 2\Phi_{\nabla^{\mathcal{H}}}\left(\operatorname{ch}\left(\operatorname{ind} D^+\right)\right) \tag{4.2.2}$$

Replacing in the formulas above $\widetilde{\theta}^{\mathcal{H}}$ by $\widetilde{\theta}^{\mathbb{A}_{u^{-1}}}$ and $\widetilde{\partial}^{\mathcal{H}}$ by $[\widetilde{\mathbb{A}}_{u^{-1}}, \cdot]$ we obtain the definition of the morphism $\Phi_{\widetilde{\mathbb{A}}} \colon CC^{entire}_{\bullet}\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})\right) \to (\Omega^*(M)[u], d_\Omega)_{\bullet}$.

Lemma 4.7. *The morphisms* $\Phi_{\widetilde{\mathbb{A}}}$, $\Phi_{\widetilde{\nabla}^{\mathcal{H}}} \colon CC^{entire}_{\bullet}\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})\right) \to (\Omega^*(B)[u], d_\Omega)_{\bullet}$ *are chain homotopic.*

Proof. This follows from the explicit formula for the chain homotopy, see Proposition 5.6 of [30]. This formula can be described as follows. Let $\mathbb{A}_{u^{-1}}(s) = s\mathbb{A}_{u^{-1}} + (1-s)\nabla^{\mathcal{H}}$. Then we can write

$$\mathbb{A}_{u^{-1}}(s) = su^{-1/2}D + \nabla^{\mathcal{H}} - \frac{su^{1/2}}{4}c(T^{\mathcal{H}}),$$

$$\theta^{\mathbb{A}_{u^{-1}}(s)} = \theta^{\mathcal{H}} + s[\nabla^{\mathcal{H}}, u^{-1/2}D - \frac{u^{1/2}}{4}c(T^{\mathcal{H}})] + s^2(u^{-1/2}D - \frac{u^{1/2}}{4}c(T^{\mathcal{H}}))^2$$

$$\text{and } \frac{d}{ds}\mathbb{A}_{u^{-1}}(s) = u^{-1/2}D - \frac{u^{1/2}}{4}c(T^{\mathcal{H}}).$$

Define $H_k\colon C_k(\Psi^{-\infty}_{\mathcal{L}}(M|B,\mathcal{E}\oplus\mathcal{E}))\to\Omega^*(B)[u]$ by

$$(4.2.3)\quad H_k(A_0,\dots,A_k)=\int_0^1 ds\left(\sum_{m=0}^{k}(-1)^m\int_{\Delta^{k+1}}dt_1\dots dt_{k+1}\,\mathrm{Str}\left(A_0e^{-ut_0\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}\right.\right.$$
$$[\widetilde{\mathbb{A}}_{u^{-1}}(s),A_1]e^{-ut_1\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}\dots[\widetilde{\mathbb{A}}_{u^{-1}}(s),A_m]e^{-ut_m\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}\;\widetilde{\frac{d}{ds}\mathbb{A}_{u^{-1}}}\;e^{-ut_{m+1}\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}\dots$$
$$\left.\left.e^{-ut_{k-1}\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}[\widetilde{\mathbb{A}}_{u^{-1}}(s),A_k]e^{-ut_{k+1}\frac{\widetilde{\theta}^{\mathbb{A}}{}_{u^{-1}}(s)}{2\pi i}}\right)\right)$$

and set $H=\sum H_k\colon CC^{entire}_{\bullet}\left(\Psi^{-\infty}_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})\right)\to\Omega^*(B)[u]$. Then

$$\Phi_{\widetilde{\mathbb{A}}}-\Phi_{\nabla^{\mathcal{H}}}=d_{\Omega}\circ H+H\circ(b+uB).$$

□

Let $\mathcal{F}$ be the algebra defined by

$$\mathcal{F}=\{F\in\Psi^0_{\mathcal{L}}(M|B;\widetilde{\mathcal{E}})^{even}\mid[D,F]\in\Psi^0_{\mathcal{L}}(M|B;\mathcal{E})\}.$$

A simple modification of the argument in [20] as done in [2], cf. also [22] shows that $\Phi_{\widetilde{\mathbb{A}}}$ extends to a morphism, also denoted $\Phi_{\widetilde{\mathbb{A}}}\colon CC^{entire}_{\bullet}(\mathcal{F})\to(\Omega^*(M)[u],d_{\Omega})$ defined by the same formula. Note that $U_D\in\mathcal{F}$, $P_D\in\mathcal{F}$ and $\begin{bmatrix}1_{\mathcal{E}}&0\\0&0\end{bmatrix}\in\mathcal{F}$.

Recall now that inner automorphisms act by identity on the entire cyclic homology, see e.g. [27] 4.1.3 or [26]. It follows that the chains $\mathrm{Ch}(P_D)$ and $\mathrm{Ch}\left(\begin{bmatrix}1_{\mathcal{E}}&0\\0&0\end{bmatrix}\right)$ are homologous in $CC^{entire}_{\bullet}(\mathcal{F})$.

We therefore obtain

$$\begin{aligned}
\Phi\left(\mathrm{ch}\left(\mathrm{ind}\,D^+\right)\right) &= \frac{1}{2}\left[\Phi_{\widetilde{\nabla}^{\mathcal{H}}}\left(\mathrm{Ch}\left(P_D-\begin{bmatrix}0&0\\0&1_{\mathcal{E}}\end{bmatrix}\right)\right)\right]\\
&= \frac{1}{2}\left[\Phi_{\widetilde{\mathbb{A}}}\left(\mathrm{Ch}\left(P_D-\begin{bmatrix}0&0\\0&1_{\mathcal{E}}\end{bmatrix}\right)\right)\right]\\
&= \frac{1}{2}\left[\Phi_{\widetilde{\mathbb{A}}}\left(\mathrm{Ch}\left(P_D\right)\right)-\Phi_{\widetilde{\mathbb{A}}}\left(\mathrm{Ch}\left(\begin{bmatrix}0&0\\0&1_{\mathcal{E}}\end{bmatrix}\right)\right)\right]\\
&= \frac{1}{2}\left[\Phi_{\widetilde{\mathbb{A}}}\left(\mathrm{Ch}\left(\begin{bmatrix}1_{\mathcal{E}}&0\\0&0\end{bmatrix}\right)\right)\right]-\frac{1}{2}\left[\Phi_{\widetilde{\mathbb{A}}}\left(\mathrm{Ch}\left(\begin{bmatrix}0&0\\0&1_{\mathcal{E}}\end{bmatrix}\right)\right)\right]\\
&= \left[\mathrm{Str}(e^{-\frac{u\theta^{\mathbb{A}}{}_{u^{-1}}}{2\pi i}})\right].
\end{aligned}$$

□

4.3. **The local index theorem.**

Theorem 4.8. *Let D be a projective family of Dirac operators on a horizontally $\mathcal{L}$-twisted Clifford module $\mathcal{E}$. We have the following equality of classes in $H^{\bullet}_{\mathcal{L}}(B)$:*

$$\left[\mathrm{Str}(e^{-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}}}})\right] = \left[u^{-\frac{k}{2}}\int_{M|B}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})\right].$$

Proof. Over each open set U_α we have $\theta^{\mathbb{A}_{u^{-1}}} = \mathbb{A}^2_{\alpha,u^{-1}} + \pi^*\omega_\alpha$. Since $\mathbb{A}^2_{\alpha,u^{-1}}$ and $\pi^*\omega_\alpha$ commute, we have

$$\mathrm{Str}\, e^{-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}}}} = e^{-\frac{u}{2\pi i}\pi^*\omega_\alpha}\,\mathrm{Str}\, e^{-\frac{u}{2\pi i}\mathbb{A}^2_{\alpha,u^{-1}}}.$$

According to Bismut's local index theorem for families [5]

$$\lim_{t\to 0}\mathrm{Str}\, e^{-\frac{1}{2\pi i}\mathbb{A}^2_{\alpha,t}} = \int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{1}{2\pi i}R^{M|B}\right)\mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{1}{2\pi i}\theta^{\mathcal{E}/\mathcal{S}}_\alpha}.$$

Moreover, by the result of Bismut and Fried [6]

$$\mathrm{Str}\, e^{-\frac{1}{2\pi i}\mathbb{A}^2_\alpha} - \int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{1}{2\pi i}R^{M|B}\right)\mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{1}{2\pi i}\theta^{\mathcal{E}/\mathcal{S}}_\alpha} = d\int_0^1 \xi_\alpha(t)dt \tag{4.3.1}$$

where

$$\xi_\alpha(t) = -\frac{1}{2\pi i}\mathrm{Str}\,\frac{d\mathbb{A}_{\alpha,t}}{dt}e^{-\frac{1}{2\pi i}\mathbb{A}^2_{\alpha,t}} \tag{4.3.2}$$

is integrable at 0.

Let $s > 0$ and let δ^B_s be the operator on $\Omega^*(B)$ which multiplies the forms of degree k by $s^{k/2}$. Then $\delta^B_s \circ (\mathbb{A}_{\alpha,t}) \circ \delta^B_{s^{-1}} = s^{1/2}\mathbb{A}_{\alpha,t/s}$. We therefore obtain

$$\begin{aligned}
\lim_{t\to 0}\mathrm{Str}\, e^{-\frac{s}{2\pi i}\mathbb{A}^2_{\alpha,t/s}} &= \delta^B_s\left(\lim_{t\to 0}\mathrm{Str}\, e^{-\frac{1}{2\pi i}\mathbb{A}^2_{\alpha,t}}\right)\\
&= \delta^B_s\int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{1}{2\pi i}R^{M|B}\right)\mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{1}{2\pi i}\theta^{\mathcal{E}/\mathcal{S}}_\alpha}\\
&= s^{-\frac{k}{2}}\int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{s}{2\pi i}R^{M|B}\right)\mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{s}{2\pi i}\theta^{\mathcal{E}/\mathcal{S}}_\alpha}
\end{aligned}$$

where, as before, $k = \dim M - \dim B$. Since both sides are polynomials in s we deduce that

$$\lim_{t\to 0}\mathrm{Str}\, e^{-\frac{u}{2\pi i}\mathbb{A}^2_{\alpha,u^{-1}t}} = u^{-\frac{k}{2}}\int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\mathrm{str}_{\mathcal{E}/\mathcal{S}}\, e^{-\frac{u}{2\pi i}\theta^{\mathcal{E}/\mathcal{S}}_\alpha}$$

Multiplying both sides by $e^{-\frac{u}{2\pi i}\pi^*\omega_\alpha}$ we obtain

$$\lim_{t\to 0}\mathrm{Str}\exp\left(-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}t}}\right)\Big|_{U_\alpha} = u^{-\frac{k}{2}}\int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})$$

and therefore

$$\lim_{t\to 0}\mathrm{Str}\exp\left(-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}t}}\right) = u^{-\frac{k}{2}}\int_{M|B}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S}).$$

Moreover by (4.3.1), we have

$$\operatorname{Str} e^{-\frac{u}{2\pi i}\mathbb{A}^2_{\alpha,u^{-1}}} - u^{-\frac{k}{2}} \int_{\pi^{-1}U_\alpha|U_\alpha} \widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right) \operatorname{str}_{\mathcal{E}/\mathcal{S}} e^{-\frac{u}{2\pi i}\theta_\alpha^{\mathcal{E}/\mathcal{S}}} = ud\int_0^1 u^{-\frac{1}{2}}\delta_u^B \xi_\alpha(t)dt \tag{4.3.3}$$

Here the right hand side is defined as follows. Write $\xi_\alpha^{[l]} \in \Omega^l(B)$ for the component of degree l of ξ_α from the equation (4.3.2). Then we have

$$u^{-\frac{1}{2}}\delta_u^B\xi_\alpha(t) = \sum u^{\frac{l-1}{2}}\xi_\alpha^{[l]}(t) = \frac{1}{2\pi i}\operatorname{Str}\frac{d\mathbb{A}_{\alpha,u^{-1}t}}{dt}\exp\left(-\frac{u}{2\pi i}\mathbb{A}^2_{\alpha,u^{-1}t}\right) \in \Omega^*(B)[u].$$

Multiplying the identity (4.3.3) by $e^{-\frac{u}{2\pi i}\pi^*\omega_\alpha}$ and using the equality

$$e^{-\frac{u}{2\pi i}\pi^*\omega_\alpha}ud(\cdot) = (ud+u^2\Omega)(e^{-\frac{u}{2\pi i}\pi^*\omega_\alpha}\cdot)$$

we obtain

$$\operatorname{Str}\exp\left(-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}}}\right)\Big|_{U_\alpha} - u^{-\frac{k}{2}}\int_{\pi^{-1}U_\alpha|U_\alpha}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\operatorname{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S}) = (ud+u^2\Omega)\Xi_\alpha$$

where $\Xi_\alpha = \int_0^1 \frac{1}{2\pi i}\operatorname{Str}\frac{d\mathbb{A}_{\alpha,u^{-1}t}}{dt}\exp\left(-\frac{u}{2\pi i}\,\theta^{\mathbb{A}_{u^{-1}t}}\big|_{U_\alpha}\right)dt$.

From the equations (4.2.1) it follows that over $U_{\alpha\beta}$ we have $\frac{d\mathbb{A}_{\alpha,u^{-1}t}}{dt} = \phi_{\alpha\beta}\left(\frac{d\mathbb{A}_{\beta,u^{-1}t}}{dt}\right)$. Hence there exists a form $\dot{\mathbb{A}}_t \in u^{-1/2}\Omega^*(B;\Psi_{\mathcal{L}}(M|B,\mathcal{E}))[u]$ such that $\dot{\mathbb{A}}_t\Big|_{U_\alpha} = \frac{d\mathbb{A}_{\alpha,u^{-1}t}}{dt}$. Therefore setting $\Xi = \int_0^1\frac{1}{2\pi i}\operatorname{Str}\dot{\mathbb{A}}_t\exp\left(-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}t}}\right)dt \in \Omega^*(B)[u]$ we can write

$$\operatorname{Str}\exp\left(-\frac{u}{2\pi i}\theta^{\mathbb{A}_{u^{-1}}}\right) - u^{-\frac{k}{2}}\int_{M|B}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\operatorname{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S}) = d_\Omega\Xi$$

and the statement of the Theorem follows. □

Combining results of the Theorems 4.6 and 4.8 we obtain the main theorem of this paper

Theorem 1.1. *Let D be a projective family of Dirac operators on a horizontally $\mathcal{L}$-twisted Clifford module $\mathcal{E}$ on a fibration $\pi: M \to B$. Then the following equality holds in $H^\bullet_{\mathcal{L}}(B)$:*

$$\left[\Phi_{\nabla^{\mathcal{H}}}(\operatorname{ch}(\operatorname{ind}D^+))\right] = \left[u^{-\frac{k}{2}}\int_{M|B}\widehat{A}\left(\frac{u}{2\pi i}R^{M|B}\right)\operatorname{Ch}_{\mathcal{L}}(\mathcal{E}/\mathcal{S})\right],$$

where $k = \dim M - \dim B$ is the dimension of the fibers.

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