ZETA-FUNCTIONS OF HARMONIC THETA-SERIES AND PRIME NUMBERS

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§1. Prime numbers and theta-series

Speaking on rational prime numbers in various arithmetical sequences, it may be noted that no essential progress have been achieved for more than century and a half since famous Dirichlet theorem on prime numbers in arithmetic progressions (1837). Absolutely mystical is still the question on prime numbers in quadratic sequences, i.e., on prime numbers of the form an^2+bn+c, where a, b, c are rational coprime integers, and $d = b^2 - 4ac$ is not a rational square. The situation is not changing despite of considerable progress of the algebraic-analytical theory of integral quadratic forms reached after Dirichlet. Our purpose here is to draw the attention of numbertheorist to some analytical aspects of the theory of quadratic forms possibly related to the problem.

In order to be more concrete, let us start from the the celebrated problem on prime numbers of the form $1+n^2$. It is well known that the problem is closely related to reductions prime modules of certain elliptic curves with complex multiplications by Gauss integers $a+\sqrt{-1}b$, say, the curve

$$y^2 = x(x^2-1). \tag{1.1}$$

Indeed, according to a formula of D.S. Gorshkov (see [Vin52, Ch.V,Question 8c]) decomposition of a prime number p of the form $4k+1$ into the sum of two squares of integral numbers can be written with help of Legendre symbol as

$$p = \left(\frac{1}{2}\sum_{x=0}^{p-1}\left(\frac{x(x^2-1)}{p}\right)\right)^2 + \left(\frac{1}{2}\sum_{x=0}^{p-1}\left(\frac{x(x^2-b)}{p}\right)\right)^2,$$

where b is a quadratic non-residue modulo p. It is easy to see that the first square of this decomposition is odd, whilst the second is even. Hence a prime $p \equiv 1 \pmod 4$ has the form $1+n^2$ if and only if

$$\frac{1}{2}\sum_{x=0}^{p-1}\left(\frac{x(x^2-1)}{p}\right) = \pm 1. \tag{1.2}$$

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It is well-known (see, e.g.,[Weil48]) that the points of projective closure of affine elliptic curve (1.1) rational over the field of p elements form a finite abelian group of order

$$N_p = p + 1 + \sum_{x=0}^{p-1} \left(\frac{x(x^2-1)}{p} \right),$$

and so p has the form $1+n^2$ if and only if $N_p = p+1 \pm 2$. Unfortunately, for the time being details of behavior of numbers N_p for different prime p is an open question, and there is no much hope to approach the problem this from side.

By the way, the criterium (1.2) can be reformulated quite elementary in the terms of simple congruences modulo p for appropriate factorials: it follows from known properties of the Legendre symbol that

$$\sum_{x=0}^{p-1} \left(\frac{x(x^2-1)}{p} \right) \equiv \sum_{x=0}^{p-1} \left(x(x^2-1) \right)^{\frac{p-1}{2}}$$

$$\equiv \sum_{x=1}^{p-1} x^{\frac{p-1}{2}} (x^2-1)^{\frac{p-1}{2}} \equiv - \begin{pmatrix} \frac{p-1}{2} \\ \frac{p-1}{4} \end{pmatrix} \pmod{p},$$

and hence the criterium (1.2) is equivalent with the congruence

$$-\frac{1}{2} \begin{pmatrix} \frac{p-1}{2} \\ \frac{p-1}{4} \end{pmatrix} = -\frac{1}{2} \frac{\left(\frac{p-1}{2}\right)!}{\left[\left(\frac{p-1}{4}\right)!\right]^2} \equiv \pm 1 \pmod{p},$$

which can be written in the form

$$4 \left[\left(\frac{p-1}{4} \right)! \right]^4 \equiv -1 \pmod{p}. \tag{1.3}$$

It looks nice, but seems to be out of use for the problem of primes in the sequence $1+n^2$, like the Wilson theorem does not help to prove that there are infinitely many primes.

Fortunately, our problem is closely related not only with reduction of elliptic curves modulo prime numbers or congruences for factorials, but also with such powerful tool of investigation of quadratic forms as modular forms for subgroups of the modular group $\Gamma = SL_2(\mathbb{Z})$. Let us consider the function defined on the upper half-plane of complex variable

$$\mathbb{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \} \qquad (i = \sqrt{-1}) \tag{1.4}$$

by a harmonic theta-series of the quadratic form $\mathbf{q} = 8(x_1^2 + x_2^2)$ (of the level $l = 32$) (see §2),

$$\begin{aligned} F(z) &= 4 \sum_{n_1, n_2 \in \mathbb{Z}} \left(n_2 + \frac{1}{4} \right) e^{\pi i z 16 \left(\left(n_1 + \frac{1}{4}\right)^2 + \left(n_2 + \frac{1}{4}\right)^2 \right)} \\ &= \sum_{\substack{m_1, m_2 \in \mathbb{Z}, \\ m_1 \equiv m_2 \equiv 1 \pmod 4}} m_2 e^{\pi i z \left(m_1^2 + m_2^2 \right)} = \sum_{n=1}^{\infty} c(n) e^{2\pi i z n}, \end{aligned} \tag{1.5}$$

where

$$c(n) = \sum_{\substack{m_1,m_2\in\mathbb{Z},\, m_1^2+m_2^2=2n,\\ m_1\equiv m_2\equiv 1 \pmod 4}} m_2 \tag{1.6}$$

are Fourier coefficients of F. The Fourier coefficients clearly satisfy relations $c(1) = 1$, and $c(n) = 0$ unless $n \equiv 1 \pmod 4$. In particular, $c(p) = 0$ if p is a prime number of the form $4k + 3$, but if p is a prime number of the form $4k + 1$ and (a_1, a_2) is one of integral solutions of the equation $x_1^2 + x_2^2 = p$, then all integral solution of the equation are $(\pm a_1, \pm a_2)$ or $(\pm a_2, \pm a_1)$, and hence all integral solution of the equation $y_1^2 + y_2^2 = 2p$ are $(\pm(a_1 - a_2), \pm(a_1 + a_2))$ or $(\pm(a_1 + a_2), \pm(a_1 - a_2))$. Since clearly one can assume that $a_1 \equiv 1 \pmod 4$ and a_2 is even, it follows then from (1.6) that

$$c(p) = \begin{cases} (a_1 + a_2) + (a_1 - a_2) = 2a_1 & \text{if } a_2 \equiv 0 \pmod 4, \\ (-a_1 + a_2) + (-a_1 - a_2) = -2a_1 & \text{if } a_2 \equiv 2 \pmod 4. \end{cases}$$

In any case a prime number p of the form $4k + 1$ has the form $1 + n^2$ if and only if

$$c(p) = \pm 2. \tag{1.7}$$

The series (1.5) converges absolutely on $\mathbb{H}$ and uniformly on compact subsets of $\mathbb{H}$. Hence the series defines a holomorphic function on $\mathbb{H}$. It is known that the function F is a modular (cusp) form of weight 2 for a congruence subgroup

$$\Gamma_0(32) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \gamma \equiv 0 \pmod{32} \right\}$$

of the modular group $\Gamma = SL_2(\mathbb{Z})$ (see, e.g., [Sch39] or [Ogg69, Th.20]).

The problem of prime numbers satisfying condition (1.7) looks similar to the problem of prime numbers p with a given value $\chi(p)$ of a Dirichlet character χ closely related to the problem of prime numbers in arithmetical progressions. In the same way as the problem on values of characters on prime numbers can not be approached in the terms of an individual character but has to take into account all characters of a given module, the problem of separation of prime numbers p with given value of the coefficient $c(p)$, perhaps, should be considered in a wider content of similar problems for another harmonic theta-series. In this paper we start to move in this direction.

Note, by the way, that the problem of prime numbers of the form $1+n^2$ is closely related to the problem of twins in the ring $\mathcal{O} = \mathbb{Z}[i]$ of Gauss integers, since if a prime p has the form $1 + n^2$, then the numbers $in - 1$ and $in + 1 = (in - 1) + 2$ are prime twins in the ring $\mathcal{O}$, although this ring obviously contains no rational prime twins. That, possibly, hints that the classical problem of prime twins for the ring of rational integers have also a quadratic nature.

Contents of the paper. In §2 basic definitions and facts on harmonic theta-functions together with their zeta-functions are reminded. §3 treats action of Hecke operators on theta-functions. §4 is devoted to action of (hereditary) Hecke operators on theta-series and deduction of corresponding Euler products. In §5 an example of harmonic theta-series and zeta-functions of multiples of the sum of two squares is considered.

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Notation. We fix the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, as usual, for the set of positive rational integers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

$\mathbb{A}_n^m$ denotes the set of all $m \times n-$matrices with elements in $\mathbb{A}$. If $\mathbb{A}$ is a ring with the identity element, $1_n = \mathbf{1}$ and $0_n = \mathbf{0}$ denote the identity element and the zero element of $\mathbb{A}_n^n$, respectively. The transpose of a matrix M is always denoted by ${}^tM$. For two matrices A and B of appropriate size we write

$$A[B] = {}^tBAB.$$

§2. Zeta-functions of harmonic theta-functions

In this section we shall remind definitions of harmonic theta-functions and corresponding zeta-functions.

Let

$$\mathbf{q}(X) = \sum_{1 \le \alpha \le \beta \le m} q_{\alpha\beta} x_\alpha x_\beta = \frac{1}{2}\, {}^tXQX = \frac{1}{2}\, Q[X] \qquad \left({}^tX = (x_1, \dots, x_m)\right)$$

be a real positive definite quadratic form in m variables with matrix

$$Q = Q(\mathbf{q}) = (q_{\alpha\beta}) + {}^t(q_{\alpha\beta}).$$

Since the form $\mathbf{q}$ is real and positive definite, there exists is a real matrix S such that $Q(\mathbf{q}) = {}^tSS$. A homogeneous polynomial of degree g in $x_1, \dots, x_m$ of the form

$$P(X) = P_Q(X) = P_0(SX),$$

where $P_0 = P_0(X)$ is an homogeneous polynomial of degree g in $x_1, \dots, x_m$ satisfying the Laplace equation

$$\sum_{1 \le \alpha \le m} \frac{\partial^2 P_0(X)}{(\partial x_\alpha)^2} = 0,$$

is called *harmonic polynomial of degree g with respect to the form* $\mathbf{q}$. The definition can be easily reformulated as follows: an homogeneous polynomial P of degree g in $x_1, \dots, x_m$ is a harmonic polynomial with respect to the form $\mathbf{q}$ with the matrix Q if it satisfies the differential equation

$$\sum_{1 \le \alpha, \beta \le m} (Q^{-1})_{\alpha\beta} \frac{\partial^2 P(X)}{\partial x_\alpha \partial x_\beta} = 0.$$

It can be verified that a polynomial of the form

$$P(X) = ({}^t\Omega Q X)^g, \tag{2.1}$$

where $\Omega \in \mathbb{C}^m$ is an *isotropic vector of form* $\mathbf{q}$ *with matrix* Q, i.e., a complex $m-$vector satisfying

$$\mathbf{q}(\Omega) = \frac{1}{2}\,{}^t\Omega Q \Omega = 0,$$

is a harmonic polynomial of order g with respect to the form $\mathbf{q}$, and each harmonic polynomial of order g with respect to the form $\mathbf{q}$ is a finite sum of such polynomials.

For a real positive definite quadratic form $\mathbf{q}$ in m variables with matrix Q and a harmonic polynomial P of order g with respect to $\mathbf{q}$, the *theta-function of* $\mathbf{q}$ (*of genus one*) *with harmonic polynomial* P *and parameters* (U, V), where $U = {}^t(u_1, \dots, u_m)$, $V = {}^t(v_1, \dots, v_m) \in \mathbb{C}^m$, is defined by the series

$$\Theta_P(z; Q, (U,V)) = \sum_{N \in \mathbb{Z}^m} P(N-V)\, e^{\pi i \left(z\, Q[N-V] + 2\cdot {}^tUQN - {}^tUQV\right)}, \tag{2.2}$$

where $z = x + iy \in \mathbb{H}$. The theta-function converges absolutely for z in the upper half-plane and converges uniformly in each half-plane $\{z \in \mathbb{C} \mid \text{Im } z \geq \epsilon\}$ with $\epsilon > 0$.

According to a specialization of the general inversion formula [An95, Lemma 5.1], the theta-function (2.2) satisfies the following *inversion formula*

$$\Theta_P(-1/z;\, Q,\, (V, -U)) = \frac{i^{m/2}}{\sqrt{\det Q}} (-z)^{\frac{m}{2}+g} \Theta_{P^*}(z;\, Q^{-1},\, Q(U,V)), \tag{2.3}$$

where

$$P^*(X) = P(Q^{-1}X)$$

is a harmonic polynomial of degree g with respect to the form $\mathbf{q}^*$ with matrix Q^{-1}.

In order to define zeta-function associated to the harmonic theta-function (2.2) we use the *Euler integral*

$$\int_0^\infty y^{s-1} e^{-\alpha y} dy = \Gamma(s)\, \alpha^{-s} \qquad (\alpha > 0, \text{ Re } s > 0),$$

where $\Gamma(s)$ is the gamma-function. For Re $s > \frac{m}{2}$ we obtain

$$\begin{aligned}
&\int_0^\infty y^{s+g/2-1} \left(\Theta_P(iy;\, Q,\, (U,V)) - e^{\pi i\, {}^tUQV} \rho(V,g)\right) dy \\
&= \sum_{N \in \mathbb{Z}^m,\, N \neq V} P(N-V)\, e^{2\pi i\, {}^tUQN - \pi i\, {}^tUQV} \int_0^\infty y^{s+g/2-1} e^{-\pi y Q[N-V]} dy \\
&= (2\pi)^{-(s+g/2)} \Gamma(s+g/2)\, e^{-\pi i\, {}^tUQV} \sum_{N \in \mathbb{Z}^m,\, N \neq V} e^{2\pi i\, {}^tUQN} \frac{P(N-V)}{\mathbf{q}(N-V)^{s+g/2}},
\end{aligned}$$

where

$$\rho(V,g)=\begin{cases}0 & \text{if } V\notin\mathbb{Z}^m \text{ or } V\in\mathbb{Z}^m \text{ and } g>0\\ P(\mathbf{0}) & \text{if } V\in\mathbb{Z}^m \text{ and } g=0\end{cases},$$

$\mathbf{q}$ – the quadratic form (2.1). The function

$$\zeta_P(s;\,Q,\,(U,V))=\sum_{N\in\mathbb{Z}^m,\,N\neq V} e^{2\pi i\,{}^tUQN}\frac{P(N-V)}{\mathbf{q}(N-V)^{s+g/2}}. \tag{2.4}$$

is called the (*Epstein*) *zeta-function of* $\mathbf{q}$ *with harmonic polynomial* P *and parameters* (U,V). The zeta function converges absolutely for Re $s>m/2$ and converges uniformly in every half-plane Re $s>m/2+\epsilon$ with $\epsilon>0$. Thus, the zeta-function is an analytic function for Re $s>m/2$.

The study of analytic continuation and the functional equation of the zeta-function is based on Riemann's method of deducing of analytical properties of the zeta-function from the integral representation and the theta-inversion formula (2.3). After standard consideration this leads to the following theorem (see, e.g. Siegel Tata-lectures [Si61/65, Ch.I, §5]).

Theorem 1. *The zeta-function* $\zeta_P(s;\,Q,\,(U,V))$ *of a real positive definite quadratic form* $\mathbf{q}$ *in* m *variables with a harmonic polynomial* P *of degree* g *and parameters* (U,V) *has an analytic continuation into the whole* $s-$*plane, which is an entire function of* s *if either* $g>0$ *or if* $g=0$ *and* QU *is not integral. If* $g=0$ *and* QU *is integral, then the zeta-function is meromorphic in the entire* $s-$*plane with the only singularity at* $s=m/2$*, where it has a simple pole with residue* $(2\pi)^{m/2}/\sqrt{\det Q}\,\Gamma(m/2)$*. In all cases* $\zeta_P(s;\,Q,\,(U,V))$ *satisfies the functional equation*

$$\begin{aligned}&(2\pi)^{-s}\Gamma(s+g/2)\,e^{-\pi i\,{}^tUQV}\zeta_P(s;\,Q,\,(U,V))\\ &=\frac{i^g}{\sqrt{\det Q}}(2\pi)^{s-m/2}\Gamma((m+g)/2-s)\,e^{\pi i\,{}^tUQV}\zeta_{P^*}\left(m/2-s;\,Q^{-1},\,(-QV,QU)\right),\end{aligned}$$

where $P^*(X)=P(Q^{-1}X)$.

§3. Hecke operators

Starting from this section we assume a positive definite quadratic form $\mathbf{q}$ in m variables is integral, i.e. has rational integral coefficients, in which case the matrix Q of $\mathbf{q}$ is *even*, i.e., has integral coefficients and even coefficients on principal diagonal. The least number $l\in\mathbb{N}$ such that the matrix lQ^{-1} is even is called the *level of the form* $\mathbf{q}$. As a particular case of [An95, Theorems 4.2-4.3] we obtain the following transformation formulas for the theta-function (2.2) of an integral positive definite quadratic form of level l: for each matrix

$$\sigma=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\Gamma_0(l)=\left\{\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in SL_2(\mathbb{Z})\;\middle|\;\gamma\equiv 0\pmod{l}\right\}$$

the theta-function satisfies functional equation

$$\Theta_P(\frac{\alpha z+\beta}{\gamma z+\delta};\, Q,\, (U,V)\,{}^t\sigma) = \mu_{\mathbf{q}}(\sigma)(\gamma z+\delta)^{\frac{m}{2}+g}\Theta_P(z;\, Q,\, (U,V)), \tag{3.1}$$

where $\mu_{\mathbf{q}}(\sigma)$ is an eight-root of the unity, which is equal to 1 if $l=1$, but if $l>1$ and m is even, it has the form

$$\mu_{\mathbf{q}}(\sigma) = \mu_{\mathbf{q}}\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right) = \chi_{\mathbf{q}}(\delta)$$

with the *character* $\chi_{\mathbf{q}}$ *of the quadratic form* $\mathbf{q}$, i.e., a real Dirichlet character modulo l satisfying conditions

$$\chi_{\mathbf{q}}(-1) = (-1)^{m/2},$$
$$\chi_{\mathbf{q}}(p) = \left(\frac{(-1)^{m/2}\det Q}{p}\right) \quad (\textit{the Legendre} \quad \textit{symbol}),$$

when p is an odd prime not dividing l, and

$$\chi_{\mathbf{q}}(2) = 2^{-m/2}\sum_{R\in\mathbb{Z}^m/2\mathbb{Z}^m} e^{\pi i Q[R]/2},$$

if l is odd.

In order to approach a natural question on Euler product factorization of zeta functions, we shall now remind the basic definitions and the simplest properties of (regular) Hecke–Shimura rings and Hecke operators for the groups $\Gamma_0(l)$ appeared above as transformation groups of theta-functions. We follow the general pattern of the theory of Hecke operators on modular forms (see, e.g., [An87, Ch. 4], or [An96, §2]) in the particular case of genus $n=1$.

Let us denote by

$$\mathcal{H}_0(l) = \mathcal{H}(\Gamma_0(l),\, \Sigma_0(l))$$

the Hecke–Shimura ring of the semigroup

$$\Sigma_0(l) = \left\{M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathbb{Z}_2^2 \;\middle|\; \det M>0,\, \gcd(\det M, l)=1,\, c\equiv 0 \pmod{l}\right\}$$

relative to the group $\Gamma_0(l)$ (over $\mathbb{C}$). The ring $\mathcal{H}_0(l)$ consists of all formal finite linear combinations with complex coefficients of the symbols $\tau(M)$, which are in one-to-one correspondence with double cosets $\Gamma_0(l)M\Gamma_0(l)\subset\Sigma_0(l)$. It is convenient to write each of the symbols $\tau(M)$, called also the *double cosets*, as the formal sum of different the left cosets it contains (more precisely, of the corresponding symbols),

$$\tau(M) = \sum_{M'\in\Gamma_0(l)\backslash\Gamma_0(l)M\,\Gamma_0(l)} (\Gamma_0(l)M') \qquad (M\in\Sigma_0(l)). \tag{3.2}$$

Then each element $T \in \mathcal{H}_0(l)$ can be also written as a finite formal linear combination of different left cosets,

$$T = \sum_{\alpha} c_\alpha(\Gamma_0(l)M_\alpha) \qquad (c_\alpha \in \mathbb{C}). \tag{3.3}$$

These linear combinations can be characterized by the the condition of invariance with respect to all right multiplication by elements of $\Gamma_0(l)$:

$$T\sigma = \sum_{\alpha} c_\alpha(\Gamma_0(l)M_\alpha\sigma) = T \quad \text{for all } \sigma \in \Gamma_0(l).$$

In this notation, the product in $\mathcal{H}_0(l)$ can be defined by

$$TT' = \sum_{\alpha} c_\alpha(\Gamma_0(l)M_\alpha)\sum_{\beta} c'_\beta(\Gamma_0(l)M'_\beta) = \sum_{\alpha,\beta} c_\alpha c'_\beta(\Gamma_0(l)M_\alpha M'_\beta).$$

The ring $\mathcal{H}_0(l)$ is a commutative $\mathbb{C}-$algebra generated over $\mathbb{C}$ by a denumerable set of algebraically independent elements. As a set of algebraically independent generators one can take, for example, double cosets of the form

$$T(p) = \tau\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right), \quad [p] = \tau\left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right), \tag{3.4}$$

where p runs over all prime numbers not dividing l (see, e.g., [An87, Theorem 3.3.23]).

In order to define Hecke operators on theta-functions we introduce certain linear spaces containing theta-functions. Let us denote $\mathfrak{F}_m$ the space of all complex-valued real-analytic functions

$$F = F(z;\, (U,V)) : \mathbb{H} \times \mathbb{C}^m \times \mathbb{C}^m \mapsto \mathbb{C},$$

where $\mathbb{H}$ is the upper half-plane of the complex variable z. For a fixed integral positive definite quadratic form $\mathbf{q}$ in an even number m of variables with matrix Q and a harmonic polynomial P with respect to $\mathbf{q}$, we define an action of the semigroup $\Sigma_0(l)$ on the spaces $\mathfrak{F}_m$ by *Petersson operators*

$$\Sigma_0(l) \ni M : F = F(z;\, (U,V)) \mapsto$$
$$F|_{\mathbf{j}}M = j_{Q,P}(M,\, z)^{-1}F(M\langle z\rangle;\, (U,V)\, {}^tM), \tag{3.5}$$

where

$$\mathbf{j} = j_{Q,P}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix},\, z\right) = \chi_{\mathbf{q}}(d)(cz+d)^{\frac{m}{2}+g}, \tag{3.6}$$

$\chi_{\mathbf{q}}$ – character of the quadratic form $\mathbf{q}$, g is degree of P, and where

$$M\langle z\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\langle z\rangle = \frac{az+b}{cz+d}.$$

It is clear that the function $cz+d$ does not vanish on $\Sigma_0(l)\times\mathbb{H}$ and hence the same is true for each of the functions $j_{Q,P}(M,\,z)$. If matrices $M=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and $M_1=\left(\begin{smallmatrix}a_1&b_1\\c_1&d_1\end{smallmatrix}\right)$ belong to $\Sigma_0(l)$ and $M'=\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)=MM_1$, then an easy direct computation shows that

$$(c\cdot M_1\langle z\rangle+d)(c_1z+d_1)=c'z+d'\qquad(z\in\mathbb{H}),$$

besides

$$\chi_{\mathbf{q}}(d')=\chi_{\mathbf{q}}(cb_1+dd_1)=\chi_{\mathbf{q}}(dd_1)=\chi_{\mathbf{q}}(d)\chi_{\mathbf{q}}(d_1),$$

since $c\equiv 0 \pmod{l}$. These relations imply that the functions $j_{Q,P}(M,\,z)$ satisfy relations of automorphic factors, i.e.,

$$j_{Q,P}(M,\,M_1\langle z\rangle))j_{Q,P}(M_1,\,z)=j_{Q,P}(MM_1,\,z)\ \text{ for all }\ M,\,M_1\in\Sigma_0(l),\ z\in\mathbb{H}.$$

This implies that the Petersson operators map the space $\mathfrak{F}_m$ into itself and satisfy the rule

$$F|_{\mathbf{j}}M|_{\mathbf{j}}M_1=F|_{\mathbf{j}}MM_1\qquad(F\in\mathfrak{F}_m,\quad M,M_1\in\Sigma_0(l)).$$

It allows us to define the standard representation $T\mapsto|_{\mathbf{j}}T$ of the Hecke-Shimura ring $\mathcal{H}_0(l)=\mathcal{H}(\Gamma_0(l),\,\Sigma_0(l))$ on the subspace

$$\mathfrak{F}_m(\Gamma_0(l))=\{F\in\mathfrak{F}_m\mid F|_{\mathbf{j}}\sigma=F\ \text{ for all }\ \sigma\in\Gamma_0(l)\}\tag{3.7}$$

of all $\Gamma_0(l)-$invariant functions by Hecke operators: the *Hecke operator* $|_{\mathbf{j}}T$ *on the space* $\mathfrak{F}_m(\Gamma_0(l))$ *corresponding to an element of the form* (3.3) is defined by

$$F|_{\mathbf{j}}T=\sum_{\alpha}c_\alpha F|_{\mathbf{j}}M_\alpha\qquad(F=F(z;\,(U,V))\in\mathfrak{F}_m(\Gamma_0(l))),\tag{3.8}$$

where $|_{\mathbf{j}}M_\alpha$ are the Petersson operators (3.5) corresponding to $\mathbf{j}=j_{Q,P}(M,\,z)$. The Hecke operators are independent of the choice of representatives $M_\alpha\in\Gamma_0(l)M_\alpha$ and map the space $\mathfrak{F}_m(\Gamma_0(l))$ into itself. It follows from the definition of multiplication in the Hecke–Shimura rings and (3.6) that Hecke operators satisfy

$$|_{\mathbf{j}}T|_{\mathbf{j}}T'=|_{\mathbf{j}}TT'\qquad\text{for all }\ T,T'\in\mathcal{H}_0(l).$$

Hence, the map $T\mapsto|_{\mathbf{j}}T$ is a linear representation of the ring $\mathcal{H}_0(l)$ on the space $\mathfrak{F}_m(\Gamma_0(l))$. The Hecke operators (3.8) on the space $\mathfrak{F}_m(\Gamma_0(l))$ are called *regular Hecke operators.*

As it follows from functional equations (3.1), the theta-function (2.2) of an integral positive definite quadratic form $\mathbf{q}$ of level l in an even number m of variables with matrix Q and a harmonic polynomial P of degree g relative to $\mathbf{q}$, considered as a function of z, and U, V, belongs to the space $\mathfrak{F}_m(\Gamma_0(l))$. Thus, this space contains the images of the theta-function under the Hecke operators corresponding to the generators (3.2) of the ring $\mathcal{H}_0(l)$. In particular, the space contains the images of the theta-function under the action of operators $|_{\mathbf{j}}T(p)$ with primes p not dividing l, i.e., the functions

$$\Theta_P(z;\,Q,\,(U,V))|_{\mathbf{j}}T(p) = \sum_{M\in\Gamma_0(l)\backslash\Gamma_0(l)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_0(l)} j_{Q,P}(M,\,z)^{-1}\Theta_P(M\langle z\rangle;Q,\,(U,V)\,{}^tM).$$

Since one can take

$$\Gamma_0(l)\backslash\Gamma_0(l)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(l)=\left\{\begin{pmatrix}1&0\\0&p\end{pmatrix},\begin{pmatrix}1&1\\0&p\end{pmatrix},\dots,\begin{pmatrix}1&p-1\\0&p\end{pmatrix},\begin{pmatrix}p&0\\0&1\end{pmatrix}\right\},$$

this image can be written in the form

$$\begin{aligned}\Theta_P(z;\,Q,\,(U,V))|_{\mathbf{j}}T(p) &= (\chi_{\mathbf{q}}(p)p^{k+g})^{-1}\sum_{b=0}^{p-1}\Theta_P\left(\frac{z+b}{p};\,Q\,(U,V)\begin{pmatrix}1&0\\b&p\end{pmatrix}\right)\\ &\quad+\Theta_P\left(pz;\,Q,\,(U,V)\begin{pmatrix}p&0\\0&1\end{pmatrix}\right)\\ &=(\chi_{\mathbf{q}}(p)p^{k+g})^{-1}\sum_{b=0}^{p-1}\Theta_P\left(\frac{z+b}{p};\,Q,\,(U+bV,pV)\right)\\ &\quad+\Theta_P\left(pz;\,Q,\,(pU,V)\right).\end{aligned}\tag{3.9}$$

The following particular case of [An09, Theorem 6.3] shows that the images of the theta-function under the Hecke operators corresponding to elements $T(p)$ with primes p in some cases can be written as finite linear combinations with constant coefficients of similar theta-functions.

Theorem 2. *Let* $\mathbf{q}$ *be an integral positive definite quadratic form in even number of variables* $m=2k$, l *– the level of* $\mathbf{q}$, $\chi_{\mathbf{q}}$ *– the character of* $\mathbf{q}$, *and let* P *be a harmonic polynomial of degree* g *relative to the form* $\mathbf{q}$. *Then, for each rational prime number* p *not dividing the level* l, *the following explicit formulas for the action of Hecke operator* $|_{\mathbf{j}}T(p)$ *with automorphic factor* $\mathbf{j}=j_{Q,P}$ *of the form* (3.5) *on the theta-function* (2.2) *of genus* 1 *with harmonic coefficient form* P *hold and arbitrary parameters* U, V*: if* $\chi_{\mathbf{q}}(p)=1$, *then*

$$\Theta_P(z;\,Q,\,(U,V))|_{\mathbf{j}}T(p) = \frac{\xi(m)}{p^{k-1}}\sum_{D\in A(Q,\,p)/\Lambda^m}\Theta_{P|p^{-1}D}(z;\,p^{-1}Q[D],\,pD^{-1}(U,V)),\tag{3.10}$$

where

$$\xi(m)=\begin{cases}1 & \text{if}\quad k=1,\\ \prod_{\alpha=1}^{k-1}(1+p^{\alpha-1}) & \text{if}\quad k>1,\end{cases}$$

$$A(Q,\mu)=\left\{D\in\mathbb{Z}^m_m\;\middle|\;\mu^{-1}Q[D]\quad\text{is even and}\ \det\mu^{-1}Q[D]=\det Q\right\}$$

is the set of all automorphes of Q *with multiplier* μ, $\Lambda^m = GL_m(\mathbb{Z})$, *and where*

$$(P|p^{-1}D)(X) = P(p^{-1}DX),$$

but if $\chi_Q(p) = -1$ *and* $k = 1$, *then*

$$\Theta_P(z;\, Q,\, (U, V))|_{\mathbf{j}}T(p) = 0. \tag{3.11}$$

It follows from the obvious relations

$$\Theta_{P|\Lambda}(z;\, Q[\Lambda],\, \Lambda^{-1}(U,\, V)) = \Theta_P(z;\, Q,\, (U,\, V)) \quad \text{for each } \Lambda \in \Lambda^m = GL_m(\mathbb{Z}),$$

where $(P|\Lambda)(X) = P(\Lambda X)$, that the sum to the right of (3.10) does not depend on particular choice of representatives $D \in A(Q,\, p)/\Lambda^m$.

Generally speaking, the set $A(Q, \mu)$ can be empty, but it is not empty, when $\mu = p$ is a prime number satisfying $\chi_{\mathbf{q}}(p) = 1$. It is clear that $A(Q, \mu)\Lambda^m = A(Q, \mu)$, and so the group Λ^m operates on each of the sets $A(Q, \mu)$ by right multiplications. Since all automorphes of $A(Q, \mu)$ are integral matrices of fixed determinants $\pm\,\mu^{m/2}$, it follows that each *set of right classes of automorphes* $A(Q, \mu)/\Lambda^m$ *modulo* Λ^m is always finite.

Standard applications of Hecke operators to Euler factorization of zeta functions of modular forms are based on consideration of common eigenfunctions of the operators. Although Theorem 2 shows that some images of theta-functions under Hecke operators are linear combinations of theta-functions, it gives no direct way to build eigenfunctions from linear combinations of theta-functions. A possible outcome can be found in a replacement of theta-functions with variable parameters U, V by corresponding theta-series obtained by suitable numerical specializations of the parameters. A variant of such specialization will be discussed in the next section.

§4. Action of hereditary Hecke operators on theta-series and Euler products

Here we shall consider action of the regular Hecke–Shimura rings $\mathcal{H}_0(l)$ by Hecke operators on theta-functions (2.3) with specialized parameters (U, V), i.e. on the corresponding *theta-series*. Resulting *hereditary Hecke operators* inherited from operators (3.8) acting on theta-functions with variable parameters are in general different from standard Hecke operators on theta-series considered only as functions in z belonging to the upper half-plane, because these theta-series are not necessarily invariant with respect to the group the form $\Gamma_0(l)$. The Hecke operator inherited from an operator $|_{\mathbf{j}}T$ will be denoted below by $|^*_{\mathbf{j}}T$.

We have seen in §2 that zeta functions of harmonic theta-functions have good analytic properties. In order to consider another essential feature of arithmetical zeta functions, the Euler product factorization, we follow classical approach to this problem initiated by E.Hecke in [He37] and based on consideration of eigenfunctions of Hecke operators acting on modular forms. In order to apply Hecke theory we have first to pass from theta-functions, which are not modular forms, to suitable modular forms.

Starting from the theta-function (2.2) of an integral positive definite quadratic forms $\mathbf{q}$ in even number of variables $m = 2k$ of level l with even matrix Q and with harmonic polynomials P of degree g relative to $\mathbf{q}$, we shall specialize parameters of the theta-function to be rational columns of the form

$$U = \mathbf{0},\ V = l^{-1}L \quad \text{with } L \in \mathbb{Z}^m \text{ satisfying congruence } QL \equiv \mathbf{0} \pmod{l}. \tag{4.1}$$

Then the theta-function turns into the theta-series

$$\Theta_P(z;\, Q|L) = \Theta_P(z;\, Q,\, (\mathbf{0},\, l^{-1}L)) = \sum_{N\in\mathbb{Z}^m} P(N - l^{-1}L)\, e^{\pi i\, z\, Q[N-l^{-1}L]} \tag{4.2}$$

with the Fourier expansion

$$\Theta_P(z;\, Q|L) = \sum_{n=0}^{\infty} r_{Q,P}(n,\, L)\, e^{\frac{2\pi\, i\, z\, n}{l}}, \tag{4.3}$$

where

$$r_{Q,P}(n,\, L) = \sum_{\substack{M\in\mathbb{Z}^m,\, M\equiv L \pmod{l}\\ Q[M]=2ln}} l^{-g} P(M) \tag{4.4}$$

are Fourier coefficients. Note that the theta-series (1.5) mentioned in §1 is proportional to the series (4.2) with $\mathbf{q} = 8(x_1^2 + x_2^2)$, $Q = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$, $l = 32$, $L = \begin{pmatrix} -8 \\ -8 \end{pmatrix}$, and $P(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = x_2$.

By formula (3.9) we obtain the following formulas for the action on the theta-series (4.2) of (hereditary) Hecke operators $|^*_{\mathbf{j}}T(p)$ with primes p not dividing the level l of $\mathbf{q}$:

$$\begin{aligned}
&\Theta_P(z;\, Q|L)|^*_{\mathbf{j}}T(p)\\
&= \frac{1}{\chi_{\mathbf{q}}(p)p^{k+g}} \sum_{b=0}^{p-1} \Theta_P\left(\frac{z+b}{p};\, Q,\, (l^{-1}bL, l^{-1}pL)\right) + \Theta_P\left(pz;\, Q,\, (\mathbf{0}, l^{-1}L)\right)\\
&= \frac{1}{\chi_{\mathbf{q}}(p)p^{k+g}} \sum_{N\in\mathbb{Z}^m} P(N - l^{-1}pL) \sum_{b=0}^{p-1} e^{\pi i\left(\frac{z+b}{p} Q[N-l^{-1}pL] + 2l^{-1}b\, {}^tLQN - l^{-2}bpQ[L]\right)}\\
&+ \sum_{N\in\mathbb{Z}^m} P(N - l^{-1}L)\, e^{\pi i p z Q[N-l^{-1}L]}\\
&= \frac{p}{\chi_{\mathbf{q}}(p)p^{k+g}} \sum_{N\in\mathbb{Z}^m,\, \mathbf{q}(N)\equiv 0 \pmod{p}} P(N - l^{-1}pL)\, e^{\frac{\pi i z}{p} Q[N-l^{-1}pL]}\\
&+ \sum_{N\in\mathbb{Z}^m} P(N - l^{-1}L)\, e^{\pi i p z Q[N-l^{-1}L]},
\end{aligned} \tag{4.5}$$

because, by readily verified identity,

$$\begin{aligned}
&\frac{z+b}{p} Q[N - l^{-1}pL] + 2l^{-1}b\, {}^tLQN - l^{-2}bpQ[L]\\
&= \frac{z}{p} Q[N - l^{-1}pL] + \frac{b}{p} Q[N - l^{-1}pL] + 2l^{-1}b\, {}^tLQN - l^{-2}bpQ[L]\\
&= \frac{z}{p} Q[N - l^{-1}pL] + \frac{b}{p} Q[N] = \frac{z}{p} Q[N - l^{-1}pL] + \frac{2b}{p} \mathbf{q}(N),
\end{aligned}$$

we have

$$\sum_{b=0}^{p-1} e^{\pi i\left(\frac{z+b}{p} Q[N-l^{-1}pL]+2l^{-1}b\,{}^tLQN-l^{-2}bpQ[L]\right)}$$

$$= e^{\frac{\pi i z}{p} Q[N-l^{-1}pL]} \sum_{b=0}^{p-1} e^{\frac{2\pi\, i\, b\, \mathbf{q}(N)}{p}} = e^{\frac{2\pi i z}{p} Q[N-l^{-1}pL]} \times \begin{cases} p & \text{if } p \mid \mathbf{q}(N), \\ 0 & \text{if } p \nmid \mathbf{q}(N). \end{cases}$$

Writing the series (4.5) as a Fourier series, in the notation (4.4) we get the formula

$$\Theta_P(z;\, Q|L)|^*_{\mathbf{j}} T(p)$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{\chi_{\mathbf{q}}(p) p^{k-1+g}} r_{Q,P}(np,\, pL) + r_{Q,P}\,(n/p,\, L) \right) e^{\frac{2\pi\, i\, z\, n}{l}}, \tag{4.6}$$

where the second term in parenthesis is present only if p divides n.

Let us consider now a linear combination of the theta-series (4.2) written in the form (4.3) with constant coefficients $\Phi(L)$ depending on L modulo l, i.e., the function

$$F = \Theta_{\Phi,\, P}(z;\, Q) = \sum_{\substack{L \in \mathbb{Z}^m / l\mathbb{Z}^m, \\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L)\, \Theta_P(z;\, Q|L)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{L \in \mathbb{Z}^m / l\,\mathbb{Z}^m, \\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L)\, r_{Q,P}(n,\, L) \right) e^{\frac{2\pi\, i\, z\, n}{l}} \tag{4.7}$$

with Fourier coefficients

$$r_{Q,P}(n,\, \Phi) = \sum_{\substack{L \in \mathbb{Z}^m / l\,\mathbb{Z}^m, \\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L)\, r_{Q,P}(n,\, L)$$
$$= \sum_{\substack{M \in \mathbb{Z}^m,\, Q[M]=2nl \\ QM \equiv \mathbf{0} \pmod{h}}} l^{-g} \Phi(M) P(M). \tag{4.8}$$

From formula (4.6) we obtain

$$F|^*_{\mathbf{j}} T(p) = \Theta_{\Phi,\, P}(z;\, Q)|^*_{\mathbf{j}} T(p) = \sum_{\substack{L \in \mathbb{Z}^m / l\,\mathbb{Z}^m, \\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L)\, \Theta_P(z;\, Q|L)|^*_{\mathbf{j}} T(p)$$

$$= \sum_{\substack{L \in \mathbb{Z}^m / l\,\mathbb{Z}^m, \\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L) \sum_{n=0}^{\infty} \left(\frac{1}{\chi_{\mathbf{q}}(p) p^{k-1+g}} r_{Q,P}(np,\, pL) + r_{Q,P}\,(n/p,\, L) \right) e^{\frac{2\pi\, i\, z\, n}{l}}$$

$$= \frac{1}{\chi_{\mathbf{q}}(p)p^{k-1+g}} \sum_{n=0}^{\infty} \sum_{\substack{L \in \mathbb{Z}^m/l\,\mathbb{Z}^m,\\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L) r_{Q,P}(np,\, pL) e^{\frac{2\pi\, i\, z\, n}{l}} + \sum_{n=0}^{\infty} r_{Q,P}(n/p,\, \Phi) e^{\frac{2\pi\, i\, z\, n}{l}}. \quad (4.9)$$

In order to compute the first sum in agreeable shape we assume now that the coefficient function Φ satisfies a homogeneity condition of the form

$$\Phi(aL) = \phi(a)\Phi(L) \qquad (\forall\, a \in \mathbb{Z},\, \gcd(a, l) = 1) \quad (4.10)$$

with a function $\phi : \mathbb{Z}/l\mathbb{Z} \to \mathbb{C}$. If Φ is not identically zero, it is easy to see that the function ϕ must satisfy the conditions

$$\phi(1) = 1, \quad \phi(ab) = \phi(a)\phi(b) \qquad \text{if } a \text{ and } b \text{ are prime to } l, \quad (4.11)$$

in particular, a value $\phi(a)$ with argument a prime to l must be a root of the unit.

Returning to (4.9), if Φ satisfies the condition (4.10), then, since p does not divide l, by replacing the sum over L by the sum over $p'L$, where p' satisfies $p'p \equiv 1 \pmod{l}$, we obtain

$$\sum_{\substack{L \in \mathbb{Z}^m/l\,\mathbb{Z}^m,\\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L) r_{Q,P}(np,\, pL) = \sum_{\substack{L \in \mathbb{Z}^m/l\,\mathbb{Z}^m,\\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(p'L) r_{Q,P}(np,\, p'pL)$$
$$= \phi(p') \sum_{\substack{L \in \mathbb{Z}^m/l\,\mathbb{Z}^m,\\ QL \equiv \mathbf{0} \pmod{l}}} \Phi(L) r_{Q,P}(np,\, L) = \overline{\phi}(p) r_{Q,P}(np,\, \Phi),$$

because the number $\phi(p') = \phi(p)^{-1} = \overline{\phi}(p)$ is complex conjugate to $\phi(p)$. Substituting the last expression in (4.9), we finally get the formula

$$F|_{\mathbf{j}}^{*}T(p) = \Theta_{\Phi,\,P}(z;\, Q)|_{\mathbf{j}}^{*}T(p)$$
$$= \sum_{n=0}^{\infty} \left(\frac{\overline{\phi}(p)}{\chi_{\mathbf{q}}(p)p^{k-1+g}} r_{Q,P}(np,\, \Phi) + r_{Q,P}\left(n/p,\, \Phi\right) \right) e^{\frac{2\pi\, i\, z\, n}{l}}. \quad (4.12)$$

Suppose now that the function $F = \Theta_{\Phi,\,P}(z;\, Q)$ of the form (4.7) is an eigenfunction for the operator $|_{\mathbf{j}}^{*}T(p)$ with the eigenvalue $\lambda_F(p)$. Then, according to (4.12), the Fourier coefficients (4.8) of F satisfy relations

$$\lambda_{F_\Phi}(p)\, r_{Q,P}(n,\, \Phi) = \frac{\overline{\phi}(p)}{\chi_{\mathbf{q}}(p)p^{k-1+g}}\, r_{Q,P}(np,\, \Phi) + r_{Q,P}\left(n/p,\, \Phi\right), \quad (4.13)$$

provided that the function Φ satisfies condition (4.10).

The relations (4.13) with numbers $n \neq 0$ differ by powers of a fixed prime p can be considered as a recursion relations and used to sum up formal power series of the form

$$R_{p,\,n}(t) = \sum_{\delta=0}^{\infty} r_{Q,P}(p^{\delta}n,\,\Phi)\,t^{\delta}.$$

Suppose that n is not divisible by p, then by (4.12) we have

$$\begin{aligned}\lambda_F(p)R_{p,\,n}(t) &= \frac{\overline{\phi}(p)}{\chi_{\mathbf{q}}(p)p^{k-1+g}}\sum_{\delta=0}^{\infty} r_{Q,P}(p^{\delta+1}n,\,\Phi)t^{\delta} + \sum_{\delta=1}^{\infty} r_{Q,P}\left(p^{\delta-1}n,\,\Phi\right)t^{\delta}\\ &= \frac{\overline{\phi}(p)}{\chi_{\mathbf{q}}(p)p^{k-1+g}t}\left(R_{p,\,n}(t) - r_{Q,P}(n,\,\Phi)\right) + tR_{p,\,n}(t),\end{aligned}$$

from which it follows that

$$\begin{aligned}R_{p,\,n}(t) &= \sum_{\delta=0}^{\infty} r_{Q,P}(p^{\delta}n,\,\Phi)\,t^{\delta} \qquad (4.14)\\ &= \left(1 - p^{k-1+g}\chi_{\mathbf{q}}(p)\phi(p)\lambda_F(p)t + p^{k-1+g}\chi_{\mathbf{q}}(p)\phi(p)t^2\right)^{-1}\, r_{Q,P}(n,\,\Phi).\end{aligned}$$

These summation formulas will be used bellow for Euler product factorization of zeta-functions corresponding to eigenfunctions of Hecke operators on spaces spanned by theta-series (4.2).

We remind that the zeta-function (2.4) of a theta-series (4.2) is defined by the Dirichlet series

$$\zeta_P(s;\,Q|L) = \sum_{N\in\mathbb{Z}^m,\,N\neq l^{-1}L} \frac{P(N-l^{-1}L)}{\mathbf{q}(N-l^{-1}L)^{s+\frac{g}{2}}} = \sum_{n=1}^{\infty}\frac{r_{Q,\,P}(n,\,L)}{(n/l)^{s+\frac{g}{2}}}, \qquad (4.15)$$

where $r_{Q,\,P}(n,\,L)$ are the Fourier coefficients (4.4) of the theta-series. Hence, the zeta-function of the linear combination (4.7) has the form

$$\zeta(s,\,F) = \sum_{\substack{L\in\mathbb{Z}^m/l\,\mathbb{Z}^m,\\ QL\equiv\mathbf{0}\pmod{l}}} \Phi(L)\zeta_P(s;\,Q|L) = \sum_{n=1}^{\infty}\frac{l^{s+\frac{g}{2}}r_{Q,\,P}(n,\,\Phi)}{n^{s+\frac{g}{2}}}, \qquad (4.16)$$

where $r_{Q,\,P}(n,\,\Phi)$ are the Fourier coefficients (4.8) of F. This zeta-function together with zeta-functions (4.15) converges absolutely for Re $s > k = m/2$, and uniformly in every half-plane Re $s > k + \epsilon$ with $\epsilon > 0$. Thus, $\zeta(s,\,F)$ is analytic function for Re $s > k$.

If we suppose that the function F is an eigenfunction for the operator $|^{*}_{\mathbf{j}}T(p)$ with the eigenvalue $\lambda_F(p)$, where p is a prime number not dividing l. Then, according to the identity (4.14), we can factor the function $\zeta(s,\,F)$ in the half-plane Re $s > k$ in the form

$$\begin{aligned}\zeta(s,\,F) &= \sum_{\delta=0}^{\infty}\sum_{n\geq 1,\,n\not\equiv 0\pmod{p}}\frac{l^{s+\frac{g}{2}}\,r_{Q,P}(p^{\delta}n,\,\Phi)}{(p^{\delta}n)^{s+\frac{g}{2}}}\\ &= \left(1 - \frac{p^{k-1+\frac{g}{2}}\chi_{\mathbf{q}}(p)\phi(p)\lambda_F(p)}{p^s} + \frac{p^{k-1}\chi_{\mathbf{q}}(p)\phi(p)}{p^{2s}}\right)^{-1}\sum_{n\not\equiv 0\pmod{p}}\frac{l^{s+\frac{g}{2}}\,r_{Q,P}(n,\,\Phi)}{n^{s+\frac{g}{2}}}.\end{aligned}$$

Assuming that function F is an eigenfunction for all operators $|_{\mathbf{j}}^{*}T(p)$ with prime p not dividing l and applying the last relation to each of the primes, we obtain.

Theorem 3. *Suppose that a linear combination F of the form (4.7) of theta-series of an integral positive definite quadratic form $\mathbf{q}$ of level l in even number of variables $m = 2k$ with a harmonic polynomial P of degree g, where coefficients Φ satisfy the condition (4.10), is a common eigenfunction of (hereditary) Hecke operators $|_{\mathbf{j}}^{*}T(p)$ for all prime numbers p not dividing l with eigenvalues $\lambda_F(p)$. Then the zeta function $\zeta(s, F)$ of F has the factorization into an absolutely and uniformly convergent in every half-plane* Re $s > k + \epsilon$ *with $\epsilon > 0$ Euler product of the form*

$$\zeta(s,\,F) = \sum_{n=1}^{\infty} \frac{l^{s+\frac{g}{2}} r_{Q,\,P}(n,\,\Phi)}{n^{s+\frac{g}{2}}} \tag{4.17}$$

$$= \prod_{p\in\mathbb{P},\,p\nmid l} \left(1 - \frac{p^{k-1+\frac{g}{2}}\chi_{\mathbf{q}}(p)\phi(p)\lambda_F(p)}{p^s} + \frac{p^{k-1}\chi_{\mathbf{q}}(p)\phi(p)}{p^{2s}}\right)^{-1} \sum_{n|l^\infty} \frac{l^{s+\frac{g}{2}}\, r_{Q,\,P}(n;\,\Phi)}{n^{s+\frac{g}{2}}},$$

where $\mathbb{P}$ is the set of all positive rational prime numbers, and the notation $n|l^\infty$ means that n divides a power of l, i.e., each prime divisor of n divides l.

§5. Eigenfunctions for theta-series of multiples of the sum of two squares

According to Theorem 3, for applications of Hecke operators to Euler factorization of zeta-functions of theta-series one has to build common eigenfunction of the operators and find corresponding eigenvalues. In the case of theta-series of one variable so fare the only way to approach this problem is to use explicit formulas, like formulas of Theorem 2, expressing images of the theta-series under Hecke operators through similar theta-series. In this section we use this approach to the simplest case of harmonic theta-series binary quadratic forms proportional to the sum of two squares. General case positive definite integral binary forms is the question of a future.

We shall consider here action of (hereditary) Hecke operators on harmonic theta-series (4.2) of a quadratic form

$$\mathbf{q}(X) = t(x_1^2 + x_2^2) \quad (t = 1, 2, \dots) \quad \text{with the matrix } Q = \begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix}$$

of level $l = l(Q) = 4t$, where a column $L \in \mathbb{Z}^2$ satisfies $QL \equiv 0 \pmod{l}$, that is $L \equiv 0 \pmod 2$, and so can be written in the form $L = -2T$, with $T \in \mathbb{Z}/2t\,\mathbb{Z}$, and where harmonic polynomial P is one of the binomials

$$P^{\pm}(X) = ((1, \pm i)X)^g = (x_1 \pm ix_2)^g \qquad \left(X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right).$$

The character $\chi_{\mathbf{q}}$ of the form $\mathbf{q}$ is defined on prime numbers by conditions $\chi_{\mathbf{q}}(p) = 0$ if $p \mid l$, and $\chi_{\mathbf{q}}(p) = (-1)^{\frac{p-1}{2}}$, otherwise.

Each such theta-series has the form

$$\Theta_P(z;\, Q| - 2T) = \sum_{N\in\mathbb{Z}^m} P(N+(2t)^{-1}T)e^{\pi i\, z\, Q[N+(2t)^{-1}T]} \tag{5.1}$$

In the notation and under the assumptions of the Theorem 2, for the theta-series (5.1) and a prime p not dividing the level $l = 4t$, we have in the case $\chi_{\mathbf{q}}(p) = 1$, i.e., if $p \equiv 1 \pmod 4$, the identity

$$\Theta_P(z;\, Q| - 2T)|^*_{\mathbf{j}}T(p) = \sum_{D\in A(Q,\,p)/\Lambda^2} \Theta_{P|p^{-1}D}(z;\, p^{-1}Q[D] \mid -2pD^{-1}T), \tag{5.2}$$

and the identity

$$\Theta_P(z;\, Q| - 2T)|^*_{\mathbf{j}}T(p) = 0 \tag{5.3}$$

if $\chi_{\mathbf{q}}(p) = -1$, i.e., $p \equiv 3 \pmod 4$.

Thus, looking for eigenfunctions, it suffices to consider only the case $p \equiv 1 \pmod 4$. Note that in this case each right coset $D\Lambda^2 \in A(Q,\, p)$ contains a representative D' with positive determinant, and such representative is unique up to a right factor from the modular group $\Gamma = SL_2(\mathbb{Z})$. Then it easy follows that the mapping $D\Lambda^2 \mapsto D'\Gamma$ defines bijection of the sets of classes $A(Q,\, p)/\Lambda^2$ and $A^+(Q,\, p)/\Gamma$, where $A^+(Q,\, \mu) = \{D \in A(Q,\, \mu) \mid \det D > 0\}$ is the set of all *proper automorph of Q with multiplier μ.* Thus, we can rewrite the formula (5.2) in the form

$$\begin{aligned}\Theta_P(z;\, Q| - 2T)|^*_{\mathbf{j}}T(p) &= \sum_{D\in A^+(Q,\,p)/\Gamma} \Theta_{P|p^{-1}D}(z;\, p^{-1}Q[D] \mid -2pD^{-1}T) \\ &= \sum_{N\in\mathbb{Z}^2}\ \sum_{D\in A^+(Q,\,p)/\Gamma} P(p^{-1}D(N+(2t)^{-1}pD^{-1}T))\, e^{\pi i\, z\, p^{-1}Q[D][N+(2t)^{-1}pD^{-1}T]}.\end{aligned} \tag{5.4}$$

Since p does not divide l, it easily follows that the matrix $p^{-1}Q[D]$ for each $D \in A^+(Q,\, p)$ has the same divisor t as matrix Q and so is of the form tQ', where Q' is an even primitive matrix of determinant $\det Q' = \det Q/t^2 = 4$, which, therefore, has the form $Q' = {}^tU \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} U$ with $U \in \Gamma$ (class number of the sum of two squares is one). Thus, representatives D of classes $A^+(Q,\, p)/\Gamma$ can be chosen so that $p^{-1}Q[D] = Q$ and the sum over $A^+(Q,\, p)/\Gamma$ can be replaced by the sum over the set $R^+(Q,\, pQ)/E^+$, where

$$R^+(Q,\, pQ) = \left\{D \in \mathbb{Z}_2^2 \mid \det D > 0,\, Q[D] = pQ\right\}$$

is the set of *proper represetations of pQ by Q* and

$$E^+ = \{U \in \Gamma \mid Q[U] = Q\} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$$

is the group of *proper units of* Q. Thus, we can rewrite (5.4) in the form

$$\Theta_P(z;\,Q|-2T)|^*_{\mathbf{j}}T(p) \tag{5.5}$$
$$= \sum_{N\in\mathbb{Z}^2}\ \sum_{D\in R^+(Q,\,pQ)/E^+} P(p^{-1}D(N+(2t)^{-1}pD^{-1}T))\, e^{\pi i\, z\, Q[N+(2t)^{-1}pD^{-1}T]}.$$

Since $p\equiv 1 \pmod 4$, it is easy to see that the set $R^+(Q,\,pQ)$ can be taken in the form

$$R^+(Q,\,pQ) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a,b\in\mathbb{Z},\ a^2+b^2=p \right\} \tag{5.6}$$

and consists of 8 matrices, whilst a set of representatives $R^+(Q,\,pQ)/E^+$ can be taken, for example, in the form

$$R^+(Q,\,pQ)/E^+$$
$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a,b\in\mathbb{Z},\ a^2+b^2=p,\ 0<a<b \right\} = \{D_1, D_2\} \tag{5.7}$$

and consists of 2 matrices.

Let us consider now the action of operator $|^*_{\mathbf{j}}T(p)$ on a linear combination of theta-series (5.1) with a coefficient function Φ, which will we shall write in the form

$$\Phi(T) = \varphi(t_1+it_2) \quad \left(T=\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in T\in\mathbb{Z}^2/2t\mathbb{Z}^2,\ t_1+it_2\in\mathcal{O}=\mathbb{Z}[i]/2t\,\mathcal{O}\right) \tag{5.8}$$

with, for example, $P(X)=P^+(X)=(x_1+i\,x_2)^g$, i.e., on the theta-series

$$F = \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(T)\Theta_P(z;\,Q|-2T) \tag{5.9}$$
$$= \sum_{N\in\mathbb{Z}^2}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(T)P(N+(2t)^{-1}T)e^{2\pi i\, z\, Q[N+(2t)^{-1}T]}$$
$$= (2t)^{-g} \sum_{\substack{n_1,n_2\in\mathbb{Z},\\ t_1,t_2\in\mathbb{Z}/2t\mathbb{Z}}} \varphi(t_1+it_2)\big((2tn_1+t_1)+i(2tn_2+t_2)\big)^g\, e^{\frac{\pi i z}{2t}\left((2tn_1+t_1)^2+(2tn_2+t_2)^2\right)}$$
$$= (2t)^{-g} \sum_{m_1,m_2\in\mathbb{Z}} \varphi(m_1+i\,m_2)(m_1+i\,m_2)^g\, e^{\frac{\pi i z}{2t}(m_1^2+m_2^2)}.$$

From formulas (5.5), we obtain

$$F|^*_{\mathbf{j}}T(p) = \left(\sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(T)\Theta_P(z;\,Q|-2T) \right) |^*_{\mathbf{j}}T(p) \tag{5.10}$$
$$= \sum_{N\in\mathbb{Z}^2}\ \sum_{D\in R^+(Q,\,pQ)/E^+}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(T)P(p^{-1}D(N+(2t)^{-1}pD^{-1}T))$$
$$\times\, e^{\pi i\, z\, Q[N+(2t)^{-1}pD^{-1}T]}$$
$$= \sum_{N\in\mathbb{Z}^2}\ \sum_{D\in R^+(Q,\,pQ)/E^+}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(p'DT)P(p^{-1}D(N+(2t)^{-1}T))e^{\pi i\, z\, Q[N+(2t)^{-1}T]},$$

where we have replaced on the last step T by $p'DT$ with an inverse p' of p modulo $2t$. Using the notation (5.8), for $D = D_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $D_2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ of the set of representatives (5.7), we have

$$\Phi(p'D_1T) = \Phi\left(p'\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \Phi\left(p'\begin{pmatrix} at_1 + bt_2 \\ -bt_1 + at_2 \end{pmatrix}\right)$$
$$= \varphi(p'(at_1 + bt_2 + i(-bt_1 + at_2))) = \varphi(p'(a+ib)(t_1 + it_2))$$

and

$$\Phi(p'D_2T) = \Phi\left(p'\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \Phi\left(p'\begin{pmatrix} at_1 - bt_2 \\ bt_1 + at_2 \end{pmatrix}\right)$$
$$= \varphi(p'(at_1 - bt_2 + i(bt_1 + at_2))) = \varphi(p'(a-ib)(t_1 + it_2)).$$

Suppose now that the function φ is multiplicative, i.e., satisfies

$$\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta) \quad (\alpha,\, \beta \in \mathcal{O}/2l\mathcal{O}), \tag{5.11}$$

in particular, $\varphi(1) = 1$, if φ is not identically zero. Then the above formulas imply relations

$$\Phi(p'D_1T) = \varphi(p'(a+ib)(t_1 + it_2)) = \varphi(p'(a+ib))\Phi(T),$$

and

$$\Phi(p'D_2T) = \varphi(p'(a-ib)(t_1 + it_2)) = \varphi(p'(a-ib))\Phi(T).$$

On the other hand, say, for the harmonic polynomial $P = P^+(X) = ((1,\, i)X)^g$, we have

$$P(p^{-1}D_1X) = \left((1,\, i)p^{-1}D_1X\right)^g = \left(p^{-1}(a-ib,\, b+ia)X\right)^g$$
$$= \left(p^{-1}(a-ib)(1,\, i)X\right)^g = \left(p^{-1}(a-ib)\right)^g P(X),$$

and, similarly,

$$P(p^{-1}D_2X) = \left(p^{-1}(a+ib)\right)^g P(X).$$

Hence, if the coefficient function (5.8) satisfies the multiplicativity condition (5.11), the formula (5.10), say, for $P = P^+$ can be rewritten in the form

$$\begin{aligned}
&F|^*_{\mathbf{j}}T(p) \\
&= \sum_{N\in\mathbb{Z}^2}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(p'D_1T)P(p^{-1}D_1(N + (2t)^{-1}T))e^{\pi i\, z\, Q[N+(2t)^{-1}T]} \\
&+ \sum_{N\in\mathbb{Z}^2}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \Phi(p'D_2T)P(p^{-1}D_2(N + (2t)^{-1}T))e^{\pi i\, z\, Q[N+(2t)^{-1}T]} \\
&= \sum_{N\in\mathbb{Z}^2}\ \sum_{T\in\mathbb{Z}^2/2t\mathbb{Z}^2} \left(\varphi(p'(a+ib))(p^{-1}(a-ib))^g + \varphi(p'(a-ib))(p^{-1}(a+ib))^g\right) \\
&\times \Phi(T)P(N + (2t)^{-1}T))e^{\pi i\, z\, Q[N+(2t)^{-1}T]} \\
&= \frac{\varphi(p')}{p^g}\left(\varphi(a+ib)(a-ib)^g + \varphi(a-ib)(a+ib)^g\right)F = \lambda_F(p)F,
\end{aligned} \tag{5.12}$$

i.e., the linear combination F is an eigenfunction of the operator $|_{\mathbf{j}}^{*}T(p)$ with the eigenvalue

$$\lambda_F(p) = \frac{\varphi(p')}{p^g}\left(\varphi(a+ib)(a-ib)^g + \varphi(a-ib)(a+ib)^g\right). \tag{5.13}$$

The above consideration allows us to apply Theorem 3 to the zeta-function

$$\zeta(s,\,F) = \sum_{\substack{N\in\mathbb{Z}^2,\,T\in\mathbb{Z}^2/2t\mathbb{Z}^2,\\ N+(2t)^{-1}T\neq\mathbf{0}}} \frac{\Phi(T)P(N+(2t)^{-1}T)}{\mathbf{q}(N+(2t)^{-1}T)^{s+\frac{g}{2}}} = \sum_{n=1}^{\infty}\frac{c_F(n)}{n^{s+\frac{g}{2}}} \tag{5.14}$$

of the linear combination (5.9) in the case of harmonic polynomials $P = P^{\pm}$, since clearly the coefficient function Φ satisfies the condition (4.10), where the function $\phi : \mathbb{Z}/2t\mathbb{Z}$ is equal to the restriction on $\mathbb{Z}\subset\mathbb{Z}[i]$ of the function φ. Then we obtain that the zeta-function (5.14) has the factorization into an absolutely and uniformly convergent in every half-plane Re $s > 1+\epsilon$ with $\epsilon > 0$ Euler product of the form

$$\zeta(s,\,F) = \prod_{p\in\mathbb{P},\,p\nmid 2t}\left(1 - \frac{p^{\frac{g}{2}}\chi_{\mathbf{q}}(p)\varphi(p)\lambda_F(p)}{p^s} + \frac{\chi_{\mathbf{q}}(p)\varphi(p)}{p^{2s}}\right)^{-1}\sum_{n|(2t)^{\infty}}\frac{c_F(n)}{n^{s+\frac{g}{2}}}, \tag{5.15}$$

where $\lambda_F(p)$ is the eigenvalue (5.13) if $p\equiv 1 \pmod 4$, and $\lambda_F(p) = 0$ if $p\equiv -1 \pmod 4$.

It turns out that Euler product factorization of the zeta-function (5.14) with multiplicative coefficient function (5.8) can be obtained in more explicit form and quite elementary without any use of Hecke operators, but just by means of interpretation of the zeta-function as an $L-$functions of the Gauss numbers field with a Hecke character. Really, the zeta-function (5.14) of the the linear combination (5.9) with the coefficient function (5.8) can be written in the form

$$\begin{aligned}\zeta(s,\,F) &= (2t)^{s-\frac{g}{2}}\sum_{m_1,m_2\in\mathbb{Z},\,(m_1,m_2)\neq(0,0)}\frac{\varphi(m_1+im_2)(m_1+im_2)^g}{(m_1+im_2)^{s+\frac{g}{2}}}\\ &= (2t)^{s-\frac{g}{2}}\sum_{\alpha\in\mathcal{O},\,\alpha\neq0}\frac{\varphi(\alpha)\alpha^g}{N(\alpha)^{\frac{g}{2}}}\frac{1}{N(\alpha)^s} = (2t)^{s-\frac{g}{2}}\sum_{\alpha\in\mathcal{O},\,\alpha\neq0}\frac{\psi(\alpha)}{N(\alpha)^s},\end{aligned} \tag{5.16}$$

where $\mathcal{O} = \mathbb{Z}[i]$ is the ring of Gauss integers, and $\psi(\alpha) = \varphi(\alpha)\alpha^g/N(\alpha)^{\frac{g}{2}}$. For $\alpha\neq 0$ all associated numbers α, $-\alpha$, $i\alpha$, $-i\alpha$ are different, have the same norm, and the sum of corresponding terms of sum (5.16) is

$$\begin{aligned}&\frac{\psi(\alpha)}{N(\alpha)^s} + \frac{\psi(-\alpha)}{N(-\alpha)^s} + \frac{\psi(i\alpha)}{N(i\alpha)^s} + \frac{\psi(-i\alpha)}{N(-i\alpha)^s}\\ &= (\psi(1)+\psi(-1)+\psi(i)+\psi(-i))\frac{\psi(\alpha)}{N(\alpha)^s} = (1+\psi(-1))(1+\psi(i))\frac{\psi(\alpha)}{N(\alpha)^s}.\end{aligned} \tag{5.17}$$

It follows that if

$$\varphi(-1) = (-1)^{g+1} \quad \text{or} \quad \varphi(i) = (-1)^{g+1} i^g, \tag{5.18}$$

then we have

$$\psi(-1) = \varphi(-1)(-1)^g = (-1)^{g+1}(-1)^g = -1$$

or

$$\psi(i) = \varphi(i) i^g = (-1)^{g+1} i^g i^g = (-1)^{g+1}(-1)^g = -1.$$

In every case each of the sums (5.17) is equal to zero, and so the whole zeta-function $\zeta(s, F)$ is identically zero. But if the zeta-function is not identically zero, then there is a nonzero sum of the form (5.17), which implies that

$$\kappa = \psi(1) + \psi(-1) + \psi(i) + \psi(-i) = 1 + \varphi(-1)(-1)^g + \varphi(i)\, i^g + \varphi(-i)(-i)^g \neq 0.$$

Since φ is multiplicative and $\varphi(1) = 1$, we get

$$\varphi(i)\kappa = \varphi(i) + \varphi(-i)\,(-1)^g + \varphi(-1) i^g + (-i)^g = (-i)^g \kappa.$$

It follows that $(\varphi(i) - (-i)^g)\kappa = 0$, i.e., $\varphi(i) = (-i)^g$. Hence we conclude that, for a principal ideal $\mathfrak{a} = \alpha\mathcal{O} = (\alpha)$ of the ring $\mathcal{O}$ with generator α, the value

$$\psi(\mathfrak{a}) = \psi(\alpha)$$

is independent of the choice of the generator α, and so each of the sums (5.17) is equal to $\kappa\psi(\alpha)/N(\alpha)^s = 4\psi((\alpha))/N((\alpha))^s$. Therefore the zeta-function can be written in the form

$$\zeta(s, F) = 4\,(2t)^{s-\frac{g}{2}} \sum_{\mathfrak{a} \neq (0)} \frac{\psi(\mathfrak{a})}{N(\mathfrak{a})^s}, \tag{5.19}$$

where $\mathfrak{a}$ ranges over all nonzero integral ideals of the ring $\mathcal{O}$, $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$, and

$$\psi(\mathfrak{a}) = \frac{\varphi(\alpha)\alpha^g}{N(\alpha)^{\frac{g}{2}}} \quad \text{if} \quad \mathfrak{a} = \alpha\mathcal{O}; \tag{5.20}$$

in addition, the function $\mathfrak{a} \mapsto \psi(\mathfrak{a})$ does not depend on the choice of generator α of the ideal $\mathfrak{a}$ and is multiplicative on the semigroup of nonzero integral ideals of the ring $\mathcal{O}$.

Using the multiplicative theory of ideals of the ring of Gauss integers, we come to Euler product factorization of the form

$$\zeta(s, F) = 4\,(2t)^{s-\frac{g}{2}} \prod_{\mathfrak{p}} \left(1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} = 4\,(2t)^{s-\frac{g}{2}} \prod_{p} \prod_{\mathfrak{p}|p} \left(1 - \frac{\psi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1},$$

where $\mathfrak{p}$ and p range over all nontrivial prime ideals of the ring and all rational prime numbers, respectively.

This observation hints that, at least in the case of zeta-functions of binary quadratic forms proportional to the sum of two squares, it suffices to take into account only Hecke $L-$functions.

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