LIE COALGEBRAS AND RATIONAL HOMOTOPY THEORY, I:
GRAPH COALGEBRAS

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1. Introduction

In this paper we develop a new, computationally friendly approach to Lie coalgebras through graph coalgebras, and we apply this approach to Harrison homology. There are two standard to presentations of a Lie algebra through "simpler" algebras. One is as a quotient of a non-associative binary algebra by Jacobi and anti-commutativity identities. Another presentation is as as embedded as Hopf algebra primitives in an associative universal enveloping algebra. The standard presentation of Lie coalgebras in the literature is dual to the second of these – as a quotient of the associative coenveloping coalgebra, namely the Hopf algebra indecomposables [12, 16]. We describe an approach to Lie coalgebras indiginous to the realm of coalgebras, dual to neither of these. We define a new kind of coalgebra structure, namely anti-commutative graph coalgebras, and we show that Lie coalgebras are quotients of these graph coalgebras.

Our approach through graph coalgebras gives a presentation for Lie coalgebras which works better than the classical presentation in two respects. First, cofree graph coalgebras come with a simple and easily computable pairing with free binary nonassociative algebras which passes to Lie coalgebras and algebras, making duality not just a theoretical statement but an explicitly computable tool. Secondly, the quotient used to create Lie coalgebras from graph coalgebras is a locally defined relation. The quotient creating Lie coalgebras from associative coalgebras is the shuffle relation, which causes global changes to an expression. As a result, proofs in the realm of Lie coalgebras are often simpler to give through graph coalgebras than through associative coalgebras, and for some important statements we have only found proofs in the graph coalgebra setting. For applications, we investigate the word problem for Lie coalgebras, and we also revisit Harrison homology. The category of graph coalgebras, and the graph cooperad on which it is based, may also be of intrinsic interest. The graph cooperad is not binary, but could play a similar role in some natural category of cooperads as is played by the tree operad for binary operads.

The plan of the paper is as follows. After defining the graph cooperad, we pair it with the tree operad to give rise to a pairing on cofree and free algebras over them. We show that upon quotienting by the kernels of the pairing, it descends to a pairing between cofree Lie coalgebras and free Lie algebras. This approach gives rise to our graphical model for the cofree Lie coalgebra on a vector space V and determines how that model pairs with the free Lie algebra on a linear dual of V. Moreover, we can deduce a formula for the linear duality between Michaelis's Lie coalgebra model [12] and the tree/bracket model for free Lie algebras. We are also able to shed new light on the structure of cofree Lie coalgebras, for example viewing them as what one gets when one starts with a graph or associative coalgebra and "kills the kernel of the cobracket."

We then lift the André-Quillen construction on a differential graded commutative algebra (DGCA) from the category of differential graded Lie coalgebras (DGLC) to anti-commutative differential graded graph coalgebras (DGGC). The Harrison model for this bar construction passes through the category of associative coalgebras, but our factorization through graph coalgebras is needed for example in developing an algebraic models for fibrations in the Lie coalgebraic formulation of rational homotopy theory. Such a result is critical in the sequel to this paper, where we define generalized Hopf invariants and show from first principals that they give a complete set of homotopy functionals in the simply-connected setting. Indeed it was an investigation of generalized Hopf invariants, which we found to be naturally indexed by graphs, which led us to the framework of this paper.

Finally, we combine these results to shed new light on Quillen's seminal work on rational homotopy theory [15]. Quillen produced a pair of adjoint functors $\mathcal{L}$ and $\mathcal{C}$ between the categories of dg-commutative coalgebras (DGCC) and dg-Lie algebras (DGLA). In the linearly dual setting, there previously were two avenues towards understanding the functors between DGCA and DGLC. One would be a formal application of linear duality to Quillen's functors. The other way to go from DGCA to DGLC explicitly was to use the Harrison complex, which from Schlessinger and Stasheff's work has the structure of a Lie coalgebra dual to Quillen's Lie algebraic functor. Our techniques allow us to explicitly calculate the linear duality between Harrison homology of a differential graded commutative algebra and Quillen's functor $\mathcal{L}$ on the corresponding linearly dual coalgebra, unifying these approaches.

In our appendices, we take the opportunity to flesh out our models and connect with other work. In particular, we give a spectral sequence for rational homotopy groups of a simply connected space, we explicitly define model structures, and we discuss minimal models.

Our work throughout is over a field of characteristic zero. We emphasize that we are adding a finiteness hypothesis, namely that our algebras and coalgebras are finite-dimensional in each positive degree, for the sake of linear duality theorems. Under this hypothesis the category of chain complexes is canonically isomorphic to that of cochain complexes, and we will use this isomorphism without further comment, by abuse denoting both categories by DG. To clarify when possible, we have endeavored to use V to denote a chain complex and W to denote a cochain complex. Many of the facts we prove are true without the finiteness hypothesis, as we may indicate.

We further restrict our work to 1-connected objects both to mirror the classical constructions of [15] and to allow ourselves to cleanly express our cofree Lie coalgebras as coinvariants rather than invariants. We plan to remove the finiteness and 1-connectivity hypotheses in the third paper in this series. Note however that though in Sullivan's rational homotopy theory it is fairly typical to quickly move to the nilpotent setting, this step requires a significant change to foundations of our work. The first author is currently writing a general theory of coalgebras over cooperads [24] so that we may proceed with such a program, where it looks like we can extend even beyond the nilpotent setting.

While we start by giving operadic definitions, we work more explicitly at the algebra and coalgebra level in later sections. One reason for this change in emphasis is a desire for explicit formulae. But the change in emphasis is necessary, since we have yet to find a purely operadic argument for the existence of the lift of the bar construction on a commutative algebra from the category of Lie coalgebras to the category of graph coalgebras. We hope to study the graph cooperad and graph coalgebras more extensively in future work. We have yet to fully understand even what general (that is, not cofree) graph coalgebras are in explicit algebraic terms.

2. The graph cooperad and the configuration pairing

We begin with constructions on the level of operads and cooperads, to give more fundamental understanding (to readers familiar with operads) and provide a general road-map for the following sections. Later proofs and constructions will be given wholly in the realm of algebras and coalgebras even when they could be inferred from operad level statements presented here, which in some important cases they cannot

be. A reader not interested in operads can skip most of this section, with the exceptions of the definitions of graphs (2.1), the configuration pairing (2.11), and the quotients defining Lie coalgebras (2.14).

Definition 2.1. The graph symmetric sequence is defined as follows.

(1) Let S be a finite set. An S-graph is a connected oriented acyclic graph with vertex set $\text{Vert}(G) = S$.
(2) For each S, let $\mathcal{G}r(S)$ be the vector space freely generated by S-graphs and write $\mathcal{G}r$ for the associated symmetric sequence of vector spaces.
(3) If $G \in \mathcal{G}r(S)$, define $|G|$ to be the cardinality of S, which we call the weight of G. Write $\mathcal{G}r(n) = \mathcal{G}r(\{1, ..., n\})$.

We outline the basic properties of the graph cooperad. For proofs and more detailed discussion, see the examples section of [24] where a more convenient notation for cooperads is developed.

Definition 2.2. A graph quotient $\phi : G \twoheadrightarrow K$ maps vertices of G to vertices of K such that edges of G are mapped to either edges of K (with the same orientation) or vertices of K, and the inverse image of each vertex of K is a non-empty connected subgraph of G.

Proposition 2.3. *The symmetric sequence $\mathcal{G}r$ has a cooperad structure induced by the map*

$$G \longmapsto \sum_{\phi : G \twoheadrightarrow K} K \otimes \Big(\bigotimes_{k \in \text{Vert}(K)} \phi^{-1}(k) \Big),$$

where $\phi : G \twoheadrightarrow K$ ranges over all graph quotient maps and $\phi^{-1}(k)$ is the connected subgraph of G mapping to vertex k.

The cooperad structure above is associative in the sense that the 2-arity structure map

$$\Big(\underset{a}{\overset{b}{\nearrow}} \Big)^* : G \longmapsto \sum_{\phi : G \twoheadrightarrow \underset{a}{\overset{b}{\nearrow}}} \phi^{-1}(a) \otimes \phi^{-1}(b)$$

is (co-)associative. The symmetric sequence $\mathcal{G}r$ has another cooperad structure which we call anti-commutative for an analogous reason.

Definition 2.4. Let $E \subset \text{Edge}(K)$. Define $\text{rev}_E(K)$ to be the graph resulting from reversing the orientations of the edges E of K.

Proposition 2.5. *The symmetric sequence $\mathcal{G}r$ has an anti-commutative cooperad structure induced by*

$$G \longmapsto \sum_{\substack{\phi : G \twoheadrightarrow K \\ E \subset \text{Edge}(K)}} (-1)^{|E|} \, \text{rev}_E(K) \otimes \Big(\bigotimes_{k \in \text{Vert}(K)} \phi^{-1}(k) \Big),$$
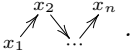
where ϕ and $\phi^{-1}(k)$ are as above.

Definition 2.6. The anti-commutative graph cooperad, denoted $\mathcal{AC}\mathcal{G}r$, is given by the symmetric sequence $\mathcal{G}r$ equipped with the anti-commutative cooperad structure of Proposition 2.5.

The associative graph cooperad, denoted $\mathcal{As}\mathcal{G}r$, is given by the symmetric sequence $\mathcal{G}r$ equipped with the associative cooperad structure of Proposition 2.3.

Remark 2.7. Coalgebras over these graph cooperads have not, to our knowledge, been studied before. We plan to study them in future work, but there are two main features we would like to highlight now. First, such graph coalgebras are not binary coalgebras. For example, $\mathcal{G}r(3)$ is twelve-dimensional, while the cooperad structure map goes to $\mathcal{G}r(2) \otimes (\mathcal{G}r(2) \otimes \mathcal{G}r(1)) \oplus \mathcal{G}r(2) \otimes (\mathcal{G}r(2) \otimes \mathcal{G}r(1))$, which is eight-dimensional so this structure map cannot be injective. Secondly, associative graph coalgebras extend associative coalgebras, as we establish in Proposition 2.8.

In the language of operads, the standard approaches to Lie algebras can be summarized by a sequence of operad maps $\mathcal{T}r \to \mathcal{L}ie \to \mathcal{A}s$. Recall that the associative operad has $\mathcal{A}s(n)$ of rank $n!$, naturally spanned by monomials in n variables with no repetition, and $\mathcal{T}r$ is the the tree operad whose structure maps are defined by grafting and which governs non-associative binary algebras (see 2.10 below). Our Lie coalgebra model follows from fitting the anti-commutative graph cooperad into the linearly dual sequence of cooperads as $\mathcal{T}r^\vee \leftarrow \mathcal{L}ie^\vee \leftarrow \mathcal{ACG}r \leftarrow \mathcal{A}s^\vee$. The following propositions are easily verified by direct calculation.

Proposition 2.8. *The associative cooperad $\mathcal{A}s^\vee$ maps to the associative graph cooperad $\mathcal{A}s\mathcal{G}r$ by sending the monomial $x_1 x_2 \cdots x_n$ to the graph* $\underset{x_1}{\overset{x_2 \quad x_n}{\nearrow \searrow \nearrow}}$.

Proposition 2.9. *The associative graph cooperad $\mathcal{A}s\mathcal{G}r$ maps to the anti-commutative graph cooperad $\mathcal{ACG}r$ via the map*

$$G \longmapsto \frac{1}{2 \#(\mathrm{Edge}(G))} \sum_{E \subset \mathrm{Edge}(G)} (-1)^{|E|} \mathrm{rev}_E(G).$$

Next, we develop the configuration pairing between graphs and trees, which allows us to explicitly compute the composition $\mathcal{ACG}r \to \mathcal{T}r^\vee$. We use this to gain a new understanding of $\mathcal{L}ie^\vee$ "in the middle." In particular we show in 2.21 that $\mathcal{L}ie^\vee$ is isomorphic as a cooperad to a quotient of $\mathcal{ACG}r$ which we call $\mathcal{E}il$. Furthermore, we show in 2.22 the standard map $\mathcal{A}s^\vee \to \mathcal{L}ie^\vee$ is equal to the composition of the maps in Propositions 2.8 and 2.9 followed by the quotient map to $\mathcal{E}il$. We first define terms.

Definition 2.10. Let S be a finite set. An S-tree is an isotopy class of acyclic graphs embedded in the upper half plane with all vertices either trivalent or univalent. Trivalent vertices are called internal vertices. One univalent vertex is distinguished as the root and embedded at the origin. The other univalent vertices are called leaves and are equipped with a labeling ismorphism $\ell : \mathrm{Leaves} \xrightarrow{\cong} S$. We will standardly conflate leaves with their labels.

Let $\mathcal{T}r(S)$ be the vector space generated by S-trees, $\mathcal{T}r$ be the associated symmetric sequence of vector spaces, and write $\mathcal{T}r(n)$ for $\mathcal{T}r(\{1,...,n\})$.

See II.1.9 in [10] for a precise definition of the operad structure maps of $\mathcal{T}r$ through grafting. The pairing between $\mathcal{G}r(n)$ and $\mathcal{T}r(n)$ was developed in [18], and arises in the study of configuration spaces. Let the height of a vertex in a tree be the number of edges between that vertex and the root. The nadir of a path in a tree is the vertex of lowest height which it traverses.

Definition 2.11. Fix a finite set S. Given an S-graph G and an S-tree T, define the map

$$\beta_{G,T} : \{\text{edges of } G\} \longrightarrow \{\text{internal vertices of } T\}$$

by sending an edge from vertex a to b in G to the vertex at the nadir of the shortest path in T between the leaves with labels a and b. The configuration pairing of G and T is

$$\langle G, T \rangle = \begin{cases} \prod_{\substack{e \text{ an edge} \\ \text{of } G}} \mathrm{sgn}\big(\beta_{G,T}(e)\big) & \text{if } \beta \text{ is surjective,} \\ 0 & \text{otherwise} \end{cases}$$

where given an edge $\underset{a}{\overset{b}{\nearrow}}$ of G, $\mathrm{sgn}\left(\beta\big(\underset{a}{\overset{b}{\nearrow}}\big)\right) = 1$ if leaf a is to the left of leaf b under the planar embedding of T; otherwise it is -1.

Example 2.12. Following is the map $\beta_{G,T}$ for a single graph G and two different trees T.

In the first example, $\text{sgn}\big(\beta(e_1)\big) = -1$ and $\text{sgn}\big(\beta(e_2)\big) = 1$. In the second example, $\text{sgn}\big(\beta(e_1)\big) = 1$ and $\text{sgn}\big(\beta(e_2)\big) = -1$. The graph and tree of the first example pair to -1; in the second example they pair to 0.

From the tree operad, the Lie operad is defined as follows.

Definition 2.13. $\mathcal{L}ie(n)$ is the quotient of $\mathcal{T}r(n)$ by the anti-symmetry and Jacobi relations:

$$\text{(anti-symmetry)} \qquad \underset{R}{\overset{T_1\ T_2}{\curlyvee}} \;=\; -\; \underset{R}{\overset{T_2\ T_1}{\curlyvee}}$$

$$\text{(Jacobi)} \qquad \underset{R}{\overset{T_1\ T_2\ T_3}{\curlyvee}} \;+\; \underset{R}{\overset{T_2\ T_3\ T_1}{\curlyvee}} \;+\; \underset{R}{\overset{T_3\ T_1\ T_2}{\curlyvee}} \;=\; 0,$$

where R, T_1, T_2, and T_3 stand for arbitrary (possibly trivial) subtrees which are not modified in these operations.

The configuration pairing respects anti-symmetry and Jacobi relations among trees. There is a similar set of relations which the configuration pairing respects among graphs.

Definition 2.14. Let $\mathcal{E}il(n)$ be the quotient of $\mathcal{G}r(n)$ by the relations

where a, b, and c stand for vertices in the graph which could possibly have other connections to other parts of the graph which are not modified in these operations. We emphasize that a, b, c are *vertices*, not subgraphs.

Sinha's paper [18] establishes the following theorem, which was first proven independently by Tourtchine [23] and, in the odd setting, Melancon and Reutenauer [11].

Theorem 2.15. *The configuration pairing $\langle G, T \rangle$ between $\mathcal{G}r(n)$ and $\mathcal{T}r(n)$ descends to a perfect equivariant pairing between $\mathcal{E}il(n)$ and $\mathcal{L}ie(n)$.*

There is an isomorphism of symmetric sequences $\mathcal{E}il(n) \cong \mathcal{L}ie^{\vee}(n)$.

The theorem is proven by first showing that the pairing vanishes on Jacobi and anti-symmetry combinations of trees as well as on arrow-reversing and Arnold combinations of graphs. These relations allow one to reduce to generating sets of "tall" trees and "long" graphs – as in the figure below. The pairing is a Kronecker pairing on these generating sets.

FIGURE 1. Tall trees and long graphs

Proposition 2.16. *The subcomplex of graph expressions generated by arrow-reversing and Arnold expressions of graphs is a coideal [7, §2.1] of $\mathcal{ACGr}$.*

Corollary 2.17. *The symmetric sequence $\mathcal{E}il$ inherits an anti-commutative cooperad structure from $\mathcal{ACGr}$.*

Definition 2.18. By abuse, write $\mathcal{E}il$ for the cooperad induced by quotienting $\mathcal{ACGr}$ by the Arnold and arrow-reversing identites.

Proposition 2.19. *The cooperad structure of $\mathcal{ACGr}$ is compatible with the operad structure of $\mathcal{Tr}$ via the configuration pairing.*

Corollary 2.20. *The cooperad structure on $\mathcal{E}il$ is compatible with the operad structure of $\mathcal{Lie}$ (inherited from that of $\mathcal{Tr}$) via the configuration pairing.*

Theorem 2.21. *As cooperads, $\mathcal{E}il \cong \mathcal{Lie}^\vee$. Quotienting by Arnold and arrow-reversing identities gives a surjection of cooperads from $\mathcal{ACGr}$ to $\mathcal{Lie}^\vee$.*

Since we would rather emphasize free and cofree algebras than the operads defining them, we will reserve the computations required for Propositions 2.16 and 2.19 for the the proofs of Propositions 3.7 and 3.14 which are the analogous statements on the level of coalgebras and algebras. A short duality computation (which we leave for the reader) now completes our operadic picture.

Proposition 2.22. *The following duality diagram of operads and cooperads commutes.*

$$
\begin{array}{ccc}
\mathcal{Lie} & \longrightarrow & \mathcal{As} \\
\Big\uparrow{\scriptstyle *} & & \Big\downarrow{\scriptstyle *} \\
\mathcal{E}il \Longleftarrow \mathcal{ACGr} \longleftarrow \mathcal{AsGr} \longleftarrow \mathcal{As}^\vee
\end{array}
$$

Algebra level consequences of this duality are discussed in Section 3.3 on coenveloping graph coalgebras.

Remark 2.23. Note that our construction of coalgebras is over cooperads rather than over operads. It is common in the literature (such as [21]) to largely eschew the use of cooperads when discussing coalgebras, instead defining coalgebras over operads briefly as follows. Recall the endomorphism operad $\text{End}(V)$ of an object V in a closed symmetric monoidal category. The endomorphism operad of V in the opposite category is called its coendomorphism operad $\text{Coend}(V)(n) = \text{Hom}(V, V^{\otimes n})$. If $\mathcal{P}$ is an operad then a $\mathcal{P}$-algebra structure on V is an operad map $\mathcal{P} \to \text{End}(V)$, and a $\mathcal{P}$-coalgebra structure on V is an operad map $\mathcal{P} \to \text{Coend}(V)$.

This relates to coalgebras over a cooperad in the following manner. A map $\mathcal{P} \to \text{Coend}(V)$ consists of equivaraint maps $\mathcal{P}(n) \to \text{Hom}(V, V^{\otimes n})$. If $\mathcal{P}(n)$ is dualizable then these are the same as equivariant maps $V \to \mathcal{P}(n)^* \otimes V^{\otimes n}$, which because V has trivial action are simply maps from V to the Σ_n-invariants of the right side. If $\mathcal{P}(n)$ is dualizable then the $\mathcal{P}(n)^*$ form a cooperad, and the structure above defines a coalgebra over this cooperad. This construction is immediately dual to the structure maps $\mathcal{P}(n) \otimes V^{\otimes n} \to V$ defining algebras over an operad. We write $\mathcal{P}^\vee$ for the cooperad $\mathcal{P}^\vee(n) = \mathcal{P}(n)^*$.

For more information about a general approach to cooperads and coalgebras over cooperads, see [24]. Developing cooperads on their own terms not only mitigates the use of linear duality, but gives a more understandable and more computable approach, at least to Lie coalgebras and Quillen's rational homotopy theory [15] as we presently develop.

Remark 2.24. While tree operad $\mathcal{Tr}$ governs binary non-associative algebras, the graph cooperads cannot govern non-associative binary coalgebras. The configuration pairing between $\mathcal{Tr}$ and $\mathcal{Gr}$ is not perfect, nor could there be a different pairing which is perfect. For example, $\mathcal{Tr}(n)$ has dimension $\frac{n(n-1)}{2} - 1$ as a $\mathbb{Q}[\Sigma_n]$-module (for $n > 1$). But as a $\mathbb{Q}[\Sigma_3]$-module $\mathcal{Gr}(3)$ is of dimension 3, and and as a $\mathbb{Q}[\Sigma_4]$-module $\mathcal{Gr}(4)$ is of dimension 8. It is also not clear what either the linear or Koszul-Moore duals (in the sense of [13]) of graph cooperads are.

3. The pairing between free tree algebras and cofree graph coalgebras

Constructing our graphical model for Lie coalgebras, we are interested in coalgebras over the anti-commutative graph cooperad $\mathcal{ACGr}$. Though we may occasionally write "anti-commutative graph coalgebra" for emphasis, in general we will write simply "graph coalgebra" to mean a coalgebra over the cooperad $\mathcal{ACGr}$. Note that below we explicitly develop only the quadratic structure of graph coalgebras since that is all that we require to understand Lie coalgebras.

3.1. **Basic manipulations of cofree graph coalgebras.** A first step in the theory of operads is the construction of free algebras. We will use the co-Schur functors associated to $\mathcal{Gr}$ and $\mathcal{Eil}$ (dual to the Schur functors of [6]) to construct explicit models for Lie coalgebras as quotients of anti-commutative graph coalgebras.

Definition 3.1. Let W be a vector space. Define the vector spaces $\overline{\mathbb{G}}(W)$ and $\overline{\mathbb{E}}(W)$ as follows.

$$\overline{\mathbb{G}}(W) \;\cong\; \bigoplus_n \left(\mathcal{Gr}(n) \otimes W^{\otimes n}\right)_{\Sigma_n}$$

$$\overline{\mathbb{E}}(W) \;\cong\; \bigoplus_n \left(\mathcal{Gr}(n) \otimes W^{\otimes n}\right)\big/_{\sim,\,\Sigma_n} \;=\; \overline{\mathbb{G}}(W)\big/_\sim$$

where $\sim$ is the relation induced by arrow-reversing and Arnold on $\mathcal{Gr}(n)$.

There is a difficulty in defining general cofree graph and Lie coalgebras similar to that of defining general cofree associative coalgebras. Recall that the cotensor coalgebra does not give cofree associative coalgebras, since in particular it is always cofinite (that is, a finite iteration of the coproduct will reduce any element to primitives). Trying to remedy this by replacing colimits by limits usually does not yield a coalgebra since this would require the tensor product to commute with infinite products. Using results of Smith [20], a cofree graph coalgebra is given in general by the largest coalgebra contained in $\prod_n \mathcal{Gr}(n) \otimes^{\Sigma_n} W^{\otimes n}$. Rather than work to get the correct definition we fall back to the time-honored tradition of restricting to 1-reduced (that is, trivial in grading zero and below) coalgebras. In this category, all coalgebras are cofinite, the cotensor coalgebra models cofree associative coalgebras, and we have the following.

Proposition 3.2. *If W is 1-reduced, then $\overline{\mathbb{G}}(W)$ is the vector space which underlies the cofree graph coalgebra on W and $\overline{\mathbb{E}}(V)$ underlies the cofree Lie coalgebra on V.*

If W is reduced and finitely generated, then so too will be $\overline{\mathbb{G}}(W)$ and $\overline{\mathbb{E}}(W)$. We leave the unreduced and infinitely generated setting for future work.

We now explicitly develop the graph and Lie coalgebra structures referred to in the previous proposition. In the ungraded case, $\overline{\mathbb{G}}(W)$ is generated by oriented, connected, acyclic graphs (of possibly infinite size) whose vertices are labeled by elements of W modulo multilinearity in the labels. Cutting a single edge separates graphs in $\overline{\mathbb{G}}(W)$, so we may define a coproduct by a summation cutting each edge in turn and tensoring the resulting graphs in the order determined by the direction of the edge which was cut – this is the coproduct encoded by $\mathcal{AsGr}$. In order to descend to the Lie coalgebra cobracket (see Corollary 3.15) we add a twisted term to the above coproduct with signs to make the result anti-cocommutative – this is the coproduct encoded by $\mathcal{ACGr}$. Explicitly, $]G[\,=\, \sum_{e \in G} (G_1^{\hat{e}} \otimes G_2^{\hat{e}} - G_2^{\hat{e}} \otimes G_1^{\hat{e}})$, where e ranges over the edges of G, and $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the connected components of the graph obtained by removing e, which points from $G_1^{\hat{e}}$ to $G_2^{\hat{e}}$.

Unfortunately, graded graph coalgebras are more complicated to represent due to the presence of Koszul signs. For example, $\overset{b}{\underset{a \quad c}{\diagdown}}$ could mean either $\left[\overset{1}{\underset{2 \quad 3}{\diagup\diagdown}} \bigotimes b \otimes a \otimes c \right]$ or $\left[\overset{1}{\underset{3 \quad 2}{\diagup\diagdown}} \bigotimes b \otimes c \otimes a \right]$, which differ by a sign of $(-1)^{|a||c|}$. The same difficulty arises when defining graded Lie algebras via the $\mathcal{Lie}$ operad (or non-associative algebras via the $\mathcal{Tr}$ operad), but the simple convention there is to choose the equivalence class

representative whose $\mathcal{L}ie(n)$ component has the ordering of its leaves consistent with the planar ordering. Because there is no general canonical choice for representativees of Σ_n-equivalence classes in $\mathcal{G}r(n)$, we are forced to write elements of $\overline{\mathbb{G}}(W)$ explicitly via representatives in $\mathcal{G}r(n) \otimes W^{\otimes n}$.

   We define the graded anti-commutative graph cobracket as follows.

Definition 3.3. The anti-commutative graph cobracket $\,]\cdot[\,: \overline{\mathbb{G}}(W) \to \overline{\mathbb{G}}(W) \otimes \overline{\mathbb{G}}(W)$ is given by

$$
\Big] G \bigotimes w_1 \otimes \cdots \otimes w_n \Big[\ = \sum_{e \in G} (-1)^{\kappa_1} \big(G_1^{\hat{e}} \bigotimes w_{\sigma^{\hat{e}}(1)} \otimes \cdots \otimes w_{\sigma^{\hat{e}}(|G_1^{\hat{e}}|)} \big) \bigotimes \big(G_2^{\hat{e}} \bigotimes w_{\sigma^{\hat{e}}(|G_1^{\hat{e}}|+1)} \otimes \cdots \otimes w_{\sigma^{\hat{e}}(n)} \big)
$$

$$
- (-1)^{\kappa_2} \big(G_2^{\hat{e}} \bigotimes w_{\sigma^{\hat{e}}(|G_1^{\hat{e}}|+1)} \otimes \cdots \otimes w_{\sigma^{\hat{e}}(n)} \big) \bigotimes \big(G_1^{\hat{e}} \bigotimes w_{\sigma^{\hat{e}}(1)} \otimes \cdots w_{\sigma^{\hat{e}}(|G_1^{\hat{e}}|)} \big),
$$

where e ranges over the edges of G and points from the connected subgraph $G_1^{\hat{e}}$ to the connected subgraph $G_2^{\hat{e}}$, $\sigma^{\hat{e}}$ is the unshuffling of vertex labels induced by separating G into $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$, and $(-1)^{\kappa_1}$, $(-1)^{\kappa_2}$ are the Koszul signs due to reordering the w_i's.

Proposition 3.4. *The anti-commutative graph cobracket $]\cdot[$ on $\overline{\mathbb{G}}(W)$ coincides with the binary coproduct arising from the 2-arity cooperad structure map of $\mathcal{ACGr}$.*

Definition 3.5. Let $\mathbb{G}(W)$ denote the cofree anti-commutative graph coalgebra on W, whose binary structure is thus given by $\overline{\mathbb{G}}(W)$ with anti-commutative graph cobracket $]\cdot[$. Similarly, let $\mathbb{E}(W)$ denote the cofree Lie coalgebra on W.

   We will commonly refer to the anti-commutative graph cobracket as merely the cobracket, since it is the only coproduct operation which we will consider on the graph complex $\overline{\mathbb{G}}(W)$. Our notation will be justified shortly by showing that the anti-commutative graph cobracket operation on $\overline{\mathbb{G}}(W)$ descends to an operation on $\overline{\mathbb{E}}(W)$ which coincides with the Lie coalgebra cobracket of $\mathbb{E}(W)$. Recall from Proposition 3.2 that $\overline{\mathbb{E}}(W)$ is the vector space underlying $\mathbb{E}(W)$.

   For horizontal brevity we will generally write $\displaystyle \mathop{G}_{w_1 \otimes \cdots \otimes w_n} := G \bigotimes w_1 \otimes \cdots \otimes w_n$ for all graphs except for the trivial one: $G = \bullet^1$.

Example 3.6. The anti-commutative graph coalgebra element $\mathop{\nearrow^2\nwarrow}_{1 \quad 3 \atop a \otimes b \otimes c}$ has cobracket:

$$
\Big]\, {}_{1 \atop a \otimes b \otimes c}^{\;\nearrow^2\nwarrow}{}_{3} \,\Big[\ = (-1)^{(|a|+|b|)|c|} \Big(c \ \otimes \ {}_{1 \atop a \otimes b}^{\;\nearrow^2} \Big) - \Big({}_{1 \atop a \otimes b}^{\;\nearrow^2} \ \otimes \ c \Big) + \Big(a \ \otimes \ {}_{2 \atop b \otimes c}^{\;\nearrow^1} \Big) - (-1)^{|a|(|b|+|c|)} \Big({}_{2 \atop b \otimes c}^{\;\nearrow^1} \ \otimes \ a \Big).
$$

Proposition 3.7. *Let $\mathrm{Arn}(W)$ be the vector subspace of $\overline{\mathbb{G}}(W)$ generated by arrow-reversing and Arnold expressions of graphs (2.14). Then $\mathrm{Arn}(W)$ is a coideal of $\mathbb{G}(W)$. That is*

$$
\big] \mathrm{Arn}(W) \big[\ \subset \mathrm{Arn}(W) \otimes \mathbb{G}(W) + \mathbb{G}(W) \otimes \mathrm{Arn}(W).
$$

Thus the cobracket descends to a well-defined operation $]\cdot[\,: \overline{\mathbb{E}}(W) \to \overline{\mathbb{E}}(W) \otimes \overline{\mathbb{E}}(W)$.

Proof. Due to the local definition of arrow-reversing and Arnold, it is enough to check the behaviour of the cobracket on an expression reversing the arrow of a graph with only two vertices and on an Arnold expression for a graph with only three vertices.

   The arrow-reversing check (neglecting Koszul signs) is:

$$
\Big]\, {}_{1 \atop a \otimes b}^{\;\nearrow^2} \ + \ {}_{1 \atop a \otimes b}^{\;\swarrow^2} \,\Big[\ = \ (a \otimes b - b \otimes a) + (b \otimes a - a \otimes b) \ = \ 0.
$$

Modulo arrow-reversing, all graphs with only three vertices are long graphs, so it suffices to check the sum of the following (again neglecting signs).

$$
\left]\begin{smallmatrix}&2&\\&\nearrow\!\!\searrow&\\1&&3\end{smallmatrix}\right[_{a\otimes b\otimes c} = \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{a\otimes b} \otimes\ c\ +\ a\ \otimes\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{b\otimes c}\ -\ c\ \otimes\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{a\otimes b}\ -\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{b\otimes c}\ \otimes\ a
$$

$$
\left]\begin{smallmatrix}&2&\\&\searrow&\\1&\!\!\leftarrow\!\!&3\end{smallmatrix}\right[_{a\otimes b\otimes c} = \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{b\otimes c} \otimes\ a\ +\ b\ \otimes\ \begin{smallmatrix}&1\\&\nearrow\\2&\end{smallmatrix}_{a\otimes c}\ -\ a\ \otimes\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{b\otimes c}\ -\ \begin{smallmatrix}&1\\&\nearrow\\2&\end{smallmatrix}_{a\otimes c}\ \otimes\ b
$$

$$
\left]\begin{smallmatrix}&2&\\&\nearrow&\\1&\!\!\leftarrow\!\!&3\end{smallmatrix}\right[_{a\otimes b\otimes c} = \begin{smallmatrix}&1\\&\nearrow\\2&\end{smallmatrix}_{a\otimes c} \otimes\ b\ +\ c\ \otimes\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{a\otimes b}\ -\ b\ \otimes\ \begin{smallmatrix}&1\\&\nearrow\\2&\end{smallmatrix}_{a\otimes c}\ -\ \begin{smallmatrix}&2\\&\nearrow\\1&\end{smallmatrix}_{a\otimes b}\ \otimes\ c
$$

$\square$

In Proposition 3.14 below, we show via duality that the operation induced on $\overline{\mathbb{E}}(W)$ by the graph cobracket agrees with the Lie coalgebra cobracket. In Proposition 3.18 below we prove also the converse of Proposition 3.7: If $]g[\in \mathrm{Arn}(W) \otimes \mathbb{G}(W) + \mathbb{G}(W) \otimes \mathrm{Arn}(W)$ then $g \in \mathrm{Arn}(W)$.

Remark 3.8. Though we cannot in general choose canonical representatives of Σ_n-classes in $\mathcal{G}r(n)$, for some classes there is a canonical choice. For long n-graphs, we chose Σ_n-representative so that the ordering of vertices is consistent with the direction of arrows. In this case we use "bar" notation

$$
a_1|a_2|\cdots|a_n := \left[\begin{smallmatrix}&2&&4&&&n\\&\nearrow\!\!\searrow&\nearrow&\searrow&\nearrow&\cdots&\nearrow\\1&&3&&&n-1&\end{smallmatrix} \bigotimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \right].
$$

Because long n-graphs span $\mathcal{E}il(n)$, the bar classes above span $\overline{\mathbb{E}}(W)$. For example, if a, b, c and d are all in even degree, then applying the Arnold and arrow-reversing identites we get

$$
\left[\begin{smallmatrix}&&4&\\&&\uparrow&\\&\nearrow\!\!2&&\\1&&&3\end{smallmatrix} \right]_{a\otimes b\otimes c\otimes d} = \left[\begin{smallmatrix}&2&&1\\&\nearrow\!\!\searrow&\nearrow&\\4&&3&\end{smallmatrix} \right]_{a\otimes b\otimes c\otimes d} - \left[\begin{smallmatrix}&1&&4\\&\nearrow\!\!\searrow&\nearrow&\\3&&2&\end{smallmatrix} \right]_{a\otimes b\otimes c\otimes d} = d|b|c|a - c|a|b|d.
$$

In terms of the bar generators of $\overline{\mathbb{E}}(W)$, the cobracket given in Proposition 3.7 is simply the anti-cocommutative coproduct (i.e. $]\cdot[= \Delta - \tau\Delta$ where τ is the twisting map $\tau(x \otimes y) = y \otimes x$). This recovers the approach taken by Michaelis [12] and Schlessinger-Stasheff [16]. We elaborate further on this approach in Section 3.3.

3.2. **Duality of free algebras and cofree coalgebras.** As in the previous section, we start with underlying vector spaces and then move on to product and coproduct structures.

Lemma 3.9. *Let G be a finite group, and let V and W be modules over a ring in which the order of G is invertible. If $\langle -, - \rangle$ is an equivariant perfect pairing between W and V, then the pairing defined between W_G and V_G by $\langle [w], [v] \rangle_G = \sum_{g \in G} \langle gw, v \rangle$ is also perfect.*

Proof. If $\langle [w], [v] \rangle_G = 0$ for all $[v] \in V_G$ then $\langle \sum_{g \in G} gw, v \rangle = 0$ for all $v \in V$. Because the pairing $\langle -, - \rangle$ is perfect, this means $\sum_{g \in G} gw = 0$ in W. Projecting to W_G implies that $|G| \cdot [w] = 0$, which by our hypotheses means $[w] = 0$. By equivariance we have $\langle [w], [v] \rangle_G = \sum_{g \in G} \langle w, gv \rangle$, so we may apply the same argument to show that there is no kernel for $\langle -, - \rangle_G$ in V_G either, yielding the result. $\square$

Let $\mathbb{T}(V)$ be the free binary non-associative algebra on V, with underlying vector space $\overline{\mathbb{T}}(V)$ given by the Schur functor $\bigoplus_n (\mathcal{T}r(n) \otimes V^{\otimes n})_{\Sigma_n}$. Define $\mathbb{L}(V)$ and $\overline{\mathbb{L}}(V)$ similarly as the free Lie algebra on V and its underlying vector space.

Definition 3.10. Given W and V vector spaces with a pairing $\langle -, - \rangle$, the configuration pairing between $\overline{\mathbb{G}}(W)$ and $\overline{\mathbb{T}}(V)$ is

$$\langle [G \textstyle\bigotimes w_1 \otimes \cdots \otimes w_n], \ [T \textstyle\bigotimes v_1 \otimes \cdots \otimes v_n] \rangle \ = \ \sum_{\sigma \in \Sigma_n} \left(\langle \sigma G, \ T \rangle \cdot \prod_{i=1}^n \langle w_{\sigma^{-1}(i)}, \ v_i \rangle \right).$$

This descends also to a configuration pairing between $\overline{\mathbb{E}}(W)$ and $\overline{\mathbb{L}}(V)$ by Theorem 2.15 (proven in [18, 23, 11]). Applying Lemma 3.9 we have the following.

Corollary 3.11. *Over a field of characteristic zero, if W and V pair perfectly then the configuration pairing between $\overline{\mathbb{E}}(V)$ and $\overline{\mathbb{L}}(W)$ is perfect.*

Example 3.12. Consider the free Lie algebra on two letters, so that V is spanned by a and b. Then we have the following pairing.

$$\left\langle \left[\underset{a^* \otimes a^* \otimes b^*}{\overset{2}{\underset{1 \quad\; 3}{\wedge}}} \right], \ \left[\underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right] \right\rangle = \left\langle \underset{a^* \otimes a^* \otimes b^*}{\overset{2}{\underset{1\;\;3}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle + (-1)^{|b||a|} \left\langle \underset{a^* \otimes b^* \otimes a^*}{\overset{3}{\underset{1\;\;2}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle$$

$$+ \ (-1)^{|a|^2} \left\langle \underset{a^* \otimes a^* \otimes b^*}{\overset{1}{\underset{2\;\;3}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle + \left\langle \underset{b^* \otimes a^* \otimes a^*}{\overset{3}{\underset{2\;\;1}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle$$

$$+ \ (-1)^{|a|^2 + |a||b|} \left\langle \underset{a^* \otimes b^* \otimes a^*}{\overset{1}{\underset{3\;\;2}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle + (-1)^{|a|^2} \left\langle \underset{b^* \otimes a^* \otimes a^*}{\overset{2}{\underset{3\;\;1}{\wedge}}}, \ \underset{a \otimes b \otimes a}{\overset{1\,2\,3}{\vee}} \right\rangle$$

$$= (-1)^{|b||a|} \left\langle \overset{3}{\underset{1\;\;2}{\wedge}}, \ \overset{1\,2\,3}{\vee} \right\rangle + (-1)^{|a|^2 + |a||b|} \left\langle \overset{1}{\underset{3\;\;2}{\wedge}}, \ \overset{1\,2\,3}{\vee} \right\rangle$$

$$= 0 + (-1)(-1)^{|a|^2 + |a||b|}$$

Remark 3.13. Melançon and Reutenauer [11] essentially showed that pairing with with bar elements in $\mathbb{G}(V^*)$ defines functionals which can alternately be defined through looking at coefficients of Lie polynomials (that is, looking at the coefficients of elements of $\mathbb{L}(V)$ in its standard embedding in the tensor algebra on V). It would be interesting to understand the functionals coming from other elements in $\mathbb{G}(V^*)$, such as those arising from Tourtchine's alternating trees [23], in a similar manner.

The configuration pairing further exhibits a duality between non-associative algebra multiplication and graph cobracket operations. This allows us to compute pairings inductively.

Proposition 3.14. *Non-associative algebra multiplication is dual to the anti-commutative graph cobracket in the configuration pairing. That is,*

$$\langle \gamma, \ (\tau_1 \tau_2) \rangle = \langle \,]\gamma[, \ \tau_1 \otimes \tau_2 \rangle$$
$$= \sum_e \langle \gamma_1^{\hat{e}}, \ \tau_1 \rangle \langle \gamma_2^{\hat{e}}, \ \tau_2 \rangle,$$

where $]\gamma[\ = \sum_e \left(\gamma_1^{\hat{e}} \otimes \gamma_2^{\hat{e}} \right)$.

Proof. Recall that non-associative algebra multiplication is induced by the $\mathcal{T}r$ operation $(T_1 T_2) = \overset{T_1 \ T_2}{\vee}$. We give a bijection between potentially non-zero terms in the summands defining $\langle \gamma, \ (\tau_1 \tau_2) \rangle$ and $\langle \,]\gamma[, \ \tau_1 \otimes \tau_2 \rangle$. In particular, we focus on those terms whose graph/tree pairing component may be non-zero.

Begin by fixing graph and tree representatives. Let $\gamma = [G \otimes \vec{w}] \in \left(\mathcal{G}r(n) \otimes W^{\otimes n} \right)_{\Sigma_n}$ (where $\vec{w} \in W^{\otimes n}$) and $\tau_i = [T_i \otimes \vec{v}_i] \in \left(\mathcal{T}r(k_i) \otimes V^{\otimes k_i} \right)_{\Sigma_{k_i}}$ (where $\vec{v}_i \in V^{\otimes k_i}$, $k_1 + k_2 = n$). Also let $\gamma_i^{\hat{e}} = [G_i^{\hat{e}} \otimes \vec{w}_i^{\hat{e}}]$ (for

$i = 1, 2$) be the graph coalgebra elements given by cutting γ at the edge e. Recall that

$$\left\langle \gamma, \ (\tau_1 \tau_2) \right\rangle = \sum_{\sigma \in \Sigma_n} \left\langle \sigma G, (T_1 T_2) \right\rangle \left\langle \sigma^{-1} \vec{w}, \ \vec{v}_1 \otimes \vec{v}_2 \right\rangle$$

$$\left\langle \gamma_1^{\hat{e}}, \ \tau_1 \right\rangle \left\langle \gamma_2^{\hat{e}}, \ \tau_2 \right\rangle = \sum_{\sigma_i \in \Sigma_{k_i}} \left\langle \sigma_1 G_1^{\hat{e}}, \ T_1 \right\rangle \left\langle \sigma_2 G_2^{\hat{e}}, \ T_2 \right\rangle \left\langle \sigma_1^{-1} \vec{w}_1^{\hat{e}}, \ \vec{v}_1 \right\rangle \left\langle \sigma_2^{-1} \vec{w}_2^{\hat{e}}, \ \vec{v}_2 \right\rangle$$

Suppose that some $\left\langle \sigma_1 G_1^{\hat{e}}, \ T_1 \right\rangle \left\langle \sigma_2 G_2^{\hat{e}}, \ T_2 \right\rangle$ is non-zero. Since $G_1^{\hat{e}}$ and $G_2^{\hat{e}}$ are the graphs resulting from cutting G at the edge e, there is a unique permuation σ which (modulo arrow-reversing at e) displays G as

$$(1) \qquad\qquad\qquad\qquad (\sigma G) = \pm \quad \overset{\overset{(\sigma_2 G_2^{\hat{e}}) + k_1}{\mathcal{A}}}{(\sigma_1 G_1^{\hat{e}})}$$

with sign $\pm$ coming from whether the arrow e was reversed when giving G this form (here $(\sigma_2 B_e) + k_1$ denotes adding k_1 to each vertex label of $(\sigma_2 B_e)$). Since the configuration pairing respects the arrow-reversing relation on graphs, it follows that

$$\left\langle \sigma G, (T_1 T_2) \right\rangle = \pm \left\langle \sigma_1 G_1^{\hat{e}}, \ T_1 \right\rangle \left\langle \sigma_2 G_2^{\hat{e}}, \ T_2 \right\rangle$$

with the same sign as in Equation 1.

Conversely, if $\left\langle \sigma G, (T_1 T_2) \right\rangle$ is non-zero then there is a corresponding non-zero $\left\langle \sigma_1 G_1^{\hat{e}}, \ T_1 \right\rangle \left\langle \sigma_2 G_2^{\hat{e}}, \ T_2 \right\rangle$. Given a subset $S \subset \{1, \ldots, n\}$ let $G|_S$ denote the full subgraph of G on the vertices with labels in S. It follows from Definition 2.11 that $\left\langle \sigma G, (T_1 T_2) \right\rangle = 0$ unless there is exactly one edge in σG between the full subgraphs $(\sigma G)|_{\{1, \ldots, k_1\}}$ and $(\sigma G)|_{\{k_1 + 1, \ldots, n\}}$. Thus these graphs must be connected and (modulo arrow-reversing at e) the graph σG must be of the form

$$(2) \qquad\qquad\qquad\qquad \sigma G = \pm \quad \overset{\overset{(\sigma G)|_{\{k_1 + 1, \ldots, n\}}}{\sigma \mathcal{A}}}{(\sigma G)|_{\{1, \ldots, k_1\}}}$$

with the sign $\pm$ reflecting whether the arrow σe was reversed when writing σG in this way. Since the configuration pairing respects the arrow-reversing relation on graphs, it follows that

$$\left\langle \sigma G, \ (T_1 T_2) \right\rangle = \pm \left\langle (\sigma G)|_{\{1, \ldots, k_1\}}, \ T_1 \right\rangle \left\langle (\sigma G)|_{\{k_1 + 1, \ldots, n\}} - k_1, \ T_2 \right\rangle$$

with the same sign as in Equation 2 (where by $G - k$ we mean to shift all labels of G down by k). We may obtain a non-zero term of the form $\left\langle \sigma_1 G_1^{\hat{e}}, T_1 \right\rangle \left\langle \sigma_2 G_2^{\hat{e}}, T_2 \right\rangle$, by setting σ_1 and σ_2 so that $\sigma_1 G_1^{\hat{e}} = (\sigma G)_{\{1, \ldots, k_1\}}$ and $\sigma_2 G_2^{\hat{e}} = (\sigma G)_{\{k_1 + 1, \ldots, n\}} - k_1$.

The remainder of the proof is straightforward. The signs and pairings between the associated tensors are equal since they are simply Koszul signs and expected pairings on both sides of the equality. $\square$

The multiplication operation for non-associative algebras induces the Lie algebra bracket upon quotienting by anti-commutativity and Jacobi relations among trees. In particular, Proposition 3.14 now implies the following.

Corollary 3.15. *The graph coalgebra cobracket agrees with the Lie coalgebra cobracket through the quotient map from cofree graph coalgebras to cofree Lie coalgebras.*

In light of this proposition, an alternate approach to exhibiting the pairing between $\mathbb{L}V$ and $\mathbb{G}W$ for dual V and W would be to define the pairing between $\mathbb{L}^n V$ and $\mathbb{G}^n W$ inductively using the bracket and cobracket.

Remark 3.16. Corollary 3.11 and Proposition 3.14 give a method for constructing functionals on Lie algebras which are not free. Any finitely generated graded Lie algebra is the homology of some free finitely generated differential graded Lie algebra. That is, $L \cong H_*(\mathbb{L}V, d)$. The complex $(\mathbb{L}V, d)$ is dual

to $(\mathbb{E}V^*, d^*)$, whose homology pairs with that of $H_*(\mathbb{L}V, d)$, namely L, through the configuration pairing. Using bar basis elements from $\mathbb{E}V^*$ one can recover the embedding of L in its universal enveloping algebra, but the approach through $\mathbb{E}V^*$ offers more flexibility.

3.3. **Coenveloping graph coalgebras.** There are four basic approaches to the free Lie algebra $\mathbb{L}(V)$ on a vector space V.

(1) $\mathbb{L}$ is the left adjoint of the forgetful functor from Lie algebras to vector spaces.
(2) $\mathbb{L}(V) \cong \bigoplus_n \mathcal{L}ie(n) \otimes_{\Sigma_n} V^{\otimes n}$, where $\sigma \in \Sigma_n$ acts on $\mathcal{L}ie(n) \otimes V^{\otimes n}$ as $\sigma \otimes \sigma^{-1}$, and the Σ_n action on $V^{\otimes n}$ is governed by the Koszul sign convention.
(3) $\mathbb{L}(V)$ is a quotient of the free non-associative algebra on V, $\bigoplus_n (\mathcal{T}r(n) \otimes V^{\otimes n})_{\Sigma_n}$, by the anti-symmetry and Jacobi relations on $\mathcal{T}r(n)$.
(4) $\mathbb{L}(V)$ is the smallest subspace of the tensor algebra on V which contains V and is closed under commutators.

So far our development of Lie coalgebras has paralleled the second and third approaches, while the adjointness properties are immediate. To complete our picture, and connect with previous work, we now focus on developing the last approach. We give a representation of $\mathbb{E}(W)$ which is dual to the Poincaré-Birkhoff-Witt embedding of $\mathbb{L}(V)$ in the tensor algebra TV. We will exhibit $\mathbb{E}(W)$ as a quotient of the cotensor coalgebra. This representation is the starting point for the seminal work of Michaelis [12] on Lie coalgebras, so we in particular identify how our graph model for cofree Lie coalgebras encompasses that approach.

Definition 3.17. Define the graded vector space $\overline{\mathbb{G}}(W) \big/ \ker \,]\cdot[$ inductively, setting $\overline{\mathbb{G}}^1(W) \big/ \ker \,]\cdot[= W$ and letting $\overline{\mathbb{G}}^n(W) \big/ \ker \,]\cdot[$ be the quotient of $\overline{\mathbb{G}}^n(W)$ by the kernel of the map

$$]\cdot[: \overline{\mathbb{G}}^n(W) \longrightarrow \left(\overline{\mathbb{G}}^{<n}(W) \big/ \ker \,]\cdot[\right) \bigotimes \left(\overline{\mathbb{G}}^{<n}(W) \big/ \ker \,]\cdot[\right).$$

Proposition 3.18. $\overline{\mathbb{E}}(W) \cong \overline{\mathbb{G}}(W) \big/ \ker \,]\cdot[.$

We encourage the interested reader to work through a direct proof of this proposition by explicitly showing the converse of Proposition 3.7. Instead we use duality and compatibility of graph cobrackets with Lie brackets.

Proof. By Proposition 3.7, $\ker(]\cdot[) \supset \mathrm{Arn}(W)$. It remains to show only that $\ker(]\cdot[) \subset \mathrm{Arn}(W)$. By Corollary 3.11 it is enough to show that the kernel of the pairing between $\mathbb{G}^n(W)$ and $\mathbb{L}^n(V)$ contains $\ker(]\cdot[)$. This follows by induction using Proposition 3.14. $\square$

Proposition 3.18 implies that $\mathbb{E}(W)$ is the quotient of $\mathbb{G}(W)$ by the largest coideal in the kernel of $\mathbb{G}(W) \to W$. This extends the definition of Lie coalgebras given by [16] as the quotient of the cotensor coalgebra TW by the largest coideal in the kernel of $TW \to W$. In particular the construction of [16] follows as an immediate corollary using the injection of operads $\mathcal{A}s^\vee \to \mathcal{A}s\mathcal{G}r$. We record this in a more computationally useful form as follows.

Corollary 3.19. $\overline{\mathbb{E}}(W)$ *is isomorphic to the quotient of the cotensor coalgebra TW by the non-primitive kernel of the anti-cocommutative coproduct.*

Proof. There is a "graphification" map g which injects TW into $\mathbb{G}W$:

$$g : w_1|w_2|\cdots|w_n \longmapsto \left[\begin{array}{c} \overset{2}{\nearrow}\overset{4}{\searrow}\overset{n}{\nearrow} \\ \underset{1}{\nearrow}\underset{3}{\searrow}\underset{n\text{-}1}{\nearrow} \\ w_1 \otimes w_2 \otimes \cdots \otimes w_n \end{array} \right].$$

By abuse, call the anti-cocommutative coproduct on TW the cobracket, and denote it $]\cdot[= \Delta - \tau\,\Delta$. As mentioned in Remark 3.8, g sends cobrackets of cotensors in TW to cobrackets of long graphs in $\mathbb{E}W$. Now apply Proposition 3.18. $\square$

Proposition 3.18 suggests a simple algorithm for checking whether a Lie coalgebra element is trivial. Inductively define the iterated cobracket on graph coalgebras $]\cdot[^n : G \to G^{\otimes n}$ by

$$]g[^n = \sum_e]g_1^{\hat{e}}[^{n-1} \otimes g_2^{\hat{e}},$$

where $]g[= \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}$. By Proposition 3.18, a necessary condition for a graph expression $g \in \mathbb{E}^n(W)$ to be trivial is for $]g[^{n-1} = 0$. In fact, this condition is also sufficient.

Proposition 3.20. *An element $g \in \mathbb{E}^n(W)$ is trivial if and only if $]g[^{n-1} = 0$.*

Proof. Applying Proposition 3.14,

$$\Big\langle g, \ \Big[[[v_1, v_2], v_3], \cdots v_n \Big] \Big\rangle \ = \ \Big\langle]g[^{n-1}, \ v_1 \otimes v_2 \otimes \cdots \otimes v_n \Big\rangle.$$

Since bracket expressions of the form $\big[[[v_1, v_2], v_3] \cdots v_n \big]$ span $\mathbb{L}W^*$ and the configuration pairing is perfect between $\mathbb{E}W$ and $\mathbb{L}W^*$, $g = 0$ if and only if $]g[^{n-1} = 0$. $\square$

Our recovery of the approaches to cofree Lie coalgebras of Michaelis [12] and Schlessinger-Stasheff [16] allows us to highlight some advantages of the graph model. Working from $\mathbb{G}(W)$ the list of relations satisfied by Lie coalgebra elements is relatively simple to describe – arrow-reversing and Arnold relations for graphs as well as symmetric group action. Once we have restricted to the bar generators, however, the relations become harder to describe. For example, below are a two relations satisfied by bar generators of $\mathbb{E}^n W$ (neglecting Koszul signs).

(3)
$$\big(w_1 | w_2 | \cdots | w_n \big) - (-1)^{n-1} \big(w_n | \cdots | w_2 | w_1 \big) = 0$$

(4)
$$\sum_{\substack{\sigma \text{ a cyclic} \\ \text{permutation of } (1, \dots, n)}} \big(w_{\sigma(1)} | \cdots | w_{\sigma(n)} \big) = 0$$

Relation (3) above comes from applying the arrow-reversing identity at every arrow of a long graph. Relation (4) is easily verified using Proposition 3.20. To complete the comparison to [16] we use our graph model to show that quotienting cotensor coalgebras by shuffle relations gives Lie coalgebras.

Proposition 3.21. *The Harrison shuffles give a spanning set of relations among bar generators of $\mathbb{E}^n W$; i.e.*

$$\sum_{\substack{\sigma \text{ a shuffle of} \\ (1,2,\dots k) \text{ into } (k+1,\dots,n)}} \big(w_{\sigma(1)} | \cdots | w_{\sigma(n)} \big) = 0.$$

Proof. Write $\mathrm{Sh}(W)$ for the vector subspace of $\mathbb{G}W$ generated by the Harrison shuffles of bar expressions. It is straightforward to show that $\mathrm{Sh}(W)$ is a coideal:

$$]\mathrm{Sh}(W)[\ \subset \ \mathrm{Sh}(W) \otimes \mathbb{G}W + \mathbb{G}W \otimes \mathrm{Sh}(W).$$

On bar expressions of either 2 or 3 elements, the Harrison shuffles are merely the arrow-reversing and Arnold relations. Thus by Proposition 3.20, $\mathrm{Sh}(W) \subset \mathrm{Arn}(W)$.

That $\mathrm{Sh}(W)$ gives all relations among bar generators is now an immediate application Proposition 3.18 and comments at the end of the first section of [16]. $\square$

Note that it is not at all clear that relations (3) and (4) above are inside the coideal of Harrison shuffles. For computational purposes, it is convenient to have a more minimal set of relations among bar generators of $\mathbb{E}W$. Directly applying the configuration pairing, we find the following set of relations.

Proposition 3.22. *The below shuffles give a spanning set of relations among bar generators of $\mathbb{E}^n W$.*

$$\big(w_1|w_2|w_3|\cdots|w_n\big) + (-1)^k \sum_{\substack{\sigma \ a \ shuffle \ of \\ (k-1,k-2,\ldots 1) \ into \ (k+1,\ldots,n)}} \big(w_k|w_{\sigma(1)}|\cdots|w_{\sigma(n-1)}\big) = 0$$

Proof. Let V have basis $v_1,\ldots,v_n$ dual to $w_1,\ldots,w_n$. Recall that $\mathbb{L}^n V$ is generated by Lie bracket expressions $\big[[v_1,v_{i_1}],v_{i_2}],\ldots v_{i_{n-1}}\big]$. From the definition of the configuration pairing, it follows that the long graph $g = w_{j_1}|\cdots|w_{j_m}|w_1|w_{j_{m+1}}|\cdots|w_{j_{n-1}}$ pairs nontrivially with a generating Lie bracket expression if and only if $(i_1, i_2,\ldots,i_{n-1})$ is a shuffle of $(j_m,\ldots,j_1)$ into $(j_{m+1},\ldots,j_{n-1})$ and in this case pairs to $(-1)^m$. We may thus express g in terms of the dual generating long graphs $w_1|w_{i_1}|\cdots|w_{i_{n-1}}$ as

$$(5) \qquad w_{j_1}|\cdots|w_{j_m}|w_1|w_{j_{m+1}}|\cdots|w_{j_{n-1}} = (-1)^m \sum_{\substack{\sigma \ \text{a shuffle of} \\ (j_m,\ldots,j_1) \ \text{into} \ (j_{m+1},\ldots,j_{n-1})}} \big(w_1|w_{\sigma(1)}|\cdots|w_{\sigma(n-1)}\big).$$

This is a complete set of relations since it expresses every long graph in terms of generating elements. Relettering so that $w_{j_1}|\cdots|w_{j_m}|w_1|w_{j_{m+1}}|\cdots|w_{j_{n-1}}$ becomes $w_1|\cdots|w_n$ we have the desired relations. $\square$

Equation 5 may be of independent interest, since it gives rise to a canonical vector space basis for cofree Lie coalgebras. To our knowledge, bases of free Lie algebras involve making choices.

4. Bar constructions to and from the category of graph coalgebras

Throughout this section we use $\mathbb{C}V$ to mean the cofree graded-cocommutative coalgebra on a vector space V. If V is reduced then $\mathbb{C}V$ is given by the symmetric invariants of the cotensor coalgebra $T^c V$ on V (where the symmetric group acts with Koszul signs). Working rationally (with V finitely generated), the norm map gives a vector space isomorphism with $\mathbb{A}V$, the free graded-commutative algebra generated by V, which is given by the symmetric coinvariants of the tensor algebra TV on V. Elsewhere in the literature this is sometimes called ΛV or SV. Our notation is inspired by the standard notation of $\mathbb{L}V$ for the free Lie algebra on V as well as our mirroring notation $\mathbb{G}W$ for the cofree graph coalgebra on W.

Note that $\mathbb{C}^0 V = \langle 1 \rangle = \mathbb{A}^0 V$, while $\mathbb{L}^0 V = 0 = \mathbb{E}^0 V$. In various instances we will take augmentation ideals of algebras (denoted $\bar{A}$) or coaugmentation coideals of coalgebras (denoted $\bar{C}$).

4.1. **The Quillen functors $\mathcal{L}$ and $\mathcal{C}$.** Recall the standard definition of the Quillen adjoint pair of functors $\mathcal{L} : \text{DGCC} \rightleftarrows \text{DGLA} : \mathcal{C}$. The functor $\mathcal{L}$ can be viewed as the cobar construction followed by taking Hopf algebra primitives; $\mathcal{C}$ can be viewed as the bar construction on the universal enveloping algebra of a Lie algebra. Topologically these are identifying the rational homotopy of a space inside the cohomology of its loopspace via the Milnor-Moore theorem. In explicit algebra, given a differential graded-cocommutative coalgebra (C, Δ_C, d_C), the functor $\mathcal{L}$ produces the free graded Lie algebra on $s^{-1}\bar{C}$ with a differential consisting of the free extension of the differential d_C plus a "twisting differential" freely induced by Δ_C. Explicitly, we have the following.

Definition 4.1. Let $\mathcal{L} : \text{DGCC} \to \text{DGLA}$ be the total complex of the bicomplex

$$\mathcal{L}(C, \Delta_C, d_C) = \big(\mathbb{L}(s^{-1}\bar{C}), d_{\mathbb{L}s^{-1}C}, d_\Delta\big),$$

where $d_{\mathbb{L}s^{-1}C}$ is the differential inherited from the differential d_C on C; and d_Δ is the free extension of the map given on the generators of $\mathbb{L}(s^{-1}\bar{C})$ by

$$d_\Delta(s^{-1}c) = \tfrac{1}{2}\sum_i (-1)^{|a_i|}[s^{-1}a_i, s^{-1}b_i] \qquad \text{where } \bar{\Delta}_C c = \sum_i a_i \otimes b_i.$$

The functor $\mathcal{C}$ is defined dually – given $(L, [\cdot,\cdot]_L, d_L)$ in DGLA, the functor $\mathcal{C}$ takes this to the cofree graded-cocommutative coalgebra primitively cogenerated by sL with a differential consisting of the cofree extension of the differential d_L plus a "twisting differential" cofreely induced by the bracket $[\cdot,\cdot]_L$.

Definition 4.2. Let $\mathcal{C} : \text{DGLA} \to \text{DGCC}$ be the total complex of the bicomplex

$$\mathcal{C}(L, \ [\cdot,\cdot]_L, \ d_L) = \big(\mathbb{C}(sL), \ d_{\mathbb{C}sL}, \ d_{[\cdot,\cdot]}\big),$$

where $d_{\mathbb{C}sL}$ is the differential inherited from d_L on L; and

$$d_{[\cdot,\cdot]}(sv_1 \cdot sv_2 \cdots sv_n) = \sum_{i<j}(-1)^{n_{ij}+|v_i|}s[v_i, v_j] \cdot sv_1 \cdots \widehat{sv_i} \cdots \widehat{sv_j} \cdots sv_n,$$

where $(-1)^{n_{ij}}$ is the Koszul sign change incurred by moving sv_i, and sv_j to the beginning of this expression.

An alternate way to view $d_{[\cdot,\cdot]}$, in parallel to Definition 4.1, is the following.

Proposition 4.3. *The differential $d_{[\cdot,\cdot]}$ is the cofree extension of the graded vector space map $[\mathbb{C}sL]_{\text{G}} \to s[L]_{\text{G}}$ given on on $\mathbb{C}^{\neq 2}sL$ by the zero map and on $\mathbb{C}^2 sL$ by $(sv_1 \cdot sv_2) \mapsto (-1)^{|v_1|}s[v_1, v_2]$.*

Adjointness of $\mathcal{L}$ and $\mathcal{C}$ follows from that of the bar and bar construction as well as that of the universal enveloping algebra and Lie primitives functors.

Remark 4.4. We will shortly construct functors $\mathcal{E} : \text{DGCA} \leftrightarrows \text{DGLC} : \mathcal{A}$ (dual to $\mathcal{L}$ and $\mathcal{C}$) as quotients of functors $\mathcal{G} : \text{DGCA} \leftrightarrows \text{DGGC} : \hat{\mathcal{A}}$ to and from graph coalgebras.

We may attempt to define a functor $\hat{\mathcal{L}} : \text{DGCC} \to \text{DGTA}$ (where DGTA denotes dg-non-associative binary algebras and $\mathbb{T}$ denotes the free such algebra) by

$$\hat{\mathcal{L}}(C, \ \Delta_C, \ d_C) = \big(\mathbb{T}(s^{-1}\bar{C}), \ d_{\mathbb{T}s^{-1}\bar{C}}, \ d_\Delta\big).$$

Unfortunately, d_Δ is not a differential on the non-associative algebra $\mathbb{T}(s^{-1}\bar{C})$ so $\hat{\mathcal{L}}$ isn't a differential complex. This is a striking difference between the non-associative algebra approach to Lie algebras and the graph coalgebra approach to Lie coalgebras we present in the next section. Indeed, we could presumably replace non-associative algebras by graph algebras in the above construction and get a functor $\hat{\mathcal{L}}$ which mapped to differential graded complexes and generalized both Adams' bar construction and Quillen's $\mathcal{L}$ functor appropriately. We leave that for furture work.

4.2. **The functor $\mathcal{G}$.** To define $\mathcal{G}$, we start with a differential graded (commutative, augmented, unital) algebra $(A, \ \mu_A, \ d_A)$, with augmentation ideal $\bar{A}$. The functor $\mathcal{G}$ produces the cofree graded anti-commutative graph coalgebra on $s^{-1}\bar{A}$ with differential consisting of the cofree extension of the differential d_A along with another part coming from the multiplication μ_A, defined by contracting edges. In order to make this precise, we must carefully define the sign associated to contracting an edge.

Definition 4.5. Let $[g]$ be a homogeneous element of $\mathbb{G}(s^{-1}\bar{A})$, namely an ordered directed graph with n vertices along with a tensor of n elements of $s^{-1}\bar{A}$ modulo the usual Σ_n-action. For every edge e of g we may construct a new ordered labeled graph $\mu_e(g)$ as follows.

Pick a representative of $[g]$ modulo Σ_n in which edge e goes from vertex number 1 to vertex number 2, with the first two entries of the associated tensor being a and b. Contract the edge from 1 to 2 in this representative to a vertex which is then given the number 1 and first entry in the tensor of $(-1)^{|a|}s^{-1}(ab)$. In this operation, the ordering of all other vertices in the graph is shifted down by one to make up for the now missing 2 (associated elements in the tensor remain the same).

$$\mu_e : \qquad \underset{s^{-1}a \otimes s^{-1}b \otimes \cdots}{\overset{\displaystyle{\underset{e \nearrow}{\overset{2}{\underset{1}{\Big|}}}}}{}} \qquad \longmapsto \qquad (-1)^{|a|} \underset{s^{-1}(ab) \otimes \cdots}{\overset{1}{}}$$

Definition 4.6. Let $\mathcal{G} : \text{DGCA} \to \text{DGGC}$ be the total complex of the bicomplex

$$\mathcal{G}(A, \ \mu_A, \ d_A) = \big(\mathbb{G}(s^{-1}\bar{A}), \ d_{\mathbb{G}s^{-1}\bar{A}}, \ d_\mu\big),$$

where $d_{\mathbb{G}s^{-1}\bar{A}}$ takes $d_{s^{-1}\bar{A}} = -s^{-1}d_A$ term-wise in the tensor associated to a graph coalgebra element and $d_\mu([g]) = \sum_e [\mu_e g]$.

This is a bicomplex by the same calculation which shows that Adams' classical bar construction is a bicomplex. Indeed, $\mathcal{G}$ extends Adams' bar construction to the category of graph coalgebras.

Proposition 4.7. *The map d_μ is compatible with the cobracket on $\mathbb{G}(s^{-1}\bar{A})$. That is, $d_\mu(\,]g[\,) = \,]d_\mu g[$.*
Moreover, d_μ is the cofree extension of the graded vector space map $\left[\mathbb{G}s^{-1}\bar{A}\right]_{\mathrm{G}} \to s^{-1}[\bar{A}]_{\mathrm{G}}$ given on
$\mathbb{G}^{\neq 2}s^{-1}\bar{A}$ *by the zero map and on on* $\mathbb{G}^2 s^{-1}\bar{A}$ *by* ${}_1\overset{2}{\nearrow}\bigotimes s^{-1}a \otimes s^{-1}b \;\longmapsto\; (-1)^{|a|}s^{-1}(ab).$

We now construct our Lie coalgebraic bar construction $\mathcal{E}$ as a quotient of our graphical bar constuction $\mathcal{G}$.

Proposition 4.8. *The differential d_μ preserves the vector subspace generated by arrow-reversing and Arnold expressions. Thus the arrow-reversing and Arnold coideal is a subcomplex of $\mathcal{G}$.*

Proof. This proposition follows immediately from the compatibility of d_μ and the cobracket and Proposition 3.7 once we show that d_μ vanishes on arrow-reversing expressions. Using the bar representation for graphs, this is shown by:
$$d_\mu\Big(s^{-1}a|s^{-1}b + (-1)^{(|a|+1)(|b|+1)}s^{-1}b|s^{-1}a\Big) \;=\; (-1)^{|a|}s^{-1}(ab) + (-1)^{|a||b|+|a|+1}s^{-1}(ba) \;=\; 0.$$
$\square$

Definition 4.9. Let $\mathcal{E}(A)$ be $\mathcal{G}(A)$ modulo the arrow-reversing and Arnold subcomplex.

Remark 4.10. In terms of the bar generators, the differentials in the definition of $\mathcal{E}$ coincide with the differentials used to define the usual algebraic (associative) bar construction, but are now defined on the quotient of the bar construction by the relations induced by arrow-reversing and Arnold. By Proposition 3.21, $\mathcal{E}(A)$ is isomorphic to the Harrison complex of the commutative algebra A equipped with the Lie coalgebra structure from [16].

4.3. **The functor $\hat{\mathcal{A}}$.** The functor $\hat{\mathcal{A}}$ is given by Adams' cobar construction applied to a graph coalgebra. Explicitly it takes the differential graded graph coalgebra $(G, \,]\cdot[_G, d_G)$ to the free graded-commutative algebra generated by sG with a differential consisting of the free extension of d_G along with another part coming from the graph cobracket.

Definition 4.11. Let $\hat{\mathcal{A}} : \text{DGGC} \to \text{DGCA}$ be the total complex of the bicomplex
$$\hat{\mathcal{A}}(G, \,]\cdot[_G, d_G) = \big(\mathbb{A}sG, \; d_{\mathbb{A}sG}, \; d_{]\cdot[_G}\big),$$
where $d_{]\cdot[}$ is the free extension of the map given on the generators of $\mathbb{A}sG$ by
$$d_{]\cdot[}(sg) = \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|} sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}, \qquad \text{for } \,]g[= \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}}.$$

Unlike in Remark 4.4, this defines a differential graded complex.

Theorem 4.12. $\hat{\mathcal{A}}(G)$ *is a bicomplex.*

Proof. We already know $d_{\mathbb{A}sG}^2 = 0$. Also, that $d_{]\cdot[}d_{\mathbb{A}sG} = d_{\mathbb{A}sG}d_{]\cdot[} = 0$ follows from anti-cocommutativity of the cobracket.

To show $d_{]\cdot[}^2 = 0$, it is enough to show that $d_{]\cdot[}^2 = 0$ on $sG \subset \mathbb{A}sG$. Furthermore it is enough to show $d_{]\cdot[}^2$ vanishes on graphs with only three vertices, since the general case is then solved by replacing vertices by graphs.
$$d_{]\cdot[}^2\left(s \,{}_1\overset{2}{\nearrow}\!\!\searrow_3 \atop a\otimes b\otimes c\right) = d_{]\cdot[}\left((-1)^{|a|+|b|}s\left({}_1\overset{2}{\nearrow}\atop a\otimes b\right)\cdot sc + (-1)^{|a|}sa \cdot s\left({}_1\overset{2}{\nearrow}\atop b\otimes c\right)\right)$$
$$= (-1)^{|a|+|b|+|a|}sa \cdot sb \cdot sc + (-1)^{|a|+(|a|+1)+|b|}sa \cdot sb \cdot sc \;=\; 0$$

The computations for $\left(\begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \right)$ are similar (though the signs involved are slightly more unpleasant). $\qquad\square$

The following proposition is an immediate consequence of Proposition 3.7.

Proposition 4.13. *Let* Arn *be the arrow-reversing and Arnold vector subspace of G. Then $d_{]\cdot[}\big(s\mathrm{Arn}\big) \subset \big(s\mathrm{Arn}\big) \cdot \big(sG\big)$.*

Note that graded anti-commutativity of the graph cobracket in G corresponds via $d_{]\cdot[}$ to graded commutativity of multiplication in $\mathbb{A}sG$.

$$]g[\;\; = \;\; \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}} \;\; = \;\; \sum_e -(-1)^{|g_1^{\hat{e}}||g_2^{\hat{e}}|} \, g_2^{\hat{e}} \otimes g_1^{\hat{e}}$$

$$d_{]\cdot[}\, sg \;\; = \;\; \tfrac{1}{2} \sum_e (-1)^{|g_1^{\hat{e}}|} \, sg_1^{\hat{e}} \cdot sg_2^{\hat{e}} \;\; = \;\; \tfrac{1}{2} \sum_e -(-1)^{|g_1^{\hat{e}}||g_2^{\hat{e}}|+|g_2^{\hat{e}}|} \, sg_2^{\hat{e}} \cdot sg_1^{\hat{e}}.$$

Corollary 4.14. *$\hat{A}$ descends to a well-defined map $\mathcal{A} : \mathrm{DGLC} \to \mathrm{DGCA}$ by $\mathcal{A}([G]) = \hat{A}(G)$.*

4.4. **Adjointness of $\mathcal{G}$ and $\hat{\mathcal{A}}$.** Let G be a DGGC and A be a DGCA, and use $[-]_{\mathrm{G}}$ to denote the forgetful functor to underlying graded vector spaces and $[-]_{\mathrm{GGC}}$, $[-]_{\mathrm{GCA}}$ to denote forgetting only differentials. It follows from the adjointness properties of $\mathbb{G}$ and $\mathbb{A}$ that the following spaces of homomorphisms are isomorphic:

(6) $\quad \mathrm{Hom}_{\mathrm{GGC}}\big([G]_{\mathrm{GGC}}, \, \mathbb{G}s^{\text{-}1}[\bar{A}]_{\mathrm{G}}\big) \cong \mathrm{Hom}_{\mathrm{G}}\big([G]_{\mathrm{G}}, \, s^{\text{-}1}[\bar{A}]_{\mathrm{G}}\big) \cong \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}}, \, [\bar{A}]_{\mathrm{G}}\big) \cong \mathrm{Hom}_{\mathrm{GCA}}\big(\mathbb{A}s[G]_{\mathrm{G}}, \, [A]_{\mathrm{GCA}}\big).$

This establishes adjointness of $\mathcal{G}$ and $\hat{\mathcal{A}}$ on the level of graded commutative algebras and graded graph coalgebras, forgetting differentials.

To display an adjointness which respects d_μ and $d_{]\cdot[}$, we translate the classical argument showing adjointness of bar and cobar constructions using twisting functions. We include the proof only to underline that the classical proof translates perfectly to this setting without any modification, even though we are now working with the much larger category of graph coalgebras.

Theorem 4.15. *The functors $\mathcal{G}$ and $\hat{\mathcal{A}}$ are an adjoint pair.*

Proof. Given G and A, a DGGC and a DGCA, we will say that a degree -1 map $\tau : [G]_{\mathrm{G}} \to [A]_{\mathrm{G}}$ is a *twisting function* if it satisfies the requirement

$$d_A \, \tau + \tau \, d_E - \tfrac{1}{2}\Big(\mu_A \circ \big(((-1)^{|\cdot|}\tau) \otimes \tau \big) \circ \,]\cdot[_G \Big) = 0.$$

We show that there are bijections between DGCA-maps $\hat{A}G \to A$, DGGC-maps $G \to \mathcal{G}A$, and twisting functions $G \to A$. In terms of Equation 6, we show that if $[f]_{\mathrm{GCA}} \in \mathrm{Hom}_{\mathrm{GCA}}(\mathbb{A}s[G]_{\mathrm{G}}, \, [A]_{\mathrm{GCA}})$ comes from applying the forgetful functor to a map $f \in \mathrm{Hom}_{\mathrm{DGCA}}\big(\hat{A}G, \, A\big)$, then the adjoint map $\tau \in \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}}, \, [\bar{A}]_{\mathrm{G}}\big)$ is in fact a twisting function. Furthermore any $\tau \in \mathrm{Hom}_{\mathrm{G}}\big(s[G]_{\mathrm{G}}, \, [\bar{A}]_{\mathrm{G}}\big)$ which is also a twisting function will be adjoint to a map $f \in \mathrm{Hom}_{\mathrm{GCA}}(\mathbb{A}s[G]_{\mathrm{G}}, \, [A]_{\mathrm{GCA}})$ in the image of the forgetful functor from $\mathrm{Hom}_{\mathrm{DGCA}}(\hat{A}E, \, A)$. This will complete one half of the argument. The half of the argument for homomorphisms $G \to \mathcal{G}A$ is similar.

Let $f : \hat{A}G \to A$ and write $\tau : s[G]_{\mathrm{G}} \to [\bar{A}]_{\mathrm{G}}$ for the adjoint of $[f]_{\mathrm{GCA}}$. Note that $\tau = [f]_{\mathrm{GCA}} \circ i$ where i is the injection map $i : s[G]_{\mathrm{G}} \hookrightarrow \mathbb{A}s[G]_{\mathrm{G}}$. The requirement that $d_A \, f = f \, d_{\hat{A}G}$ ensures that τ gives a twisting

function. Explicitly, let $sg \in s[G]_\mathrm{G}$, then

$$
\begin{aligned}
0 &= d_A\, f\, i(sg) - f\, d_{\hat{A}G}\, i(sg) \\
&= d_A\, f\, i(sg) - f\,(d_{\mathbb{A}sG} + d_{]\cdot[})\, i(sg) \\
&= d_A\, \tau(sg) - f\,(-s\, d_G g) - f\left(\tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}\right) \qquad \text{where} \quad]g[= \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}} \\
&= d_A\, \tau(sg) + \tau(s\, d_G g) - \tfrac{1}{2}\sum_e (-1)^{|a_i|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}).
\end{aligned}
$$

Conversely, let $\tau : s[G]_\mathrm{G} \to [\bar{A}]_\mathrm{G}$ give a twisting function $G \to A$ and let $f : \mathbb{A}s[G]_\mathrm{G} \to A$ be the adjoint of τ given by free extension. To show that $d_A\, f = f\,(d_{\mathbb{A}sG} + d_{]\cdot[})$ it is enough to check on generators $sg \in \mathbb{A}s[G]_\mathrm{G}$. On generators we have

$$
\begin{aligned}
d_A\, f(sg) &= d_A\, \tau(sg) \\
f\, d_{\mathbb{A}sG}(sg) &= -\tau(s\, d_G g) \\
f\, d_{]\cdot[}(sg) &= f\left(\tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|} sg_1^{\hat{e}} \cdot sg_2^{\hat{e}}\right), \qquad \text{where} \quad]g[= \sum_e g_1^{\hat{e}} \otimes g_2^{\hat{e}} \\
&= \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}).
\end{aligned}
$$

However, since τ is a twisting function we know that

$$
d_A\, \tau(sg) + \tau(s\, d_G g) - \tfrac{1}{2}\sum_e (-1)^{|g_1^{\hat{e}}|}\, \tau(sg_1^{\hat{e}}) \cdot \tau(sg_2^{\hat{e}}) = 0.
$$

Substitution yields the desired equality.

We only sketch the bijection between $G \to \mathcal{G}A$ and twisting functions $G \to A$, since it is given similarly. Let $f : G \to \mathcal{G}A$ and write $\tau : [G]_\mathrm{G} \to s^{-1}[\bar{A}]_\mathrm{G}$ for the adjoint of $[f]_\mathrm{GGC} : [G]_\mathrm{GGC} \to \mathbb{G}s^{-1}[\bar{A}]_\mathrm{G}$. Note that $\tau = \pi \circ [f]_\mathrm{GGC}$ where π is the projection map $\pi : \mathbb{G}s^{-1}[\bar{A}] \twoheadrightarrow s^{-1}[\bar{A}]$. By direct computation, the requirement that $\pi\, f\, d_G = \pi\, d_{\mathcal{G}A}\, f$ is equivalent to the condition that τ is a twisting function. $\qquad\square$

The adjointness of our duals of Quillen's functors $\mathcal{L}$ and $\mathcal{C}$ now follows.

Corollary 4.16. *The functors $\mathcal{E}$ and $\mathcal{A}$ are an adjoint pair.*

Finally, we summarize our results as follows.

Theorem 4.17. *The functor $\mathcal{E} : \mathrm{DGCA} \to \mathrm{DGLC}$ factors through the category of differential graded anti-commutative graph coalgebras.*

4.5. **Pairings of Quillen functors.** Our graphical approach to the Lie coalgebraic bar construction not only gives rise to the factorization of the previous section, but allows us to explicitly understand canonical linear dualities of Lie algebraic and coalgebraic Quillen functors.

Theorem 4.18. *The diagram*

(7)
$$
\begin{array}{ccc}
\mathrm{DGCC} & \overset{\mathcal{L}}{\underset{\mathcal{C}}{\rightleftarrows}} & \mathrm{DGLA} \\
{\scriptstyle *}\Big\uparrow\Big\downarrow{\scriptstyle *} & & {\scriptstyle *}\Big\uparrow\Big\downarrow{\scriptstyle *} \\
\mathrm{DGCA} & \underset{\mathcal{E}}{\overset{\mathcal{A}}{\rightleftarrows}} & \mathrm{DGLC}
\end{array}
$$

displays a duality of adjoint pairs of functors. In particular, the square sub-diagrams obtained by starting at any corner and mapping to the opposite are commutative up to canonical isomorphism. In particular, if C is a differential graded-cocommutative coalgebra which is linearly dual to a differential graded-commutative algebra A, then $\mathcal{E}(A)$ is linearly dual to $\mathcal{L}(C)$ through the configuration pairing.

This result refines the work of Schlessinger-Stasheff by identifying the configuration pairing as giving rise to the canonical duality between the Lie algebraic and coalgebraic bar constructions.

Proof. We treat separately the commutativity of the squares which constitute the theorem. The first two are restated as follows.

If L is a differential graded Lie algebra which is linearly dual to a differential graded Lie coalgebra E, then $\mathcal{C}(L)$ is linearly dual to $\mathcal{A}(E)$.

Write $L = (L_\bullet, d_L, [\cdot, \cdot])$ and $E = (E^\bullet, d_E, \ ]\cdot[)$. By definition, we need to establish the duality of the bicomplexes $\mathcal{C}(L) = \big(\mathbb{C}sL, \ d_{\mathbb{C}sL}, \ d_{[\cdot,\cdot]}\big)$ and $\mathcal{A}(E) = \big(\mathbb{A}sE, \ d_{\mathbb{A}sE}, \ d_{]\cdot[}\big)$. Using standard multiplication/comultiplication duality the duality between $L_\bullet$ and $E^\bullet$ induces an algebra/coalgebra duality between $\mathbb{C}sL$ and $\mathbb{A}sE$. Furthermore, since d_L and d_E are linearly dual, their cofree/free extensions $d_{\mathbb{C}sL}$ and $d_{\mathbb{A}sE}$ will be as well. It remains to show that the maps $d_{[\cdot,\cdot]}$ and $d_{]\cdot[}$ are dual. However, these are also cofree/free extensions, namely of the maps

$$\mathbb{C}sL \longrightarrow sL \quad \text{by} \quad sa \cdot sb \longmapsto (-1)^{|a|}s[a, b]$$

$$sE \longrightarrow \mathbb{A}sE \quad \text{by} \quad s\gamma \longmapsto \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat{e}}|} s\gamma_1^{\hat{e}} \cdot s\gamma_2^{\hat{e}} \qquad \text{where} \quad]\gamma[\, = \sum_e \gamma_1^{\hat{e}} \otimes \gamma_2^{\hat{e}}.$$

We verify the duality of these restrictions explicitly, using compatibility of pairings with our assorted multiplications and comultiplications.

$$\big\langle s\gamma, \ (-1)^{|a|}s[a, b]\big\rangle = \big\langle \gamma, \ (-1)^{|a|}[a, b]\big\rangle$$

$$= \big\langle]\gamma[, \ (-1)^{|a|}a \otimes b\big\rangle$$

$$= (-1)^{|a|}\sum_e \big\langle \gamma_1^{\hat{e}}, \ a\big\rangle \big\langle \gamma_2^{\hat{e}}, \ b\big\rangle$$

$$\Big\langle \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat{e}}|} s\gamma_1^{\hat{e}} \cdot s\gamma_2^{\hat{e}}, \ sa \cdot sb\Big\rangle = \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat{e}}|}\big\langle s\gamma_1^{\hat{e}} \otimes s\gamma_2^{\hat{e}}, \ \Delta(sa \cdot sb)\big\rangle$$

$$= \tfrac{1}{2}\sum_e (-1)^{|\gamma_1^{\hat{e}}|}\Big(\big\langle s\gamma_1^{\hat{e}}, \ sa\big\rangle \big\langle s\gamma_2^{\hat{e}}, \ sb\big\rangle$$

$$+ (-1)^{(|a|+1)(|b|+1)}\big\langle s\gamma_1^{\hat{e}}, \ sb\big\rangle \big\langle s\gamma_2^{\hat{e}}, \ sa\big\rangle\Big)$$

$$= \sum_e (-1)^{|\gamma_1^{\hat{e}}|}\big\langle s\gamma_1^{\hat{e}}, \ sa\big\rangle \big\langle s\gamma_2^{\hat{e}}, \ sb\big\rangle$$

The equality of the last two lines above uses anti-cocommutativity of the cobracket $]\gamma[$ as well as the fact that, for the pairings to be nonzero, the degrees of $\gamma_1^{\hat{e}}$ and a must match, as must the degrees of $\gamma_2^{\hat{e}}$ and b.

Since each of the above pairings are 0 unless $|\gamma_1^{\hat{e}}| = |a|$, we have equality, establishing the first half of the theorem.

The proof of the second half of the theorem proceeds in the same manner as that of the first half. Briefly, if we write $A = (A^\bullet, d_A, \mu)$ and $C = (C_\bullet, d_C, \Delta)$, then the duality of the bicomplexes defining $\mathcal{E}(A)$ and $\mathcal{L}(C)$ is immediate, given by the configuration pairing as stated, except for that of the differentials d_μ and d_Δ. But d_μ and d_Δ are also cofree/free extensions, namely of the maps

$$\mathbb{E}s^{-1}\bar{A} \longrightarrow s^{-1}\bar{A} \quad \text{by} \quad \overset{2}{\underset{1}{\bigwedge}}\otimes s^{-1}a \otimes s^{-1}b \longmapsto (-1)^{|a|}s^{-1}(ab)$$

$$s^{-1}\bar{C} \longrightarrow \mathbb{L}s^{-1}\bar{C} \quad \text{by} \quad s^{-1}\gamma \longmapsto \sum_i (-1)^{|\alpha_i|}[s^{-1}\alpha_i, \ s^{-1}\beta_i] \qquad \text{where} \quad \bar{\Delta}\gamma = \sum_i \alpha_i \otimes \beta_i$$

The duality of these restrictions follows from direct calculation, as before. $\qquad\qquad\square$

Note that the statements given in the previous proof do not require our underlying finiteness hypotheses. If we start with a linearly dual pair of an algebra and coalgebra, the functors $\mathcal{L}$ and $\mathcal{E}$ will produce a linearly dual Lie algebra and coalgebra. The finite generation hypotheses only ensure that our vertical linear duality maps are isomorphisms.

Appendix A. Application to computing rational homotopy groups

We will now collect a number of facts and constructions that were either in the literature (Schlessinger-Stasheff, Bausfield-Gugenheim) or were "in the air" during the formative years of rational homotopy theory. We are starting to see that a significant pay-off will be obtained when moving to the non-simply-connected case, where our graph coalgebra approach can give rise to additional understanding of fundamental groups themselves, rather than having the fundamental group act on a (minimal) model. Such results will be the focus of future work. For the sake of reference, we collect first results in the simply-connected setting here.

We discovered the functor $\mathcal{E}$ in the process of defining functionals on homotopy groups, which in the literature are referred to as homotopy periods. Combining our results with the standard translation from spaces to differential graded algebras shows that this formalism is a perfect setting for homotopy periods. Let $A_*(X)$ be a DGCC model for the rational space X, most often given by the PL chains functor [4]. By Quillen's theorem, we know that $H_*(\mathcal{L}(A_*(X)))$ is isomorphic to $\pi_*(X) \otimes \mathbb{Q}$. Let $A^*(X)$ be the linear dual to $A_*(X)$, in other words the PL cochains functor, and let $\pi_\mathbb{Q}^*(X) = \mathrm{Hom}(\pi_*(X), \mathbb{Q})$.

Corollary A.1. *The homology of $\mathcal{E}(A^*(X))$ is isomorphic to $\pi_\mathbb{Q}^*(X)$.*

The standard way to recover homotopy data from cochains to this point has been essentially to replace $A^*(X)$ with a quasi-isomorphic $\mathcal{A}(E)$ for some Lie coalgebra E, from which it follows by Quillen's theorem that $\pi^*(X) \cong E$ (see also Corollary C.2 below). Our approach has a number of properties which will be useful in some settings.

In the sequel to this paper [19], we develop geometry underlying Corollary A.1, defining homotopy periods for any cycles in $\mathcal{E}(A_{PL}^*(X))$. This geometry unifies and generalizes approaches of Hopf, Whitehead, Boardman-Steer, Sullivan, Novikov, Chen and Hain, and can yield $\mathbb{Z}$ and $\mathbb{Z}/p$-valued homotopy periods.

Finally, we may employ the spectral sequence of a bicomplex, which yields the following.

Corollary A.2. *If X is a finite complex, there is a spectral sequence converging to $\pi^*(X)$ with E^1 given by $\mathcal{E}(H^*(X))$. This spectral sequence collapses at E^2 if X is formal.*

After Corollary C.2 we show that this spectral sequence is isomorphic to one constructed by Halperin and Stasheff [8] using deformations of minimal models.

Appendix B. Model structures

We now note that the adjointness results of Section 4.4 preserve model structures, so that $\mathcal{E}$, $\mathcal{A}$ and also $\mathcal{G}$, $\hat{\mathcal{A}}$ form Quillen adjoint pairs. Because we are in the finitely generated setting, we get only model structures, not closed model structures.

All categories in this section are reduced appropriately.

Theorem B.1 (Quillen [15])**.** *A model category structure on DGLA is given by the following:*
- *Weak equivalences are the quasi-isomorphisms.*
- *Fibrations are the level-wise surjections above the bottom degree.*
- *Cofibrations are determined by left lifting; they are the free GLA-maps.*

A model category structure on DGCC is given by the following:
- *Weak equivalences are the quasi-isomorphisms.*
- *Cofibrations are the levelwise injections.*
- *Fibrations are determined by right lifting.*

Recall that $\otimes$ gives finite products in GCC, since our coalgebras are counital, coaugmented. Note that all DGCC's are cofibrant and all DGLA's are fibrant.

Remark B.2. By the results of Quillen [15], these give model category structures even with the finiteness assumptions removed. Though Quillen did not show that these model categories are closed when finiteness hypotheses are removed, in particular that infinite limits exist in the coalgebra setting, there are now a number of proofs in the literature.

In the course of developing algebraic models for rational homotopy theory, Quillen established the following (see [15, Thm 5.3]).

Theorem B.3 (Quillen). *The functors $\mathcal{L}$: DGCC $\leftrightarrows$ DGLA : $\mathcal{C}$ are a Quillen adjoint pair. That is, $\mathcal{L}$ preserves cofibrations and trivial cofibrations; $\mathcal{C}$ preserves fibrations and trivial fibrations.*

Furthermore, $\mathcal{L}$ and $\mathcal{C}$ give a Quillen equivalence. That is, if C is a cofibrant DGCC and L is a fibrant DGLA, then a map $\mathcal{L}(C) \to L$ is a weak equivalence if and only if the adjoint map $C \to \mathcal{C}(L)$ is a weak equivalence.

We now give parallels to these results in our algebra–Lie coalgebra setting. In the following, we continue to restrict to finitely generated, reduced objects.

Definition B.4. We will say that a DGCA-map $f : A \to B$ is a free GCA-map if as a GCA-map, it is an inclusion of a graded algebra with free cokernel, as displayed in the diagram:

$$
\begin{array}{ccc}
[A]_{\mathrm{GLA}} & \xrightarrow{\;[f]_{\mathrm{GLA}}\;} & [B]_{\mathrm{GLA}} \\
& \searrow & \;\;\|\wr \\
& & [A]_{\mathrm{GLA}} \otimes \mathbb{A}W.
\end{array}
$$

In GCA, $\otimes$ is the categorical coproduct, since our algebras are unital.

We will say that a DGGC-map $f : D \to E$ is a cofree GGC-map if as a GGC-map, it is a projection of graded coalgebras with cofree kernel, as displayed in the diagram:

$$
\begin{array}{ccc}
[D]_{\mathrm{GGC}} & \xrightarrow{\;[f]_{\mathrm{GGC}}\;} & [E]_{\mathrm{GGC}} \\
\|\wr & \nearrow & \\
[E]_{\mathrm{GGC}} \circledast \mathbb{G}W. & &
\end{array}
$$

By $\circledast$ we mean the "cofree product" – the categorical product of graph coalgebras – given by the categorical equalizer of the pair of maps

$$
G \circledast K := \mathrm{Eq}\Big(\mathbb{G}(G \oplus K) \rightrightarrows \mathbb{G}(\mathbb{G}G \oplus \mathbb{G}K) \Big)
$$

coming from $\mathbb{G}$ being a cotriple and from G, K being graph coalgebras.

Theorem B.5. *A model category structure on DGCA is given by the following:*
- *Weak equivalences are the quasi-isomorphisms.*
- *Fibrations are the levelwise surjections.*
- *Cofibrations are determined by left lifting; they are the free GCA maps.*

A model category structure on DGGC is given by the following:
- *Weak equivalences are the quasi-isomorphisms.*
- *Cofibrations are injections above degree one.*
- *Fibrations are determined by right lifting; they are the cofree GGC maps.*
    A model category structure on DGLC is given similarly.

While it is possible to merely mimic the original proof of Quillen from [15], we may instead infer this from the literature on model categories.

Proof Sketch. The stated model category structure on DGCA is standard in the literature – it is given by lifting the projective model structure on (reduced) cochains. See [9] and [17, 4.1].

To see that the cofibrations are indeed the free maps may be done in the same way as Quillen shows the corresponding fact in DGLA (see [15, Prop 5.5, p256]) by attatching cells using pushouts of cofibrations. In this manner one may show that all cofibrations are retracts of free maps. However, subalgebras of free algebras are again free; so such maps must themselves be free.

The listed model category structure on DGGC is implied by general operad theory work of [1, Thm 3.2.3]. That fibrations are indeed the cofree maps follows in the finitely generated case from the dual of the corresponding statement about cofibrations in DGLA.   □

Remark B.6. As in the DGCC and DGLA settings, the structures given in Theorem B.5 (minus the description of fibrations in DGGC) give closed model category structures when finiteness assumptions are removed. There is a discrepancy between this situation and that of [1], which defines cooperads using direct sums and orbits instead of products and fixed points.

Lemma B.7. *The model structures of* DGLA *and* DGLC *and of* DGCA *and* DGCC *given in Theorem B.1 and Theorem B.5 are linearly dual. That is, each vertical linear duality isomorphism sends fibrations to cofibrations, cofibrations to fibrations, and weak equivalences to weak equivalences.*

While we have generally chosen to give self-contained arguments, for showing that $\mathcal{E}$ and $\mathcal{A}$ give a Quillen equivalence we stray from this choice for the sake of brevity. We may deduce the following result from Lemma B.7, our main Theorem 4.18, and Quillen's Theorem as stated in Theorem B.3.

Theorem B.8. *The functors $\mathcal{G}$ and $\hat{\mathcal{A}}$ are a Quillen adjoint pair.*
The functors $\mathcal{E}$ and $\mathcal{A}$ are a Quillen adjoint pair. Further, $\mathcal{E}$ and $\mathcal{A}$ are a Quillen equivalence.

Appendix C. Minimal models

We end with some brief notes about minimal models, originally due to Sullivan [22, 4]. In our language, a minimal model in DGCA is an object of the form $(\mathbb{A}W, d)$ where $dW \subset \mathbb{A}^{\geq 2}W$. Sullivan's theorem [22] is that every DGCA supports a quasi-isomorphism from a minimal model $(\mathbb{A}W, d) \xrightarrow{\cong} A$, and furthermore the minimal model $(\mathbb{A}W, d)$ is unique up to isomorphism. Minimal models in DGCA are useful because the Postnikov tower of a rational one-reduced space is encoded transparently in its minimal model as the increasing filtration by free sub-algebras.

Baues and Lemaire [2] note that the property satisfied by the differential of a minimal model may be more concisely stated as $(-)^{\mathrm{ind}} \circ d = 0$. Further, they show that making the analogous definition in DGLA also agrees with the naive definition, namely $(\mathbb{L}V, d)$ with $dV \subset \mathbb{L}^{\geq 2}V$. These minimal models have existence and uniqueness properties similar to those of Sullivan's minimal models in DGCA, but because of the switch from cochains to chains their construction is more difficult – see [2]. From the point of view of topology, minimal models in DGLA encode the Eckmann-Hilton homology decomposition of a rational space.

One lemma in the proof of the uniqueness of minimal models of algebras is interesting in its own right. We say that a DGCA is a "differential free graded algebra" if it has the form $(\mathbb{A}V, d)$, and similarly for a "differential free graded Lie algebra". Then we have the following [2, Prop 1.5].

Proposition C.1 (Sullivan, Baues-Lemaire)*. A map f of differential free graded (Lie) algebras is a quasi-isomorphism if and only if the induced DG-map $(f)^{ind}$ on indecomposables is a quasi-isomorphism.*

We apply this proposition to the units of the adjunctions $\mathcal{A}\mathcal{E} \to \mathbb{1}_{\mathrm{DGCA}}$ and $\mathcal{L}\mathcal{C} \to \mathbb{1}_{\mathrm{DGLA}}$.

Corollary C.2. *If A is a differential free graded algebra, then $[\mathcal{E}A]_{\mathrm{DG}} \simeq s(A)^{ind}$. Similarly, if L is a differential free graded Lie algebra, then $[\mathcal{C}L]_{\mathrm{DG}} \simeq s(L)^{ind}$.*

In particular if A is a DGCA minimal model, then $H^*\mathcal{E}A \cong s(A)^{ind}$ as a graded vector space. Similarly, if L is a DGLA minimal model then $H_*\mathcal{C}L \cong s(L)^{ind}$.

We use this corollary to recover the Halperin-Stasheff spectral sequence for calculating the linear dual of homotopy groups of a finite complex, as described in 4.14 of [8], from our Corollary A.2. The main construction of [8] is that of a filtered model for (A, d_A) as a deformation of a minimal model for $(H_*(A), 0)$, which in our notation would be called $(\mathbb{A}Z, D)$ and $(\mathbb{A}Z, d)$ respectively. When $A = A^*(X)$, the results of Section 8 of [22] imply that $H_*(Z, D) \cong \pi^*(X)$. Because D and d differ by terms of lower filtration, there is a spectral sequence starting with $H_*(Z, d)$ and converging to $H_*(Z, D) \cong \pi^*(X)$.

By Corollary C.2, we have $H_*(Z, d) \cong H_*(\mathcal{E}(H^*(X)))$, so this spectral sequence has the same E^2 term as that of Corollary A.2. Indeed, we may relate these two spectral sequences by comparing them both to equivalent spectral sequences for $\mathcal{E}(\mathbb{A}Z, D)$, which on one hand is quasi-isomorphic to $\mathcal{E}(A, d_A)$ simply because $\mathcal{E}$ is quasi-isomorphism invariant; and on the other hand is quasi-isomorphic to (Z, D) by Corollary C.2. Our approach through $\mathcal{E}(A, d_A)$ seems to have better functorality properties, a more transparent cobracket structure, and greater flexibility in addition to the conjectured relationship with Hopf invariants.

Natural notions of minimal models in coalgebras are obtained by duality. Explicitly we require them to be cofree with differentials satisfying $d \circ (-)^{\mathrm{pr}} = 0$.

Definition C.3. A minimal model in DGCC is a coalgebra of the form $(\mathbb{C}V, d)$ where $dV = 0$.
A minimal model in DGLC is a coalgebra of the form $(\mathbb{E}W, d)$ where $dW = 0$.

We may speak of "differential cofree graded (Lie) coalgebras" similarly to obtain duals to Proposition C.1 and Corollary C.2.

Proposition C.4. *A map f of differential cofree graded (Lie) coalgebras is a quasi-isomorphism if and only if the induced DG-map $(f)^{pr}$ on primitives is a quasi-isomorphism.*

Corollary C.5. *If C is a differential cofree graded coalgebra, then $[\mathcal{L}C]_{\mathrm{DG}} \simeq s^{\text{-}1}(C)^{pr}$. Similarly, if E is a differential cofree graded $\mathcal{E}il$ coalgebra, then $[\mathcal{A}E]_{\mathrm{DG}} \simeq s^{\text{-}1}(E)^{pr}$.*

In particular if C is a DGCC minimal model, then $H_*\mathcal{L}C \cong s^{\text{-}1}(C)^{pr}$ as a graded vector space. Similarly, if E is a DGLC minimal model, then $H^*\mathcal{A}E \cong s^{\text{-}1}(E)^{pr}$.

Minimal models in all cases are unique up to isomorphism for each object, an Bousfield-Gugenheim even give a functorial construction of them [3]. Minimal models of algebras are cofibrant replacements, and minimal models of coalgebras are fibrant replacements. There are other standard functorial fibrant and cofibrant replacements, namely in each setting by applying the appropriate pair of adjoint horizontal arrows from the diagram of Theorem 4.18. These generally differ from minimal models, and as indicated by our discussion of the Halperin-Stasheff spectral sequence the interplay between the two approaches can be enlightening.

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