Locally nilpotent derivations of affine domains

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Abstract

Any locally nilpotent derivation of an affine domain is equivalent to a restriction of a Jacobian type derivation of a polynomial ring.

Introduction.

Let us recall that a locally nilpotent derivation is a generalization of a partial derivative of a polynomial ring. It is a derivation of a ring which when applied sufficiently many times to a given element of a ring sends it to the zero.

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Locally nilpotent derivations (let us call them lnd) are useful though rather elusive objects. Though on "majority" of rings we do not have them at all, when we have them it is rather hard to find them and even harder to find all of them or to give some qualitative statements. Even for polynomial rings we do not know much. Of course it is rather easy to understand the situation for a polynomial ring with one variable. For two variable case it is not easy and requires a considerable effort (see [Re] or [ML2]). For three variables the kernel of a non-zero lnd is a polynomial ring in two variables ([Mi]) and there is a description of all weighted homogeneous derivations, i. e. derivations which send forms which are homogeneous relative to a given set of variables supplied with given positive weights to homogeneous forms ([Fr1], [Da2], [DR]). There are examples of lnd on polynomial rings with more than four variables with non-finitely generated kernels ([Ro], [Fr2], [DF1]) and though it is not known whether such derivations exist in the case of four generators, it is know that here there is no uniform bound on the number of generators of the kernel of an lnd ([DF2]). There are also several classes of surfaces for which we know everything as far as lnds are concerned ([ML2], [ML3], [ML4], [Da3], [Wi]) and a class of threefolds for which we know a lot ([ML1], [KML]), and that is all.

The purpose of this paper is to give a standard form for an lnd on the affine domains. This form is somewhat analogous to a matrix representation of a linear operator. Hopefully this representation of lnds will be helpful in the future research.

Definitions and claim.

Let $\mathbb{C}$ be the field of complex numbers and let A be a domain over $\mathbb{C}$. A $\mathbb{C}$-linear mapping $\partial : A \to A$ is called a derivation if it distributes a product of elements of A according to the Leibnitz law: $\partial(ab) = \partial(a)b + a\partial(b)$.

Constants $A^\partial = \{a \in A | \partial(a) = 0\}$ of ∂ form a subalgebra of A.

A derivation is locally nilpotent if for any $a \in A$ there exists a natural $n = n(a)$ for which $\partial^n(a) = 0$. Let us denote by $\mathrm{LND}(A)$ the set of all lnd of A.

Let $\mathbb{C}_n = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring with n generators. A class of derivations on $\mathbb{C}_n$ can be obtained as follows. Take $f_1, \dots, f_{n-1} \in \mathbb{C}_n$. Define $\partial(g) = \mathrm{J}(f_1, \dots, f_{n-1}, g)$ where J stands for the Jacobian of its arguments, that is for the determinant of the corresponding Jacobi matrix. Sometimes these derivations are called Jacobian derivations. A Jacobian derivation is also a derivation in every parameter f_i, which helps to make computations.

Now we are ready to formulate the main result.

Theorem. Let I be a prime ideal of $\mathbb{C}_n$, R be the factor ring $\mathbb{C}_n/I$, and $\pi : \mathbb{C}_n \to R$ be the corresponding projection. Let $\partial \in \mathrm{LND}(R)$. Then there exist elements $f_1, \dots, f_{n-1} \in \mathbb{C}_n$ and non-zero elements $a, b \in R^\partial$ for which $a\partial(\pi(f)) = b\pi(\mathrm{J}(f_1, \dots, f_{n-1}, f))$ for any $f \in C_n$.

The proof consists of eight lemmas. To make it less boring for the reader, here is an outline of what is going to happen.

Let us assume that the transcendence degree of R is $n - k$.

In the first five Lemmas we will see that it is possible to find k elements $i_1, \ldots, i_k \in I$ so that $\mathrm{J}(i_1, \ldots, i_k, f_{k+1}, \ldots, f_n) \in I$ if and only if the elements $f_{k+1}, \ldots, f_n$ are algebraically dependent over I (this is probably obvious to a geometer). Only in the last three Lemmas the derivation ∂ appears.

Exposition is made as elementary as possible to make this technique as popular as possible, possibly a misguided approach!

First five Lemmas.

Let I be an ideal of $\mathbb{C}_n$. Choose a non-zero element $p \in I$ of the minimal possible total degree. We are not concerned, of course, with the case $p \in \mathbb{C}$. Up to renumbering of generators $x_1, \ldots, x_n$ we may assume that p contains x_1. If we consider p as a polynomial in x_1 with coefficients in $\mathbb{C}[x_2, \ldots, x_n]$ then the coefficients of p with positive degrees of x_1 do not belong to I since their degrees are too small. By looking at all elements of I as polynomials of x_1 we can find an element $p_1 \in I$ satisfying the following two conditions:

The coefficient q_1 of the highest degree of x_1 of p_1 is not in I.

The degree d of p_1 relative to x_1 is the smallest degree possible provided the first condition is satisfied.

The degree d is positive because otherwise $p_1 \in I$ is the coefficient of x_1^0 and the first condition is not satisfied.

Let $I_1 = I \cap \mathbb{C}[x_2, \ldots, x_n]$. Find now $p_2 \in I_1$ with the same property relative to x_2 (again up to a possible renumbering) and so on until we reach k for which $I_k = 0$. In the process we will obtain polynomials $p_1, p_2, \ldots, p_k$.

Lemma 1. Any elements $f_k, \dots, f_n \in \mathbb{C}_n$ are algebraically dependent over I.

Proof. We may assume that $f_k, \dots, f_n$ are algebraically independent since otherwise the statement is clear. Let us consider the linear subspace V_N of $\mathbb{C}_n$ of all polynomials of $f_k, \dots, f_n$ with total degree relative to $f_k, \dots, f_n$ not exceeding N. The linear dimension of V_N over $\mathbb{C}$ is at least $(\frac{N}{n})^{n-k+1}$.

Denote by deg the total degree of elements of $\mathbb{C}_n$ relative to $x_1, \dots, x_n$ and by $\deg_j$ the degree relative to x_j. If $D = \max(\deg(f_k), \dots, \deg(f_n))$ then $\deg(f) \leq DN$ for any $f \in V_N$. Also let $e_i = \deg(p_i)$.

Take $p = ax_1^m + \dots$ with $\deg(p) \leq DN$. Recall $p_1 = q_1 x_1^d + \dots$. (Here $\dots$ are the terms of the smaller degree in x_1.)

Let $r_{1,1} = q_1 p - a x_1^{m-d} p_1$ if $m \geq d$. Then $\deg_1(r_{1,1}) < \deg_1(p)$. If $m < d$ take $r_{1,1} = q_1 p$. Since $\deg(a x_1^{m-d}) \leq \deg(p)$ and $\deg(q_1) < \deg(p_1)$ in both cases $\deg(r_{1,1}) \leq DN + e_1$. Since $DN \geq \deg_1(p)$ after DN steps like this we obtain r_1 with $\deg_1(r_1) < \deg_1(p_1)$ and $\deg(r_1) \leq DN(1 + e_1)$.

Now apply the same procedure to r_1 and p_2. After $DN(1 + e_1)$ steps we obtain r_2 with $\deg_2(r_2) < \deg_2(p_2)$ and $\deg(r_2) \leq DN(1 + e_1)(1 + e_2)$. Also $\deg_1(r_2) \leq \deg_1(r_1)$ since $\deg_1(q_2) = \deg_1(p_2) = 0$.

Considering further on the pairs r_2, p_3, etc., we will get r_k with $\deg(r_k) \leq DN(1 + e_1) \dots (1 + e_k)$ and $\deg_i(r_k) < \deg_i(p_i)$.

Take $\mathbf{m} = q_1 q_2^{(1+e_1)} \dots q_k^{(1+e_1)\dots(1+e_{k-1})}$. We observed that $\mathbf{m}^{DN} p = \sum \alpha_i p_i + r_k$ where $\deg_j(r_k) < \deg_j(p_j)$ for $j = 1, \dots, k$ and $\deg(r_k) \leq DN(1 + e_1) \dots (1 + e_k)$ if $\deg(p) \leq DN$. Denote r_k by $r_k(p)$. It is clear that when N is fixed the restriction of $p \to r_k(p)$ on V_N is a linear mapping of V_N into $\mathbb{C}_n$.

Since the degrees of $r_k(p)$ relative to $x_1, \dots, x_k$ are bounded and $\deg(r_k(p)) \leq DN(1+e_1)\dots(1+e_k)$, all remainders belong to a subspace R_N of $\mathbb{C}_n$ with dimension not exceeding cN^{n-k} where c is a constant which does not depend on N. So $\dim(V_N) > \dim(R_N)$ for a sufficiently large N and a linear mapping from V_N to R_N should have a kernel. It implies that there is a non-zero $f \in V_N$ with $r_k(f) = 0$. So $\mathbf{m}^{DN} f = \sum \alpha_i p_i \in I$ and since I is prime and all $q_i \notin I$ we conclude that $f \in I$.

Remark. If $f \in I$ then $r_k(f) = 0$ for any N. Indeed, in this case $r = r_k(f) \in I$. Since $\deg_1(r) < \deg_1(p_1)$, from the definition of p_1 all coefficients of r as a monomial in x_1 must be in I_1. Looking at these coefficients as monomials in x_2 and using $\deg_2(r) < \deg_2(p_2)$ and definition of p_2 we see that they in turn have all coefficients in I_2, etc.. Since $I_k = 0$ we see that $r = 0$.

Lemma 2. $\mathrm{J}(p_1, p_2, \dots, p_k, x_{k+1}, \dots, x_n) \notin I$.

Proof. Indeed, $\mathrm{J}(p_1, p_2, \dots, p_k, x_{k+1}, \dots, x_n) = \frac{\partial p_1}{\partial x_1}\frac{\partial p_2}{\partial x_2}\dots\frac{\partial p_k}{\partial x_k}$ since only p_1 depends on x_1, only p_2 depends on x_2, etc.. But $\frac{\partial p_i}{\partial x_i} \notin I$ for all i by definition of p_i, so $\frac{\partial p_1}{\partial x_1}\frac{\partial p_2}{\partial x_2}\dots\frac{\partial p_k}{\partial x_k} \notin I$.

Lemma 3. $\mathrm{J}(f_1, \dots, f_{k+1}, \dots, f_n) \in I$ if $f_1, \dots, f_{k+1} \in I$.

Proof. By Lemma 1 and Remark to Lemma 1 we can find a monomial $\mathbf{M}$ such that $\mathbf{M} f_i = \sum \alpha_{i,j} p_j$ for $i \leq k+1$. Therefore $\mathrm{J}(\mathbf{M}f_1, \dots, \mathbf{M}f_{k+1}, f_{k+2}, \dots, f_n) =$

$\mathbf{M}^{k+1}\mathrm{J}(f_1,\ldots,f_{k+1},\ldots,f_n) + \Delta = \sum \mathrm{J}(\alpha_{1,j_1}p_{j_1},\ldots,\alpha_{k+1,j_{k+1}}p_{j_{k+1}},\ldots,f_n)$. Each summand $\mathrm{J}(\alpha_{1,j_1}p_{j_1},\ldots,\alpha_{k+1,j_{k+1}}p_{j_{k+1}},\ldots,f_n) \in I$ since there is only k different p_i's. Also $\Delta \in I$ because it is a linear combination of Jacobians with coefficients containing f_i's where $i \leq k+1$.

So $\mathbf{M}^{k+1}\mathrm{J}(f_1,\ldots,f_{k+1},\ldots,f_n) = \sum \mathrm{J}(\alpha_{1,j_1}p_{j_1},\ldots,\alpha_{k+1,j_{k+1}}p_{j_{k+1}},\ldots,f_n) - \Delta \in I$. Since I is prime we see that $\mathrm{J}(f_1,\ldots,f_{k+1},\ldots,f_n) \in I$.

Lemma 4. $\mathrm{J}(f_1,f_2,\ldots,f_k,f_{k+1},\ldots,f_n) \in I$ if $f_1,f_2,\ldots,f_k \in I$ and $f_{k+1},\ldots,f_n$ are dependent over I.

Proof. Let P be a non-zero polynomial of $f_{k+1},\ldots,f_n$ with value in I of minimal total degree possible. We may assume (up to renumbering) that P depends on f_n. So $\mathrm{J}(f_1,f_2,\ldots,f_k,f_{k+1},\ldots,f_{n-1},P) \in I$ by Lemma 3. On the other hand $\mathrm{J}(f_1,f_2,\ldots,f_k,f_{k+1},\ldots,f_{n-1},P) = \frac{\partial P}{\partial f_n}\mathrm{J}(f_1,f_2,\ldots,f_k,f_{k+1},\ldots,f_n)$. Since $\frac{\partial P}{\partial f_n} \notin I$ it implies that $\mathrm{J}(f_1,f_2,\ldots,f_k,f_{k+1},\ldots,f_n) \in I$.

Lemma 5. $\mathrm{J}(p_1,p_2,\ldots,p_k,f_{k+1},\ldots,f_n) \notin I$ if $f_{k+1},\ldots,f_n$ are independent over I.

Proof. Assume that $\mathrm{J}(p_1,p_2,\ldots,p_k,x_{k+1},\ldots,x_{k+i},f_{k+i+1},\ldots,f_n) \notin I$ and that i is minimal possible. From Lemma 2 we know that $i \leq n-k$. Elements $x_{k+1},\ldots,x_{k+i},f_{k+i+1},\ldots,f_n$ are algebraically independent over I by Lemma 4. Let us consider now $x_{k+1},\ldots,x_{k+i},f_{k+i},f_{k+i+1},\ldots,f_n$. By Lemma 1 they are algebraically dependent over I. So let us take a non-zero polynomial P of $x_{k+1},\ldots,x_{k+i},f_{k+i},f_{k+i+1},\ldots,f_n$ with value in I of minimal total degree possible. Its degree relative to f_{k+i} is positive as we noticed above. Since by assumption of the Lemma $f_{k+i}, f_{k+i+1}, \ldots, f_n$ are algebraically independent over I, polynomial P (up to renumbering) depends on x_{k+i}.

From Lemma 3 follows that $I \ni \mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i-1}, P, f_{k+1+i}, \ldots, f_n) = \frac{\partial P}{\partial x_{k+i}} \mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i}, f_{k+i+1}, \ldots, f_n) + \frac{\partial P}{\partial f_{k+i}} \mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i-1}, f_{k+i}, \ldots, f_n)$.

Since $\frac{\partial P}{\partial x_{k+i}} \notin I$, $\frac{\partial P}{\partial f_{k+i}} \notin I$, and $\mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i}, f_{k+i+1}, \ldots, f_n) \notin I$, we see that $\mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i-1}, f_{k+i}, \ldots, f_n) \notin I$.

To avoid a contradiction we must assume that $\mathrm{J}(p_1, p_2, \ldots, p_k, x_{k+1}, \ldots, x_{k+i}, f_{k+i+1}, \ldots, f_n)$ does not contain x_j's at all which proves the Lemma.

Additional definitions and facts about lnd.

Let A be a domain of characteristic zero, $F = \mathrm{Frac}(A)$, and $\partial \in \mathrm{LND}(A)$.

Derivation ∂ can be extended on F by $\partial(ab^{-1}) = \partial(a)b^{-1} - a\partial(b)b^{-2}$. Denote this extension also by ∂.

Field F contains an element s for which $\partial(s) = 1$, s is transcendental over F^{∂}, $A \subset F^{\partial}[s]$, and $F = F^{\partial}(s)$. So every element of A can be looked at as a polynomial in s and we can define $\deg_{\partial}(a)$ for $a \in A$ as the degree of this polynomial. As usual, $deg_{\partial}(0) = -\infty$.

Element $s = \frac{t}{\partial(t)}$ for some $t \in A$ with $\partial(t) \in A^{\partial}$. Also $F^{\partial} = \mathrm{Frac}(A^{\partial})$ and since the transcendence degree of F over F^{∂} is one we have $\mathrm{trdeg}(A^{\partial}) = \mathrm{trdeg}(A) - 1$ if $\mathrm{trdeg}(A) < \infty$.

All these facts can be found in [Es].

Two lnds of A are *equivalent* if they determine the same degree function on A.

Remaining Lemmas.

Recall that $R = \mathbb{C}_n/I$, π is the projection of $\mathbb{C}_n$ on R, and $\partial \in \mathrm{LND}(R)$. So all of the above is applicable to our setting.

We will be looking at elements of R also as polynomials from $F^\partial[s]$ where F is the field of fractions of R, $s \in F$, and $\partial(s) = 1$ and denote by deg the corresponding degree.

Let $i_1, \ldots, i_k \in I$ and $f_{k+1}, \ldots, f_{n-1} \in \mathbb{C}_n$. From Lemma 3 it follows that $\epsilon(\pi(f)) = \pi(\mathrm{J}(i_1, \ldots, i_k, f_{k+1}, \ldots, f_{n-1}, f)$ is a well-defined derivation on R since replacement of f by $f + i$ in the right side does not change the left side if $i \in I$.

Let us chose now some algebraically independent $r_{k+1}, \ldots, r_{n-1} \in R^\partial$. It is possible since $\mathrm{trdeg}(R^\partial) = n - k - 1$. Let $p_{k+1}, \ldots, p_{n-1} \in \mathbb{C}_n$ be any elements for which $\pi(p_i) = r_i$. Denote the derivation $\pi(\mathrm{J}(i_1, \ldots, i_k, p_{k+1}, \ldots, p_{n-1}, f))$ of R by $\epsilon(\mathbf{i}, \mathbf{r})(g)$ where $g = \pi(f)$. Again from Lemma 3 it is clear that $\epsilon(\mathbf{i}, \mathbf{r})(g)$ does not depend on a particular choice of $p_{k+1}, \ldots, p_{n-1}$. As always we will think about these derivations as derivations of F.

From Lemma 5 we know that for some choices of $\mathbf{i}$ these derivations are non-zero.

Lemma 6. Non-zero derivations $\epsilon(\mathbf{i}, \mathbf{r})$ and $\epsilon(\mathbf{i}, \mathbf{s})$ are linearly dependent over F^∂.

Proof. The claim is obvious if $\mathbf{r}$ and $\mathbf{s}$ are the same. Let us assume that $\epsilon(\mathbf{i},\mathbf{r})$ and $\epsilon(\mathbf{i},\mathbf{s})$ do not satisfy the claim and that the number of common elements of $\mathbf{r}$ and $\mathbf{s}$ is maximal possible under this condition. Up to renumbering we may assume that $s_{n-1} \notin \{r_{k+1},\ldots,r_{n-1}\}$. Elements $r_{k+1},\ldots,r_{n-1},s_{n-1}$ are algebraically dependent. Let us take a non-zero polynomial P of $r_{k+1},\ldots,r_{n-1},s_{n-1}$ with value zero of minimal total degree possible. Its degree relative to s_{n-1} is positive. P also must depend on at least one $r_j \notin \{s_{k+1},\ldots,s_{n-1}\}$ since the elements $\{s_{k+1},\ldots,s_{n-1}\}$ are algebraically independent. Again up to renumbering we may assume that it is r_{n-1}.

Let $p_{k+1},\ldots,p_{n-1} \in \mathbb{C}_n$ be co-images of $r_{k+1},\ldots,r_{n-1}$ and let q be a co-image of s_{n-1}. So $P(p_{k+1},\ldots,p_{n-1},q) \in I$ and $I \ni \mathrm{J}(i_1,\ldots,i_k,p_{k+1},\ldots,p_{n-2},P,f) = \frac{\partial P}{\partial p_{n-1}}\mathrm{J}(i_1,\ldots,i_k,p_{k+1},\ldots,p_{n-1},f)+\frac{\partial P}{\partial q}\mathrm{J}(i_1,\ldots,i_k,p_{k+1},\ldots,p_{n-2},q,f)$ for any $f \in \mathbb{C}_n$ by Lemma 3. Therefore $\pi(\frac{\partial P}{\partial p_{n-1}}\mathrm{J}(i_1,\ldots,i_k,p_{k+1},\ldots,p_{n-1},f) + \frac{\partial P}{\partial q}\mathrm{J}(i_1,\ldots,i_k,p_{k+1},\ldots,p_{n-2},q,f)) = 0$. Both $\pi(\frac{\partial P}{\partial p_{n-1}})$ and $\pi(\frac{\partial P}{\partial q})$ are polynomials of ∂-constants $r_{k+1},\ldots,r_{n-1},s_{n-1}$, and so $\pi(\frac{\partial P}{\partial p_{n-1}})$, $\pi(\frac{\partial P}{\partial q}) \in R^\partial \setminus 0$ by the definition of P. So the corresponding derivations are linearly dependent over F^∂, are both non-zero, and are linearly independent with $\epsilon(\mathbf{i},\mathbf{s})$.

Since $\{s_{k+1},\ldots,s_{n-1}\}$ has more common elements with $\{\pi(p_{k+1}),\ldots,\pi(p_{n-2}),\pi(q)\}$ $= \{r_{k+1},\ldots,r_{n-2},s_{n-1}\}$ than with $\{r_{k+1},\ldots,r_{n-1}\}$ we reached a contradiction which proves the Lemma.

Remark. Since $\mathbf{r}$ is a transcendence basis of R^∂, the kernel of $\epsilon(\mathbf{i},\mathbf{r})$ is F^∂ by Lemma 4.

This Lemma shows that the derivations $\epsilon(\mathbf{i},\mathbf{r})$ essentially do not depend on $\mathbf{r}$. So let us fix a basis $\mathbf{r}$ and omit it from the notation for ϵ: $\epsilon(\mathbf{i}, \mathbf{r}) = \epsilon(\mathbf{i})$.

From Lemma 5 we know that $\epsilon(\mathbf{i})$ is a non-zero derivation for some choices of $\mathbf{i}$. Recall $s \in F$ satisfying $\partial(s) = 1$. If $\epsilon(\mathbf{i}) \neq 0$ then $\epsilon(\mathbf{i})(s) \neq 0$ since otherwise the kernel of $\epsilon(\mathbf{i})$ is too large. So there are choices of $i_1, \ldots, i_k \in I$ for which $q = \epsilon(\mathbf{i})(s) \neq 0$ and $\deg(q)$ is minimal possible. Let us fix such a q.

Lemma 7. For any $j_1, \ldots, j_k \in I$ the element $\epsilon(\mathbf{j})(s)$ is divisible by q as an element of $F^\partial[s]$.

Proof. If the claim is wrong choose a pair $\mathbf{i}$, $\mathbf{j}$ so that $q = \epsilon(\mathbf{i})(s)$, $p = \epsilon(\mathbf{j})(s)$ is not divisible by q, and the number d of common elements of $\mathbf{i}$ and $\mathbf{j}$ is maximal possible.

For any $h_1, \ldots, h_k \in I$ denote $\epsilon(\mathbf{h})(s)$ by $\Delta(h_1, \ldots, h_k)$. Since $\Delta(h_1, \ldots, h_k) = \pi(\mathrm{J}(h_1, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s)$ it is skew-symmetric. Also $\mathrm{J}(fh_1, h_2, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s) = h_1\mathrm{J}(f, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s) + f\mathrm{J}(h_1, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s)$ for any $f \in \mathbb{C}_n$. So $\pi(\mathrm{J}(fh_1, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s)) = \pi(f)\pi(\mathrm{J}(h_1, \ldots, h_k, p_{k+1}, \ldots, p_{n-1}, s))$ since $\pi(h_1) = 0$. This can be written with a slight abuse of notations as $\Delta(gh_1, h_2, \ldots, h_k) = g\Delta(h_1, \ldots, h_k)$ where $g = \pi(f)$ because any co-image of g in the left side gives the same right side. Since Δ is skew-symmetric $\Delta(h_1, h_2, \ldots, gh_j, \ldots, h_k) = g\Delta(h_1, \ldots, h_k)$ for any j.

It is easy to see now that $\delta(i) = \Delta(i_1, \ldots, i_{k-1}, i)$ is divisible by q for any $i \in I$. Consider q and $\delta(i)$ as elements of $F^\partial[s]$ and choose $u, v \in F^\partial[s]$ so that $w = uq + v\delta(i)$ is the greatest common divisor of q and $\delta(i)$. If q does not divide $\delta(i)$ then $\deg(w) < \deg(q)$. Since $F^\partial = \mathrm{Frac}(R^\partial)$ and

$s = \frac{t}{\partial(t)}$ where $t \in R$ and $\partial(t) \in R^{\partial}$ we may assume that $u, v \in R$. But $\Delta(i_1, \ldots, i_{k-1}, ui_k + vi) = w$ which leads to a contradiction with the definition of q.

Without loss of generality we may assume that $j_k \notin \{i_1, \ldots, i_k\}$ and that $i_k \notin \{j_1, \ldots, j_k\}$. As we already know $\Delta(i_1, \ldots, i_{k-1}, j_k) = rq$ for some $r \in F^{\partial}[s]$. If $r \notin R$ find $f \in R^{\partial}$ so that $fr \in R$. For $j = fj_k - fri_k$ we have $\Delta(i_1, \ldots, i_{k-1}, j) = 0$. Now, $\Delta(j_1, \ldots, j_{k-1}, j + i_k) = f\Delta(j_1, \ldots, j_k) + (1 - fr)\Delta(j_1, \ldots, j_{k-1}, i_k)$. The element $\Delta(j_1, \ldots, j_{k-1}, i_k)$ is divisible by q since the number of common elements of $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k-1}, i_k\}$ is $d+1$. On the other hand $\Delta(i_1, \ldots, i_{k-1}, j + i_k) = q$ since $\Delta(i_1, \ldots, i_{k-1}, j) = 0$. The sets $\{j_1, \ldots, j_{k-1}, j + i_k\}$ and $\{i_1, \ldots, i_{k-1}, j + i_k\}$ also have $d+1$ common elements, so q divides $\Delta(j_1, \ldots, j_{k-1}, j + i_k)$. Therefore $f\Delta(j_1, \ldots, j_k) = \Delta(j_1, \ldots, j_{k-1}, j + i_k) - (1-r)\Delta(j_1, \ldots, j_{k-1}, i_k)$ is divisible by q. Since $f \in R^{\partial}$ we can conclude that $\Delta(j_1, \ldots, j_k)$ is divisible by q in $F^{\partial}[s]$.

Lemma 8. $q \in F^{\partial} \setminus 0$.

Proof. Let us show that $1 = \pi(\mathrm{J}(x_1, \ldots, x_n))$ is divisible by q in $F^{\partial}[s]$. Write $\pi(x_a) = \sum f_{a,b} s^b$ where $f_{a,b} \in F^{\partial}$.

As we know it is possible to find a $v \in R^{\partial} \setminus 0$ so that $v\pi(x_a) = \sum g_{a,b} t^b$ where $g_{a,b} \in R^{\partial}$ and $t \in R$. We can lift these expressions into $\mathbb{C}_n$: $V x_a = \sum G_{a,b} T^b + i_a$ where V, $G_{a,b}$, and T are corresponding co-images and $i_a \in I$.

Let $A = \mathbb{C}_n[V^{-1}]$ i. e. A is a subring of the field of fractions of $\mathbb{C}_n$ with denominators which are powers of V. Projection π can be defined on A and it will take it to $R[v^{-1}]$, which is a subring of $F^{\partial}[s]$.

So $\mathrm{J}(x_1, \dots, x_n) = \sum \mathrm{J}(y_1, \dots, y_n)$ where y_a is either $\frac{G_{a,b}}{V} T^b$ or $\frac{i_a}{V}$. Since $\mathrm{J}(\dots, HT^b, \dots) = T^b \mathrm{J}(\dots, H, \dots) + bT^{b-1} H \mathrm{J}(\dots, T, \dots)$ and $\mathrm{J}(\dots, \frac{G}{V}, \dots) = V^{-1} \mathrm{J}(\dots, G, \dots) - GV^{-2} \mathrm{J}(\dots, V, \dots)$ we can write $\mathrm{J}(x_1, \dots, x_n) = \sum u_m \mathrm{J}(z_1, \dots, z_n)$ where $u_m \in \mathbb{C}[T, G_{1,0}, \dots, G_{n,N}, i_1, \dots, i_n, V^{-1}]$ and z_j is either $G_{a,b}$, or T, or i_a.

Since $\pi(u_m) \in F^\partial[s]$ it suffices to show that q divides $\pi(\mathrm{J}(z_1, \dots, z_n))$. If T appears in this Jacobian in two positions then it is zero, so we may assume that there is at most one T in it. Also by Lemma 3 at most k positions have elements from I. If exactly k positions have elements from I then q divides $\pi(\mathrm{J}(z_1, \dots, z_n))$ by Lemmas 6 and 7. So we may assume that at least $n - k$ positions have elements $G_{a,b}$. Since $\pi(G_{a,b}) \in F^\partial$ these elements are algebraically dependent over I.

Assume that $\pi(\mathrm{J}(z_1, \dots, z_n))$ is not divisible by q and rewrite it so that the elements from I are in the beginning and T is in the last position if at all. So $\mathrm{J}(z_1, \dots, z_n) = \mathrm{J}(i_1, \dots, i_m, h_{m+1}, \dots, h_{n-1}, U)$ where $i_j \in I$, $\pi(h_j) \in F^\partial$, $\pi(U) \in F^\partial \bigcup t$ and $m < k$. Assume also that the number of elements from I is maximal possible under this condition.

Let P be a non-zero polynomial of $h_{m+1}, \dots, h_{n-1}$ with value in I and of minimal total degree possible. We may assume (up to renumbering) that P depends on h_{m+1}. Therefore $\mathrm{J}(i_1, \dots, i_m, P, h_{m+2}, \dots, h_{n-1}, U) = \frac{\partial P}{\partial h_{m+1}} \mathrm{J}(i_1, \dots, i_m, h_{m+1}, \dots, h_{n-1}, U)$ is divisible by q. Since $\pi(\frac{\partial P}{\partial h_{m+1}}) \in R^\partial \backslash 0$ it implies that $\mathrm{J}(i_1, \dots, i_m, h_{m+1}, \dots, h_{n-1}, U)$ is divisible by q. So q is a unit in $F^\partial[s]$, i. e. $q \in F^\partial$.

Find now any $i_1, \dots, i_k \in I$ for which $\epsilon(\mathbf{i})(s) = q$ and take $\epsilon = \epsilon(\mathbf{i})$. It is clear that $\epsilon = \epsilon(s)\partial$. Since $q = \frac{a}{b}$ where $a, b \in R^{\partial}\setminus 0$ the Theorem is proved.

Remarks. Some forms of this Theorem appeared before. In [ML2] it was proved for any lnd of $\mathbb{C}_n$, and in [Da1] a more precise result was found for $n = 3$. Also in [KML] a somewhat confusing form of it was proved for hypersurfaces. It should be mentioned that for hypersurfaces and complete intersections the choice of elements from the corresponding ideal is essentially unique.

Example. The following example shows that in general we can give with a Jacobian only an equivalent lnd. Let ∂ be an lnd on $\mathbb{C}_4$ given by $\partial(z) = y$, $\partial(y) = x$, $\partial(x) = a$, and $\partial(a) = 0$. Since ∂ is a homogeneous operator any invariant of ∂ is the sum of homogeneous invariants. It is clear that there is just one invariant of degree 1. So $J(f_1, f_2, f_3, x) \neq a$ if $f_i \in \ker(\partial)$. Indeed f_i may be assumed homogeneous and then only one of them can have degree 1. Therefore $\deg(J(x, f_1, f_2, f_3)) > 1$.

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