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THE FREIHEITSSATZ FOR NOVIKOV ALGEBRAS*

LEONID MAKAR-LIMANOV[1], UALBAI UMIRBAEV[2]

Abstract. We prove the Freiheitssatz for Novikov algebras in characteristic zero. It is also proved that the variety of Novikov algebras is generated by a Novikov algebra on the space of polynomials $k[x]$ in a single variable x over a field k with respect to the multiplication $f \circ g = \partial(f)g$. It follows that the base rank of the variety of Novikov algebras equals 1.

Keywords: Novikov algebras, Freiheitssatz, identities.

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1. Introduction

In 1930 W. Magnus proved one of the most important theorems of the combinatorial group theory (see [8]): *Let* $G = \langle x_1, x_2, \ldots, x_n | r = 1\rangle$ *be a group defined by a single cyclically reduced relator* r. *If* x_n *appears in* r, *then the subgroup of* G *generated by* $x_1, \ldots, x_{n-1}$ *is a free group, freely generated by* $x_1, \ldots, x_{n-1}$. He called it *the Freiheitssatz* ("freedom/independence theorem" in German). In the same paper W. Magnus proved the decidability of the word problem for groups with a single defining relation. The Freiheitssatz for solvable and nilpotent groups was researched by many authors (see, for example [13]).

In 1962 A. I. Shirshov [14] established the Freiheitssatz for Lie algebras and proved the decidability of the word problem for Lie algebras with a single defining relation. These results recently were generalized in [7] for right-symmetric algebras. In 1985 L. Makar-Limanov [9] proved the Freiheitssatz for associative algebras of characteristic zero and in [10] it was also proved for Poisson algebras of characteristic zero. Note that the question of decidability of the word problem for associative algebras and Poisson algebras with a single defining relation and the Freiheitssatz for associative algebras in a positive characteristic remain open. The Freiheitssatz for Poisson algebras in a positive characteristic is not true [10].

In this paper we prove the Freiheitssatz for Novikov algebras over fields of characteristic zero. There are two principal methods of proving the Freiheitssatz: one, employing the combinatorics of free algebras, applied in [7, 8, 13, 14], and the other, related to the study of algebraic and differential equations, applied in [9, 10]. The latter is used here.

Recall that an algebra A over a field k is called *right-symmetric* if it satisfies the identity

$$(xy)z - x(yz) = (xz)y - x(zy). \tag{1}$$

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[1] The Weizmann Institute of Science, Rehovot, Israel, University of Michigan, Ann Arbor, and Wayne State University, Detroit, MI 48202, USA,
e-mail: lml@math.wayne.edu

[2] Eurasian National University, Astana, Kazakhstan and Wayne State University, Detroit, MI 48202, USA,
e-mail: umirbaev@math.wayne.edu

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In other words, the associator $(x, y, z) = (xy)z - x(yz)$ is symmetric in y and z. The variety of right-symmetric algebras is Lie-admissible, i.e., each right-symmetric algebra A with the operation $[x, y] = xy - yx$ is a Lie algebra. A right-symmetric algebra A is called *Novikov* ([2], [12], [6]), if it satisfies also the identity

$$x(yz) = y(xz). \tag{2}$$

Let $k[x]$ be the polynomial algebra in a single variable x over a field k of characteristic 0. There are two interesting multiplications on $k[x]$ (see, for example [3, 4, 5]):

$$f * g = f \int_0^x g dx$$

and

$$f \circ g = \partial(f)g, \quad \partial = \frac{d}{dx}.$$

The algebra $\langle k[x], *\rangle$ is a free dual Leibniz algebra freely generated by 1 and it was proved in [11] that the variety of dual Leibniz algebras is generated by $\langle k[x], *\rangle$. The algebra $A = \langle k[x], \circ\rangle$ is a Novikov algebra [3] and it is the main object of this paper. We prove that the variety of Novikov algebras is generated by A. It follows that the base rank of the variety of Novikov algebras is equal to 1.

The paper is organized as follows. In Section 2 we prove that all identities of A are corollaries of (1)–(2). In Section 3, using the homomorphisms of free Novikov algebras into A and some results on differential equations from [10], we prove the Freiheitssatz.

2. Identities

Let k be a field of characteristic 0. Denote by $\mathfrak{N}$ the variety of Novikov algebras over k and denote by $\mathrm{N}\langle X\rangle$ the free Novikov algebra freely generated by $X = \{x_1, x_2, \ldots, x_n\}$. Put $x_1 < x_2 < \ldots < x_n$. In [3, 5] several constructions of a linear basis of $\mathrm{N}\langle X\rangle$ are given. We use a linear basis of $\mathrm{N}\langle X\rangle$ given in [5] in terms of Young diagrams.

Recall that a Young diagram is a set of boxes (we denote them by bullets) with non-increasing numbers of boxes in each row. Rows and columns are numbered from the top to the bottom and from the left to the right. Let k be the number of rows and r_i be the number of boxes in the ith row. The total number of boxes, $r_1 + \cdots + r_k$, is called the *degree* of the Young diagram.

To get a Novikov diagram, we need to add one box (call it "a nose") to a Young diagram. Namely, we need to add one more box to the first row, i.e.,

$$\begin{array}{cccccc} \bullet & \cdots & \bullet & \bullet & \bullet & \\ \bullet & \cdots & \bullet & \bullet & & \\ \vdots & \cdots & \vdots & \vdots & & \\ \bullet & \cdots & \bullet & & & \end{array} \quad \mapsto \quad \begin{array}{ccccccc} \bullet & \cdots & \bullet & \bullet & \bullet & * \\ \bullet & \cdots & \bullet & \bullet & & \\ \vdots & \cdots & \vdots & \vdots & & \\ \bullet & \cdots & \bullet & & & \end{array}$$

The number of boxes in a Novikov diagram is also called its *degree*. So, the difference between the degrees of a Novikov diagram and the corresponding Young diagram is 1.

To construct Novikov tableaux on X we need to fill Novikov diagrams by elements of X. Denote by $a_{i,j}$ the element of X in the box (i, j), that is the cross of the i-th row and the j-th column. The *filling rules* are

(F1) $a_{i,1} \geq a_{i+1,1}$, if $r_i = r_{i+1}, i = 1, 2, \ldots, k - 1$;

(F2) the sequence of elements $a_{k,2}, \ldots, a_{k,r_k}, a_{k-1,2}, \ldots, a_{k-1,r_{k-1}}, \ldots, a_{1,2}, \ldots, a_{1,r_1}, a_{1,r_1+1}$ is non-decreasing.

In particular, all boxes beginning from the second place in each row are labeled by non-decreasing elements of X.

To any Novikov tableau

$$T = \begin{matrix} a_{1,1} & \cdots & \cdots & a_{1,r_1-1} & a_{1,r_1} & a_{1,r_1+1} \\ a_{2,1} & \cdots & a_{2,r_2-1} & a_{2,r_2} & & \\ \vdots & \cdots & \vdots & \vdots & & \\ a_{k,1} & \cdots & a_{k,r_k} & & & \end{matrix} \tag{3}$$

associate a non-associative word

$$W_T = W_k(W_{k-1}(\ldots(W_2W_1)\ldots)), \tag{4}$$

in the alphabet X where

$$W_1 = (\ldots((a_{1,1}a_{1,2})a_{1,3})\ldots a_{1,r_1})a_{1,r_1+1},$$

$$W_i = (\ldots((a_{i,1}a_{i,2})a_{i,3})\ldots a_{i,r_i-1})a_{i,r_i}, \quad 1 < i \leq k.$$

The set of all non-associative words associated with Novikov tableaux composes a linear basis of the free Novikov algebra $\mathrm{N}\langle X\rangle$ [5].

Recall that $A = \langle k[x], \circ\rangle$ is the Novikov algebra on the space of the polynomial algebra $k[x]$ with respect to multiplication $\circ$. For any $s = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of all nonnegative integers, we define a homomorphism

$$\overline{s} : \mathrm{N}\langle X\rangle \longrightarrow A = \langle k[x], \circ\rangle$$

given by $\overline{s}(x_i) = x^{s_i}$ for all $1 \leq i \leq n$.

Consider the polynomial algebra $k[\lambda_1, \ldots, \lambda_n]$ in the variables $\lambda_1, \ldots, \lambda_n$. Put $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $k[\lambda] = k[\lambda_1, \ldots, \lambda_n]$. Put also $x^{k[\lambda]} = \{x^{f(\lambda)} | f(\lambda) \in k[\lambda]\}$. Define a multiplication on $x^{k[\lambda]}$ by

$$x^{f(\lambda)}x^{g(\lambda)} = x^{f(\lambda)+g(\lambda)}.$$

Obviously, $x^{k[\lambda]}$ is a multiplicative copy of the additive group of $k[\lambda]$. Denote by G the group algebra of $x^{k[\lambda]}$ over $k[\lambda]$. It is easy to check that there exists a unique $k[\lambda]$-linear derivation

$$D : G \longrightarrow G$$

such that $D(x^{f(\lambda)}) = f(\lambda)x^{f(\lambda)-1}$ for all $f(\lambda) \in k[\lambda]$. With respect to

$$a \circ b = D(a)b, \quad a, b \in G,$$

G is a Novikov algebra again. Denote by $A(\lambda)$ the Novikov k-subalgebra of G generated by $x^{\lambda_1}, \ldots, x^{\lambda_n}$. The algebra $A(\lambda)$ looks like an algebra of general matrices (see, for example [1]).

Let

$$\overline{\lambda} : \mathrm{N}\langle X\rangle \longrightarrow A(\lambda)$$

be an epimorphism of Novikov algebras defined by $\overline{\lambda}(x_i) = x^{\lambda_i}$ for all $1 \leq i \leq n$. Note that $\overline{\lambda}$ is a "general" element for the set of all homomorphisms $\overline{s}$, where $\overline{s} \in \mathbb{Z}_+^n$. A homomorphism $\overline{s}$ is called a *specialization* of $\overline{\lambda}$.

Now we fix a Novikov tableau T and its associated non-associative word W_T from (3)–(4). Denote by deg the standard degree function on $\mathrm{N}\langle X\rangle$ and by $\deg_{x_i}$ the degree function with respect to x_i for all $1 \leq i \leq n$. Denote by d the degree of T and by d_i the number of occurrences of x_i in T. Obviously, $d = \deg W_T$, $d_i = \deg_{x_i} W_T$, and

$$\overline{\lambda}(W_T) = f_T(\lambda)x^{g_T(\lambda)}$$

for some $f_T(\lambda), g_T(\lambda) \in k[\lambda] = k[\lambda_1, \ldots, \lambda_n]$.

Our first aim is to calculate the polynomials $f_T(\lambda)$ and $g_T(\lambda)$. For this reason we change the tableau T from (3) by substituting λ_i instead of x_i for all $1 \leq i \leq n$. Denote the new tableau by $T(\lambda)$. Then denote by $\lambda_{i,j}$ the element in the box (i, j) of $T(\lambda)$. In fact, we have just changed all $a_{i,j}$ to $\lambda_{i,j}$ in (3).

Lemma 2.1. *The following statements are true:*

(a) $g_T(\lambda) = (d_1\lambda_1 + \ldots + d_n\lambda_n - d + 1)$;

(b) $f_T(\lambda) = f_1 f_2 \ldots f_k$ *where*

$$f_i = \lambda_{i,1}(\lambda_{i,1} + \lambda_{i,2} - 1) \ldots (\lambda_{i,1} + \ldots + \lambda_{i,r_i} - r_i + 1),\ 1 \leq i \leq k.$$

Proof. Direct calculation gives that

$$\overline{\lambda}(W_1) = \overline{\lambda}((\cdots((a_{1,1}a_{1,2})a_{1,3})\cdots a_{1,r_1})a_{1,r_1+1}) =$$
$$= \overline{\lambda}((\cdots((x^{\lambda_{1,1}} \circ x^{\lambda_{1,2}}) \circ x^{\lambda_{1,3}}) \circ \cdots \circ x^{\lambda_{1,r_1}}) \circ x^{\lambda_{1,r_1+1}}) =$$
$$= \lambda_{1,1}(\lambda_{1,1} + \lambda_{1,2} - 1) \ldots (\lambda_{1,1} + \ldots + \lambda_{1,r_1} - r_1 + 1) x^{(\lambda_{1,1} + \ldots + \lambda_{1,r_1} + \lambda_{1,r_1+1} - r_1)}.$$

Using this and leading an induction on k we get

$$\overline{\lambda}(W_k) = \lambda_{k,1}(\lambda_{k,1} + \lambda_{k,2} - 1) \ldots (\lambda_{k,1} + \ldots + \lambda_{k,r_k-1} - r_k + 2) x^{(\lambda_{k,1} + \ldots + \lambda_{k,r_k} - r_k + 1)}$$

and

$$\overline{\lambda}(W_{k-1}(W_{k-2}\cdots(W_2W_1)\cdots)) = f_1 f_2 \ldots f_{k-1} x^s,$$

where $s = \sum_{i<k,j} \lambda_{i,j} - d + r_k + 1$. Consequently,

$$\overline{\lambda}(W_T) = \overline{\lambda}(W_k) \circ \overline{\lambda}(W_{k-1}(W_{k-2}\cdots(W_2W_1)\cdots)) =$$
$$= \partial(\overline{\lambda}(W_k))\overline{\lambda}(W_{k-1}(W_{k-2}\cdots(W_2W_1)\cdots)) =$$
$$= f_k x^{(\lambda_{k,1} + \ldots + \lambda_{k,r_k} - r_k)} f_1 f_2 \ldots f_{k-1} x^s = f_T x^t,$$

where $t = \lambda_{k,1} + \ldots + \lambda_{k,r_k} - r_k + s = \sum_{i,j} \lambda_{i,j} - d + 1 = g_T(\lambda)$. □

Lemma 2.2. *A Novikov tableau T is uniquely defined by the polynomials $f_T(\lambda)$ and $g_T(\lambda)$.*

Proof. For any linear form l of the type

$$l = t_1\lambda_1 + \ldots + t_n\lambda_n - t_1 - \ldots - t_n + 1 \tag{5}$$

we put $\alpha(l) = t_1 + \ldots + t_n$ and $\widehat{l} = t_1\lambda_1 + \ldots + t_n\lambda_n$. Let s_i be the number of boxes in the i-th column of the Young diagram corresponding to T. It follows from Lemma 2.1(b) that s_i is equal to the number of all divisors l of f_T of the form (5) with $\alpha(l) = i$, counted together with multiplicity. So, the Young diagram and the Novikov diagram corresponding to T are uniquely defined.

By Lemma 2.1(a), the degree of T and the number of occurrences of x_i in T are also uniquely defined by $g_T(\lambda)$. It follows from Lemma 2.1(b) that x_i occurs in the first column of T m-times if and only if $\lambda_i^m | f_T$ and $\lambda_i^{m+1} \nmid f_T$. Consequently, the elements of all columns of T, except the first one, are uniquely defined by the filling rule (F2).

So, the only question to answer is that how to arrange the elements of the first row. Let $l_1, \ldots, l_s$ be all divisors of f_T of the form (5) with maximal $\alpha = \alpha(l_1) = \ldots = \alpha(l_s)$. By Lemma 2.1(b), $l_1, \ldots, l_s$ correspond to the first s rows of T and the first s rows of the Young diagram corresponding to T have lengths $r_1 = \ldots = r_s = \alpha$. We have

$$\sum_{1 \leq i \leq s} \sum_{1 \leq j \leq r_i} \lambda_{i,j} = \widehat{l_1} + \ldots + \widehat{l_s}.$$

Suppose that

$$\sum_{1\le i\le s}\lambda_{i,1}=\widehat{l_1}+\ldots+\widehat{l_s}-\sum_{1\le i\le s}\sum_{2\le j\le r_i}\lambda_{i,j}=\sum_{i=1}^{n}t_i\lambda_i.$$

Obviously $t_i\ge 0$, $t_1+\ldots+t_n=s$, and

$$(a_{1,1},\ldots,a_{s,1})=(\underbrace{x_n,\ldots,x_n}_{t_n},\ldots,\underbrace{x_1,\ldots,x_1}_{t_1})$$

by the filling rule (F1). So, the first s rows of the Novikov tableaux T are uniquely determined. Consequently, the polynomials $f_1,\ldots,f_s$ are also uniquely determined. Using the polynomial $f_T/(f_1\ldots f_s)$ and continuing the same discussions, we can uniquely determine T. □

Denote by $\mathbb{T}_n$ the set of all Novikov tableaux of degree n on $X=\{x_1,\ldots,x_n\}$ without repeated elements. Then $\{W_T|T\in\mathbb{T}_n\}$ is a linear basis of the space of all multi-linear homogeneous of degree n elements of the free Novikov algebra $\mathrm{N}\langle X\rangle$ [5].

Corollary 2.1. *Suppose that $T\in\mathbb{T}_n$. Then T is uniquely defined by f_T.*

Let $u=\lambda_1^{k_1}\ldots\lambda_n^{k_n}$ be an arbitrary monomial in $k[\lambda]=k[\lambda_1,\ldots,\lambda_n]$. Put $|u|=k_1+\ldots+k_n$. Put also $\gamma(u)=(s_1,\ldots,s_n)$ if $u=\lambda_{\sigma(1)}^{s_1}\ldots\lambda_{\sigma(n)}^{s_n}$ where σ is a permutation on $\{1,\ldots,n\}$ and $s_1\ge s_2\ge\ldots\ge s_n$. We define a linear order $\preceq$ on the set of all monomials of $k[\lambda]$. If u and v are two monomials then put $u\preceq v$ if $|u|<|v|$ or $|u|=|v|$ and $\gamma(u)$ is preceeds to $\gamma(v)$ with respect to the lexicographical order (from left to right) on $\mathbb{Z}_+^n$. If $|u|=|v|$ and $\gamma(u)=\gamma(v)$ then $u\preceq v$ is defined arbitrarily. For any $f\in k[\lambda]$ denote by $\widetilde{f}$ its highest term with respect to $\preceq$.

The statement of the next corollary trivially follows from Lemma 2.1(b).

Corollary 2.2. *Suppose that $T\in\mathbb{T}_n$ and $(a_{1,1},a_{2,1},\ldots,a_{k,1})=(x_{i_1},x_{i_2},\ldots,x_{i_k})$ in (3). Then,*

$$\widetilde{f_T}=\lambda_{i_1}^{r_1}\lambda_{i_2}^{r_2}\ldots\lambda_{i_k}^{r_k}\quad and\quad \gamma(\widetilde{f_T})=(r_1,r_2,\ldots,r_k).$$

Corollary 2.3. *The set of polynomials $f_T\in k[\lambda]$, where T runs over $\mathbb{T}_n$, is linearly independent over k.*

Proof. Suppose that $(a_{1,1},a_{2,1},\ldots,a_{k,1})=(x_{i_1},x_{i_2},\ldots,x_{i_k})$ in (3). Then, $\gamma(\widetilde{f_T})=(r_1,r_2,\ldots,r_k)$ by Corollary 2.2. It follows that the Novikov diagram corresponding to T is uniquely determined by $\widetilde{f_T}$. Moreover, x_{i_s} is the first element of the row with length r_s. Then the filling rule (F1) uniquely determines the elements of the first row of T. The filling rule (F2) determines uniquely the other part of T.

So, the mapping $T\mapsto\widetilde{f_T}$ associates different tableaux to different basis elements of $k[\lambda]$. Consequently, the set of polynomials $\widetilde{f_T}$, where T runs over $\mathbb{T}_n$, is linearly independent. This proves the lemma. □

In characteristic 0 any identity is equivalent to the set of multi-linear homogeneous identities [15]. Any nontrivial multi-linear homogeneous Novikov identity of degree n can be written as

$$\sum_{T\in\mathbb{T}_n}\alpha_T W_T=0 \tag{6}$$

where $\alpha_T\in k$ and at least one of α_T is nonzero.

Theorem 2.1. *The Novikov algebra $A=\langle k[x],\circ\rangle$ does not satisfy any nontrivial Novikov identity.*

Proof. Suppose that A satisfies a nontrivial identity of the form (6). Consider the homomorphism $\overline{\lambda}$. Applying $\overline{\lambda}$ to the left hand side of (6) we get

$$\overline{\lambda}(\sum_{T\in\mathbb{T}_n}\alpha_T W_T)=\sum_{T\in\mathbb{T}_n}\alpha_T f_T x^{g_T}=(\sum_{T\in\mathbb{T}_n}\alpha_T f_T)x^{\lambda_1+\ldots+\lambda_n-n+1}$$

since $g_T(\lambda)=\lambda_1+\ldots+\lambda_n-n+1$ for all T. By Corollary 2.3, $\sum_T \alpha_T f_T$ is a nontrivial polynomial from $k[\lambda]$. Then it is not difficult to find $s=(s_1,\ldots,s_n)\in\mathbb{Z}_+^n$ such that $\sum_T \alpha_T f_T(s_1,\ldots,s_n)\neq 0$. This means that the image of the left hand side of (6) under the homomorphism $\overline{s}$ is not equal to 0. Consequently, (6) is not a nontrivial identity of A. □

Corollary 2.4. *The variety of Novikov algebras* $\mathfrak{N}$ *is generated by* $A=\langle k[x],\circ\rangle$, *i.e.,* $\mathfrak{N}=\mathrm{Var}\,A$.

Recall that the least natural number n such that the variety $Var(\mathrm{N}\langle x_1,x_2,\ldots,x_n\rangle)$ of algebras generated by $\mathrm{N}\langle x_1,x_2,\ldots,x_n\rangle$ is equal to $\mathfrak{N}$ is called the *base rank* $rb(\mathfrak{N})$ of the variety $\mathfrak{N}$ (see, for example [11]).

Corollary 2.5. *The base rank of the variety of Novikov algebras is equal to one.*

Proof. Consider the ideal I of the polynomial algebra $k[x]$ generated by x^2. It is easy to check that $\langle I,\circ\rangle$ is a Novikov algebra generated by x^2. In the proof of Theorem 2.1, we can easily chose $s=(s_1,\ldots,s_n)$ such that $s_i\geq 2$ for all i. Consequently, $\langle I,\circ\rangle$ does not satisfy any nontrivial Novikov identity. Then, $\mathfrak{N}=\mathrm{Var}\,\langle I,\circ\rangle$. We have $\mathrm{Var}(\mathrm{N}\langle x_1\rangle)\supseteq\mathrm{Var}\,\langle I,\circ\rangle$ since $\langle I,\circ\rangle$ is a homomorphic image of $\mathrm{N}\langle x_1\rangle$. Therefore, $\mathfrak{N}=\mathrm{Var}(\mathrm{N}\langle x_1\rangle)$. □

3. The Freiheitssatz

To prove the Freiheitssatz we need the following corollary of Proposition 1 from [10].

Corollary 3.1. [10] *Let* $f(x,t_{\alpha_1},t_{\alpha_2},\ldots,t_{\alpha_m})\in k[x,t_{\alpha_1},t_{\alpha_2},\ldots,t_{\alpha_m}]$ *and* $\alpha_1<\alpha_2<\ldots<\alpha_m$ *be nonnegative integers. Suppose that there exists* $(c,c_{\alpha_1},c_{\alpha_2},\ldots,c_{\alpha_m})\in k^{1+m}$ *so that* $f(c,c_{\alpha_1},c_{\alpha_2},\ldots,c_{\alpha_m})=0$ *and* $\frac{\partial f}{\partial t_{\alpha_m}}(c,c_{\alpha_1},c_{\alpha_2},\ldots,c_{\alpha_m})\neq 0$. *Then the differential equation*

$$f(x,\partial^{\alpha_1}(T),\partial^{\alpha_2}(T),\ldots,\partial^{\alpha_m}(T))=0$$

has a solution in the formal power series algebra $k[[x-c]]$.

Note that in the formulation of this corollary, the variables $x,t_{\alpha_1},t_{\alpha_2},\ldots,t_{\alpha_m}$ are independent variables, ∂ is the standard derivation $\frac{d}{dx}$ of $k[[x-c]]\supseteq k[x]$, and ∂^{α_i} is the α_ith power of ∂.

If $f\in\mathrm{N}\langle x_1,\ldots,x_n\rangle$, then we denote $\mathrm{id}(f)$ the ideal of $\mathrm{N}\langle x_1,\ldots,x_n\rangle$ generated by f.

Theorem 3.1. (Freiheitssatz) *Let* $\mathrm{N}\langle x_1,\ldots,x_n\rangle$ *be the free Novikov algebra over a field* k *of characteristic* 0 *in the variables* $x_1,\ldots,x_n$. *If* $f\in\mathrm{N}\langle x_1,\ldots,x_n\rangle$ *and* $f\notin\mathrm{N}\langle x_1,\ldots,x_{n-1}\rangle$, *then* $\mathrm{id}(f)\cap\mathrm{N}\langle x_1,\ldots,x_{n-1}\rangle=0$.

Proof. Without loss of generality we may assume that k is algebraically closed and that $f(x_1,\ldots,x_{n-1},0)\neq 0$. The theorem will be proved if for f and any nonzero $g\in\mathrm{N}\langle x_1,\ldots,x_{n-1}\rangle$ there exist a Novikov algebra B and a homomorphism $\theta:\mathrm{N}\langle x_1,\ldots,x_n\rangle\to B$ of Novikov algebras such that $\theta(g)\neq 0,\theta(f)=0$.

Let $\hat{f}$ be the highest homogeneous part of f with respect to x_n. By Theorem 2.1, there exists a homomorphism $\phi:\mathrm{N}\langle x_1,\ldots,x_n\rangle\to A=\langle k[x],\circ\rangle$ such that $\phi((gf)\hat{f})\neq 0$. Denote by $Z_1,Z_2,\ldots,Z_{n-1}$ the images of $x_1,x_2,\ldots,x_{n-1}$ under ϕ, by Z a general element of A, and consider the equation

$$f(Z_1,Z_2,\ldots,Z_{n-1},Z)=0$$

in A. Using the definition of the multiplication in A, we can rewrite the last equation in the form

$$h(x, \partial^{\alpha_1}(Z), \partial^{\alpha_2}(Z), \ldots, \partial^{\alpha_r}(Z)) = 0, \tag{7}$$

where $h = h(x, t_{\alpha_1}, \ldots, t_{\alpha_r})$ is a polynomial in the variables $x, t_{\alpha_1}, \ldots, t_{\alpha_r}$. Since $f \notin \mathrm{N}\langle x_1, \ldots, x_{n-1}\rangle$ the polynomial h essentially depends on $t_{\alpha_1}, \ldots, t_{\alpha_r}$, i.e. $r > 0$ in (4).

Assume that $\alpha_1 < \ldots < \alpha_r$ and that h is irreducible. If h is not irreducible we can replace it with its irreducible factor which contains t_{α_r}. We assert that there exists $L = (c, c_{\alpha_1}, \ldots, c_{\alpha_r}) \in k^{1+r}$ such that $h(L) = 0$ and $\frac{\partial h}{\partial t_{\alpha_r}}(L) \neq 0$. If this is not true then by Hilbert's Nulstellenssatz h divides $(\frac{\partial h}{\partial t_{\alpha_r}})^s$ for some $s > 0$. But then, since h is irreducible, h divides $(\frac{\partial h}{\partial t_{\alpha_r}})$, which is clearly impossible.

Therefore we can use Corollary 3.1 and find a solution Z_n of the differential equation (7) in the formal power series algebra $k[[x-c]]$. Note that $B = \langle k[[x-c]], \circ\rangle$ is a Novikov algebra and A is a subalgebra of B. Take a homomorphism of Novikov algebras $\theta : \mathrm{N}\langle x_1, \ldots, x_n\rangle \to B$ defined by

$$\theta(x_1) = Z_1, \theta(z_2) = Z_2, \ldots, \theta(z_{n-1}) = Z_{n-1}, \theta(x_n) = Z_n.$$

Then $\theta_{|\mathrm{N}\langle x_1,\ldots,x_{n-1}\rangle} = \phi_{|\mathrm{N}\langle x_1,\ldots,x_{n-1}\rangle}$ and $\theta(f) = 0$. □

In many cases the Freiheitssatz is formulated directly in the language of freeness.

Corollary 3.2. (Freiheitssatz) *Let* $\mathrm{N}\langle x_1, \ldots, x_n\rangle$ *be the free Novikov algebra over a field* k *of characteristic* 0 *in the variables* $x_1, \ldots, x_n$. *Suppose that* $f \in \mathrm{N}\langle x_1, \ldots, x_n\rangle$ *and* $f \notin \mathrm{N}\langle x_1, \ldots, x_{n-1}\rangle$. *Then the subalgebra of the quotient algebra* $\mathrm{N}\langle x_1, \ldots, x_n\rangle/\mathrm{id}(f)$ *generated by* $x_1 + \mathrm{id}(f), \ldots, x_{n-1} + \mathrm{id}(f)$ *is a free Novikov algebra with free generators* $x_1 + \mathrm{id}(f), \ldots, x_{n-1} + \mathrm{id}(f)$.

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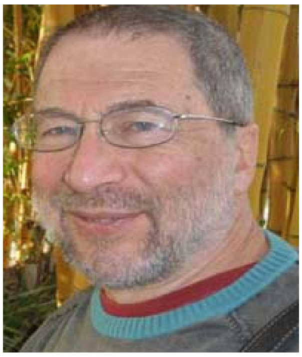

Leonid Makar-Limanov was born in 1945, Moscow, Russia. He got Ph.D. degree in 1970, Moscow State University, Professor at the Department of Mathematics, Wayne State University, Detroit, USA; visiting Fulbright Scientist at the Department of Mathematics, Weizmann Institute of Science, Rehovot, Israel; visiting scholar, Department of Mathematics, University of Michigan, Ann Arbor, USA.

Ualbai Umirbaev, for a photograph and biography, see TWMS J. Pure Appl. Math., V.1, N.1, 2011, p.86