HYPERBOLIC EISENSTEIN SERIES FOR GEOMETRICALLY FINITE HYPERBOLIC SURFACES OF INFINITE VOLUME.

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Abstract. Let M be a geometrically finite hyperbolic surface of infinite volume. After writing down the spectral decomposition for the Laplacian on 1-forms of M, we generalize the Kudla and Millson's construction of hyperbolic Eisenstein series (Invent Math 54:199-211, 1979) and other related results (see theorems 3.1, 4.2, 5.1).

Introduction.

The spectrum of the Laplace-Beltrami operator for a compact Riemann surface is discrete, it is no more the case when M is not compact. For example when you withdraw one point from M, it appears a continuous part in the spectrum whose spectral measure is described by an Eisenstein series. The study of the limiting behavior of the spectrum of the Laplace-Beltrami operator for a degenerating family of Riemann surfaces with finite area hyperbolic metrics have been used to explain this apparition (see for example [22], [15], [13]) and one of the motivation of this paper is the interest in a question of L.Ji in [15], p.308, concerning the approximation of Eisenstein series by suitable eigenfunctions of a degenerating family of hyperbolic Riemann surface. We hope to surround it via hyperbolic Eisenstein series (for results on degenerating Eisenstein series see, for example [18], [19], [8], [9]). What we really do here is to develop the suggestion in [17], to construct hyperbolic Eisenstein series and harmonic dual form in the infinite volume case: in this context we verify the convergence of hyperbolic Eisenstein series and the fact that it permits to realize a harmonic dual form to a simple closed geodesic on a geometrically finite hyperbolic surface of infinite volume (theorem 3.1) and in a similar way of an infinite geodesic joining a pair of punctures (theorem 4.2). We obtain a degeneration of hyperbolic Eisenstein series to horocyclic ones (theorem 5.1 and corollary 5.2):

Theorem 0.1. *Let $(S_l)_l$ a degenerating family of Riemann surfaces with infinite area hyperbolic metric and $\Omega_{c_l} = \Omega_l$ the hyperbolic Eisenstein series associated to the pinching geodesic c_l. For $\operatorname{Re} s > 0$, the family of 1-forms $\frac{1}{l^{s+1}}\Omega_l(s, \pi_l(.))$ converge uniformly on compact subsets of S_0 to $\frac{\Gamma(1+\frac{s}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{s}{2})} \operatorname{Im} \mathcal{E}_\infty(s+1, .)$.*

1. Preliminary definitions

Let us recall the standard analytic and geometric notations which will be used. In this paper a surface is a connected orientable two-dimensional manifold, without boundary unless

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otherwise specified. We denote by H the hyperbolic upper half-plane endowed with its standard metric of constant gaussian curvature -1. A topologically finite surface is a surface homeomorphic to a compact surface with finitely many points exised and a geometrically finite hyperbolic surface M is a topologically finite, complete Riemann surface of constant curvature -1. We will require that M is of infinite volume, then there exists a finitely generated, torsion free, discrete subgroup, Γ, of $PSL(2,\mathbb{R})$, unique up to conjugation, such that M is the quotient of H by Γ acting as Möbius transformations, Γ is a fuchsian group of the second kind and Γ has no elliptic elements different from the identity. The group Γ admits a finite sided polygonal fundamental domain in H. We recall now the description of the fundamental domain of $M = \Gamma\backslash H$ (see [1]). Let $L(\Gamma)$ be the limit set of Γ and $O(\Gamma) = \mathbb{R}\cup\{\infty\} - L(\Gamma)$. As $L(\Gamma)$ is closed in $\mathbb{R}\cup\{\infty\}$, $O(\Gamma)$ is open and so can be written as a countable union of $O(\Gamma) = \cup_{\alpha\in A} O_\alpha$ where the O_α are disjoint open intervals in $\mathbb{R}\cup\{\infty\}$. Then let $\Gamma_\alpha = \{\gamma \in \Gamma, \gamma(O_\alpha) = O_\alpha\}$.
This is an elementary hyperbolic subgroup of Γ. The fixed points of Γ_α are exactly the end-points of O_α. There is a finite subset $\{\alpha(1), \alpha(2), ..., \alpha(n_f)\} \subset A$ so that, for $\alpha \in A$, O_α is conjugate to precisely one $O_{\alpha(j)}$ $(1 \le j \le n_f)$. Let λ_α be the half-circle, lying in H, joining the end-points of O_α. Let Δ_α be the region in H bounded by O_α and λ_α. The Δ_α $(\alpha \in A)$ are mutually disjoint.
Let P be the set of parabolic vertices of Γ, and for $p \in P$ let Γ_p be the parabolic subgroup of Γ fixing p. There is a finite subset $\{p(1), p(2), ..., p(n_c)\} \subset P$ so that Γ_p is conjugate to precisely one $\Gamma_{p(j)}$ $(1 \le j \le n_c)$. A circle lying in H and tangent to ∂H at p is called a horocycle at p. We can construct an open disc C_p determined by a horocycle at $p \in P$ so that:

$$\begin{array}{rc} (i) & \text{if } p,\, q \in P, p \neq q, \text{then } C_p \cap C_q = \emptyset, \\ (ii) & g(C_p) = C_{g(p)}\, (g \in \Gamma), \\ (iii) & C_p \cap \Delta_\alpha = \emptyset\, (p \in P, \alpha \in A). \end{array}$$

If we consider the set $H - (\bigcup_{p\in P} C_p \cup \bigcup_{\alpha\in A} \Delta_\alpha)$, we see that it is invariant under Γ. We can find a finite-sided fundamental domain D for the action of Γ on this set; D is relatively compact in H.

Proposition 1.1. *There is a fundamental domain D for Γ of the form*

$$D = K^* \cup \cup_{j=1}^{n_f} D_{\alpha(j)} \cup_{k=1}^{n_c} D^*_{p(k)}$$

where
1) K^* is relatively compact in H.
2) $D_{\alpha(j)}$ is a standard fundamental domain of $\Gamma_{\alpha(j)}$ on $\Delta_{\alpha(j)}$
3) $D^*_{p(k)}$ is a standard fundamental domain for $\Gamma_{p(k)}$ on $C_{p(k)}$.

We should note that $p \neq 0$ if and only if Γ is of the second kind.

The Nielsen region of the group Γ is the set $\tilde{N} = H - (\cup_{\alpha\in A}\Delta_\alpha)$, the truncated Nielsen region of Γ is $\tilde{K} = \tilde{N} - (\cup_{p\in P} C_p)$, $K = \Gamma\backslash\tilde{K}$ is called the compact core of M. So the surface $M = \Gamma\backslash H$ can be decomposed into a finite area surface with geodesic boundary N, called the Nielsen region, on which infinite area ends F_i are glued: the funnels. The Nielsen region N is iteself decomposed into a compact surface K with geodesic and horocyclic boundary on which non compact, finite area ends C_i are glued: the cusps. We have $M = K \cup C \cup F$, where $C = C_1 \cup ... \cup C_{n_c}$ and $F = F_1 \cup ... \cup F_{n_f}$.

A hyperbolic transformation $T \in PSL(2,\mathbb{R})$ generates a cyclic hyperbolic group $\langle T \rangle$. The quotient $C_l = \langle T \rangle \backslash H$ is a hyperbolic cylinder of diameter $l = l(T)$. By conjugation, we can identify the generator T with the map $z \mapsto e^l z$, and we define Γ_l to be the corresponding cyclic group. A natural fundamental domain for Γ_l would be the region $\mathcal{F}_l = \{1 \leq |z| \leq e^l\}$. The $y-$axis is the lift of the only simple closed geodesic on C_l, whose length is l. The standard funnel of $l > 0$, F_l, is the half hyperbolic cylinder $\Gamma_l \backslash H$, $F_l = (\mathbb{R}^+)_r \times (\mathbb{R} \backslash \mathbb{Z})_x$ with the metric $ds^2 = dr^2 + l^2 \cosh^2(r) dx^2$.

We can always conjugate a parabolic cyclic group $\langle T \rangle$ to the group Γ_∞ generated by $z \mapsto z+1$, so the parabolic cylinder is unique up to isometry. A natural fundamental domain for Γ_∞ is $\mathcal{F}_\infty = \{0 \leq \operatorname{Re} z \leq 1\} \subset H$. The standard cusp C_∞ is the half parabolic cylinder $\Gamma_\infty \backslash H$, $C_\infty = ([0,\infty[)_r \times (\mathbb{R}\backslash\mathbb{Z})_x$ with the metric $ds^2 = dr^2 + e^{-2r} dx^2$. The funnels F_i and the cusps C_i are isometric to the preceding standard models. We define the function r as the distance to the compact core K and the function ρ by

$$\rho(r) = \begin{cases} 2e^{-r} & \text{in F} \\ e^{-r} & \text{in C} \end{cases} .$$

We will adopt $(\rho, t) \in (0, 2] \times \mathbb{R}/l_j\mathbb{Z}$ as the standard coordinates for the funnel F_j, where t is arc length around the central geodesic at $\rho = 2$.

For the cusp our standard coordinates $(\rho, t) \in (0, 1] \times \mathbb{R}/\mathbb{Z}$ are based on the model defined by the cyclic group Γ_∞. The cusp boundary is $y = 1$, so that $y = e^r$ and $\rho = 1/y$. We set $t = x(\mod \mathbb{Z})$.

2. Hyperbolic Eisenstein series on a geometrically finite hyperbolic surface of infinite volume.

2.1. **Return to Kudla and Millson hyperbolic Eisenstein series' definition.** In the following, M will denote an arbitrary Riemann surface and $L^2(M)$, the Hilbert space of square integrable 1-forms with inner product

$$(w_1, w_2) = \frac{1}{2} \int_M w_1 \wedge *\overline{w_2} \,,$$

and correspondig norm $||.||$.

Let c be a simple closed curve on M. We may associate with c a real smooth closed differential n_c with compact support such that

$$\int_c \omega = \int_M \omega \wedge n_c \,,$$

for all closed differentials ω. Since every cycle c on M is a finite sum of cycles corresponding to simple closed curves, we conclude that to each such c, we can associated a real closed differential n_c with compact support such that (2.1) holds.

Let a and b be two cycles on the Riemann surface M. We define the intersection number of a and b by

$$a.b = \int_M n_a \wedge n_b \,.$$

In [17], Kudla and Millson construct the harmonic 1-form dual to a simple closed geodesic on a hyperbolic surface of finite volume M in terms of Eisenstein series. Let us recall the definition:

Definition 2.1. *Let η be a simple closed geodesic or an infinite geodesic joining p and q. An 1-form α is dual to η if for any closed 1-form with compact support, ω*

$$\int_M \omega \wedge \alpha = \int_\eta \omega \ .$$

Or equivalently: for any closed oriented cycle c', we have

$$\int_{c'} \alpha = \eta \ . \ c'. \tag{1}$$

Kudla and Millson construct a meromorphic family of forms on M, called hyperbolic Eisenstein series associated to an oriented simple closed geodesic c. Let $\tilde{c}$ be a component of the inverse image of c in the covering $H \to M$ and Γ_1 the stabilizer of $\tilde{c}$ in Γ. The hyperbolic Eisenstein series are expressed in Fermi coordinates in the following way for $\operatorname{Re} s > 0$:

$$\Omega_c(s,z) = \Omega(s,z) = \frac{1}{k(s)} \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \frac{dx_2}{(\cosh x_2)^{s+1}}, \quad . \tag{2}$$

with

$$k(s) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)} \ .$$

By applying an element of $SL(2,\mathbb{R})$ we may assume $\tilde{c}$ is the $y-$ axis in H and that Γ_1 is generated by $\gamma_1 : z \mapsto e^l z$. The Fermi coordinates (x_1, x_2) associated to $\tilde{c}$ are related to euclidean polar coordinates by

$$\begin{aligned} r &= e^{x_1} \\ \sin\theta &= \frac{1}{\cosh x_2} \end{aligned}$$

At the end of their paper they do the remark that "it is also interesting to consider the infinite volume case".

2.2. **The infinite volume case.** We are going to verify that this definition retains a meaning in the case of a geometrically finite hyperbolic surface of infinite volume, $M = \Gamma \backslash H$.

Proposition 2.1. *The hyperbolic Eisenstein series $\Omega(s,z)$ converges for $\operatorname{Re} s > 0$, uniformly on compact subsets of H, is bounded on M and represents a C^∞ closed form which is dual to c. Moreover it is an analytic function of s in $\operatorname{Re} s > 0$.*

The proof in the infinite volume case is as straightforward as in these in the finite volume case ([17], [10]), but for the convenience of the reader we give some details.

Lemma 2.1. *Let K a compact of the fundamental domain D of Γ, there exists $\eta > 0$ such that for all $z_0 \in K$, $(B(\gamma z, \eta))_{\gamma \in \Gamma_1 \backslash \Gamma}$ are disjoints.*

Let choose for fundamental domain of Γ_1, $\mathcal{D}_1 = \{z \in H : 1 \leq |z| \leq \beta\}$. After passing to ordinary Euclidean polar coordinates (r,θ) we obtain with $\sigma = \mathrm{Re}\, s$:

$$\begin{aligned} ||\Omega(s,z)|| &\leq \frac{1}{|k(s)|}\sum_{\Gamma_1\backslash\Gamma}\frac{1}{(\mathrm{ch}\, x_2(\gamma z))^{\sigma+1}} \\ &\leq \frac{1}{|k(s)|}\sum_{\Gamma_1\backslash\Gamma}\left(\frac{y}{r}\right)^{\sigma+1}(\gamma z) \leq \frac{1}{|k(s)|}\sum_{[\gamma]\in\Gamma_1\backslash\Gamma,\gamma z\in\mathcal{D}_1} y^{\sigma+1}(\gamma z) \end{aligned}$$

Moreover for all $z \in K$: $\int_{B(z,\eta)} y^{\sigma+1}\frac{dxdy}{y^2} = \Lambda_\eta y(z)^{\sigma+1}$. We note as in [17]: $R(T_1,T_2) = \{P \in \mathcal{D}_1 : T_1 < x_2(P) < T_2\}$. So for $T > 2\eta$:

$$\frac{1}{|k(s)|}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}\frac{1}{(\mathrm{ch}(x_2(\gamma z)))^{\sigma+1}} \leq \frac{1}{|k(s)|}\sum_{[\gamma]\in\Gamma_1\backslash\Gamma,\gamma z\in\mathcal{D}_1-R(-T,T)} y^{\sigma+1}(\gamma z)$$

We need the following:

Lemma 2.2. *Let $\gamma \in \Gamma_1\backslash\Gamma$, z, $\zeta \in H$ such that for $\gamma z \notin R(-T,T)$, $\gamma\zeta \in B(\gamma z,\eta)$ then $\gamma\zeta \notin R(-T+2\eta, T-2\eta)$.*

Proof. As orthogonal projection P is 1- lipschitzien, we have $d(P\gamma z, P\gamma\zeta) \leq d(\gamma z,\gamma\zeta) \leq \eta$. If $x_2(\gamma z) \geq T$:

$$T \leq d(\gamma z, P\gamma z) \leq d(\gamma z,\gamma\zeta) + d(\gamma\zeta, P\gamma\zeta) + d(P\gamma z, P\gamma\zeta)\,.$$

Then

$$T - 2\eta \leq d(\gamma\zeta, P\gamma\zeta)\,.$$

□

Then

$$\begin{aligned} \sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)} y^{\sigma+1}(\gamma z) &= \frac{1}{\Lambda_\eta}\sum_{\gamma\in\Gamma_1\backslash\Gamma,\gamma z\notin R(-T,T)}\int_{B(\gamma z,\eta)} y^{\sigma+1}\frac{dxdy}{y^2} \\ &\leq \frac{1}{\Lambda_\eta}\int_{R^c(-T+2\eta,T-2\eta)} y^{\sigma+1}\frac{dxdy}{y^2} \end{aligned}$$

where $R^c(-T+2\eta, T-2\eta)$ is the complementary in $\mathcal{D}_1$ of $R(-T+2\eta, T-2\eta)$. Note that if $\gamma z \notin R(-T,T)$ then $y(\gamma z) \leq \dfrac{\beta}{\mathrm{ch}\, T}$, so:

$$\begin{aligned} \sum_{\gamma\in\Gamma_1\backslash\Gamma\gamma z\notin R(-T,T)} y^{\sigma+1}(\gamma z) &\leq \frac{\beta}{\Lambda_\eta}\int_0^{\frac{\beta}{\mathrm{ch}(T-2\eta)}} y^{\sigma-1}\,dy \\ &\leq \frac{\beta}{\Lambda_\eta\sigma}\left(\frac{\beta}{\mathrm{ch}(T-2\eta)}\right)^{\sigma}\,. \end{aligned}$$

From this follow the uniform convergence of $\Omega(s,z)$ on compact subsets of H, uniformly on compact subsets of the half plane $\mathrm{Re}\, s > 0$. We next show that $\Omega(s,z)$ is bounded on D.
For this we use a "very useful (and well-worn) fundamental lemma"(sic), see [14], p. 178, [12], p.27:

Proposition 2.2. *For any fuschian group Γ, there exists $\mathcal{C}(q,\Gamma)$, such that for all $z \in H$*

$$\sum_{\gamma\in\Gamma} \frac{y(\gamma z)^q}{[1+|\gamma z|]^{2q}} \leq \mathcal{C}(q,\Gamma)$$

The constant $\mathcal{C}(q,\Gamma)$ depending only of q and Γ.

Let $z \in H$, there exists a system of representant S of $\Gamma_1\backslash\Gamma$ such that for all $\gamma \in S,\ |\gamma z| \leq \beta$. Then:

$$\begin{aligned}
\sum_{\Gamma_1\backslash\Gamma} \frac{y^{\sigma+1}(\gamma z)}{(1+\beta)^{2(\sigma+1)}} &\leq \sum_{\Gamma_1\backslash\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1+|\gamma z|)^{2(\sigma+1)}} \\
&\leq \sum_{\Gamma} \frac{y(\gamma z)^{\sigma+1}}{(1+|\gamma z|)^{2(\sigma+1)}} \\
&\leq \mathcal{C}(\sigma+1,\Gamma)
\end{aligned}$$

and the result.

The fact that $\Omega(s,z)$ is dual to c follows straightly from the construction of Kudla and Millson.

3. Spectral decomposition and analytic continuation.

The aim is to realize the injection $H_c^1 \to \mathcal{H}^1$, where H_c^1 is the first de Rham's cohomology group with compact support of M and $\mathcal{H}^1$ is the space of L^2 harmonic 1-forms of M. Recall that in our context $\dim \mathcal{H}^1 = \infty$ (see [2], p. 27).
We are going to prove, as in [17], the analytic continuation of the hyperbolic Eisenstein series. The essential difference with the finite volume case is the spectral decomposition of $L^2(M)$.

3.1. **Spectral theory.** For any non-compact geometrically finite hyperbolic surface M, the essential spectrum of the (positive) Laplacian Δ_M defined by the hyperbolic metric on M (the Laplacian on functions) is $[1/4,\infty)$ and this is absolutely continuous. The discrete spectrum consists of finitely many eigenvalues in the range $(0,1/4)$. In the finite-volume case one may also have embedded eigenvalues in the continuous spectrum, but these do not occur for infinite-volume surfaces. Then if M as infinite volume, the discrete spectrum of Δ_M is finite (possibly empty). The exponent of convergence δ of a fuchsian group Γ is defined to be the abscissa of convergence of the Dirichlet series:

$$\delta = \inf\{s>0, \sum_{T\in\Gamma} e^{-sd(z,Tw)} < \infty\}$$

for some $z, w \in \Gamma$.

Let Γ be a fuchsian group of the second kind and $L(\Gamma)$ be its limit set, then $0<\delta<1$ with $\delta > 1/2$ if Γ has parabolic elements. Patterson and Sullivan showed that δ is the Hausdorff dimension of the limit set when Γ is geometrically finite. Furthermore, if $\delta > 1/2$, then $\delta(1-\delta)$ is the lowest eigenvalue of the Laplacian Δ_M. The connection to spectral theory was later extended to the case $\delta \leq 1/2$ by Patterson. In this case, the discrete spectrum of Δ_M is empty and δ is the location of the first resonance. For a detailed account of the spectral theory of infinite area surfaces, we refer the reader to [1].

3.2. **Tensors and automorphic forms.** This section introduces the notations used in the following subsection 3.3, section 5. Let M be a Riemann surface of finite Euler characteristic and carrying a metric ds^2 of constant curvature -1. let z be a local conformal variable and $ds^2 = \rho|dz|^2$. Let T^q denote the space of tensors $f(z)(dz)^q$ for q integer, on M. The covariant derivative ∇ sends T^q into $T^q \otimes (T^1 \oplus \bar{T}^1)$ and can be decomposed accordingly as $\nabla = \nabla^q_{\bar{z}} \oplus \nabla^z_q$, with $\nabla^q_{\bar{z}} : T^q \to T^Q \otimes T^1 \cong T^{q+1}$, $\nabla^q = \rho^q \partial \rho^{-q}$, $\nabla_q = \rho^{-1}\bar{\partial}$, $\nabla^z_q : T^q \to T^Q \otimes \bar{T}^1 \cong T^{q-1}$. The Laplacians Δ^+_q and Δ^{-1}_q on T^q are defined by $\Delta^+_q = -\nabla_{q+1}\nabla^q$, $\Delta^-_q = -\nabla_{q-1}\nabla^q$. Thus $\Delta_0 = \Delta^\pm_0$ is the Laplacian on functions. The operators $\Delta^\pm_q$ are non-negative selfadjoint. We recall now the link between $q-$forms and automorphic forms of weight $2q$[1]. We make use of the uniformization theorem. M may be realized as $H\backslash\Gamma$, where H is the upper half plane and γ a discrete subgroup of $PSL(2,\mathbb{R})$. Let $\tilde{\Gamma} \subset SL(2,\mathbb{R})$ be the groupe covering Γ under the projection $SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$. Using notably the notations in [11] and [7], let

Definition 3.1. *Set*

$$j_\gamma(z) = \frac{(cz+d)^2}{|cz+d|^2} = \frac{cz+d}{c\bar{z}+d} = \left(\frac{\gamma' z}{|\gamma' z|}\right)^{-1} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let $\mathcal{F}_q$ be the space of all functions $f : H \to \mathbb{C}$ with

$$f(\gamma z) = j_\gamma(z)^q f(z), \quad \gamma \in \Gamma\,,$$

and if $\mathcal{D} = \Gamma\backslash H$ is the fundamental domain of Γ, define the Hilbert space $\mathcal{H}_q = \{f \in \mathcal{F}_q, \langle f, f\rangle_\mathcal{D} = \int_\mathcal{D} |f(z)|^2\, d\mu(z) < \infty\}$ with $d\mu(z) = \frac{dxdy}{y^2}$ and the inner product $\langle f, g\rangle = \int_\mathcal{D} f(z)\overline{g(z)} d\mu(z)$. $\mathcal{F}_q$ is isometric to T^q through the correspondance I:

$$T^q \ni f \to y^q f\,.$$

Under this correspondence, the operators ∇^z_q, $\nabla^q_{\bar{z}}$ go over to the Maasz operators $L_q : \mathcal{F}_q \to \mathcal{F}_{q-1}$, $K_q : \mathcal{F}_q \to \mathcal{F}_{q+1}$ according to the diagram

$$\begin{array}{ccccc} T^{q-1} & \overset{\nabla^z_q}{\leftarrow} & T^q & \overset{\nabla^q_{\bar{z}}}{\rightarrow} & T^{q+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{q-1} & \overset{L_q}{\leftarrow} & \mathcal{F}_q & \overset{K_q}{\rightarrow} & \mathcal{F}_{q+1} \end{array}$$

where $L_q = (\bar{z}-z)\frac{\partial}{\partial \bar{z}} - q$ and $K_q = (z-\bar{z})\frac{\partial}{\partial z} + q$. We have also:

$$\begin{aligned} L_q &= -2iy^{1+q}\frac{\partial}{\partial \bar{z}} y^{-q} = \overline{K_{-q}} \\ K_q &= 2iy^{1-q}\frac{\partial}{\partial z} y^q = \overline{L_{-q}}. \end{aligned}$$

We note

$$\begin{aligned} -L_{q+1}K_q &= -\Delta_{2q} + q(q+1) \\ -K_{q-1}L_q &= -\Delta_{2q} + q(q-1) \end{aligned}$$

with

$$\Delta_{2q} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - 2iqy\frac{\partial}{\partial x}\,.$$

[1]Some authors called them of weight q or $-2q$

These second order differential operators are self-adjoint on $\mathcal{F}_q$. Furthermore the isometry I conjugates Δ_q^+ with $-\Delta_{2q}+q(q+1)$ and Δ_q^- with $-\Delta_{2q}+q(q-1)$.
We are first interested in the case $q=2$. Let Δ_{Diff} the (positive) Laplacian on 1-forms on a geometrically finite hyperbolic surface, $\Delta_{\text{Diff}} = d\delta+\delta d$, $\delta = -*d*$ with $*$ the Hodge operator. In the following we note $\Delta_{\text{Diff}} = \Delta$. If ω is a 1-form in the holomorphic cotangent bundle, $\omega = f(z)\,dz$, then we define the image by the isometry I, $I(w) = I(f\,dz) = yf(z) = \tilde{f}(z)$. We have the relation: if $y\Delta(f\,dz) = -(\Delta_2\tilde{f})dz$, in other words, with the preceding notations $\Delta = \Delta_1^-$.

3.3. **Generalized eigenfunctions.** We are going to give the spectral expansion in eigenforms of Δ; we use [7], [20], [1]. For a finitely generated group of the second kind, for each cusp and for each funnel of the quotient there is a corresponding Eisenstein series, this is what we are going to develop now.

Proposition 3.1. *For* $\operatorname{Re} s > \delta$, *the kernel of the resolvent* $G_s(z,w,1)$ *for the self-adjoint operator* Δ_2 *acting on the Hilbert space* $L_{2,2}$ *of automorphic forms of weight 2, is given by the convergent series*

$$G_s(z,w,1) = \sum_{\gamma\in\Gamma} j_\gamma(w) g_s(z,\gamma w,1)$$

with $g_s(z,w,1) = -\dfrac{w-\bar{z}}{z-\bar{w}}\dfrac{\Gamma(s+1)\Gamma(s-1)}{4\pi\Gamma(2s)}\sigma^{-s}F(s+1,s-1,s;\sigma^{-1})$ *and* F *is the Gauss hypergeometric function.*

For the funnel case, we identify z' with the standard coordinates (ρ',t') in the funnel F_j, and define

$$E^f_{j,1}(s,z,t') = \lim_{\rho'\to 0} \rho'^{-s} G_s(z,z',1)\,, \tag{3}$$

for $j=1,...,n_f$. In the cusp C_j, with standard coordinates $z'=(\rho',t')$, we set

$$E^c_{j,1}(s,z) = \lim_{\rho'\to 0} \rho'^{1-s} G_s(z,z',1)\,, \tag{4}$$

for $j=1,...,n_c$.
Let

$$P(z,\zeta) = \operatorname{Im}(z)/|z-\zeta|^2$$

where $z\in H$ and $\zeta\in\mathbb{R}$ be the Poisson kernel. For $b\in O(\Gamma) = \mathbb{R}\cup\{\infty\} - L(\Gamma)$ define the Eisenstein series ([20], [3])

$$E_b(z,s,k) = \sum_{\gamma\in\Gamma} j(\gamma,z)^k P(\gamma(z),b)^s (\gamma(z),b)^k\,,$$

where $j(\gamma,z) = \gamma'(z)/|\gamma'(z)|$ and $(z,b) = (\bar{z}-b)/(z-b)$.
For the standard funnel F_l which corresponds to the region $\operatorname{Re} z \geq 0$ in the model $C_l = \Gamma_l\backslash H$, we have (see [7] p.200):

$$\begin{aligned}\frac{1}{1-2s}E^f_{l,1}(s,z,x') &= \lim_{z'\to x'} (\operatorname{Im} z')^{-s} G_s(z,z',1)\\ &= -\frac{4^s}{4\pi}\frac{\Gamma(s+1)\Gamma(s-1)}{\Gamma(2s)}E_{x'}(z,s,1).\end{aligned}$$

We note then

$$\mathcal{E}_{f_l}(s,z,x') = \frac{E^f_{l,1}(s,z,x')}{y}\,dz.$$

Recall 3.1. *Recall the definition of the classical Eisenstein series. The stability group of a cusp $\mathfrak{a}$ is an infinite cyclic group generated by a parabolic motion,*

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\,\mathfrak{a} = \mathfrak{a}\} = \langle \gamma_{\mathfrak{a}} \rangle \ ,$$

say. There exists $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ such that $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$, $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We call $\sigma_{\mathfrak{a}}$ a scaling matrix of the cusp $\mathfrak{a}$, it is determinated up to composition with a translation from the right. The Eisenstein series for the cusp $\mathfrak{a}$ is then defined by:

$$E_{\mathfrak{a}}(z,s) = \sum_{\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \ ,$$

where s is a complex variable, $\mathrm{Re}\, s > 1$.

An Eisenstein series of weight 2 associated to a cusp $\mathfrak{a}$ is the 1-form, defined for $\mathrm{Re}\, s > 1$, *by:*

$$\mathcal{E}_{\mathfrak{a}}(s,z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\ d(\sigma_{\mathfrak{a}}^{-1}\gamma z) = \frac{E_{\mathfrak{a},1}(s,z)}{y}\ dz \ ,$$

with

$$E_{\mathfrak{a},1}(s,z) = y \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} y(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\ ((\sigma_{\mathfrak{a}}^{-1}\gamma)'z) \ .$$

We now verify that it corresponds to the defining formula (4).
For the standard cusp, we write

$$G_s(z,z',1) = \sum_{\Gamma_\infty\backslash\Gamma} \left(\frac{c\bar{z}+d}{cz+d}\right) G_s^{\Gamma_\infty}(\gamma z, z', 1)\,,$$

where $G_s^{\Gamma_\infty}(\gamma z, z', 1)$ is the resolvent kernel of the standard cusp for automorphic forms of weight 2. We use then [7] p. 155 (38), p.177 for $\mathrm{Im}\, z' > \mathrm{Im}\,\gamma z$, p. 172 (see also [1] p.72, p.102) to conclude that

$$\lim_{y'\to\infty} y'^{s-1} G_s(z,z',1) = \sum_{\Gamma_\infty\backslash\Gamma} \left(\frac{c\bar{z}+d}{cz+d}\right) \frac{(\mathrm{Im}\,\gamma z)^s}{1-2s} = \frac{1}{1-2s} E_{\infty,1}(s,z)\,.$$

With the preceding notations we then have (see for example [7], [20]). For $w = f(z)\,dz$ square integrable, we have

$$\begin{aligned} w(z) \;&=\; \sum_{i=1}^{m} (w)_{\lambda_i}(z) + \frac{1}{4\pi i}\sum_{j=1}^{n_c} \int_{-\infty}^{+\infty} \langle w, \mathcal{E}_{c_j}(1/2+it,.)\rangle \mathcal{E}_{c_j}(1/2+it,z)\,dt \;+ \\ &\frac{1}{4\pi i}\sum_{j=1}^{n_f} \int_{-\infty}^{+\infty} \left[\int_1^{\lambda_f^2} \langle w, \mathcal{E}_{f_j}(1/2+it,.,b)\rangle \mathcal{E}_{f_j}(1/2+it,z,b)\,db\right] dt. \end{aligned}$$

Remark 3.1. *One can easily deduce the formula for an arbitrary square integrable 1-form*

$$\Omega(z) = f\,dz + g\,d\bar{z} = \sum_{i=1}^{m} (\Omega)_{\lambda_i}(z) +$$
$$\frac{1}{4\pi i}\sum_{j=1}^{n_c}\int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2+it,.)\rangle \mathcal{E}_{c_j}(1/2+it,z) \;+ \langle \Omega, \mathcal{E}_{c_j}(1/2+it,.)_{-1}\rangle \mathcal{E}_{c_j}(1/2+it,z)_{-1}\,dt +$$
$$\frac{1}{4\pi i}\sum_{j=1}^{n_f}\int_{-\infty}^{+\infty}\int_{1}^{\lambda_f^2} \langle \Omega, \mathcal{E}_{f_j}(1/2+it,.,b)\rangle \mathcal{E}_{f_j}(1/2+it,z,b) \;+ \langle \Omega, \mathcal{E}_{f_j}(1/2+it,.,b)_{-1}\rangle \mathcal{E}_{f_j}(1/2+it,z,b)_{-1}\,db\,dt\,,$$

where, with obvious notations, $\mathcal{E}_{-1} = \overline{\mathcal{E}} = \frac{\overline{E}}{y} d\bar{z}$*. To simplify we will write*

$$\Omega(z) = (\Omega)_{\lambda_i}(z) + \frac{1}{4\pi i}\int_{-\infty}^{+\infty} \langle \Omega, \mathcal{E}_{c_j}(1/2+it,.)_{\pm}\rangle \mathcal{E}_{c_j}(1/2+it,z)_{\pm}\,dt +$$
$$\frac{1}{4\pi i}\int_{-\infty}^{+\infty}\left[\int_{1}^{\lambda_f^2} \langle \Omega, \mathcal{E}_{f_k}(1/2+it,.,b)_{\pm}\rangle \mathcal{E}_{f_k}(1/2+it,z,b)_{\pm}\,db\right] dt\,.$$

3.4. **Harmonic dual form.** We are now going to see

Proposition 3.2. *The hyperbolic Eisenstein series* Ω_c *are square integrable.*

Proof. We consider a fundamental domain D contained in $\{z, 1 \le |z| \le e^l\}$ in which the segment (i, ie^l) represents the geodesic c. We note $C_\lambda = \{z \in D, d(z,c) = \lambda\}$ and $F_\lambda = \{z \in D, d(z,c) \ge \lambda\}$. Without loss of generality we can suppose that there is only one funnel on M and no cusps. Let V_λ the volume of $F_\lambda - F_{\lambda+1}$ there exists a constant c_1 such that $V_\lambda \ge c_1(sh(\lambda+1) - sh(\lambda))$. For $\operatorname{Re} s = \sigma > 0$,

$$||\Omega_c(s,z)|| = ||\Omega(s,z)|| \le \frac{1}{|k(s)|}\sum_{\Gamma_1\backslash\Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$$

Let $\eta(z) = \sum_{\Gamma_1\backslash\Gamma} \frac{1}{(\cosh x_2(\gamma z))^{\sigma+1}}$, we have

$$\begin{aligned}
\int_D ||\Omega_c(s,z)||^2 d\mu(z) &\le \frac{1}{|k(s)|^2}\int_D \eta^2(z) d\mu(z)\\
&\le \frac{1}{|k(s)|^2}\int_{1\le x_1 \le e^l, -\infty < x_2 < +\infty} \eta(z)\frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2\, dx_1\, dx_2\\
&\le \frac{M}{|k(s)|^2}\int_{1\le x_1 \le e^l, -\infty < x_2 < +\infty} \frac{1}{(\cosh x_2(z))^{\sigma+1}} \cosh x_2\, dx_1\, dx_2
\end{aligned}$$

where $M > 0$ is such that $\forall z \in H$, $\eta(z) \le M$.
The last integral is $\frac{\Gamma(\frac{1}{2})\Gamma(\frac{\sigma}{2})}{\Gamma(\frac{1}{2}+\frac{\sigma}{2})}(e^l - 1)$ and the result. □

As before we have:

$$\Delta(\Omega(s,z)) + s(s+1)\Omega(s,z) = s(s+1)\Omega(s+2,z)\,.$$

This formula has a consequence that, for fixed s with $\mathrm{Re}\, s > 0$, the function $\Delta^k(\Omega(s,z))$ is again square integrable for any $k > 0$.
Set $\mathrm{Re}\, s > 0$, with our convention of notations

$$\Omega(s,z) = \Omega_0(z) + a_i(s)\varphi_i(z) + \frac{1}{4\pi i}\int_{-\infty}^{+\infty} h^c_\pm(s,t)\mathcal{E}_c(1/2+it,z)_\pm\, dt$$

$$+\frac{1}{4\pi i}\int_{-\infty}^{+\infty}\left[\int_1^{\lambda_f^2} H^f_\pm(s,t,b)\mathcal{E}_f(1/2+it,z,b)_\pm\, db\right] dt\,. \tag{5}$$

We obtain

$$H(s,t,b)[1/4+t^2+s(s+1)] = s(s+1)H(s+2,t,b)\,,$$

where H corresponds to any $H^\pm_f$. From this we get a continuation of H to the region $\mathrm{Re}\, s > -1/2$ and we note that we have for all t and all b $H(0,t,b) = 0$.
Moreover for $\mathrm{Re}\, s > -1/2$, $\mathrm{Re}(s+2) > 0$ and we may substitute in (5) to obtain a continuation of $\Omega(s,z)$ to $\mathrm{Re}\, s > -1/2$. Thus we have proved the following theorem

Theorem 3.1. *$\Omega(s,z)$ has a meromorphic continuation to $\mathrm{Re}\, s > -1/2$ with $s = 0$ a regular point and $\Omega(0,z)$ is a harmonic form which is dual to c.*

Remark 3.2. *1) Another way to see this:*
write $\Omega(s,z) = (\Delta + s(s+1))^{-1}(s(s+1)\Omega(s+2,z))$ and use the meromorphic continuation of the resolvent (see for example [1], [21]*).*
2) With an analogue study of [17] *(see also* [16]*) we can obtain a total description of the singularities of the hyperbolic Eisenstein series.*

4. The case of an infinite geodesic joining two points.

Without loss of generality we suppose the two cusps p and q to be 0 and ∞ respectively and, as the lift of the geodesic, we take the imaginary axis. Let η be the infinite geodesic $]p,q[$, can we do the same construction as Kudla and Millson? As in the finite volume case, the problem reduces to study the following series for $\mathrm{Re}\, s > 1$:

$$\hat{\eta}^s(z) = \frac{1}{k(s-1)}\sum_{\gamma\in\Gamma}\gamma * \left[\left(\frac{y}{|z|}\right)^{s-1}\mathrm{Im}(z^{-1}dz)\right] = \mathrm{Im}(\theta^s(z))\ , \tag{6}$$

where

$$\theta^s(z) = \frac{1}{k(s-1)}\sum_{\gamma\in\Gamma}\gamma * \left[\left(\frac{y}{|z|}\right)^{s-1}\frac{dz}{z}\right]\ ,$$

and $k(s-1) = \dfrac{\Gamma(1/2)\Gamma(s/2)}{\Gamma(1/2+s/2)}$.

4.1. **Some useful estimations.** As usual we can suppose $\Gamma_\infty = \langle z \mapsto z+1\rangle$ to be the stabilizer of ∞ in Γ and the stabilizer of 0, Γ_0 is then generated by $z \mapsto \frac{z}{-c_0^2 z+1}$ (for some non zero constant c_0).
First of all we note that, contrary to the finite volume case, we have $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\gamma z)$ convergent (see proposition 3.1 and formula 4). Another way to see this "by hand":

Remark 4.1. *We know that for* $\operatorname{Re} s > \delta$, $\sum_{T\in\Gamma} e^{-sd(i,Tz)}$ *converges, moreover there exists a constant* $C > 0$ *such that* $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}^s(\gamma z) \leq C \sum_{T\in\Gamma} e^{-sd(i,Tz)}$*, as in our case* $\delta < 1$*, we have the result.*
As $\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left|\operatorname{Im}^s(\gamma z)\frac{c\bar{z}+d}{cz+d}\right| = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}^{\operatorname{Re} s}(\gamma z)$ *we also deduce the convergence of* $E_{\infty,1}(1,z)$*.*

With the notations of recall 3.1 and ρ the standard coordinate for the cusp $\mathfrak{a}$, we write the results of [1](p.110): $E_{\mathfrak{a}}(s,.) = \rho^{-s}(1-\chi_0(\rho)) + 0(\rho_f^s\rho_c^{s-1})$, ρ is decomposed as $\rho_f\rho_c$ with $\rho_f = \rho$ in the funnels and $\rho_c = \rho$ in the cusps and we define $\chi_0 \in \mathcal{C}_0^\infty(X)$ such that

$$\chi_0 = \left\{ \begin{array}{ll} 1, & r \leq 0 \\ 0, & r \geq 1 \end{array} \right. .$$

Lemma 4.1. *We have the following asymptotic behaviors for* $\operatorname{Re} s > \delta$*:*
1) in a funnel for all cusp $\mathfrak{a}$, $E_{\mathfrak{a}}(s,z)$ *is square integrable;*
2) at $\mathfrak{a} = \infty$, $E_\infty(s,z) - y^s = O(y^{1-s})$ *and* $E_0(s,z) = O(y^{1-s})$*;*
3) near $\mathfrak{a} = 0$, $E_0(s,z) - y^s/(c_0^2|z|^2)^s = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$ *and* $E_\infty(s,z) = O(y^{1-s}/(c_0^2|z|^2)^{s-1})$*.*

4.2. **Convergence of the Hyperbolic Eisenstein series and analytic continuation.** The calculus and results to prove the convergence of (6) adapt easily from the finite volume case. For the convenience of the lecture we recall the essential points.
We have $|| \sum_{\gamma\in\Gamma} \gamma * \left(\frac{y}{|z|}\right)^{s-1} \operatorname{Im}(z^{-1}dz)|| \leq \sum_{\gamma\in\Gamma} \left(\frac{y}{|z|}\right)^{\sigma}(\gamma z)$, where $\sigma = \operatorname{Re} s > 1$ and if we denote by $\mathcal{S} = \sum_{\gamma\in\Gamma} \left(\frac{y}{|z|}\right)^{\sigma}(\gamma z)$, we have

$$\mathcal{S} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \sum_{n\in\mathbb{Z}} \frac{1}{|\gamma z + n|^\sigma} .$$

Let S_z a system of representatives of $\Gamma_\infty\backslash\Gamma$ such that $|\operatorname{Re}\gamma z| \leq 1/2$, then

$$||\mathcal{S}|| \leq \sum_{\gamma\in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + 2\sum_{\gamma\in S_z} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma} .$$

We have

$$\begin{aligned} \sum_{\gamma\in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &= \sum_{\Gamma\backslash S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n\in\mathbb{Z}} \frac{1}{|ne\gamma z + 1|^\sigma} \\ &= \sum_{\gamma\in\Gamma\backslash S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} + \sum_{\gamma\in\Gamma\backslash S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} \sum_{n\in\mathbb{Z}^*} \frac{1}{|ne|^\sigma[(x(\gamma z) + 1/ne)^2 + y^2(\gamma z)]^{\sigma/2}}. \end{aligned}$$

and for K a compact set in H there exists m in H such that

$$\forall z \in K, \forall\gamma \in \Gamma_0\backslash S_z,\ |\gamma z| \geq |m| \text{ and } \operatorname{Im}\gamma z \geq \operatorname{Im} m\,.$$

So

$$\begin{aligned} \sum_{\gamma\in S_z} \frac{y^\sigma(\gamma z)}{|\gamma z|^\sigma} &\leq \sum_{\gamma\in\Gamma_0\backslash S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} + \sum_{\gamma\in\Gamma_0\backslash S_z} \frac{y^\sigma(\gamma z)}{|m|^\sigma} \sum_{n\in\mathbb{Z}^*} \frac{1}{|ne|^\sigma(\operatorname{Im} m)^\sigma} \\ &\leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) + 2\sum_{n\in\mathbb{N}^*} \frac{1}{(ne)^\sigma} \frac{1}{|m|^\sigma(\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \ ; \end{aligned}$$

and finally

$$||\mathcal{S}|| \leq \frac{1}{|m|^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) + 2\sum_{n\in\mathbb{N}^*} \frac{1}{(ne)^\sigma} \frac{1}{|m|^\sigma(\operatorname{Im} m)^\sigma} \sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) +$$

$$2\sum_{\Gamma_\infty\backslash\Gamma} y^\sigma(\gamma z) \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^\sigma},$$

the uniform convergence on all compact of H and all compact of $\operatorname{Re} s > 1$.

From this ultimate inequality we conclude that θ^s is square integrable in the funnels as the Eisenstein series $\mathcal{E}_\infty$. To conclude, we have the following theorem:

Theorem 4.1. *For* $\operatorname{Re} s > 1$*, the Eisenstein series associated to the geodesic* $\eta = (p, q)$ *converge uniformly on all compact sets. It represents a* $\mathcal{C}^\infty$ *closed form which is dual to* η*. For* $\operatorname{Re} s > 1$ *it satisfies the differential functional equation:*

$$\Delta\hat{\eta}^s = s(1-s)[\hat{\eta}^s - \hat{\eta}^{s+2}].$$

Now we want to prove the analytic continuation of $\hat{\eta}^s$ at $s = 1$. For this, first of all, we are going to show that $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ is square integrable. What we have to do is to investigate the Fourier expansion of θ^s, at each inequivalent cusp : 0 and ∞, and to show that $y|\theta^s(z)|$ is bounded.We have the following proposition whose proof is identical in the finite volume (see [5]):

Proposition 4.1. *At* ∞

$$\theta^s(z) = (\frac{1}{i} + O(1/y))\ dz\ ,$$

and at 0

$$\theta^s(z) = (-\frac{1}{ic_0^2 z^2} + O(1/y))\ dz\ .$$

By proposition 4.1 and lemma 4.1, we conclude:

Proposition 4.2. *The 1-forms* $\theta^s(z) - 1/i(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ *and* $\hat{\eta}^s(z) + \operatorname{Re}(\mathcal{E}_\infty(1, z) - \mathcal{E}_0(1, z))$ *are square integrable.*

Finally as in [5]:

Theorem 4.2. *The 1-form* $\hat{\eta}^s$ *has a meromorphic continuation to* $\operatorname{Re} s > 1/2$*, with* $s = 1$ *a regular point and* $\hat{\eta}$ *is the harmonic dual form to* η*.*

5. A case of degeneration.

A family of degenerating hyperbolic surfaces consists of a manifold M and a family $(g_l)_{l>0}$ of Riemannian metrics on M that meet the following assumptions: M is an oriented surface of negative Euler characteristic, and the metrics g_l are hyperbolic, chosen in such a way that there are finitely many closed curves c_i, geodesic with respect to all metrics, with the length l_i of each curve converging to 0 as l decreases. On the complement of the distinguished curves, the sequence of metrics is required to converge to a hyperbolic metric. More precisely there are finitely many disjoint open subsets $C_i \subset M$ that are diffeomorphic to cylinders $F_i \times J_i$

where $J_i \subset \mathbb{R}$ is a neighborhood of 0. The complement of $\bigcup_i C_i$ is relatively compact. The restriction of each metric g_l to $C_i = F_i \times J_i$ is a product metric

$$(x, a) \longmapsto (l_i^2 + a^2)dx^2 + (l_i^2 + a^2)^{-1}da^2$$

and $l_i \to 0$ as $l \to 0$ (the curves $F_i \times \{0\} \subset C_i$ are closed geodesics of length l_i with respect to g_l). Let M_l denote the surface M equipped with the metric g_l if $l > 0$, and let $M_0 = M \setminus \bigcup_i c_i$ carry the limit metric $\lim_{l\to 0} g_l$. Note that M_0 is a complete hyperbolic surface by definition, which contains a pair of cusps for each i. Here consider a family of surfaces $S_l = \Gamma_l \backslash H$ degenerating to the surface S with only one geodesic c_l being pinched, Γ_l containing the transformation $\sigma_l(z) = e^l z$ corresponding to c_l. Let K_l be S_l minus C_l the standard collar for c_l. There exist homeomorphisms f_l from $S_l \backslash c_l$ to S_0, with f_l tending to isometries C^2-uniformly on the compact core $K_l \subset S_l$, define $\pi_l = f_l^{-1}$. Suppose that p is one of the two cusps of S arising from pinching c_l. Let $S_0 = \Gamma \backslash H$ be the component of S containing p and conjugate Γ to represent the cusp by the translation $w \mapsto w + 1$, in the following $p = \infty$. Let for $\operatorname{Re} s > 1$ $\alpha_l(s, z) = \sum_{\gamma \in \langle \sigma_l \rangle \backslash \Gamma_l} \gamma * \left[\left(\frac{y}{|z|} \right)^{s-1} \operatorname{Im}(z^{-1}dz) \right]$ such that the hyperbolic Eisenstein series $\Omega_{c_l} = \Omega_l$ is related by $\Omega_l(s, z) = \frac{1}{k(s)} \alpha_l(s + 1, z)$. Without loss of generality we suppose S_l having only one funnel F_1. With the notations of the beginning, $S_l = K \cup (C_1 \cup ... \cup C_{n_c}) \cup F_1$ and c_l is the one geodesic of the boundary of the compact core K, we consider the specific case of p the limit of the right side of the c_l-collar contained in $S_l \backslash F_1$.

Theorem 5.1. *Let* $\operatorname{Re} s > 1$*, the family of 1-forms* $\frac{1}{l^s} \alpha_l(s, \pi_l(.))$ *converge uniformly on compact subsets of* S_0 *to* $\operatorname{Im} \mathcal{E}_\infty(s, .)$.

It is a particular case of the theorem 5.2 bellow.

The sketch of the proof of this theorem is the same as in the finite volume case ([4], [22], see also [8]), but for the convenience of the reader we recall some material and results.

The following lemma can be found for example in [1]. The neighborhood of points within distance a of a geodesic γ,

$$G_a = \{z \in K, d(z, \gamma) \le a\},$$

is isometric for small a to a half-collar $[0, a] \times S^1$, $ds^2 = dr^2 + l^2 \cosh^2 r \, d\theta^2$.

Lemma 5.1. *Suppose that* γ *is a simple closed geodesic of length* $l(\gamma)$ *on a geometrically finite hyperbolic surface* M*. Then* γ *has a collar neighborhood of half-width d, such that*

$$\sinh(d) = \frac{1}{\sinh(l(\gamma)/2)} .$$

As a consequence, if η *is any other closed geodesic intersecting* γ *transversally (still assuming* γ *is simple), then the lengths of the two geodesics satisfy the inequality*

$$\sinh(l(\eta)/2) \ge \frac{1}{\sinh(l(\gamma)/2)} .$$

Lemma 5.2. *Let* γ *be a simple closed geodesic of length* l *on a complete hyperbolic surface* M*. If* α *is a simple closed geodesic that does not intersect* γ*, then*

$$\cosh d(\gamma, \alpha) \ge \coth(l/2) .$$

A standard collar for a length l geodesic is a cylinder isometric to $\mathcal{C}\backslash\langle z\mapsto e^l z\rangle$ with $\mathcal{C}=\{z=re^{i\theta},1\le r\le e^l,l<\theta<\pi-l\}\subset H$ with the restriction of the hyperbolic metric, and $\langle z\mapsto e^l z\rangle$ the cyclic group generated by the transformation $z\mapsto e^l z$. There is a constant c_0 (the short geodesic constant) such that each closed geodesic on S of length at most c_0 has a neighborhood isometric to the standard collar and each cusp for S has a neighborhood isometric to the standard cusp; furthermore, the collars for short geodesics and the cusp regions are all mutually disjoint.

Now to study the right side of the c_l-collar let $w=\frac{1}{l}\log z$, $z\in H$, and conjugate Γ_l by the map w to obtain $\tilde{\Gamma}_l$ acting on $\mathcal{S}_l=\{w,0<\operatorname{Im}w<\pi/l\}$. The hyperbolic metric on $\mathcal{S}_l$ is $ds_l^2=\left(\frac{l|dw|}{\sin(l\operatorname{Im}w)}\right)^2$, which tends uniformly on compact subsets to $\left(\frac{|dw|}{\operatorname{Im}w}\right)^2$. $\tilde{\Gamma}_l$ is a (non Möbius) group of deck transformations acting on $\mathcal{S}_l$; the quotient $\tilde{\Gamma}_l\backslash\mathcal{S}_l$ is S_l. Let $\hat{f}_l$ be the restriction of f_l to the component $\mathcal{S}_l^{(r)}$ of $S_l\backslash c_l$ containing the right half-collar for c_l. Let F_l be a lift of $\hat{f}_l$ to the universel covers $\mathcal{A}_l$ and H, where $\mathcal{A}_l$ is the simply connected component of $H\backslash\pi^{-1}(c_l)$ which contained the standard right collar $\{z=re^{i\theta},1\le r\le e^l,l<\theta<\pi/2\}$. More precisely [4](p.350), [22]:

Lemma 5.3. *The simply connected component $\mathcal{A}_l$ contains $\{z=re^{i\theta},1\le r\le e^l,lc(l)<\theta<\pi/2\}$ where $c(l)\to 0$, $l\to 0$.*

Start with the standard Γ fundamental domain $\mathcal{F}=\{w,0\le\operatorname{Re}w<1,\operatorname{Im}w\ge\operatorname{Im}A(w),\forall A\in\Gamma\}$. Set $D_l=F_l^{-1}(\mathcal{F})$, then D_l is a fundamental domain of S_l. Divide the cosets of $\langle\sigma_l\rangle\backslash(\Gamma_l-\langle\sigma_l\rangle)$ into two classes $D=\{[A],A\in\Gamma_l,\inf\operatorname{Re}A(D_l)>0\}$ and $G=\{[A],A\in\Gamma_l,\sup\operatorname{Re}A(D_l)<0\}$.

Then $\hat{f}_l$ has a lift $\tilde{f}_l$, a homeomorphism from a sub domain of $\mathcal{S}_l$ to H: $\tilde{f}_l=F_l\circ w^{-1}:w(\mathcal{A}_l)\to H.\hat{f}_l$ induces a group homomorphism $\rho_l:\Gamma\to\tilde{\Gamma}_l$ by the rule $A\mapsto\tilde{f}_l^{-1}A\tilde{f}_l$, $A\in\Gamma$. We call $\rho_l(A)\in\tilde{\Gamma}_l$ the element corresponding to $A\in\Gamma$. Now by our normalizations for $\tilde{\Gamma}_l$ and Γ the translation $w\mapsto w+1$ corresponds to itself. If we specify the further normalization $\tilde{f}_l(i)=i$ then the lifts $\tilde{f}_l$ are uniquely determined and then we have [22]

Lemma 5.4. *The $\tilde{f}_l$ tend uniformly on compact subsets to the identity, and thus for $A\in\Gamma$, the corresponding elements $\rho_l(A)$ tend uniformly on compact subsets to A.*

Divide the cosets $\langle z+1\rangle\backslash(\tilde{\Gamma}_l-\langle z+1\rangle)$ into two classes, the left and the right: for $\mathcal{F}_l=\tilde{f}_l^{-1}(\mathcal{F})$, $L=wGw^{-1}=\{[A],A\in\tilde{\Gamma}_l,\inf\operatorname{Im}A(\mathcal{F}_l)>\pi/2l\}$ and $R=wDw^{-1}=\{[A],A\in\tilde{\Gamma}_l,\sup\operatorname{Im}A(\mathcal{F}_l)<\pi/2l\}$ (the line $\{\operatorname{Im}w=\pi/2l\}$ is a lift of c_l, and we write $[A]$ for the $\langle z+1\rangle$ coset of A). In particular the cosets $\langle z+1\rangle\backslash(\Gamma-\langle z+1\rangle)$ correspond to the right cosets of $\tilde{\Gamma}_l$: $\{[\rho_l(A)],A\in\Gamma,\langle w\mapsto w+1\rangle\}\subset R$. Then we can write, where χ_+ is the characteristic function of $\{\operatorname{Re}z>0\}$ and χ_- the one of $\{\operatorname{Re}z\le 0\}$,

$$\begin{aligned}A_{l,q}(s,z)&=\sum_{\langle\sigma_l\rangle\backslash\Gamma_l}\left(\frac{\gamma'(z)}{\gamma(z)}\right)^q\sin^{s-q}\theta(\gamma z)\,dz^q\\&=y(z)^{s-q}(\chi_++\sum_D\frac{\gamma'(z)^q}{\gamma(z)^q}\frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}})dz^q+y(z)^{s-q}(\chi_-+\sum_G\frac{\gamma'(z)^q}{\gamma(z)^q}\frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}})dz^q\end{aligned}$$

and the q-form on $\mathcal{S}_l^{(r)}$: $a_{l,q}^R(s,z)=y(z)^{s-q}(\chi_++\sum_D\frac{\gamma'(z)^q}{\gamma(z)^q}\frac{|\gamma'(z)|^{s-q}}{|\gamma(z)|^{s-q}})dz^q=l^q(\sin l\operatorname{Im}w)^{s-q}(\chi+\sum_R(\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q})dw^q$ where χ is the characteristic function of $w(\{\operatorname{Re}z>0\})$.

Theorem 5.2. *Let $S_l = \Gamma_l \backslash H$ a family of geometrically finite hyperbolic surfaces degenerating to the surface S with only one geodesic c_l being pinched and S_l having only one funnel F_1; Γ_l contains the transformation $\sigma_l(z) = e^l z$ corresponding to c_l and the right half-collar for σ_l is in $S_l \backslash F_1$. Let S_0 be as above. For $q \in \mathbb{N}$, we associate to the pinching geodesic c_l, the q-form defined for $\operatorname{Re} s > 1$ by:*

$$A_{l,q}(s,z) = \sum_{\langle \sigma_l \rangle \backslash \Gamma_l} \left(\frac{\gamma'(z)}{\gamma(z)} \right)^q \sin^{s-q} \theta(\gamma z)\, dz^q .$$

Let $w = \frac{1}{l} \log z$ and write again with a little misuse of notation $A_{l,q}(s,w) = a_{l,q}(s,w)\, dw^q$ and the corresponding q-automorphic form (for $\tilde{\Gamma}_l$) $\tilde{a}_{l,q}(s,w) = \operatorname{Im}(w)^q a_{l,q}(s,w)$. Note $\tilde{a}^R_{l,q}$ the right half of $\tilde{a}_{l,q}$. Then we have $\frac{1}{l^s} \tilde{a}^R_{l,q}(s, \pi_l(.))$ converge uniformly on compact subsets of S_0 and on compact subsets of $\operatorname{Re} s > 1$ to the Eisenstein series for weight q:

$$E_{\infty,q}(s,z) = \sum_{\Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s \left(\frac{c\bar{z} + d}{cz + d} \right)^q .$$

Before proving this theorem we give some complements and its corollaries.
Let for $\operatorname{Re} s > 1$, $b_q(s) = e^{i\pi q/2} \int_0^\pi \sin^{s-2} u e^{-iqu}\, du$. Note that $b_1(s) = k(s-1)$. The function b_q has the following properties (see e.g. [16]): $b_q(s+2) = \dfrac{s(s-1)}{s^2 - q^2} b_q(s)$, b_q admits a meromorphic continuation to all $s \in \mathbb{C}$, more precisely

$$b_q(s) = \pi 2^{-s+2} \frac{\Gamma(s+1)}{\Gamma(\frac{s+q}{2}+1)\Gamma(\frac{s-q}{2}+1)} .$$

In order to be consistent with the definition of Kudla and Millson's hyperbolic Eisenstein series, we may use the normalized $q-$automorphic forms

$$\Xi_{l,q}(s,z) = \frac{1}{b_q(s)} A_{l,q}(s,z). \tag{7}$$

We recall that the series (7) converges absolutely and locally uniformly for any $z \in H$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, and that it is invariant with respect to Γ. A straightforward computation shows that the series $A_{l,q}(s,z)$ satisfies the differential functional equation:

$$\Delta_q^{\pm} A_{l,q}(s,z) - s(1-s) A_{l,q}(s,z) = (s+q)(s-q) A_{l,q}(s+2,z),$$

and the series (7)

$$\Delta_q^{\pm} \Xi_{l,q}(s,z) - s(1-s) \Xi_{l,q}(s,z) = s(s-1) \Xi_{l,q}(s+2,z).$$

Proposition 5.1. *The series $A_{l,q}(s,z)$(resp. $\Xi_{l,q}(s,z)$) admits a meromorphic continuation to all of $\mathbb{C}$.*

There are different ways to prove this; one is to use the differential functional equation (7) and to apply the method developped in [17] (see also [16]). More precisely let $\tilde{A}_{l,q}(s,z)$ (resp. $\tilde{\Xi}_{l,q}(s,z)$) the $q-$automorphic form associated to $A_{l,q}(s,z)$ (resp. $\Xi_{l,q}(s,z)$). We have

$$\Delta_{2q} \tilde{A}_{l,q}(s,z) + s(1-s) \tilde{A}_{l,q}(s,z) = (s+q)(q-s) \tilde{A}_{l,q}(s+2,z),$$

and

$$\Delta_{2q} \Xi_{l,q}(s,z) + s(1-s) \tilde{\Xi}_{l,q}(s,z) = s(1-s) \tilde{\Xi}_{l,q}(s+2,z); \tag{8}$$

in other words

$$\tilde{\Xi}_{l,q} = s(1-s)\int_{\mathcal{D}} G_s(z,z',q)\tilde{\Xi}_{l,q}(s+2,z')\,d\mu(z')\,.$$

We precise another calculation we will develop further. We can rewrite $\tilde{A}_{l,q}(s,z)$ as

$$\tilde{A}_{l,q}(s,z) = \sum_{\Gamma_l\backslash\Gamma}\left(\frac{c\bar{z}+d}{cz+d}\right)^q\left(\frac{\overline{\gamma z}}{\gamma z}\right)^{q/2}\left(\frac{\operatorname{Im}\gamma z}{|\gamma z|}\right)^s$$

and using the Fourier development of $G_s(z,z',q)$ and the expansion of $\int_1^{e^l} G_s(z,iy',q)\,d\ln y'$ we obtain the result (see [7] corollary 4.2 p.188).

Theorem 5.3. *Let λ_k, $1\leq k\leq n$ the eigenvalues of the Laplace-Beltrami operator on S_0 such that $0<\lambda_1<\lambda_2<...<\lambda_n<1/4\leq\lambda_{n+1}$ and the corresponding $s_k=\frac{1}{2}+\sqrt{\frac{1}{4}-\lambda_k}$ with $\operatorname{Re}s\geq 1/2$. The family $(\frac{1}{l^s}\tilde{a}^R_{l,q}(s,\pi_l(.)))_l$ converge uniformly on compact subsets of S_0 and on compact subsets of $\Omega=\{\operatorname{Re}s>1/2\}\backslash\{s_1,...,s_n\}$ to $E_{\infty,q}(.,s)$.*

Corollary 5.1. *In the case the geodesic c_l is not separating $(\frac{1}{l^s}\tilde{a}_{l,q}(s,\pi_l(.)))_l$ converge uniformly on compact subsets of S_0 and Ω to $E_{\infty,q}-E_{0,q}$.*

Corollary 5.2. *In the case c_l is the geodesic boundary of a funnel $(\frac{1}{l^s}\tilde{a}_{l,q}(s,\pi_l(.)))_l$ converge uniformly on compact subsets of S_0 and Ω to $E_{\infty,q}$.*

Remark 5.1. *We refer also to the result of J.Fay [6] final remark p.201-202.*

Proof. (theorem 5.2)

We have $\frac{1}{l^s}\tilde{a}^R_{l,q}(s,w)=\frac{1}{l^{s-q}}(\operatorname{Im}w)^q(\sin l\operatorname{Im}w)^{s-q}(\chi+\sum_R(\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q})=$ $\frac{1}{l^{s-q}}(\operatorname{Im}w)^q(\sin l\operatorname{Im}w)^{s-q}(\chi+\sum_{G_l}(\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q})+\frac{1}{l^{s-q}}(\operatorname{Im}w)^q(\sin l\operatorname{Im}w)^{s-q}(\sum_{R-G_l}(\tilde{\gamma}'w)^q|\tilde{\gamma}'w|^{s-q})$.

Because of lemma 5.4, the principal problem lies in estimating the second sum: it remains to show that

$$\lim_{l\to 0}\sum_{R-G_l}|\tilde{\gamma}'w|^\sigma=0\,,$$

where $\sigma=\operatorname{Re}s$. It is enough to demonstrate the convergence for β a relatively compact set in the fundamental domain $\mathcal{F}$. Given $\epsilon>0$ denote G the set of cosets and representatives for $\langle z+1\rangle\backslash\Gamma$ such that $\sup\operatorname{Im}A(\mathcal{F})<\epsilon$ for $[A]\notin G$ and let G_l be the corresponding cosets of $\langle z+1\rangle\backslash\tilde{\Gamma}_l$ with corresponding representatives. The set G is finite.
The cosets G of Γ satisfy (modulo the $\langle z+1\rangle$ action) $\{0\leq\operatorname{Re}w<1,\operatorname{Im}w>\epsilon\}\subset\cup_{A\in G}A(\mathcal{F})$; thus for l sufficiently small the cosets G_l satisfy (modulo the $\langle z+1\rangle$ action) $\{0\leq\operatorname{Re}w<1,2\epsilon<\operatorname{Im}w<\pi/2l\}\subset\cup_{A\in G_l}A(\mathcal{F}_l)$ (a consequence of the convergence on compact subsets of the $\tilde{f}_l$ and that $\mathcal{F}_l$ contains the right-half-collar for c_l , $\{0\leq\operatorname{Re}w<1,c(l)\leq\operatorname{Im}w<\pi/2l\}$). Now for a right coset $[A]\in R-G_l$ then $A(\mathcal{F}_l)$ lies below the c_l geodesic $\{\operatorname{Im}w=\pi/2l\}$ and is disjoint modulo the $\langle z+1\rangle$ action from $\{0\leq\operatorname{Re}w<1,2\epsilon<\operatorname{Im}w<\pi/2l\}$, since the latter is covered by the G_l cosets. Thus for $[A]\in R-G_l$, modulo the $\langle z+1\rangle$ action, then $A(\mathcal{F}_l)\subset\{0\leq\operatorname{Re}w<1,\operatorname{Im}w<2\epsilon\}$. For $w\in\beta$ we write

$$\sum_{R-G_l}|\tilde{\gamma}'w|^\sigma=\sum_{w^{-1}(R-G_l)w}|z_l\frac{\gamma'z_l}{\gamma z_l}|^\sigma\leq|z_l|^\sigma\sum_{w^{-1}(R-G_l)w}|\gamma'z_l|^\sigma\,,$$

where $w = \frac{1}{l}\log z_l$ and $\gamma\mathcal{F}_l \subset w^{-1}(\{0 \le \operatorname{Re} w < 1, \operatorname{Im} w < 2\epsilon\})$. We deduce

$$\sum_{R-G_l} |\tilde{\gamma}' w|^\sigma \le \frac{1}{\sin^\sigma(l \operatorname{Im} w)} \sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l)\,.$$

Let $\epsilon_0 \in]0, \sinh^{-1} 1[$ such that for $l \le l_0(\epsilon)$ and $w \in \beta$, $B(z_l, \epsilon_0) \subset \mathcal{F}_l$. Now y^s is an eigenfunction of all the invariant integral operators on H. Let $k(z, z')$ be the point-pair invariant defined by $k(z, z') = 1$ or 0 according as the distance between z and z' is smaller than ϵ_0. Then there exists Λ_{ϵ_0} independent of z_0 so that

$$\int_{B(z_0,\epsilon)} y^\sigma \frac{dxdy}{y^2} = \int_H k(z_0, z) y^\sigma \frac{dxdy}{y^2}$$

and

$$\int_{B(z_0,\epsilon)} y^\sigma \frac{dxdy}{y^2} = \Lambda_{\epsilon_0} y(z_0)^\sigma\,.$$

So

$$\sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma = \frac{1}{\Lambda_{\epsilon_0}} \sum_{w^{-1}(R-G_l)w} \int_{B(\gamma z_l,\epsilon_0)} y^\sigma \frac{dxdy}{y^2}\,.$$

Now [22]:

Lemma 5.5. *The multiplicity of the projection map $H \to H\backslash\Gamma$ restricted to $B(z_0, \eta)$ with $2\eta < c_0$ is at most $M\rho^{-2}(z_0)$ where M a constant and $\rho(z_0)$ the injectivity radius at z_0.*

Proof. If $B(z_0, \eta) \cap B(\gamma z_0, \eta) \ne \emptyset$, $\gamma \in \Gamma$, then $d(z_0, \gamma z_0) < 2\eta < c_0$ and z_0 is in a cusp region or the collar for a short geodesic. Let $c = c_0/2$, then we have $\rho(z_0) < c$. Set $m(\eta)$ the multiplicity of the projection restricted to $B(z_0, \eta)$. As $2\eta + \rho(z_0) < 3c$ and the $B(\gamma z_0, \rho(z_0))$ are disjoints we have

$$m(\eta)\mu(B(z_0, \rho(z_0)) \le \mu(B(z_0, 3c))\,.$$

So $m(\eta) \le \frac{\cosh 3c - 1}{\cosh \rho(z_0) - 1} \le 2(\cosh 3c - 1)\rho(z_0)^{-2}$. □

Then we have

$$\sum_{w^{-1}(R-G_l)w} \operatorname{Im}^\sigma(\gamma z_l)^\sigma \le \frac{A(c_0)}{\Lambda_{\epsilon_0}} \rho^{-2}(z_l) \frac{e^{l\sigma} - 1}{\sigma} \frac{(2\epsilon l)^{\sigma-1}}{\sigma - 1}\,.$$

Moreover as $B(z_l, \epsilon_0) \subset \mathcal{F}_l$, $\rho(z_l) \ge \epsilon_0$ and the conclusion.

□

Proof. (corollary 5.2) We remark that σ_l being the geodesic in the funnel, then $D = \langle\sigma_l\rangle\backslash(\Gamma_l - <\sigma_l\rangle)$ and $G = \emptyset$.

□

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