DEGREE BOUNDS FOR TYPE-A WEIGHT RINGS AND GELFAND–TSETLIN SEMIGROUPS

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Abstract. A weight ring in type A is the coordinate ring of the GIT quotient of the variety of flags in $\mathbb{C}^n$ modulo a twisted action of the maximal torus in $\mathsf{SL}(n, \mathbb{C})$. We show that any weight ring in type A is generated by elements of degree strictly less than the Krull dimension, which is at worst $O(n^2)$. On the other hand, we show that the associated semigroup of Gelfand–Tsetlin patterns can have an essential generator of degree exponential in n.

Contents

1. Introduction

Given a pair λ, μ of weights for $\mathsf{SL}_n(\mathbb{C})$ with λ dominant, let $V_\lambda[\mu]$ denote the μ-isotropic component of the irreducible representation V_λ with highest weight λ. The *weight ring* $R(\lambda, \mu)$ is the graded ring $\bigoplus_{N=0}^{\infty} V_{N\lambda}[N\mu]$; it is the projective coordinate ring of the GIT quotient of the flag variety modulo the μ–twisted action of the maximal torus $\mathbb{T}$ in $\mathsf{SL}_n(\mathbb{C})$. We define the *weight variety* $W(\lambda, \mu)$ as

$$W(\lambda, \mu) \coloneqq \operatorname{Proj} R(\lambda, \mu).$$

Remark 1.0.1. Weight varieties for arbitrary reductive Lie groups (not just those of type A) were studied by A. Knutson in his Ph.D. thesis [11]. Knutson also studied the symplectic geometry of these spaces.

Our first theorem (Theorem 3.0.13 below) is that $R(\lambda, \mu)$ is generated in degree strictly less than the Krull dimension of $R(\lambda, \mu)$, provided that the degree-one piece $V_\lambda[\mu]$ is nonzero.

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The basic idea behind the proof is to show that the degree-one piece contains a system of parameters, and that the a-invariant of $R(\lambda, \mu)$ is negative. (The a-invariant is the degree of the Hilbert series, which is a rational function.) The theorem then follows from the fact that $R(\lambda, \mu)$ is Cohen–Macaulay.

Remark 1.0.2. The a-invariant is negative in all types; however, for types other than type A, the degree-one piece sometimes fails to contain a system of parameters. This condition is equivalent to the condition that all semistable flags lie in the supports of degree-one $\mathbb{T}$-invariants; see [9] for a counterexample in $G = \mathrm{SO}_5(\mathbb{C})$.

It is well known (cf. [6,7,12]) that $R(\lambda, \mu)$ has a flat degeneration to the semigroup algebra $R'(\lambda, \mu)$ of the semigroup of Gelfand–Tsetlin patterns associated to semistandard tableaux of shape $m\lambda$ and content $m\mu$ for $m \geq 0$. In particular, the ring $R'(\lambda, \mu)$ is the graded ring associated to a filtration of $R(\lambda, \mu)$ by natural numbers. Generators for $R'(\lambda, \mu)$ can be lifted to generators of $R(\lambda, \mu)$, so one might hope that $R'(\lambda, \mu)$ is relatively simple. Unfortunately, we find pairs λ, μ for which $R'(\lambda, \mu)$ has essential generators of degree exponential in n.

Our second main result (Theorem 5.0.16 below) is that, in the case where $n = 3k$ is a multiple of 3, the semigroup algebra $R'(k\varpi_3, 0)$ has an essential generator of degree approximately $(\sqrt{2})^n$. This is in striking contrast to the lower bound of $2n - 8$ for $R(k\varpi_3, 0)$ that follows from our first theorem (since the Krull dimension of $R(k\varpi_3, 0)$ is $2n - 7$). This case is particularly interesting, because, via the Gelfand–MacPherson correspondence, it is the moduli space of n-tuples of points in the projective plane. This is a remarkable example of how a semigroup algebra produced by a promising toric degeneration can fail to serve as an effective proxy for the original ring.

Our motivation for studying the semigroup of Gelfand–Tsetlin patterns was to imitate the method of [10], which studied the case of n points on the projective line. Here one takes λ to be a multiple of the second fundamental weight ϖ_2. It was shown in [10] that the associated semigroup of Gelfand–Tsetlin patterns is generated in degree ≤ 2. We had hoped to use the same method in the case of n points in the projective plane, but the second theorem indicates why this is not the right approach. However, there might still be another toric degeneration that yields a bound better than the one in Theorem 3.0.13, perhaps among those discovered by Caldero [2].

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2. A description of the weight ring $R(\lambda, \mu)$

In this section, we give an explicit description of $R(\lambda, \mu)$. Let $n \geq 2$, and let B denote the Borel subgroup of $\mathsf{SL}_n(\mathbb{C})$ consisting of the upper-triangular matrices in $\mathsf{SL}_n(\mathbb{C})$. Fix a nontrivial dominant weight λ of $\mathsf{SL}_n(\mathbb{C})$. We represent λ as a *partition*, *i.e.*, as a weakly

decreasing sequence $(\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, with $\lambda_1 \geq 1$ and $\lambda_n = 0$. Let μ be a weight of $\mathsf{SL}_n(\mathbb{C})$ such that $V_\lambda[\mu]$ is nonzero. Thus μ may be expressed as a sequence $(\mu_1, \ldots, \mu_n)$ of nonnegative integers such that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$.

By the Borel–Weil construction, the irreducible representation with highest weight λ is the finite-dimensional vector space

$$V_\lambda = \big\{\text{holomorphic } f\colon \mathsf{SL}_n(\mathbb{C}) \to \mathbb{C} \;\mid\; f(gb) = e^\lambda(b) f(g) \text{ for all } g \in \mathsf{SL}_n(\mathbb{C}) \text{ and } b \in B\big\},$$

where $e^\lambda(b) \coloneqq \prod_{i=1}^n b_{ii}^{\lambda_i}$ for $b = (b_{ij})_{1 \leq i,j \leq n} \in B$. The action of $\mathsf{SL}_n(\mathbb{C})$ on V_λ is given by $(g \cdot f)(h) = f(g^{-1}h)$ for $g, h \in \mathsf{SL}_n(\mathbb{C})$. We define

$$R(\lambda) \coloneqq \bigoplus_{N=0}^{\infty} V_{N\lambda}.$$

Multiplication in $R(\lambda)$ is the usual multiplication of functions $\mathsf{SL}_n(\mathbb{C}) \to \mathbb{C}$.

Remark 2.0.3. As we will review in the next section, the dominant weight λ determines a line bundle $L_\lambda \to \mathsf{SL}_n(\mathbb{C})/B$ such that the space $\Gamma(\mathsf{SL}_n(\mathbb{C})/B, L_\lambda)$ of sections is isomorphic to V_λ. The multiplication in $R(\lambda)$ coincides with multiplication of the corresponding sections. The ring $R(\lambda)$ is the coordinate ring of the partial flag variety.

We define

$$V_\lambda[\mu] \coloneqq \{f \in V_\lambda \mid f(tg) = e^\mu(t) f(g) \text{ for all } t \in \mathbb{T},\, g \in \mathsf{SL}_n(\mathbb{C})\}.$$

We now define a μ-twisted action of $\mathbb{T} \subset \mathsf{SL}_n(\mathbb{C})$ on $R(\lambda)$. For $f \in V_{N\lambda} = R(\lambda)_N$ of degree N, the action of $t \in \mathbb{T}$ on f is given by

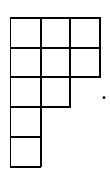
$$(t \cdot f)(g) \coloneqq e^{N\mu}(t) f(t^{-1}g).$$

Relative to this twisted action, the $\mathbb{T}$-invariant subring of $R(\lambda)$ is exactly

$$R(\lambda, \mu) \coloneqq \bigoplus_{N=0}^{\infty} V_{N\lambda}[N\mu].$$

Remark 2.0.4. There is a unique $\mathsf{SL}_n(\mathbb{C})$-linearization of L_λ. This defines a canonical $\mathbb{T}$-linearization of L_λ by restriction $T \hookrightarrow \mathsf{SL}_n(\mathbb{C})$. The above action of $\mathbb{T}$ coincides with the canonical $\mathbb{T}$-linearization twisted by μ.

A fundamental fact from the representation theory of $\mathsf{SL}_n(\mathbb{C})$ is that V_λ has a basis indexed by semistandard tableaux of shape λ. A *Young diagram* of shape λ is a left-justified arrangement of $\lambda_1 + \cdots + \lambda_n$ boxes with λ_i boxes in the ith row. For example, if $\lambda = (3, 3, 2, 1, 1, 0)$, then the Young diagram of shape λ is



A *semistandard tableaux* of shape λ is a filling of each box in a Young diagram of shape λ with a number from 1 through n such that the rows are weakly increasing and the columns are strictly increasing. For example, if $\lambda = (3, 3, 2, 1, 1, 0)$ then

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 1 & 5 \\ \hline 2 & 4 & 6 \\ \hline 3 & 5 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

is a semistandard tableau of shape λ.

Such a tableau τ determines a basis vector $b_\tau \in V_\lambda$ as follows. Write $\mathrm{len}(I)$ for the length of a column I of τ. We identify I with the $\mathrm{len}(I)$-tuple of its entries, read from top to bottom. If $I = (i_1, \ldots, i_{\mathrm{len}(I)})$, let $\det_I \colon \mathsf{SL}_n(\mathbb{C}) \to \mathbb{C}$ be the function that returns the determinant of the $\mathrm{len}(I) \times \mathrm{len}(I)$ submatrix consisting of rows $i_1, \ldots, i_{\mathrm{len}(I)}$ and columns $1, 2, \ldots, \mathrm{len}(I)$. The basis vector b_τ is then defined by

$$b_\tau \coloneqq \prod_{\text{columns } I \text{ of } \tau} \det{}_I \,.$$

Hence, in the example above, $b_\tau = \det_{1,2,3,5,6} \det_{1,4,5} \det_{5,6}$.

We can also describe the $\mathbb{T}$-isotropic subspace $V_\lambda[\mu]$ in terms of semistandard tableaux. The *content* of a tableau τ is $\mu = (\mu_1, \ldots, \mu_n)$ if μ_i is the number of boxes in τ that contain the number i for $1 \le i \le n$. The subspace $V_\lambda[\mu]$ is the span of the b_τ such that τ has shape λ and content μ.

3. The first theorem: generators of $R(\lambda, \mu)$

We will derive an upper bound on the degree in which $R(\lambda, \mu)$ is generated. We generally follow the method of [14] (also explained in [3]). The idea is to find a homogeneous system of parameters, together with an upper bound on the a-invariant of the ring; this yields an upper bound on the degree of a generating set.

We begin by referencing a result of [9] and showing why this implies the existence of a system of parameters in degree one. First we must introduce the notion of semistability. Recall that B is the Borel subgroup of $\mathsf{SL}_n(\mathbb{C})$, and $\mathsf{SL}_n(\mathbb{C})/B$ is the flag variety. A function $f \in V_\lambda$ defines a section of the line bundle $L_\lambda \coloneqq \mathsf{SL}_n(\mathbb{C}) \times_B \mathbb{C} \to \mathsf{SL}_n(\mathbb{C})/B$, where $\mathsf{SL}_n(\mathbb{C}) \times_B \mathbb{C}$ denotes the quotient of $\mathsf{SL}_n(\mathbb{C}) \times \mathbb{C}$ by the equivalence relation $(gb, e^\lambda(b)z) \sim (g, z)$. The projection $L_\lambda \to \mathsf{SL}_n(\mathbb{C})/B$ is given by sending the equivalence class of (g, z) to gB. We define the μ-twisted linearization of $\mathbb{T}$ on L_λ by $t \cdot (g, z) \coloneqq (t^{-1}g, e^\mu(t)z)$. Given $f \in V_\lambda$, we define a global section s_f of L_λ by $s_f(gB) = (g, f(g))$. The map $f \mapsto s_f$ is an isomorphism $V_\lambda \cong \Gamma(\mathsf{SL}_n(\mathbb{C})/B, L_\lambda)$. The μ-twisted torus action on L_λ defines an action on global sections, which coincides with the μ-twisted action on V_λ that we earlier defined.

A flag gB is *semistable* if there is a positive integer N and a $\mathbb{T}$-invariant global section s of $L_{N\lambda}$ such that $s(gB) \neq 0$. That is, gB is semistable if and only if there is an $N > 0$ and

an $f \in V_{N\lambda}[N\mu]$ such that $f(g) \neq 0$. It was shown in [9] that we may take $N = 1$. That is, gB is semistable if and only if there exists an $f \in V_\lambda[\mu]$ such that $f(g) \neq 0$. We shall use this fact to show that there is a system of parameters within $V_\lambda[\mu]$ for $R(\lambda, \mu)$.

We now recall some basic facts from commutative algebra. Our main references on Cohen–Macaulay rings and modules are [1,13]. Let k be an algebraically closed field. Suppose that A is a $\mathbb{Z}_{\geq 0}$-graded finitely-generated k-algebra with $A_0 = k$. Let $\mathfrak{m}$ denote the graded ideal generated by the positive degree homogeneous elements of A. Then $\mathfrak{m}$ is the unique graded ideal such that all other graded ideals are contained within it. A *homogeneous system of parameters* for A is a set of homogeneous elements $x_1, \ldots, x_s$ such that s is the Krull-dimension of A and the ideal $(x_1, \ldots, x_s)$ is $\mathfrak{m}$-primary. By [1, Theorem 1.5.17] we have that $x_1, \ldots, x_s$ is a homogeneous system of parameters if and only if A is an integral extension of the subalgebra $k[x_1, \ldots, x_s]$, and that this is the case if and only if A is a finitely-generated $k[x_1, \ldots, x_s]$-module.

Let the *null cone* $\mathcal{N}$ be the subvariety of points in $\operatorname{Spec} R(\lambda)$ at which all positive-degree homogeneous elements of $R(\lambda, \mu)$ vanish. The result of [9] translates into the following:

Proposition 3.0.5. *The elements of $V_\lambda[\mu]$ suffice to cut out the null cone set-theoretically. That is, $\mathcal{N}$ is exactly the set of points at which all elements of $V_\lambda[\mu]$ vanish.*

Now suppose that $I \subset R(\lambda)$ is the ideal of elements vanishing on the null cone. By the above proposition, I is the radical closure in $R(\lambda)$ of the ideal generated by $V_\lambda[\mu] \subset R(\lambda)$. Recall that $R(\lambda, \mu)$ is the ring of polynomials in $R(\lambda)$ that are invariant under the μ-twisted action of $\mathbb{T}$. Since $\mathbb{T}$ is linearly reductive, there is a canonical projection $\pi : R(\lambda) \to R(\lambda, \mu)$, called the *Reynolds operator*, which is $R(\lambda, \mu)$-linear. Following Hilbert (cf. [3, Prop. 3.1]), we have the following result.

Proposition 3.0.6. *The invariant ring $R(\lambda, \mu)$ is a finitely-generated module over the subalgebra generated by $V_\lambda[\mu] \subset R(\lambda, \mu)$.*

Proof. Let J and S be the ideal and subalgebra, respectively, generated by $V_\lambda[\mu]$ in $R(\lambda)$. Then, since $I = \operatorname{Rad}(J)$, we have that $I^m \subset J$ for some $m > 0$. Since $\mathbb{T}$ is linearly reductive, the invariant ring $R(\lambda, \mu)$ is finitely generated. Thus, there exist homogeneous $y_1, \ldots, y_t \in R(\lambda, \mu)$ such that $y_1, \ldots, y_t$ generate $R(\lambda, \mu)$. Suppose that $h_1, \ldots, h_\ell$ span $V_\lambda[\mu]$. We have that each y_i^m belongs to the ideal J, and so $y_i^m = \sum_{j=1}^{\ell} f_j h_j$ for some homogeneous $f_j \in R(\lambda)$. Now we apply the Reynolds's operator π to obtain $y_i^m = \sum_{j=1}^{\ell} \pi(f_j) h_j$. Each coefficient $\pi(f_j)$ is a homogeneous invariant of degree less than y_i^m. It follows that $R(\lambda, \mu)$ is generated as an S-module by monomials $\mathbf{m} = \prod_{i=1}^{t} y_i^{e_i}$, where each $e_i < m$. There are only a finite number of such monomials, proving the claim. □

We now have the following: (cf. Proposition 3.2 of [3])

Proposition 3.0.7. *The degree-one piece $V_\lambda[\mu]$ of $R(\lambda,\mu)$ contains a homogeneous system of parameters.*

Proof. We already know from the previous proposition that $R(\lambda,\mu)$ is a finitely-generated module over the subalgebra generated by its degree-one piece $V_\lambda[\mu]$. Let s be the Krull dimension of $R(\lambda,\mu)$. It is easy to show that s generic elements $f_1,\ldots,f_s$ in $V_\lambda[\mu]$ are algebraically independent and that $R(\lambda,\mu)$ is an integral extension of $\mathbb{C}[f_1,\ldots,f_s]$. □

Proposition 3.0.8. *The weight ring $R(\lambda,\mu)$ is Cohen–Macaulay.*

Proof. The argument is the same as that given in [13, Corollary 14.25] to show that $R(\lambda)$ is Cohen–Macaulay. We know that $R(\lambda,\mu)$ has a Gröbner degeneration to a semigroup algebra of Gelfand–Tsetlin patterns (see Section 4). Such semigroup algebras are the invariant subrings of polynomial rings by the action of a torus (cf. [4]). By the theorem of Hochster [8], the subring of torus invariants in a polynomial ring is Cohen–Macaulay. A general principle regarding Gröbner degenerations is that any good property of the special fiber is shared by the general fiber. This is true in particular for the Cohen–Macaulay property [13, Corollary 8.31]. □

Proposition 3.0.9. *If $f_1,\ldots,f_s$ is a homogeneous system of parameters for $R(\lambda,\mu)$, then $R(\lambda,\mu)$ is a free $\mathbb{C}[f_1,\ldots,f_s]$-module.*

Proof. Since $f_1,\ldots,f_s$ are algebraically independent, the ring $\mathbb{C}[f_1,\ldots,f_s]$ is regular, and so this proposition follows from [1, Proposition 2.2.11]. □

For a graded module M, let $H(M;t) \coloneqq \sum_{d=0}^{\infty} \dim(M_d)\, t^d$ denote the *Hilbert series* of M. It is well known that, if M is finitely generated, then $H(M;t)$ is a rational function in t. Let $a(M)$ be the degree of $H(M;t)$ as a rational function. The number $a(M)$ is called the *a-invariant* of M.

Fix a homogeneous system of parameters $f_1,\ldots,f_s \in V_\lambda[\mu]$ for $R(\lambda,\mu)$. Let $S = \mathbb{C}[f_1,\ldots,f_s]$ be the subalgebra generated by the f_i. For brevity of notation, we will write $R \coloneqq R(\lambda,\mu)$. Let $\mathbf{f}$ denote the s-tuple $f_1,\ldots,f_s$. By Theorems 13.37(5) and 13.37(6) of [13], R is a free S-module, and

$$H(R/\mathbf{f}R;t) = H(R;t)(1-t)^s.$$

But we can easily compute $H(R/\mathbf{f}R;t)$. Suppose that $R = Sy_1 \oplus \cdots \oplus Sy_m$. Let $k \coloneqq \max_j(\deg y_j)$. Now, $H(R/\mathbf{f}R;t)$ is the polynomial $p(t) = \sum_{i=0}^{k} h_d t^d$, where h_d is the number of y_j such that $\deg y_j = d$. Therefore, we have proved the following.

Proposition 3.0.10. *The ring $R(\lambda,\mu)$ is generated in degree $\leq k = \dim R(\lambda,\mu) + a(R(\lambda,\mu))$.*

Proposition 3.0.11. *The a-invariant of $R(\lambda,\mu)$ is negative.*

Proof. Let $R \coloneqq R(\lambda, \mu)$. The dimension of the d-th graded piece R_d of R is equal to the number of semistandard tableaux of shape $d\lambda$ and content $d\mu$; this coincides with the number of integer lattice points in the d-th dilate of the rational polytope $GT(\lambda, \mu)$ (see Definition 4.0.14 below). As a result of the theory of lattice point enumeration for rational polytopes (see, *e.g.*, [15, Chapter 4]), we may conclude that the Hilbert series $H(R; t) = \sum_{d=0}^{\infty} f(d)t^d$ is a rational function of negative degree. □

Remark 3.0.12. In fact, in all types, given a pair of weights λ, μ with λ dominant, the dimension of the $d\mu$-weight space in the irreducible representation $V_{d\lambda}$ with highest weight $d\lambda$ equals the number of integer lattice points in the d-th dilate of a certain polytope (for example the string polytope associated with the reduced word for the longest Weyl element). And so, in all types, the a-invariant of weight rings is negative.

The above propositions imply our first theorem:

Theorem 3.0.13. *The algebra $R(\lambda, \mu)$ is generated in degree strictly less than the Krull dimension of $R(\lambda, \mu)$.*

Proof. This follows immediately from Proposition 3.0.10 and Proposition 3.0.11. □

Finally, we point out that the Krull dimension of $R(\lambda, \mu)$ is one more than the dimension of the GIT quotient of the flag variety by $\mathbb{T}$. This is at most the dimension of the flag variety itself, which is $n(n-1)/2$. In the case of n points in projective space $\mathbb{P}^{m-1}$, where λ is a multiple of the m-th fundamental weight ϖ_m for $\mathsf{SL}_n(\mathbb{C})$, the Krull dimension of $R(\lambda, \mu)$ is at most $n(m-1) - (m^2 - 1) + 1$.

4. The toric degeneration to Gelfand–Tsetlin patterns

A *Gelfand–Tsetlin pattern*, or GT pattern, is a triangular array $\mathbf{x} = (x_{ij})_{1 \le i \le j \le n}$ of real numbers satisfying the *interlacing inequalities* $x_{i,j+1} \ge x_{ij} \ge x_{i+1,j+1}$. We express $\mathbf{x}$ as a triangular array by arranging the entries as follows:

$$
\begin{array}{ccccccccc}
x_{1n} & & x_{2n} & & x_{3n} & & \cdots & & x_{nn} \\
 & & & & \cdots\cdots\cdots\cdots & & & & \\
 & & x_{13} & & x_{23} & & x_{33} & & \\
 & & & x_{12} & & x_{22} & & & \\
 & & & & x_{11} & & & &
\end{array}
$$

Given a semistandard tableaux τ with entries from 1 through n, let $\tau(j)$ be the tableau obtained from τ by deleting all boxes containing indices strictly larger than j. Hence, $\tau(n) = \tau$. Let $\lambda(j)$ denote the shape of $\tau(j)$. One obtains an integral GT pattern $\mathbf{x}(\tau) =$

$(x(\tau)_{ij})_{1\le i\le j\le n}$ by letting $x(\tau)_{ij} = \lambda(j)_i$. If τ has shape λ and content $\mu = (\mu_1, \ldots, \mu_n)$, then the resulting GT pattern $\mathbf{x}(\tau)$ has top row λ, and, for $1 \le j \le n$,

$$\sum_{i=1}^{j} x(\tau)_{ij} = \mu_1 + \cdots + \mu_j.$$

We denote this assignment by $\Phi\colon \tau \to \mathbf{x}(\tau)$. It is easy to see that it is a bijection from semistandard tableaux of shape λ and content μ to integral GT patterns with top row λ and row sums equal to the partial sums of μ. The GT patterns with a fixed top row and fixed row sums constitute a rational polytope.

Definition 4.0.14. The *GT polytope* $GT(\lambda, \mu)$ is the set of *real* GT patterns $(x_{ij})_{1\le i\le j\le n}$ with top row λ and with row sums $\sum_{i=1}^{j} x_{ij} = \mu_1 + \cdots + \mu_j$ for $1 \le j \le n$.

Let $S(\lambda, \mu)$ denote the graded semigroup of integer GT patterns (under addition) that lie in $GT(N\lambda, N\mu)$ for some nonnegative integer N. Gonciulea and Lakshmibai have described a Gröbner degeneration of the ring $R(\lambda) = \bigoplus_{N=0}^{\infty} V_{N\lambda}$ to a semigroup algebra $R'(\lambda)$ as the special fiber [7]. It was shown in [12] (and also in [13, Corollary 14.24]) that this semigroup is isomorphic to the semigroup of integral GT patterns with top row $N\lambda$ for some nonnegative integer N. This construction also applies to the subring $R(\lambda, \mu)$ by restricting to $\mathbb{T}$-invariants, as we now describe. See [6] for details.

The resulting degenerated ring $R'(\lambda, \mu)$ has the same underlying graded vector space as $R(\lambda, \mu)$. The semistandard tableaux of shape $N\lambda$ and content $N\mu$, $N > 0$, index a basis for $R'(\lambda, \mu)_N$. Let $b'_\tau \in R'(\lambda, \mu)$ denote the basis element indexed by τ. The basis element b'_τ is the leading term of $b_\tau \in R(\lambda, \mu)$ for a certain filtration of $R(\lambda, \mu)$ (see [7] and [6, 12]). The filtration has the special property that, if τ_1, τ_2 are any two semistandard tableaux, and if $b_{\tau_1} b_{\tau_2} = \sum_\tau c^\tau b_\tau$, where the sum is over semistandard tableaux, then the term $b_{\Phi^{-1}(\mathbf{x}(\tau_1)+\mathbf{x}(\tau_2))}$ appears on the right-hand side with coefficient equal to 1. Furthermore, all other terms $c^\tau b_\tau$ have strictly smaller filtration level. Thus, in $R'(\lambda, \mu)$, the multiplication rule becomes

$$b'_{\tau_1} b'_{\tau_2} = b'_{\Phi^{-1}(\mathbf{x}(\tau_1)+\mathbf{x}(\tau_2))}.$$

Therefore $R'(\lambda, \mu)$ is isomorphic to $\mathbb{C}[S(\lambda, \mu)]$, the semigroup algebra of GT patterns under addition of patterns.

Given an m-tuple of rational numbers $q_1, \ldots, q_m$, define $\mathrm{den}(q_1, \ldots, q_m)$ to be the least positive integer N such that $Nq_i \in \mathbb{Z}$ for each i, $1 \le i \le m$. We call this the *denominator* of the m-tuple. Now, if some vertex $\mathbf{x}$ of the polytope $GT(\lambda, \mu)$ has denominator $N > 1$, then the integer point $N\mathbf{x}$ is an essential generator of the semigroup $\mathbb{C}[S(\lambda, \mu)]$, since $\mathbf{x}$ cannot be written as a sum of other integral patterns in $S(\lambda, \mu)$. In the next section we show the existence of such a vertex with large denominator for the case where n is a multiple of 3, $\lambda = \frac{n}{3}\varpi_3$, and $\mu = (1, 1, \ldots, 1)$.

5. The second theorem: $3k$ points on $\mathbb{P}^2$ and a nasty GT pattern

Suppose that $n = 3k$, where $k \geq 2$ is an integer. Let $\lambda = k\varpi_3$ be a multiple of the third fundamental weight for $\mathsf{SL}_n(\mathbb{C})$. Thus, as a partition, $\lambda = (k, k, k, 0, \ldots, 0) \in \mathbb{R}^{3k}$. Now let μ be the "democratic" weight dominated by λ. That is, we represent μ by the composition $(1, \ldots, 1) \in \mathbb{R}^{3k}$. With this choice of λ and μ, the projective variety $\operatorname{Proj} R(\lambda, \mu)$ is the moduli space of equally weighted $3k$-tuples of points in projective space $\mathbb{P}^2$ (see [5] for more details).

We now construct a GT pattern that we claim will be a vertex of $GT(\lambda, \mu)$. Define the sequences $\{T_j^{(1)}\}$ and $\{T_j^{(2)}\}$ by the coupled recurrence relations

$$\begin{aligned} T_0^{(1)} &= k & T_0^{(2)} &= k - 1/2 \\ T_j^{(1)} &= T_{j-1}^{(2)} - 1 \quad (j \geq 1) & T_j^{(2)} &= \frac{1}{2}\left(T_j^{(1)} + T_{j-1}^{(1)}\right) \quad (j \geq 1). \end{aligned} \tag{5.1}$$

Solving this system of recurrence relations yields the closed-form expressions

$$T_j^{(1)} = k - \frac{2}{3}j + \frac{5}{9}\left(\frac{-1}{2}\right)^j - \frac{5}{9} \tag{5.2}$$

$$T_j^{(2)} = k - \frac{2}{3}j - \frac{5}{18}\left(\frac{-1}{2}\right)^j - \frac{2}{9}. \tag{5.3}$$

Let $N = k + \lfloor k/2 \rfloor - 2$. We will construct a triangular array $\mathbf{x}$ by filling in the entries of $\mathbf{x}$ in blocks from the upper left to the lower right using the values $T_j^{(1)}$ and $T_j^{(2)}$. Begin by filling the entries in the upper left of the triangular array as follows.

$$\begin{array}{ccccc} x_{1n} & & x_{2n} & & x_{3n} \\ & x_{1,n-1} & & x_{2,n-1} & \\ & & x_{1,n-2} & & \\ & & & x_{1,n-3} & \end{array} = \begin{array}{ccccc} k & & k & & k \\ & k & & k & \\ & & k & & k - \frac{1}{2} \\ & & & k & \end{array}$$

We then proceed from the upper left to the lower right of the triangular array by filling in blocks of entries as follows. For $1 \leq j \leq N - 1$, let

$$\begin{array}{ccc} & x_{3,n-2j} & \\ x_{2,n-2j-1} & & x_{3,n-2j-1} \\ x_{1,n-2j-2} & & x_{2,n-2j-2} \\ & x_{1,n-2j-3} & \end{array} = \begin{array}{ccc} & T_j^{(1)} & \\ T_j^{(1)} & & T_j^{(1)} \\ T_j^{(2)} & & T_j^{(1)} \\ & T_j^{(1)} & \end{array}$$

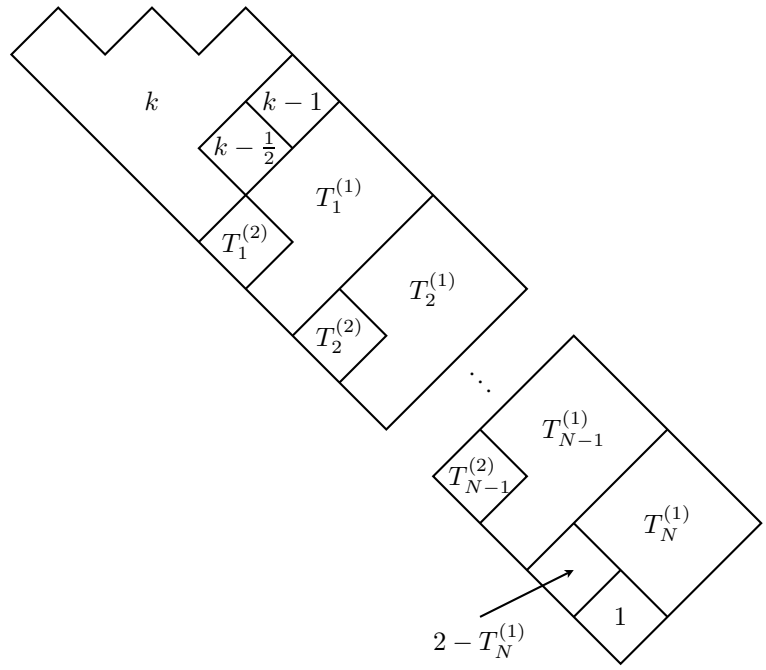


FIGURE 1. Tiling of $\mathbf{x}$ when k is even

If k is even, the final entries at the bottom of the array are filled in as follows.

$$\begin{array}{ccccc} & & x_{34} & & \\ & x_{23} & & x_{33} & \\ x_{12} & & x_{22} & & \\ & x_{11} & & & \end{array} = \begin{array}{ccc} & T_N^{(1)} & \\ T_N^{(1)} & & T_N^{(1)} \\ 2-T_N^{(1)} & & T_N^{(1)} \\ & 1 & \end{array}$$

On the other hand, if k is odd, then the final entries are filled in as follows:

$$\begin{array}{ccccc} & x_{35} & & & \\ x_{24} & & x_{34} & & \\ x_{13} & x_{23} & & x_{33} & \\ x_{12} & & x_{22} & & \\ & x_{11} & & & \end{array} = \begin{array}{ccc} & T_N^{(1)} & \\ T_N^{(1)} & & T_N^{(1)} \\ T_N^{(1)} & & T_N^{(1)} & & 3-2T_N^{(1)} \\ T_N^{(1)} & & 2-T_N^{(1)} \\ & 1 & \end{array}$$

All the remaining entries of the triangular array are assigned the value 0.

Proposition 5.0.15. *The triangular array constructed above is a vertex of $GT(\lambda,\mu)$ with denominator 2^N.*

Proof. To show that $\mathbf{x} \in GT(\lambda,\mu)$, we first check that $\mathbf{x}$ is a GT-pattern. In this case, the interlacing inequalities to be verified are

$$\left.\begin{array}{c} T^{(1)}_{j-1} > T^{(1)}_{j} > 0 \\ T^{(1)}_{j-2} > T^{(2)}_{j-1} > T^{(1)}_{j-1} \end{array}\right\} \quad \text{for } 2 \le j \le N,$$

$$\left.\begin{array}{c} T^{(1)}_{N-1} > 2 - T^{(1)}_{N} > T^{(1)}_{N} \\ 2 - T^{(1)}_{N} > 1 > T^{(1)}_{N} \end{array}\right\} \quad \text{if } k \text{ is even,}$$

$$\left.\begin{array}{c} T^{(1)}_{N} > 3 - 2T^{(1)}_{N} > 0 \\ T^{(1)}_{N} > 2 - T^{(1)}_{N} > 3 - 2T^{(1)}_{N} \\ T^{(1)}_{N} > 1 > 2 - T^{(1)}_{N} \end{array}\right\} \quad \text{if } k \text{ is odd.}$$

These are all straightforward consequences of the closed-form expressions (5.2) and (5.3) for $T^{(1)}_j$ and $T^{(2)}_j$, respectively, so $\mathbf{x}$ is a GT-pattern. Thus, to show that $\mathbf{x} \in GT(\lambda,\mu)$, we need only establish that the row-sums of $\mathbf{x}$ are correct. This amounts to showing that

$$\begin{aligned} T^{(2)}_{j-1} + T^{(1)}_{j-1} + T^{(1)}_{j} &= 3k - 2j \\ T^{(1)}_{j-1} + 2T^{(1)}_{j} &= 3k - 2j - 1 \end{aligned} \tag{5.4}$$

for $2 \le j \le N$. These equalities may be shown using induction and the recursive definition (5.1) of $T^{(1)}_j$ and $T^{(2)}_j$. It is clear from equation (5.4) that $T^{(1)}_j$ has denominator 2^j when written as a reduced fraction. Hence, $\mathbf{x}$ has denominator 2^N, as claimed.

It remains only to show that $\mathbf{x}$ is a vertex of $GT(\lambda,\mu)$. We prove this by showing that, for any triangular array ε, if $\mathbf{x} \pm \varepsilon \in GT(\lambda,\mu)$, then $\varepsilon = 0$. This is most easily seen by partitioning the entries of $\mathbf{x}$ so that entries that are equal and adjacent are grouped together. We call each group of entries in this partition a *tile*. See Figure 1 for a depiction of the case when k is even. Each tile is labeled with the value shared by the entries that it contains.

Suppose that $\mathbf{x} \pm \varepsilon \in GT(\lambda,\mu)$. Note that, after the addition of $\pm\varepsilon$, the entries in each tile must still share a value, and the row-sums must be unchanged. We prove inductively that the entries in each tile cannot have changed, proceeding from the upper left to the lower right.

The entries in the tile labeled $T^{(1)}_0 = k$ cannot have changed because the top row is fixed. For the same reason, the 0 entries in $\mathbf{x}$ are also fixed. Proceeding by induction, the entries in the tile labeled $T^{(1)}_j$ cannot have changed because there is a row on which this is the only tile besides the tile labeled $T^{(1)}_{j-1}$ and the tile of 0s, which have already been fixed. Hence, the entries in all the tiles labeled $T^{(1)}_j$, $0 \le j \le N$, are fixed under the addition of $\pm\varepsilon$.

Finally, for $1 \leq j \leq N-1$, the tile labeled $T_j^{(2)}$ lies on a row in which the other entries, $T_j^{(1)}$, $T_{j+1}^{(1)}$, and 0, have been shown to be fixed, so the entry in this tile is also fixed under the addition $\pm\varepsilon$. Therefore, we conclude that $\varepsilon = 0$, so that $\mathbf{x}$ is a vertex, as claimed. □

The following theorem is an immediate consequence.

Theorem 5.0.16. *The Gelfand–Tsetlin algebra $R'(k\varpi_3, \mu)$ has essential generators of degree exceeding $2^{n/2-3}$.*

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