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The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes

Marco Schlichting

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Abstract We prove localization and Zariski-Mayer-Vietoris for higher Grothendieck-Witt groups, alias hermitian K-groups, of schemes admitting an ample family of line-bundles. No assumption on the characteristic is needed, and our schemes can be singular. Along the way, we prove Additivity, Fibration and Approximation theorems for the hermitian K-theory of exact categories with weak equivalences and duality.

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M. Schlichting (✉)
Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA
e-mail: mschlich@math.lsu.edu

M. Schlichting
Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

1 Introduction

A classical invariant of a scheme X is its Grothendieck-Witt group $GW_0(X)$ of symmetric bilinear spaces over X. According to Knebusch [16, Sect. 4], this is the Abelian group generated by isometry classes $[\mathcal{V},\varphi]$ of vector bundles $\mathcal{V}$ over X equipped with a non-singular symmetric bilinear form $\varphi : \mathcal{V} \otimes_{O_X} \mathcal{V} \to O_X$ modulo the relations $[(\mathcal{V},\varphi) \perp (\mathcal{V}',\varphi')] = [\mathcal{V},\varphi] + [\mathcal{V}',\varphi']$ and $[\mathcal{M},\varphi] = [\mathcal{H}(\mathcal{N})]$ for every metabolic space $(\mathcal{M},\varphi)$ with Lagrangian subbundle $\mathcal{N} = \mathcal{N}^{\perp} \subset \mathcal{M}$ and associated hyperbolic space $\mathcal{H}(\mathcal{N})$. Grothendieck-Witt groups naturally occur in $\mathbb{A}^1$-homotopy theory [19] and are to oriented Chow groups what algebraic K-theory is to ordinary Chow groups; see [3, 6, 11].

Using a hermitian version of Quillen's Q-construction, we have defined in [25] the higher Grothendieck-Witt groups $GW_i(X)$, $i \in \mathbb{N}$, of a scheme X, generalizing the group $GW_0(X)$. The purpose of this article is to prove the following Mayer-Vietoris principle for open covers.

Theorem 1 *Let $X = U \cup V$ be a scheme with an ample family of line-bundles (e.g., quasi-projective over an affine scheme, or regular separated noetherian) which is covered by two open quasi-compact subschemes $U, V \subset X$. Then there is a long exact sequence where $i \in \mathbb{Z}$*

$$\begin{aligned}\cdots GW_{i+1}(U \cap V) &\to GW_i(X)\\ &\to GW_i(U) \oplus GW_i(V) \to GW_i(U \cap V) \to GW_{i-1}(X) \cdots\end{aligned}$$

This is a special case of our Theorem 16 which also includes versions for skew-symmetric forms, for forms with coefficients in line-bundles other than O_X and for certain non-commutative schemes. Note that we don't need the common assumption $\frac{1}{2} \in \Gamma(X, O_X)$, and X can be singular!

Theorem 1 is a consequence of two theorems, Localization and Zariski-excision. To explain the implication, let X be a scheme, L a line bundle on X, $n \in \mathbb{Z}$ an integer, and $Z \subset X$ a closed subscheme with open complement U. With this set of data, we associate in Definition 7 a topological space $GW^n(X \text{ on } Z, L)$ which, for $Z = X$, $n = 0$ and $L = O_X$, yields the Grothendieck-Witt space $GW(X)$ introduced in [25] whose homotopy groups are the higher Grothendieck-Witt groups $GW_i(X)$ in Theorem 1; see Corollary 1. The space $GW^n(X \text{ on } Z, L)$ is the Grothendieck-Witt space (as in Definition 3) of an exact category with weak equivalences and duality, namely, the exact category of bounded chain complexes of vector bundles on X which are (cohomologically) supported in Z, equipped with the set of quasi-isomorphisms as weak equivalences and duality $E \mapsto \mathrm{Hom}(E, L[n])$, where $L[n]$ denotes the complex which is L in degree $-n$. If $Z = X$, we write $GW^n(X, L)$ for $GW^n(X \text{ on } Z, L)$. The non-negative part of Theorem 1 is a consequence of the following two theorems (proved in Theorems 10 and 11). They are extended to negative Grothendieck-Witt groups in Sect. 10 (Theorems 14 and 15).

Theorem 2 (Localization) *Let X be a scheme with an ample family of line-bundles, let $U \subset X$ be a quasi-compact open subscheme with closed complement $Z = X - U$. Let L be a line bundle on X, and $n \in \mathbb{Z}$ an integer. Then there is a homotopy fibration*

$$GW^n(X \text{ on } Z, L) \longrightarrow GW^n(X, L) \longrightarrow GW^n(U, j^*L).$$

Theorem 3 (Zariski excision) *Let $j : U \subset X$ be quasi-compact open subscheme of a scheme X which has an ample family of line-bundles. Let $Z \subset X$ be a closed subset such that $Z \subset U$. Then restriction of vector-bundles induces a homotopy equivalence for all $n \in \mathbb{Z}$ and all line bundles L on X*

$$GW^n(X \text{ on } Z, L) \xrightarrow{\sim} GW^n(U \text{ on } Z, j^*L).$$

Theorems 1–3 have well-known analogs in algebraic K-theory proved by Thomason in [30] based on the work of Waldhausen [31] and Grothendieck et al. [4]. In fact, our Theorems 10, 14, 11, 15 and 16—special cases of which are Theorems 2, 3 and 1—are generalizations of the corresponding theorems in Thomason's work. More recently, Balmer [1] and Hornbostel [10] proved results reminiscent of our Theorems 1–3. Both need X to be regular noetherian and separated and they need 2 to be a unit in the ring of regular functions on X. Balmer works with (triangular) Witt-groups instead of Grothendieck-Witt groups, and Hornbostel works with Karoubi's hermitian K-groups of rings extended to regular separated schemes using Jouanolou's device of replacing such a scheme by an affine vector-bundle torsor.

Neither Balmer's nor Hornbostel's methods can be generalized to cover our Theorems 2, 3 and 1. This is because the assumption $\frac{1}{2} \in \Gamma(X, O_X)$ is ubiquitous in their work, the analog of Theorem 1 for Balmer's triangular Witt groups fails to hold for singular quasi-projective schemes (see [27] for a counter example even with $\frac{1}{2} \in \Gamma(X, O_X)$), Hornbostel imposes homotopy invariance which doesn't hold for singular schemes, and his proof uses Karoubi's Fundamental Theorem [14] which fails to hold for higher Grothendieck-Witt groups when $\frac{1}{2} \notin \Gamma(X, O_X)$ (see [26] for a counter example). Instead, we generalize Thomason's work [30]. His proofs of the K-theory analogs of Theorems 2, 3 and 1 are based on Waldhausen's Fibration Theorem [31, 1.6.4] and on "invariance of K-theory under derived equivalences" [30, Theorem 1.9.8] which itself is a consequence of Waldhausen's Approximation Theorem [31, 1.6.7]. We prove in Theorem 6 the analog of Waldhausen's Fibration Theorem for higher Grothendieck-Witt groups. Its proof, however, is not a formal consequence of Additivity (proved for higher Grothendieck-Witt groups in Sect. 3), contrary to the K-theory situation. Our proof relies on the author's cone construction in [25]. "Invariance under derived equivalences" as well as the naive generalization of Waldhausen's Approximation Theorem fail to hold for higher Grothendieck-Witt groups when "2 is not a unit" (see [26] for a counter example). We prove in Theorems 8 and 9 versions of Waldhausen's Approximation Theorem for higher Grothendieck-Witt groups. Though not as general as one might wish, they are enough to show Theorems 2, 3 and 1 and their generalizations in Sects. 9 and 10.

Prerequisites. The article can be read independently of [30] and [31], though, of course, much of our inspiration derives from these two papers. The reader is advised to have some background in homotopy theory in the form of [8, I–IV] and in the theory of triangulated categories in the form of [15, 20], [21, Sects. 1–2]. Also, we will frequently use results from [25].

2 The Grothendieck-Witt space

In this section we introduce the Grothendieck-Witt group and the Grothendieck-Witt space of an exact category with weak equivalences and duality (Definitions 1 and 3), and we show in Proposition 2 that the Grothendieck-Witt space of an exact category with duality defined here is equivalent to the one defined in [25]. We start with recalling definitions from [25]. Note that our terminology (for "category with duality", "duality preserving functor") sometimes differs from standard terminology as in [17, 23].

2.1 Categories with duality, $\mathcal{C}_h$ and form functors

A *category with duality* is a triple $(\mathcal{C}, *, \eta)$ with $\mathcal{C}$ a category, $* : \mathcal{C}^{op} \to \mathcal{C}$ a functor, $\eta : 1 \to **$ a natural transformation, called *double dual identification*, such that $1_{A^*} = \eta_A^* \circ \eta_{A^*}$ for all objects A in $\mathcal{C}$. If η is a natural isomorphism, we say that the duality is *strong*. In case η is the identity (in which case $** = id$), we call the duality *strict*.

A *symmetric form* in a category with duality $(\mathcal{C}, *, \eta)$ is a pair (X, φ) where $\varphi : X \to X^*$ is a morphism in $\mathcal{C}$ satisfying $\varphi^* \eta_X = \varphi$. A map of symmetric forms $(X, \varphi) \to (Y, \psi)$ is a map $f : X \to Y$ in $\mathcal{C}$ such that $\varphi = f^* \circ \psi \circ f$. Composition of such maps is composition in $\mathcal{C}$. For a category with duality $(\mathcal{C}, *, \eta)$, we denote by $\mathcal{C}_h$ the *category of symmetric forms in* $\mathcal{C}$. It has objects the symmetric forms in $\mathcal{C}$ and maps the maps between symmetric forms.

A *form functor* from a category with duality $(\mathcal{A}, *, \alpha)$ to another such category $(\mathcal{B}, *, \beta)$ is a pair (F, φ) with $F : \mathcal{A} \to \mathcal{B}$ a functor and $\varphi : F* \to *F$ a natural transformation, called *duality compatibility morphism*, such that $\varphi^*_A \beta_{FA} = \varphi_{A^*} F(\alpha_A)$ for every object A of $\mathcal{A}$. There is an evident definition of composition of form functors; see [25, 3.2]. The category $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ of functors $\mathcal{A} \to \mathcal{B}$ is a category with duality, where the dual $F^\sharp$ of a functor F is $*F*$, and double dual identification $\eta_F : F \to F^{\sharp\sharp}$ at an object A of $\mathcal{A}$ is the map $\beta_{F(A^{**})} \circ F(\alpha_A) = F(\alpha_A)^{**} \circ \beta_{FA}$. To give a form functor (F, φ) is the same as to give a symmetric form $(F, \hat{\varphi})$ in the category with duality $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ in view of the formulas $\varphi_A = F(\alpha_A)^* \circ \hat{\varphi}_{A^*}$ and $\hat{\varphi}_A = \varphi_{A^*} \circ F(\alpha_A)$. A natural transformation $(F, \varphi) \to (G, \psi)$ of form functors is a map $(F, \hat{\varphi}) \to (G, \hat{\psi})$ of symmetric forms in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$.

A *duality preserving functor* between categories with duality $(\mathcal{A}, *, \alpha)$ and $(\mathcal{B}, *, \beta)$ is a functor $F : \mathcal{A} \to \mathcal{B}$ which commutes with dualities and double dual identifications, that is, we have $F* = *F$ and $F(\alpha) = \beta_F$. In this case, (F, id) is a form functor. We will consider duality preserving functors F as form functors (F, id). Note that our use of the phrase "duality preserving functor" may differ from its use by other authors!

2.2 Exact categories with weak equivalences

Recall that an *exact category* is an additive category $\mathcal{E}$ equipped with a family of sequences of maps in $\mathcal{E}$, called *conflations* (or *admissible short exact sequences*, or simply *exact sequences*),

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

satisfying a list of axioms; see [22], [15, Sect. 4], [25, 2.1]. The map i in an exact sequence is called inflation (or admissible monomorphism) and may be depicted as $\rightarrowtail$, and the map p is called *deflation* (or admissible epimorphism) and may be depicted as $\twoheadrightarrow$ in diagrams. Unless otherwise stated, all exact categories in this article will be (essentially) small.

An *exact category with weak equivalences* is a pair $(\mathcal{E}, w)$ with $\mathcal{E}$ an exact category and $w \subset \mathrm{Mor}\,\mathcal{E}$ a set of morphisms, called weak equivalences, which contains all identity morphisms, is closed under isomorphisms, retracts, push-outs along inflations, pull-backs along deflations, composition and the 2 out of three property for composition (if 2 of the 3 maps among a, b, ab are in w then so is the third). A weak equivalence is usually depicted as $\xrightarrow{\sim}$ in diagrams. A functor $F : \mathcal{A} \to \mathcal{B}$ between exact categories with weak equivalences $(\mathcal{A}, w)$ and $(\mathcal{B}, w)$ is called *exact* if it sends conflations to conflations and weak equivalences to weak equivalences.

For an exact category with weak equivalences $(\mathcal{E}, w)$, we will write $w\mathcal{E}$ for the subcategory of weak equivalences in $\mathcal{E}$. Its objects are the objects of $\mathcal{E}$ and its maps the maps in w. Also, we will regard an exact category $\mathcal{E}$ (without specifying weak equivalences) as an exact category with weak equivalences $(\mathcal{E}, i)$ where i is the set of isomorphisms in $\mathcal{E}$.

2.3 Witt and Grothendieck-Witt groups of exact categories with weak equivalences and duality

An *exact category with weak equivalences and duality* is a quadruple $(\mathcal{E}, w, *, \eta)$ with $(\mathcal{E}, w)$ an exact category with weak equivalences and $(\mathcal{E}, *, \eta)$ a category with duality such that $* : (\mathcal{E}^{op}, w) \to (\mathcal{E}, w)$ is an exact functor (in particular, $*(w) \subset w$) and $\eta : id \to **$ is a natural weak equivalence, that is, $\eta_X \in w$ for all objects X in $\mathcal{E}$. We may simply say $\mathcal{E}$ or $(\mathcal{E}, w)$ is an exact category with weak equivalences and duality if the remaining data are understood. Note that if $\mathcal{E}$ is an exact category with weak equivalences and duality, the category $w\mathcal{E}$ of weak equivalences in $\mathcal{E}$ is a category with duality.

A symmetric form (X, φ) in $(\mathcal{E}, w, *, \eta)$ is called *non-singular* if φ is a weak equivalence. In this case, we call the pair (X, φ) a *symmetric space* in $(\mathcal{E}, w, *, \eta)$. A form functor $(F, \varphi) : (\mathcal{A}, w, *, \eta) \to (\mathcal{B}, w, *, \eta)$ is called *exact* if $F : (\mathcal{A}, w) \to (\mathcal{B}, w)$ is exact. It is called *non-singular*, if the duality compatibility morphism $\varphi : F* \to *F$ is a natural weak equivalence.

An *exact category with duality* is an exact category with weak equivalences and duality where the set of weak equivalences is the set of isomorphisms. In particular, the double dual identification has to be a natural isomorphism.

Definition 1 The *Grothendieck-Witt group*

$$GW_0(\mathcal{E}, w, *, \eta)$$

of an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ is the free Abelian group generated by isomorphism classes $[X, \varphi]$ of symmetric spaces (X, φ) in $(\mathcal{E}, w, *, \eta)$, subject to the following relations

(a) $[X, \varphi] + [Y, \psi] = [X \oplus Y, \varphi \oplus \psi]$
(b) if $g : X \to Y$ is a weak equivalence, then $[Y, \psi] = [X, g^*\psi g]$, and

(c) if $(E_\bullet, \varphi_\bullet)$ is a symmetric space in the category of exact sequences in $\mathcal{E}$, that is, a map

$$\begin{array}{cccccc} E_\bullet : & E_{-1} & \overset{i}{\rightarrowtail} & E_0 & \overset{p}{\twoheadrightarrow} & E_1 \\ \wr\downarrow \varphi_\bullet & \wr\downarrow \varphi_{-1} & & \wr\downarrow \varphi_0 & & \wr\downarrow \varphi_1 \\ E_\bullet^* : & E_1^* & \underset{p^*}{\rightarrowtail} & E_0^* & \underset{i^*}{\twoheadrightarrow} & E_{-1}^* \end{array}$$

of exact sequences with $(\varphi_{-1}, \varphi_0, \varphi_1) = (\varphi_1^*\eta, \varphi_0^*\eta, \varphi_{-1}^*\eta)$ a weak equivalence, then

$$[E_0, \varphi_0] = \left[E_{-1} \oplus E_1, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_{-1} & 0 \end{pmatrix} \right].$$

Remark 1 If in Definition 1, the set of weak equivalences is the set of isomorphisms, then we recover the classical Grothendieck-Witt group of an exact category with duality; see for instance [25, 2.2]. In this case, relation (c) says that the class $[E_0, \varphi_0]$ of a metabolic space (E_0, φ_0) with Lagrangian $i : E_{-1} \rightarrowtail E_0$ is equivalent in the Grothendieck-Witt group to the class of the hyperbolic space $\mathcal{H}(E_{-1})$ of the Lagrangian E_{-1}. In particular, if $\mathcal{E}$ is the category $\mathrm{Vect}(X)$ of vector bundles on X, $*$ is the duality functor $E \mapsto \mathrm{Hom}(E, O_X)$ and η is the usual canonical double dual identification, the group $GW_0(\mathcal{E}, i, *, \eta)$ is Knebusch's Grothendieck-Witt group $GW_0(X)$ of a scheme X, denoted $L(X)$ in [16].

Definition 2 The *Witt* group

$$W_0(\mathcal{E}, w, *, \eta)$$

of an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ is the free Abelian group generated by isomorphism classes $[X, \varphi]$ of symmetric spaces (X, φ) in $(\mathcal{E}, w, *, \eta)$, subject to the relations (a), (b) in Definition 1 and

(c′) if $(E_\bullet, \varphi_\bullet)$ is a symmetric space in the category of exact sequences in $(\mathcal{E}, w, *, \eta)$, then $[E_0, \varphi_0] = 0$.

The hermitian $S_\bullet$-construction of [29], [12, 1.5], which gives rise to the Grothendieck-Witt space in Definition 3, is the edgewise subdivision of Waldhausen's $S_\bullet$-construction [31]. We review the relevant definitions and start with the edgewise subdivision of a simplicial object; see [31, 1.9 Appendix], [28, Appendix 1].

2.4 Edgewise subdivision

Let Δ be the category with objects $[n]$, $n \in \mathbb{N}$, the totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms all order preserving maps. Let $\underline{n}$ be the totally ordered set

$$\underline{n} = \{n' < (n-1)' < \cdots < 0' < 0 < \cdots < n\}.$$

It is (uniquely) isomorphic to $[2n+1]$. The assignment $T : [n] \mapsto \underline{n}$ defines an endo-functor $\Delta \to \Delta$ where a map $\theta : [n] \to [m]$ is sent to the map $T(\theta) : \underline{n} \to \underline{m}$ given by $p \mapsto \theta(p)$, $p' \mapsto \theta(p)'$. For a simplicial object $X_\bullet$, the *edge-wise subdivision* $X^e_\bullet$ of $X_\bullet$ is the simplicial object $X_\bullet \circ T$. The inclusion $[n] \hookrightarrow \underline{n} : i \mapsto i$ defines a map $X^e \to X$ of simplicial objects. It is known [28, Appendix 1] that for a simplicial set $X_\bullet$, the topological realization of $X_\bullet$ and of its edge-wise subdivision $X^e_\bullet$ are homeomorphic. We need the following (well-known) variant.

Lemma 1 *For any simplicial set $X_\bullet$, the map $X^e_\bullet \to X_\bullet$ is a homotopy equivalence.*

Proof Let $X_\bullet$ be a simplicial set. For a small category $\mathcal{C}$, write $X_{\mathcal{C}}$ for the set $\mathrm{Hom}(N_*\mathcal{C}, X_\bullet)$ of simplicial maps from the nerve $N_*\mathcal{C}$ of $\mathcal{C}$ to $X_\bullet$. Note that X^e is the simplicial set $[n] \mapsto X_{\underline{n}}$. We define bisimplicial sets $X^e_{\bullet\bullet}$ and $X_{\bullet\bullet}$ by the formulas $X^e_{m,n} = X_{\underline{m}\times[n]}$ and $X_{m,n} = X_{[m]\times[n]}$. Consider the following diagram of bisimplicial sets

$$\begin{array}{ccccc}
X^e_{0\bullet} & \xrightarrow{\sim} & X^e_{\bullet\bullet} & \xleftarrow{\sim} & X^e_{\bullet 0} \\
\downarrow\wr & & \downarrow & & \downarrow \\
X_{0\bullet} & \xrightarrow{\sim} & X_{\bullet\bullet} & \xleftarrow{\sim} & X_{\bullet 0}
\end{array}$$

in which the horizontal maps are the canonical inclusions of horizontally respectively vertically constant bisimplicial sets, and the vertical maps are induced by the inclusions $[m] \subset \underline{m}$. Once we show that all arrows labeled $\xrightarrow{\sim}$ are homotopy equivalences, we are done, because the right vertical map can be identified with $X^e_\bullet \to X_\bullet$.

The left vertical map $X^e_{0\bullet} \to X_{0\bullet}$ is a homotopy equivalence since it can be identified with the map $X^I_\bullet \to X_\bullet$ which is evaluation at 0, where I is the standard simplicial interval $I = N_*\underline{0} \cong N_*[1]$. In order to see that the upper right horizontal map $X^e_{\bullet 0} \to X^e_{\bullet\bullet}$ is a homotopy equivalence, it suffices to prove that for every n, the map $X^e_{\bullet 0} \to X^e_{\bullet n}$ is a homotopy equivalence of simplicial sets. Since the map $[0] \to [n] : 0 \mapsto 0$ induces a retraction

$X^e_{\bullet n} \to X^e_{\bullet 0}$, we have to show that the composition $X^e_{\bullet n} \to X^e_{\bullet 0} \to X^e_{\bullet n}$ is homotopic to the identity. The unique natural transformation from the constant functor $[n] \to [n] : i \mapsto 0$ to the identity functor $[n] \to [n]$ defines a functor $h : [1] \times [n] \to [n]$ such that the restrictions to $\{i\} \times [n] \to [n]$, $i = 0, 1$ are the constant respectively the identity functor. We have a map of simplicial sets

$$I^e \times X^e_{\bullet n} \to X^e_{\bullet n} \tag{1}$$

which in degree m sends the pair $(\xi, f) \in I_{\underline{m}} \times X_{\underline{m}\times[n]}$ to the composition

$$N_*(\underline{m} \times [n]) \xrightarrow{(1,\xi)\times 1} N_*(\underline{m} \times [1] \times [n]) \xrightarrow{(1\times h)} N_*(\underline{m} \times [n]) \xrightarrow{f} X_\bullet.$$

Since the two points $\{0, 1\} = \{0, 1\}^e \subset I^e$ are path connected in I^e, the map (1) defines the desired homotopy. The other horizontal homotopy equivalences in the diagram are similar, and we omit the details. □

2.5 Waldhausen's $S_\bullet$-construction

We recall Waldhausen's $S_\bullet$-construction [31, Sect. 1.3]. Let $\mathrm{Ar}[n]$ denote the category whose objects are the arrows of the category $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are the commutative squares in $[n]$. For an exact category with weak equivalences $(\mathcal{E}, w)$, Waldhausen defines $S_n\mathcal{E} \subset \mathrm{Fun}(\mathrm{Ar}[n], \mathcal{E})$ as the full subcategory of the category $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{E})$ of functors

$$A : \mathrm{Ar}[n] \to \mathcal{E} : (p \le q) \mapsto A_{p,q}$$

for which $A_{p,p} = 0$ and $A_{p,q} \rightarrowtail A_{p,r} \twoheadrightarrow A_{q,r}$ is a conflation whenever $p \le q \le r$, $p, q, r \in [n]$. The category $S_n\mathcal{E}$ is an exact category with weak equivalences where a sequence $A \to B \to C$ of functors $\mathrm{Ar}[n] \to \mathcal{E}$ in $S_n\mathcal{E}$ is exact if $A_{p,q} \rightarrowtail B_{p,q} \twoheadrightarrow C_{p,q}$ is exact in $\mathcal{E}$, and a map $A \to B$ of functors in $S_n\mathcal{E}$ is a weak equivalence if $A_{p,q} \to B_{p,q}$ is a weak equivalence in $\mathcal{E}$ for all $p \le q \in [n]$.

The cosimplicial category $n \mapsto \mathrm{Ar}[n]$ makes the assignment $n \mapsto S_n\mathcal{E}$ into a simplicial exact category with weak equivalences. According to [30, 31], the K-theory space $K(\mathcal{E}, w)$ of an exact category with weak equivalences $(\mathcal{E}, w)$ is the space

$$K(\mathcal{E}, w) = \Omega|wS_\bullet\mathcal{E}|.$$

2.6 The hermitian $S_\bullet$-construction

The category $[n]$ has a unique structure of a category with strict duality $[n]^{op} \to [n] : i \mapsto n - i$. This induces a strict duality on the category $\mathrm{Ar}[n]$ of arrows in $[n]$. For an exact category with weak equivalences and duality

$(\mathcal{E}, w, *, \eta)$, the category $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{E})$ is therefore a category with duality (see Sect. 2.1). This duality preserves the subcategory $S_n\mathcal{E} \subset \mathrm{Fun}(\mathrm{Ar}[n], \mathcal{E})$, and makes $S_n\mathcal{E}$ into an exact category with weak equivalences and duality. It turns out that the simplicial structure maps of $n \mapsto S_n\mathcal{E}$ are not compatible with dualities. However, its edgewise subdivision

$$S^e_\bullet\mathcal{E} : n \mapsto S^e_n\mathcal{E} = S_{2n+1}\mathcal{E}$$

is a simplicial exact category with weak equivalences and duality; the simplicial structure maps being duality preserving. Considering $S^e_n\mathcal{E}$ as a full subcategory of $\mathrm{Fun}(\mathrm{Ar}(\underline{n}), \mathcal{E})$, the dual A^* of an object $A : \mathrm{Ar}(\underline{n}) \to \mathcal{E}$ satisfies $(A^*)_{p,q} = A^*_{q',p'}$ for $p \le q \in \underline{n}$ with the understanding that p'' means p. The double dual identification $A \to A^{**}$ at $(p \le q)$ is $\eta_{A_{p,q}}$.

2.7 The Grothendieck-Witt space

Definition 3 Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. By Sect. 2.6, the assignment $n \mapsto S^e_n\mathcal{E}$ defines a simplicial exact category $S^e_\bullet\mathcal{E}$ with weak equivalences and duality. The subcategories of weak equivalences define a simplicial category with duality $n \mapsto wS^e_n\mathcal{E}$. Taking associated categories of symmetric forms (see Sect. 2.1), we obtain a simplicial category $(wS^e_\bullet\mathcal{E})_h$.

The composition $(wS^e_\bullet\mathcal{E})_h \to wS^e_\bullet\mathcal{E} \to wS_\bullet\mathcal{E}$ of simplicial categories, in which the first arrow is the forgetful functor $(X, \varphi) \mapsto X$, and the second is the canonical map $X^e_\bullet \to X_\bullet$ of simplicial objects (see Sect. 2.4), yields a map of classifying spaces

$$|(wS^e_\bullet\mathcal{E})_h| \to |wS_\bullet\mathcal{E}| \tag{2}$$

whose homotopy fibre (with respect to a zero object[1] of $\mathcal{E}$ as base point of $wS_\bullet\mathcal{E}$) is defined to be the *Grothendieck-Witt space*

$$GW(\mathcal{E}, w, *, \eta)$$

of $(\mathcal{E}, w, *, \eta)$. If $(*, \eta)$ are understood, we may simply write $GW(\mathcal{E}, w)$ instead of $GW(\mathcal{E}, w, *, \eta)$. We define the *higher Grothendieck-Witt groups* of $(\mathcal{E}, w, *, \eta)$ as the homotopy groups

$$GW_i(\mathcal{E}, w, *, \eta) = \pi_i GW(\mathcal{E}, w, *, \eta), \quad i \ge 1,$$

and show in Proposition 3 below that $\pi_0 GW(\mathcal{E}, w, *, \eta) \cong GW_0(\mathcal{E}, w, *, \eta)$. Therefore, our definition here extends that in Definition 1.

[1] We assume that every exact category comes with a choice of a zero object and exact functors are to preserve that choice.

Remark 2 Orthogonal sum makes the spaces $(wS^e_\bullet\mathcal{E})_h$ and $GW(\mathcal{E}, w, *, \eta)$ into commutative H-spaces. Since the commutative monoid of connected components of these spaces are groups (see Proposition 3 and Remark 5 below), both spaces are actually commutative H-groups.

2.8 Functoriality

A non-singular exact form functor $(F, \varphi) : (\mathcal{A}, w, *, \eta) \to (\mathcal{B}, w, *, \eta)$ between exact categories with weak equivalences and duality induces maps

$$(F, \varphi) : (wS^e_\bullet\mathcal{A})_h \to (wS^e_\bullet\mathcal{B})_h : (A, \alpha) \mapsto (FA, \varphi_A F(\alpha)), \quad \text{and}$$
$$F : \quad wS_\bullet\mathcal{A} \to wS_\bullet\mathcal{B} : \quad A \mapsto FA$$

of simplicial categories compatible with composition of form functors. Taking homotopy fibres of $(wS^e_\bullet)_h \to wS_\bullet$, we obtain an induced map

$$GW(F, \varphi) : GW(\mathcal{A}, w, *, \eta) \to GW(\mathcal{B}, w, *, \eta)$$

of associated Grothendieck-Witt spaces. For the next lemma, recall that a natural transformation of form functors $(F, \varphi) \to (G, \psi)$ is a map of associated symmetric forms in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. It is a natural weak equivalence if $FA \to GA$ is a weak equivalence for all objects A of $\mathcal{A}$.

Lemma 2 *Let $(F, \varphi) \xrightarrow{\sim} (G, \psi)$ be a natural weak equivalence of non-singular exact form functors $(\mathcal{A}, w, *, \eta) \to (\mathcal{B}, w, *, \eta)$ between exact categories with weak equivalences and duality. Then, on associated Grothendieck-Witt spaces, (F, φ) and (G, ψ) induce homotopic maps $GW(\mathcal{A}, w, *, \eta) \to GW(\mathcal{B}, w, *, \eta)$.*

Proof The natural weak equivalence $(F, \varphi) \xrightarrow{\sim} (G, \psi)$ induces natural transformations of functors $(wS^e_n\mathcal{A})_h \to (wS^e_n\mathcal{B})_h$ and $wS_n\mathcal{A} \to wS_n\mathcal{B}$. These natural transformations define functors $[1] \times (wS^e_n\mathcal{A})_h \to (wS^e_n\mathcal{B})_h$ and $[1] \times wS_n\mathcal{A} \to wS_n\mathcal{B}$ whose restrictions to $0, 1 \in [1]$ are the two given functors. They are compatible with the simplicial structure and induce, after topological realization, the homotopy between $GW(F, \varphi)$ and $GW(G, \psi)$. □

2.9 Hyperbolic categories

We will associate to every exact category with weak equivalences $(\mathcal{E}, w)$ a category with weak equivalences and duality $(\mathcal{H}\mathcal{E}, w)$ such that the Grothendieck-Witt space of $(\mathcal{H}\mathcal{E}, w)$ is equivalent to the K-theory space of $(\mathcal{E}, w)$. In this sense, Grothendieck-Witt theory is a generalization of algebraic K-theory.

Let $\mathcal{C}$ be a category. Its hyperbolic category is the category with strict duality $\mathcal{HC} = (\mathcal{C} \times \mathcal{C}^{op}, *)$ where $(X, Y)^* = (Y, X)$. For any category with duality $\mathcal{A}$ there is a functor $\mathcal{A}_h \to \mathcal{A} : (X, \varphi) \mapsto X$ that “forgets the forms”. We define the functor $(\mathcal{HC})_h \to \mathcal{C}$ as the composition of the functor $(\mathcal{HC})_h \to \mathcal{HC}$ and the projection $\mathcal{HC} = \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}$ onto the first factor.

Lemma 3 *For any small category $\mathcal{C}$, the functor $(\mathcal{HC})_h \to \mathcal{C}$ is a homotopy equivalence.*

Proof The category $(\mathcal{HC})_h$ of symmetric forms in $\mathcal{HC}$ is isomorphic to the category whose objects are maps $f : X \to Y$ in $\mathcal{C}$ and where a map from f to $f' : X' \to Y'$ is a pair of maps $a : X \to X'$, $b : Y' \to Y$ in $\mathcal{C}$ such that $f = bf'a$. Composition is composition in $\mathcal{C}$ of the a's and b's. The functor $(\mathcal{HC})_h \to \mathcal{C}$ in the lemma sends the object $f : X \to Y$ to X and the map (a, b) to a. Write F for this functor, and let A be an object of the target category $\mathcal{C}$. We will show that the comma categories $(A \downarrow F)$ are contractible. By Quillen's Theorem A [22, Sect. 1], this implies the lemma.

The category $(A \downarrow F)$ has objects sequences $A \stackrel{x}{\to} X \stackrel{f}{\to} Y$ of maps in $\mathcal{C}$. A morphism from (x, f) to $A \stackrel{x'}{\to} X' \stackrel{f'}{\to} Y'$ is a pair $a : X \to X'$, $b : Y' \to Y$ of maps in $\mathcal{C}$ such that $x' = ax$ and $f = bf'a$. In particular, the category $(A \downarrow F)$ is non-empty as $(1_A, 1_A)$ is one of its objects. Let $\mathcal{C}_0 \subset (A \downarrow F)$ be the full subcategory of objects (x, f) with $x = 1_A$. The inclusion has a right adjoint $(A \downarrow F) \to \mathcal{C}_0 : (x, f) \mapsto (1_A, fx)$ with counit of adjunction $(1_A, fx) \to (x, f)$ given by the pair of maps $x : A \to X$ and $1 : Y \to Y$. It follows that the inclusion $\mathcal{C}_0 \subset (A \downarrow F)$ is a homotopy equivalence. Since the category $\mathcal{C}_0$ has a terminal object, namely $(1_A, 1_A)$, the categories $\mathcal{C}_0$ and $(A \downarrow F)$ are contractible. □

If $(\mathcal{E}, w)$ is an exact category with weak equivalences, we make $\mathcal{HE}$ into an exact category with weak equivalences and (strict) duality by declaring a map $(a, b) : (X, Y) \to (X', Y')$ in $\mathcal{HE}$ to be a weak equivalence if $a : X \to X'$ and $b : Y' \to Y$ are weak equivalences in $\mathcal{E}$. Note that $w\mathcal{HE} = \mathcal{H}w\mathcal{E}$ as categories with strict duality.

Proposition 1 *Let $(\mathcal{E}, w)$ be an exact category with weak equivalences, then there is a natural homotopy equivalence*

$$GW(\mathcal{HE}, w) \simeq K(\mathcal{E}, w).$$

Proof Consider the commutative diagram of simplicial categories

$$
\begin{array}{ccccc}
(wS^e_\bullet \mathcal{H}\mathcal{E})_h & \xrightarrow{=} & (\mathcal{H}wS^e_\bullet \mathcal{E})_h & \overset{\sim}{\searrow} & \\
\downarrow & & \downarrow & & \\
wS^e_\bullet \mathcal{H}\mathcal{E} & \xrightarrow[=]{} & wS^e_\bullet \mathcal{E} \times (wS^e_\bullet \mathcal{E})^{op} & \xrightarrow[p_1]{} & wS^e_\bullet \mathcal{E} \\
\wr\downarrow & & \downarrow\wr & & \downarrow\wr \\
wS_\bullet \mathcal{H}\mathcal{E} & \xrightarrow{=} & wS_\bullet \mathcal{E} \times (wS_\bullet \mathcal{E})^{op} & \xrightarrow{p_1} & wS_\bullet \mathcal{E}
\end{array}
$$

where the upper vertical maps are the functors that "forget the forms". The lower vertical maps are induced by the inclusion $[n] \hookrightarrow \underline{n}$ and are thus homotopy equivalences, by Lemma 1. The diagonal map is a homotopy equivalence by Lemma 3. By the "octahedron axiom" for homotopy fibres applied to the upper right triangle, it follows that the homotopy fibre of the composition of the left vertical maps is equivalent to the loop space of the fibre of $p_1 : wS_\bullet\mathcal{E} \times (wS_\bullet\mathcal{E})^{op} \to wS_\bullet\mathcal{E}$ which is the loop of the fibre of $(wS_\bullet\mathcal{E})^{op} \to$ (point) which is $K(\mathcal{E}, w)$. □

Remark 3 Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. The functor

$$F : (\mathcal{E}, w, *, \eta) \to (\mathcal{H}\mathcal{E}, w) : X \mapsto (X, X^*)$$

together with $(1, \eta_X) : (X^*, X^{**}) \to (X^*, X)$ as duality compatibility morphism is called *forgetful form functor*. It is a non-singular exact form functor between exact categories with weak equivalences and duality. The map $(wS^e_\bullet\mathcal{E})_h \to wS_\bullet\mathcal{E}$ defining the Grothendieck-Witt space factors as

$$(wS^e_\bullet\mathcal{E})_h \overset{F}{\to} (wS^e_\bullet\mathcal{H}\mathcal{E})_h \overset{\sim}{\to} wS_\bullet\mathcal{E},$$

where the second map is the homotopy equivalence in the diagram of the proof of Proposition 1 (going right, diagonally and down). It follows that the Grothendieck-Witt space $GW(\mathcal{E}, w, *, \eta)$ is naturally homotopy equivalent to the homotopy fibre of

$$(wS^e_\bullet\mathcal{E})_h \overset{F}{\to} (wS^e_\bullet\mathcal{H}\mathcal{E})_h.$$

2.10 The hermitian Q-construction

We finish the section with a comparison result between the definition of the Grothendieck-Witt space of an exact category with duality $(\mathcal{E}, i, *, \eta)$ in terms of the hermitian $S_\bullet$-construction and the definition given in [25, Definition 4.4] in terms of the hermitian Q-construction. We begin with recalling the relevant definitions.

Recall from [22] that for an exact category $\mathcal{E}$ its Q-construction is the category with objects the objects of $\mathcal{E}$ and maps $X \to Y$ equivalence classes of diagrams

$$X \overset{p}{\twoheadleftarrow} U \overset{i}{\rightarrowtail} Y \tag{3}$$

with p a deflation and i an inflation. The datum (U, i, p) is equivalent to (U', i', p') if there is an isomorphism $g : U \to U'$ in $\mathcal{E}$ such that $p = p' \circ g$ and $i = i' \circ g$. The composition in $Q\mathcal{E}$ of maps $X \to Y$ and $Y \to Z$ represented by the data (U, i, p) and (V, j, q) is given by the datum $(U \times_Y V, p\bar{q}, j\bar{i})$ where $\bar{q} : U \times_Y V \to U$ and $\bar{i} : U \times_Y V \to V$ are the canonical projections to U and V, respectively.

For an exact category with duality $(\mathcal{E}, *, \eta)$, the hermitian Q-construction is the category $Q^h(\mathcal{E}, *, \eta)$ with objects the symmetric spaces (X, φ) in $\mathcal{E}$. A map $(X, \varphi) \to (Y, \psi)$ is a map $X \to Y$ in Quillen's Q-construction, that is, an equivalence class of diagrams (3), such that the square of maps $p, i, p^*\varphi$ and $i^*\psi$ is commutative and biCartesian. Composition of maps is as in Quillen's Q-construction. For more details, we refer the reader to [25, Definition 4.1] and the references in [25, Remark 4.2].

In [25, Definition 4.4], we defined the Grothendieck-Witt space of $(\mathcal{E}, *, \eta)$ as the homotopy fibre of the forgetful functor $Q^h\mathcal{E} \to Q\mathcal{E} : (X, \varphi) \mapsto X$. The following proposition reconciles this definition with the one given in Definition 3. This allows us to freely use the results proved in [25].

Proposition 2 *For an exact category with duality $(\mathcal{E}, *, \eta)$, there are natural zigzags of homotopy equivalences between $|Q^h\mathcal{E}|$ and $|(iS^e_\bullet\mathcal{E})_h|$ and between the homotopy fibre of the forgetful functor $|Q^h\mathcal{E}| \to |Q\mathcal{E}|$ and the Grothendieck-Witt space of Definition 3.*

Proof For the first homotopy equivalence, the proof is the same as in [31, 1.9 Appendix]. In detail, let $iQ^h_\bullet\mathcal{E}$ be the simplicial category which in degree n is the category $iQ^h_n\mathcal{E}$ whose objects are sequences $X_0 \to X_1 \to \cdots \to X_n$ of maps in $Q^h\mathcal{E}$ and a map in $iQ^h_n\mathcal{E}$ is an isomorphism of such sequences. As n varies, $iQ^h_n\mathcal{E}$ defines a simplicial category where face and degeneracy maps are defined as in the usual nerve construction. The nerve of $iQ^h_\bullet\mathcal{E}$ as a bisimplicial set is isomorphic to the nerve of the simplicial category which in degree m are the sequences $X_0 \to X_1 \to \cdots \to X_m$ of isomorphisms in $Q^h\mathcal{E}$ and where maps are maps of sequences in $Q^h\mathcal{E}$ (which are not necessarily isomorphisms). The latter simplicial category is degree-wise equivalent to $Q^h\mathcal{E}$ (via the embedding of $Q^h\mathcal{E}$ as the constant sequences). Thus the latter simplicial category (and therefore also $iQ^h_\bullet\mathcal{E}$) is homotopy equivalent to $Q^h\mathcal{E}$.

Every object $(A_{p,q})_{p\leq q\,\in\,\underline{n}}$ in $(S^e_n\mathcal{E})_h$ defines a string of maps

$$A_{0',0} \to A_{1',1} \to \cdots \to A_{n',n}$$

in $Q^h\mathcal{E}$. This defines a map $(iS^e_\bullet\mathcal{E})_h \to iQ^h_\bullet\mathcal{E}$ of simplicial categories which is degree-wise an equivalence. Therefore, this map defines a homotopy equivalence on topological realizations.

For the second homotopy equivalence, consider the commutative diagram of topological spaces

$$\begin{array}{ccccccc}
|(iS^e_\bullet\mathcal{E})_h| & \xleftarrow{1} & |(iS^e_\bullet\mathcal{E})_h| & \xrightarrow{\sim} & |iQ^h_\bullet\mathcal{E}| & \xleftarrow{\sim} & |Q^h\mathcal{E}| \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
|iS_\bullet\mathcal{E}| & \xleftarrow{\sim} & |iS^e_\bullet\mathcal{E}| & \xrightarrow{\sim} & |iQ_\bullet\mathcal{E}| & \xleftarrow{\sim} & |Q\mathcal{E}|
\end{array}$$

in which the lower right two horizontal maps are defined in a similar way as their hermitian analogs above them (see [31, 1.9 Appendix]), the three right vertical maps are "forgetful" functors and all maps labeled $\stackrel{\sim}{\to}$ are homotopy equivalences. It follows that the homotopy fibre of the first vertical map is equivalent to the homotopy fibre of the last vertical map. □

3 Additivity theorems

Additivity theorems are fundamental in algebraic K-theory. They imply, for instance, Waldhausen's Fibration Theorem [31, 1.6.4] which is the basis for the K-theory version of Theorem 2. In this section, we prove the analogs of Additivity for higher Grothendieck-Witt theory. In order to formulate them, we recall the concept of "admissible short complexes" from [25, Sect. 7].

3.1 Admissible short complexes and their homology

Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. A *short complex* in $\mathcal{E}$ is a complex

$$A_\bullet: \qquad 0 \to A_1 \stackrel{d_1}{\to} A_0 \stackrel{d_0}{\to} A_{-1} \to 0 \quad (d_0 \circ d_1 = 0) \tag{4}$$

in $\mathcal{E}$ concentrated in degrees $-1, 0, 1$. It is *admissible* if d_1 and d_0 are inflation and deflation, respectively, and the map $A_1 \to \ker(d_0)$ (or equivalently $\operatorname{coker}(d_1) \to A_{-1}$) is an inflation (deflation). A sequence $A_\bullet \to B_\bullet \to C_\bullet$ of admissible short complexes is *exact* if $A_i \to B_i \to C_i$ is exact in $\mathcal{E}$;

a map $A_\bullet \to B_\bullet$ is a weak equivalence if $A_i \to B_i$ is a weak equivalence in $\mathcal{E}$, $i = -1, 0, 1$. The dual of the complex (4) is the (admissible short) complex

$$A^*_{-1} \stackrel{d^*_0}{\to} A^*_0 \stackrel{d^*_1}{\to} A^*_1,$$

and the double dual identification $\eta_{A_\bullet} : A_\bullet \to (A_\bullet)^{**}$ is η_{A_i} in degrees $i = -1, 0, 1$. We denote by $(s\mathrm{Cx}(\mathcal{E}), w, *, \eta)$ the exact category with weak equivalences and duality of admissible short complexes in $\mathcal{E}$.

If $(A_\bullet, \alpha_\bullet)$ is a symmetric form in $s\mathrm{Cx}(\mathcal{E})$, we write $H_0(A_\bullet, \alpha_\bullet)$ for its zeroth homology symmetric form. It has $H_0(A_\bullet) = \ker(d_0)/\mathrm{im}(d_1)$ as underlying object and is equipped with the form $\bar{\alpha}$ which is the unique symmetric form such that $\bar{\alpha}_{|\ker(d_0)} = \alpha_{|\ker(d_0)}$. This makes $H_0 : s\mathrm{Cx}(\mathcal{E}) \to \mathcal{E}$ into a non-singular exact form functor for every exact category with weak equivalences and duality $\mathcal{E}$. For more details, we refer the reader to [25, Sect. 7].

In the special case of an exact category with duality (where all weak equivalences are isomorphisms), the following two theorems were proved in [25, Theorems 7.1, 7.2]. The K-theory version is due to Waldhausen in [31, Theorem 1.4.2 and Proposition 1.3.2], in view of the equivalence $S^e_1\mathcal{E} \to s\mathrm{Cx}(\mathcal{E}) : A \mapsto (A_{1'0'} \to A_{1'1} \to A_{01})$ of exact categories with weak equivalences and duality.

Theorem 4 (Additivity for short complexes) *Let* $(\mathcal{E}, w, *, \eta)$ *be an exact category with weak equivalences and duality. Then the non-singular exact form functor* $H_0 : s\mathrm{Cx}(\mathcal{E}) \to \mathcal{E}$ *together with the exact functor* $ev_1 : s\mathrm{Cx}(\mathcal{E}) \to \mathcal{E} : A_\bullet \mapsto A_1$ *induce homotopy equivalences* (H_0, ev_1):

$$(wS^e_\bullet\, s\mathrm{Cx}\,\mathcal{E})_h \stackrel{\sim}{\longrightarrow} (wS^e_\bullet\mathcal{E})_h \times wS_\bullet\mathcal{E},$$

$$GW(s\mathrm{Cx}\,\mathcal{E}, w, *, \eta) \stackrel{\sim}{\longrightarrow} GW(\mathcal{E}, w, *, \eta) \times K(\mathcal{E}, w).$$

For the next theorem, recall that a form functor $(\mathcal{A}, *, \eta) \to (\mathcal{B}*, \eta)$ between categories with duality is nothing else than a symmetric form (F, φ) in the category with duality of functors $(\mathrm{Fun}(\mathcal{A}, \mathcal{B}), \sharp, \eta)$; see Sect. 2.1.

Theorem 5 (Additivity for functors) *Let* $(\mathcal{A}, w, *, \eta)$ *and* $(\mathcal{B}, w, *, \eta)$ *be exact categories with weak equivalences and duality. Given a non-singular exact form functor* $(F_\bullet, \varphi_\bullet) : (\mathcal{A}, w, *, \eta) \to s\mathrm{Cx}(\mathcal{B}, w, *, \eta)$, *that is, a commuta-*

tive diagram of exact functors $F_i : (\mathcal{A}, w) \to (\mathcal{B}, w)$

$$\begin{array}{lccccc}
F_\bullet : & F_1 & \overset{d_1}{\rightarrowtail} & F_0 & \overset{d_0}{\twoheadrightarrow} & F_{-1} \\
\wr\downarrow \varphi_\bullet & \wr\downarrow \varphi_1 & & \wr\downarrow \varphi_0 & & \wr\downarrow \varphi_{-1} \\
F_\bullet^\sharp : & F_{-1}^\sharp & \underset{d_0^\sharp}{\rightarrowtail} & F_0^\sharp & \underset{d_1^\sharp}{\twoheadrightarrow} & F_1^\sharp
\end{array}$$

with $d_0 d_1 = 0$, $F_1 \rightarrowtail \ker d_0$ *an inflation, and* $(\varphi_1, \varphi_0, \varphi_{-1}) = (\varphi_{-1}^\sharp \eta, \varphi_0^\sharp \eta, \varphi_1^\sharp \eta)$. *Then the two non-singular exact form functors*

$$(F_0, \varphi_0) \quad \textit{and} \quad H_0(F_\bullet, \varphi_\bullet) \perp \left(F_1 \oplus F_{-1}, \begin{pmatrix} 0 & \varphi_{-1} \\ \varphi_1 & 0 \end{pmatrix} \right) \tag{5}$$

induce homotopic maps on Grothendieck-Witt spaces

$$GW(\mathcal{A}, w, *, \eta) \to GW(\mathcal{B}, w, *, \eta).$$

We will reduce the proofs of the Additivity Theorems 4 and 5 to the case of exact categories with dualities dealt with in [25, Sect. 7]. This will be done with the help of the Simplicial Resolution Lemma 5 below. For that, we need to replace an exact category with weak equivalences and duality (where the double dual identification is a natural weak equivalence) by one with a strong duality (where the double dual identification is a natural isomorphism) without changing its hermitian K-theory. This will be done in the strictification Lemma 4 below.

We introduce notation for Lemma 4. Let *ExWeDu* be the category of small exact categories with weak equivalences and duality; and non-singular exact form functors as maps. Recall that a category with duality $(\mathcal{A}, *, \eta)$ has a strict duality if $\eta = id$ (and in particular, $** = id$). Let $\mathit{ExWeDu}^{\mathrm{str}}$ be the category of small exact categories with weak equivalences and strict duality; and duality preserving functors as maps. Write $\mathrm{lax} : \mathit{ExWeDu}^{\mathrm{str}} \subset \mathit{ExWeDu}$ for the natural inclusion.

Lemma 4 (Strictification lemma) *There is a (strictification) functor*

$$\mathrm{str} : \mathit{ExWeDu} \to \mathit{ExWeDu}^{\mathrm{str}} : (\mathcal{A}, w, *, \eta) \mapsto (\mathcal{A}_w^{\mathrm{str}}, w, \sharp, id)$$

and natural transformations $(\Sigma, \sigma) : id \to \mathrm{lax} \circ \mathrm{str}$ *and* $(\Lambda, \lambda) : \mathrm{lax} \circ \mathrm{str} \to id$ *such that the compositions* $(\Sigma, \sigma) \circ (\Lambda, \lambda)$ *and* $(\Lambda, \lambda) \circ (\Sigma, \sigma)$ *are weakly equivalent to the identity form functor. In particular, for any exact category* *with weak equivalences and duality* $(\mathcal{A}, w, *, \alpha)$, *we have a homotopy equivalence of Grothendieck-Witt spaces*

$$GW(\Sigma, \sigma) : GW(\mathcal{A}, w, *, \alpha) \xrightarrow{\sim} GW(\mathcal{A}_w^{\mathrm{str}}, w, \sharp, id).$$

Proof The construction of the strictification functor is as follows. Let $(\mathcal{A}, w, *, \alpha)$ be an exact category with weak equivalences and duality. The objects of $\mathcal{A}_w^{\mathrm{str}}$ are triples (A, B, f) with A, B objects of $\mathcal{A}$ and $f : A \xrightarrow{\sim} B^*$ a weak equivalence in $\mathcal{A}$. A morphism from (A_0, B_0, f_0) to (A_1, B_1, f_1) is a pair (a, b) of morphisms $a : A_0 \to A_1$ and $b : B_1 \to B_0$ in $\mathcal{A}$ such that $f_1 a = b^* f_0$. Composition is composition of the a's and b's in $\mathcal{A}$. A map (a, b) is a weak equivalence if a and b are weak equivalences in $\mathcal{A}$. A sequence $(A_0, B_0, f_0) \to (A_1, B_1, f_1) \to (A_2, B_2, f_2)$ is exact if $A_0 \to A_1 \to A_2$ and $B_2 \to B_1 \to B_0$ are exact in $\mathcal{A}$. The duality $\sharp : (\mathcal{A}_w^{\mathrm{str}})^{op} \to \mathcal{A}_w^{\mathrm{str}}$ is defined by $(A, B, f : A \to B^*)^\sharp = (B, A, f^* \alpha_B)$ on objects, and by $(a, b)^\sharp = (b, a)$ on morphisms. The double dual identification is the identity natural transformation. The category thus constructed $\mathcal{A}_w^{\mathrm{str}} = (\mathcal{A}_w^{\mathrm{str}}, w, \sharp, id)$ is an exact category with weak equivalences and strict duality. We may write $\mathcal{A}^{\mathrm{str}}$, or $\mathcal{A}_w^{\mathrm{str}}$ for $(\mathcal{A}_w^{\mathrm{str}}, w, \sharp, id)$ if the remaining data are understood. If $(F, \varphi) : (A, w, *, \alpha) \to (\mathcal{B}, v, *, \beta)$ is a non-singular exact form functor, its image under the strictification functor is the functor $F^{\mathrm{str}} : \mathcal{A}_w^{\mathrm{str}} \to \mathcal{B}_v^{\mathrm{str}}$ given by $F^{\mathrm{str}} : (A, B, f) \mapsto (FA, FB, \varphi_B F(f))$ on objects, and by $(a, b) \mapsto (Fa, Fb)$ on morphisms. One checks that F^{str} preserves composition and commutes with dualities.

The natural transformations $(\Sigma, \sigma) : id \to \mathrm{lax} \circ \mathrm{str}$ and $(\Lambda, \lambda) : \mathrm{lax} \circ \mathrm{str} \to id$ are defined as follows. The functor $\Sigma : \mathcal{A} \to \mathcal{A}_w^{\mathrm{str}}$ sends an object A to (A, A^*, α_A) and a morphism a to (a, a^*). The duality compatibility morphism $\sigma : \Sigma(A^*) \to \Sigma(A)^\sharp$ is the map $(1, \alpha_A) : (A^*, A^{**}, \alpha_{A^*}) \to (A^*, A, 1)$. The functor $\Lambda : \mathcal{A}^{\mathrm{str}} \to \mathcal{A}$ sends the object (A, B, f) to A and a morphism (a, b) to a, and is equipped with the duality compatibility morphism $\lambda : \Lambda[(A, B, f)^\sharp] \to [\Lambda(A, B, f)]^*$ the map $f^* \alpha_B : B \to A^*$.

The composition $(\Sigma, \sigma) \circ (\Lambda, \lambda)$ sends (A, B, f) to (A, A^*, α_A) and a morphism (a, b) to (a, a^*). It is equipped with the duality compatibility morphism $(f^* \alpha_B, f) : (B, B^*, \alpha_B) \to (A^*, A, 1)$. There is a natural weak equivalence of form functors $(\Sigma, \sigma) \circ (\Lambda, \lambda) \xrightarrow{\sim} id$ given by the map $(1, f^* \alpha_B) : (A, A^*, \alpha_A) \to (A, B, f)$. Regarding the other composition, we have $(\Lambda, \lambda) \circ (\Sigma, \sigma) = id$.

By Lemma 2, the form functor $(\Sigma, \sigma) : \mathcal{A} \to \mathcal{A}_w^{\mathrm{str}}$ induces a homotopy equivalence of Grothendieck-Witt spaces with homotopy inverse (Λ, λ). □

Remark 4 Here is a slight generalization of Lemma 4. If v is another set of weak equivalences for $(\mathcal{A}, *, \alpha)$ such that $w \subset v$, then $(\mathcal{A}_w^{\mathrm{str}}, v, \sharp, id)$ is also an exact category with weak equivalences and strict duality. The form functors

$(\Lambda, \lambda) : (\mathcal{A}^{\mathrm{str}}_w, v) \to (\mathcal{A}, v)$ and $(\Sigma, \sigma) : (\mathcal{A}, v) \to (\mathcal{A}^{\mathrm{str}}_w, v)$ are still exact and non-singular with compositions that are naturally weakly equivalent to the identity functors. In particular, we have a homotopy equivalence

$$GW(\Sigma, \sigma) : GW(\mathcal{A}, v, *, \alpha) \xrightarrow{\sim} GW(\mathcal{A}^{\mathrm{str}}_w, v, \sharp, id).$$

The next lemma will sometimes allow us to replace an exact category with weak equivalences and duality by a simplicial exact category with duality (where weak equivalences and double dual identification are isomorphisms).

Notation for the Simplicial Resolution lemma Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality, and let $\mathcal{D}$ be an arbitrary (small) category. Recall from Sect. 2.1 that the category $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ of functors $\mathcal{D} \to \mathcal{E}$ is a category with duality. It is an exact category with weak equivalences and duality if we declare maps $F \to G$ (sequences $F_{-1} \to F_0 \to F_1$) of functors $\mathcal{D} \to \mathcal{E}$ to be a weak equivalence (conflation) if $F(A) \to G(A)$ is a weak equivalence ($F_{-1}(A) \to F_0(A) \to F_1(A)$ is a conflation) in $\mathcal{E}$ for all objects A of $\mathcal{D}$. We write $\mathrm{Fun}_w(\mathcal{D}, \mathcal{E}) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{E})$ for the full subcategory of those functors $F : \mathcal{D} \to \mathcal{C}$ for which the image $F(d)$ of all maps d of $\mathcal{D}$ are weak equivalences in $\mathcal{E}$: $F(d) \in w\mathcal{E}$. The category $\mathrm{Fun}_w(\mathcal{D}, \mathcal{E})$ inherits from $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ the structure of an exact category with weak equivalences and duality. In particular, for $n \in \mathbb{N}$ and $(\mathcal{E}, w, *, \eta)$ an exact category with weak equivalences and strong duality, the category $\mathrm{Fun}_w(\underline{n}, \mathcal{E})$ is an exact category with weak equivalences and strong duality. It has objects strings of weak equivalences and maps commutative diagrams in $\mathcal{E}$. Varying n, the categories $\mathrm{Fun}_w(\underline{n}, \mathcal{E})$ define a simplicial exact category with weak equivalences and strong duality. Recall that the symbol i stands for the set of isomorphisms in a category.

Lemma 5 (Simplicial Resolution lemma) *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and strong duality. Then there are homotopy equivalences*

$$|(wS^e_\bullet\mathcal{E})_h| \simeq |n \mapsto (iS^e_\bullet \mathrm{Fun}_w(\underline{n}, \mathcal{E}))_h|,$$

$$GW(\mathcal{E}, w) \simeq |n \mapsto GW(\mathrm{Fun}_w(\underline{n}, \mathcal{E}), i)|$$

which are functorial for exact form functors (F, φ) for which φ is an isomorphism.

Proof We start with some general remarks. Let $\mathcal{C}$ be a category, and recall that i stands for the set of isomorphisms in $\mathcal{C}$. Since $\mathcal{C} = \mathrm{Fun}_i([0], \mathcal{C})$, inclusion of degree-zero simplices yields a map of simplicial categories $\mathcal{C} \to (n \mapsto \mathrm{Fun}_i([n], \mathcal{C}))$ which is degree-wise an equivalence of categories, and thus induces a homotopy equivalence after topological realization. Using the equality of bisimplicial sets

$$p, q \mapsto N_p \mathrm{Fun}_i([q], \mathcal{C}) = N_q i\mathrm{Fun}([p], \mathcal{C}),$$

where $N_\bullet$ stands for the nerve of a category, we obtain a homotopy equivalence $|\mathcal{C}| \xrightarrow{\sim} |n \mapsto i\mathrm{Fun}([n], \mathcal{C})|$. Since the topological realizations of $p \mapsto i\mathrm{Fun}([p], \mathcal{C})$ and of $p \mapsto i\mathrm{Fun}([p]^{op}, \mathcal{C})$ are isomorphic, we have a homotopy equivalence

$$|\mathcal{C}| \xrightarrow{\sim} |n \mapsto i\mathrm{Fun}([n]^{op}, \mathcal{C})|. \tag{6}$$

The homotopy equivalence is natural in the category $\mathcal{C}$.

Let $(\mathcal{C}, *, \eta)$ be a category with strong duality. There is an equivalence of categories $i\mathrm{Fun}([n]^{op}, \mathcal{C}_h) \to (i\mathrm{Fun}(\underline{n}, \mathcal{C}))_h$ which sends an object

$$(X_n, \varphi_n) \xrightarrow{f_n} (X_{n-1}, \varphi_{n-1}) \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} (X_0, \varphi_0)$$

of the left hand category to the object

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 \xrightarrow{\varphi_0} X_0^* \xrightarrow{f_1^*} X_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_n^*} X_n^*$$

equipped with the form $(\eta_{X_n}, \dots, \eta_{X_0}, 1, \dots, 1)$. A map $(g_n, \dots, g_0)$ (which is an isomorphism compatible with forms) is sent to $(g_n, \dots, g_0, (g_0^*)^{-1}, \dots, (g_n^*)^{-1})$. The equivalence is functorial in $[n] \in \Delta$ and thus induces a homotopy equivalence after topological realization. Together with (6), we obtain a homotopy equivalence of topological spaces

$$|\mathcal{C}_h| \xrightarrow{\sim} |n \mapsto (i\mathrm{Fun}(\underline{n}, \mathcal{C}))_h| \tag{7}$$

which is natural for categories with strong duality $(\mathcal{C}, *, \eta)$ and form functors (F, φ) between them for which φ is an isomorphism.

For an exact category with weak equivalences and strong duality $(\mathcal{E}, w, *, \eta)$, we apply the homotopy equivalence (7) to the form functor $wS_p^e\mathcal{E} \to wS_p^e\mathcal{H}\mathcal{E}$ induced by the forgetful form functor $\mathcal{E} \to \mathcal{H}\mathcal{E}$ (see Remark 3). Varying p, we obtain a map of homotopy equivalences after topological realization

$$\begin{array}{ccc} |(wS_\bullet^e\mathcal{E})_h| & \xrightarrow{\sim} & |n \mapsto (iS_\bullet^e\mathrm{Fun}_w(\underline{n}, \mathcal{E}))_h| \\ \downarrow & & \downarrow \\ |(wS_\bullet^e\mathcal{H}\mathcal{E})_h| & \xrightarrow{\sim} & |n \mapsto (iS_\bullet^e\mathcal{H}\mathrm{Fun}_w(\underline{n}, \mathcal{E}))_h|. \end{array}$$

The top row gives the first homotopy equivalence of the lemma. By Remark 3, the left vertical homotopy fibre of the diagram is $GW(\mathcal{E}, w, *, \eta)$. In view of

Bousfield-Friedlander's Theorem [5, B4], [8, Theorem IV 4.9], the homotopy fibre of the right vertical map is the simplicial realization of the degree-wise homotopy fibres. By Remark 3, this is

$$|n \mapsto GW(\mathrm{Fun}_w(\underline{n}, \mathcal{E}), i, *, \eta)|. \qquad \square$$

Before proving the Additivity Theorems, we give a first application of the Simplicial Resolution Lemma and show that $\pi_0 GW(\mathcal{E}, w, *, \eta)$ is the Grothendieck-Witt group as in Definition 1.

Proposition 3 (Presentation of GW_0) *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. There is a natural isomorphism*

$$GW_0(\mathcal{E}, w, *, \eta) \cong \pi_0 GW(\mathcal{E}, w, *, \eta).$$

Proof In view of relation (b) of Definition 1 and Lemma 2, weakly equivalent non-singular exact form functors $(F, \varphi) \xrightarrow{\sim} (G, \psi)$ induce the same map on GW_0 and on $\pi_0 GW$. Therefore, we can replace $(\mathcal{E}, w)$ by its strictification $(\mathcal{E}_w^{\mathrm{str}}, w)$ given in Lemma 5. So we can assume the duality on $(\mathcal{E}, w)$ to be strong which will allow us to use the Simplicial Resolution Lemma 5. For a bisimplicial set $X_{\bullet\bullet}$, there is a co-equalizer diagram

$$\pi_0 |X_{1\bullet}| \underset{d_0}{\overset{d_1}{\rightrightarrows}} \pi_0 |X_{0\bullet}| \longrightarrow \pi_0 |X_{\bullet\bullet}|. \tag{8}$$

This is well known and follows from an examination of the usual skeletal filtration of $|X_{\bullet\bullet}| = |n \mapsto X_{n,n}| = |n \mapsto X_{n\bullet}|$—which the reader can find in [8, Diagram IV.1 (1.6)], for instance—using the fact that the functor $\pi_0 : \Delta^{op}\mathrm{Sets} \to \mathrm{Sets}$ preserves push-out diagrams as it is left adjoint to the inclusion functor $\mathrm{Sets} \to \Delta^{op}\mathrm{Sets}$. By the Simplicial Resolution Lemma 5 and the fact (proven in [25, Proposition 4.11] together with Proposition 2) that the proposition holds when the set of weak equivalences is the set of isomorphisms, we deduce that $\pi_0 GW(\mathcal{E}, w)$ is the co-equalizer of the diagram

$$GW_0\left[\mathrm{Fun}_w(\underline{1}, \mathcal{E})\right] \underset{d_0}{\overset{d_1}{\rightrightarrows}} GW_0\left[\mathrm{Fun}_w(\underline{0}, \mathcal{E})\right]$$

of Grothendieck-Witt groups of exact categories with duality. It suffices therefore to display $GW_0(\mathcal{E}, w, *, \eta)$ as the co-equalizer of the same diagram. Evaluation at the object $0'$ of $\underline{0}$ defines a non-singular exact form functor $F : (\mathrm{Fun}_w(\underline{0}, \mathcal{E}), i) \to (\mathcal{E}, w)$ which sends the object $f : X_{0'} \to X_0$ to $F(f) = X_{0'}$ and has duality compatibility morphism $F(f^*) \to F(f)^*$ the map $f^* : X_0^* \to X_{0'}^*$. The form functor induces a map $GW_0(\mathrm{Fun}_w(\underline{0}, \mathcal{E}), i) \to$

$GW_0(\mathcal{E}, w)$ which equalizes d_0 and d_1 in view of the relation (b) in Definition 1. We therefore obtain a map from the co-equalizer of the diagram to $GW_0(\mathcal{E}, w)$. To construct its inverse, consider the map from the free Abelian group generated by isomorphism classes $[X, \varphi]$ of symmetric spaces in $(\mathcal{E}, w)$ to $GW_0(\mathrm{Fun}_w(\underline{0}, \mathcal{E}), i)$ sending (X, φ) to $\varphi : X \to X^*$ equipped with the non-singular form $(\eta_X, 1)$. This map is surjective and factors through relations (a) and (c) of Definition 1. The map induces a surjective map to the co-equalizer which factors through relation (b) of Definition 1 and thus induces a surjective map from $GW_0(\mathcal{E}, w, *, \eta)$ to the co-equalizer. Since composition with the map from the co-equalizer to $GW_0(\mathcal{E}, w, *, \eta)$ is the identity, the claim follows. □

Remark 5 Using the co-equalizer diagram (8), one can show directly the isomorphism $\pi_0|(wS^e_\bullet\mathcal{E})_h| \cong W_0(\mathcal{E}, w, *, \eta)$ without the need of the Simplicial Resolution lemma.

Proof of Theorem 4 In view of the strictification Lemma 4 we can assume the duality on $\mathcal{E}$ to be strong. By the Simplicial Resolution Lemma 5 the proof reduces further to the case of an exact category with duality (in which all weak equivalences are isomorphisms). This case was proved in [25, Theorems 7.2, Corollary 7.7] in view of Proposition 2. □

Proof of Theorem 5 Theorem 5 is a formal consequence of Theorem 4. Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. Consider the form functors $s\mathrm{Cx}(\mathcal{E}) \to \mathcal{H}\mathcal{E} : E_\bullet \mapsto (E_1, E^*_{-1})$ and $\mathcal{H}\mathcal{E} \to s\mathrm{Cx}(\mathcal{E}) : (E_1, E_{-1}) \mapsto (E_1 \to E_1 \oplus E^*_{-1} \to E^*_{-1})$ with duality compatibility morphisms $(1, \eta) : (E^*_{-1}, E^{**}_1) \to (E^*_{-1}, E_1)$ and $(\eta, \left(\begin{smallmatrix} 0 & 1 \\ \eta & 0 \end{smallmatrix}\right), 1) : (E_{-1} \to E_{-1} \oplus E^*_1 \to E^*_1) \to (E^{**}_{-1} \to E^*_1 \oplus E^{**}_{-1} \to E^*_1)$. Together with the form functor $H_0 : s\mathrm{Cx}(\mathcal{E}) \to \mathcal{E}$ and the duality preserving functor $\mathcal{E} \to s\mathrm{Cx}(\mathcal{E}) : E \mapsto (0 \to E \to 0)$ they define non-singular exact form functors

$$\mathcal{E} \times \mathcal{H}\mathcal{E} \to s\mathrm{Cx}(\mathcal{E}) \to \mathcal{E} \times \mathcal{H}\mathcal{E}$$

whose composition is weakly equivalent to the identity functor. By the Additivity Theorem for short complexes (Theorem 4), the second form functor induces a homotopy equivalence in hermitian K-theory. It follows that the two form functors induce inverse homotopy equivalences on Grothendieck-Witt spaces and on hermitian $S_\bullet$ constructions. Therefore, the compositions

$$\mathcal{A} \xrightarrow{(F_\bullet, \varphi_\bullet)} s\mathrm{Cx}(\mathcal{B}) \xrightarrow{ev_0} \mathcal{B} \quad \text{and}$$
$$\mathcal{A} \xrightarrow{(F_\bullet, \varphi_\bullet)} s\mathrm{Cx}(\mathcal{B}) \to \mathcal{B} \times \mathcal{H}\mathcal{B} \to s\mathrm{Cx}(\mathcal{B}) \xrightarrow{ev_0} \mathcal{B}$$

induce homotopic maps in hermitian K-theory. These compositions are (weakly equivalent to) the maps in (5). □

Remark 6 In the proof of Proposition 4 below, we will need the following relative version of Theorem 5. Let $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$ be duality preserving exact inclusions of exact categories with duality. Recall that there is an equivalence of categories with duality $s\mathrm{Cx}(\mathcal{B}) \cong S_1^e(\mathcal{B})$. With this in mind, we will give the relative version in terms of S_1^e. Let $(F, \varphi) : \mathcal{A} \to S_1^e \mathcal{B}$ be a non-singular exact form functor sending $\mathcal{A}_0$ to $S_1^e \mathcal{B}_0$. Then the homotopy

$$(F_{1'1}, \varphi_{1'1}) \sim (F_{0'0}, \varphi_{0'0}) \perp \mathcal{H}(F_{1'0'})$$

of Theorem 5 is a homotopy of pairs

$$(GW(\mathcal{A}), GW(\mathcal{A}_0)) \to (GW(B), GW(\mathcal{B}_0)),$$

that is, the homotopy sends $GW(\mathcal{A}_0)$ to $GW(\mathcal{B}_0)$.

This follows directly from the proof of Theorem 5 given just above. We only have to see that the map $s\mathrm{Cx}(\mathcal{B}) \to \mathcal{B} \times \mathcal{HB} \to s\mathrm{Cx}(\mathcal{B})$ and the identity map induce maps on Grothendieck-Witt spaces which are homotopic as maps of pairs. For that, recall that the category of morphisms of topological spaces has a model category structure in which a map $(f_0, f_1) : (X_0 \to X_1) \to (Y_0 \to Y_1)$ of morphisms is a weak equivalence (fibration) if f_0 and f_1 are homotopy equivalences (Hurewicz fibrations), and the map (f_0, f_1) is a cofibration if f_0 and f_1 are cofibrations and if the induced map $Y_0 \sqcup_{X_0} X_1 \to Y_1$ is a cofibration. In particular, every object is fibrant and the object $(X_0 \to X_1)$ is cofibrant if $X_0 \to X_1$ is a cofibration of spaces. In our case, the objects $(X_0 \to X_1)$ and $(Y_0 \to Y_1)$, which are the Grothendieck-Witt spaces of the inclusions

$$s\mathrm{Cx}(\mathcal{B}_0) \subset s\mathrm{Cx}(\mathcal{B}) \quad \text{and} \quad \mathcal{B}_0 \times \mathcal{HB}_0 \subset \mathcal{B} \times \mathcal{HB},$$

are fibrant and cofibrant in the category of morphisms of spaces. From the proof of Theorem 5, the maps $(X_0 \to X_1) \to (Y_0 \to Y_1)$ and $(Y_0 \to Y_1) \to (X_0 \to X_1)$ are weak equivalence of pairs, and they are inverse to each other in the homotopy category of pairs. Since both objects are fibrant and cofibrant, the composition of the maps $(X_0 \to X_1) \to (Y_0 \to Y_1) \to (X_0 \to X_1)$ is homotopic to the identity through a homotopy of pairs.

Remark 7 Iterated application of Theorem 5 implies homotopy equivalences

$$(wS_\bullet^e S_n^e \mathcal{E})_h \xrightarrow{\sim} \begin{cases} (wS_\bullet^e \mathcal{E})_h \times \prod_{p=1}^k wS_\bullet \mathcal{E}, & n = 2k+1 \\ \prod_{p=1}^k wS_\bullet \mathcal{E}, & n = 2k. \end{cases}$$

This allows us to identify $(wS^e_\bullet S^e_\bullet\mathcal{E})_h$ with the Bar construction of the H-group $wS_\bullet\mathcal{E}$ acting on $(wS^e_\bullet\mathcal{E})_h$ and leads to a homotopy fibration

$$(wS^e_\bullet\mathcal{E})_h \to (wS^e_\bullet S^e_\bullet\mathcal{E})_h \to wS_\bullet S_\bullet\mathcal{E}$$

in which the first map is "inclusion of degree zero simplices" and the second map is the "forgetful map" $(E,\varphi)\mapsto E$ followed by the canonical homotopy equivalence $X^e_\bullet \to X_\bullet$. In particular, the iterated hermitian $S_\bullet$-construction $(wS^e_\bullet S^e_\bullet\mathcal{E})_h$ is not a delooping of $(wS^e_\bullet\mathcal{E})_h$ contrary to the K-theory situation, compare [31, Proposition 1.5.3 and remark thereafter].

Remark 8 Define the *Witt-theory space* $W(\mathcal{E},w,*,\eta)$ as the colimit of the top row in the sequence of homotopy fibrations

$$\begin{array}{ccccccccc}
GW(\mathcal{E},w) & \longrightarrow & (wS^e_\bullet\mathcal{E})_h & \longrightarrow & (wS^e_\bullet S^e_\bullet\mathcal{E})_h & \longrightarrow & (wS^e_\bullet S^e_\bullet S^e_\bullet\mathcal{E})_h & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & wS_\bullet\mathcal{E} & & wS_\bullet S_\bullet\mathcal{E} & & wS_\bullet S_\bullet S_\bullet\mathcal{E} & &
\end{array}$$

Since the spaces in the second row get higher and higher connected, we see that $\pi_0 W(\mathcal{E},w,*,\eta) = W_0(\mathcal{E},w,*,\eta)$ and $\pi_1 W(\mathcal{E},i,*,\eta) = W_{form}(\mathcal{E},*,\eta)$ where the group $W_{form}(\mathcal{E},*,\eta)$ is the Witt-group of formations in $(\mathcal{E},*,\eta)$, that is, the cokernel of the hyperbolic map $K_0(\mathcal{E}) \to GW_{form}(\mathcal{E}) : [X] \mapsto [\mathcal{H}X, X, X^*]$ to the Grothendieck-Witt group of formations $GW_{form}(\mathcal{E})$ defined in [25, 4.3].

If $\mathcal{E}$ is a $\mathbb{Z}[\frac{1}{2}]$-linear category and "complicial", we show in [26] that the Witt-theory space $W(\mathcal{E},w,*,\eta)$ is the infinite loop space associated with (the (-1)-connected cover of) Ranicki's $\mathbb{L}$-theory spectrum, and its homotopy groups

$$\pi_i W(\mathcal{E},w,*,\eta) = W^{-i}(w^{-1}\mathcal{E},*,\eta), \quad i \geq 0,$$

are Balmer's Witt groups $W^{-i}(w^{-1}\mathcal{E},*,\eta)$ of the triangulated category with duality $(w^{-1}\mathcal{E},*,\eta)$. At this point, I don't know how to calculate $\pi_n W(\mathcal{E},i,*,\eta)$, $n\geq 2$, for (complicial) $(\mathcal{E},w,*,\eta)$ when $\mathcal{E}$ is not $\mathbb{Z}[\frac{1}{2}]$-linear.

4 Change-of-weak-equivalences and cofinality

In this section we prove in Theorems 6 and 7 the higher Grothendieck-Witt theory analogs of Waldhausen's Fibration Theorem [31, 1.6.4] and of Thomason's Cofinality Theorem [30, 1.10.1].

Waldhausen's K-theory version of Theorem 6 below needs a "cylinder functor". The purpose of the next definition is to define the higher Grothendieck-Witt theory analog. We first fix some notation. For an exact category with weak equivalences $(\mathcal{E}, w)$, write $\mathcal{E}^w \subset \mathcal{E}$ for the full subcategory of w-acyclic objects, that is, those objects E of $\mathcal{E}$ for which the unique map $0 \to E$ is a weak equivalence. The category $\mathcal{E}^w$ is closed under extensions in $\mathcal{E}$, and thus inherits an exact structure from $\mathcal{E}$ such that the inclusion $\mathcal{E}^w \subset \mathcal{E}$ is fully exact.

Definition 4 Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality. A *symmetric cone* on $(\mathcal{E}, w, *, \eta)$ is given by the following data:

(a) exact functors $P : \mathcal{E} \to \mathcal{E}^w$, and $C : \mathcal{E} \to \mathcal{E}^w$,
(b) a natural deflation $p_E : PE \twoheadrightarrow E$ and a natural inflation $i_E : E \rightarrowtail CE$,
(c) a natural map $\gamma_E : P(E^*) \to (CE)^*$ such that $i_E^* \gamma_E = p_{E^*}$ for all objects E of $\mathcal{E}$.

It is convenient to define $\bar{\gamma}_E : C(E^*) \to (PE)^*$ by $\bar{\gamma}_E = P(\eta_E)^* \circ \gamma_{E^*}^* \circ \eta_{C(E^*)}$. Then $p_E^* = \bar{\gamma}_E \circ i_{E^*}$, and $\gamma_E = C(\eta_E)^* \circ \bar{\gamma}_{E^*}^* \circ \eta_{P(E^*)}$. In other words, the sequence $P \to id \to C$ defines an exact form functor from $\mathcal{E}$ to the category of sequences $E_{-1} \twoheadrightarrow E_0 \rightarrowtail E_1$ in $\mathcal{E}$ with duality compatibility map $(\gamma, 1, \bar{\gamma})$.

For examples of symmetric cones, see Sect. 7.5.

The proof of the next theorem will occupy most of this section.

Theorem 6 (Change-of-weak-equivalences) *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality which has a symmetric cone. Let v be another set of weak equivalences in $\mathcal{E}$ containing w and which is closed under the duality. Then the duality $(*, \eta)$ on $\mathcal{E}$ makes $(\mathcal{E}^v, w)$, $(\mathcal{E}^v, v)$, $(\mathcal{E}, v)$ into exact categories with weak equivalences and duality such that the commutative square of duality preserving inclusions*

$$\begin{array}{ccc} (\mathcal{E}^v, w) & \longrightarrow & (\mathcal{E}^v, v) \\ \downarrow & & \downarrow \\ (\mathcal{E}, w) & \longrightarrow & (\mathcal{E}, v) \end{array}$$

induces a homotopy Cartesian square of associated Grothendieck-Witt spaces. Moreover, the upper right corner has contractible Grothendieck-Witt space.

Remark 9 A square of homotopy commutative H-groups (such as Grothendieck-Witt spaces) is homotopy Cartesian if and only if the map between, say, horizontal homotopy fibres is a homotopy equivalence, and the map of Abelian groups between horizontal cokernels of π_0's is a monomorphism.

Remark 10 In Theorem 6, the map $GW_0(\mathcal{E}, w, *, \eta) \to GW_0(\mathcal{E}, v, *, \eta)$ is not surjective, in general, contrary to the K-theory situation in [31, Fibration Theorem 1.6.4].

4.1 Idempotent completion

We will reduce the proof of Theorem 6 to idempotent complete exact categories with weak equivalences and duality. Recall that the *idempotent completion* $\tilde{\mathcal{E}}$ of an exact category $\mathcal{E}$ has objects pairs (A, p) with $p = p^2 : A \to A$ an idempotent in $\mathcal{E}$. A map $(A, p) \to (B, q)$ is a map $f : A \to B$ in $\mathcal{E}$ such that $f = fp = qf$. Composition is composition of maps in $\mathcal{E}$. The idempotent completion $\tilde{\mathcal{E}}$ has a canonical structure of an exact category such that the inclusion $\mathcal{E} \subset \tilde{\mathcal{E}} : A \mapsto (A, 1)$ is fully exact (see [30, Theorem A.9.1], where "idempotent completion" is called "Karoubianisation"). Any duality $(*, \eta)$ on $\mathcal{E}$ extends to a duality $(A, p)^* = (A^*, p^*)$ on $\tilde{\mathcal{E}}$ with double dual identification $\eta_A \circ p : (A, p) \to (A, p)^{**}$. If $(\mathcal{E}, w, *, \eta)$ is an exact category with duality, call a map in the idempotent completion $\tilde{\mathcal{E}}$ *weak equivalence* if it is a retract of a weak equivalence in $\mathcal{E}$. Then $(\tilde{\mathcal{E}}, w, *, \eta)$ is an exact category with weak equivalences and duality. Note that the natural inclusion $\widetilde{\mathcal{E}^w} \subset (\tilde{\mathcal{E}})^w$ is an equivalence of categories if $(\mathcal{E}, w, *, \eta)$ has a (symmetric) cone. This is because for an object X in $(\tilde{\mathcal{E}})^w$, the weak equivalence $0 \to X$ in $\tilde{\mathcal{E}}$ is, by definition, a retract of a weak equivalence $f : Y \to Z$ in $\mathcal{E}$, and, by functoriality, also a retract of $0 \to C(f)$, where $C(f)$ is the push-out of $i_Y : Y \to CY$ along f. Since $C(f)$ is in $\mathcal{E}^w$, the object X is (isomorphic to an object) in $\widetilde{\mathcal{E}^w}$.

Lemma 6 *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and strong duality which has a symmetric cone. Then the commutative diagram of exact categories with duality*

$$\begin{array}{ccc} (\mathcal{E}^w, i) & \longrightarrow & (\tilde{\mathcal{E}}^w, i) \\ \downarrow & & \downarrow \\ (\mathcal{E}, i) & \longrightarrow & (\tilde{\mathcal{E}}, i) \end{array}$$

induces a homotopy Cartesian square of Grothendieck-Witt spaces.

Proof By the Cofinality Theorem in [25, Corollary 5.2], the horizontal homotopy fibres of associated Grothendieck-Witt spaces are contractible. Therefore, it suffices to show that the map $GW_0(\tilde{\mathcal{E}}^w)/GW_0(\mathcal{E}^w) \to GW_0(\tilde{\mathcal{E}})/GW_0(\mathcal{E})$

between the cokernels of horizontal π_0's is injective. For an exact category with duality $\mathcal{A}$, the quotient $GW_0(\tilde{\mathcal{A}})/GW_0(\mathcal{A})$ is the Abelian monoid of isometry classes of symmetric spaces in $\tilde{\mathcal{A}}$ modulo the submonoid of symmetric spaces in $\mathcal{A}$ [25, 5.2]. In particular, a symmetric space in $\tilde{\mathcal{A}}$ yields the zero class in $GW_0(\tilde{\mathcal{A}})/GW_0(\mathcal{A})$ if and only if it is stably in $\mathcal{A}$.

Let (A,α) be a symmetric space in $\tilde{\mathcal{E}}^w$ whose class in $GW_0(\tilde{\mathcal{E}})/GW_0(\mathcal{E})$ is zero. Then there are symmetric spaces (X,φ) and (Y,ψ) in $\mathcal{E}$ and an isometry $(A,\alpha)\perp(Y,\psi)\cong(X,\varphi)$. In particular, there is an exact sequence $0\to A\to X\xrightarrow{f} Y\to 0$ in $\tilde{\mathcal{E}}$. The push-out $C(f)$ of f along the $\mathcal{E}$-inflation $X\rightarrowtail CX$ is in $\mathcal{E}$. In the exact sequence $0\to A\to CX\to C(f)\to 0$, we have A and CX in $\tilde{\mathcal{E}}^w$. Therefore, $C(f)$ is also in $\tilde{\mathcal{E}}^w$, hence $C(f)\in\tilde{\mathcal{E}}^w\cap\mathcal{E}=\mathcal{E}^w$. Choose $\bar{A}\in\tilde{\mathcal{E}}^w$ such that $A\oplus\bar{A}\in\mathcal{E}^w$. Then $\bar{A}\oplus CX$ is in $\mathcal{E}^w$ as it is an extension of $A\oplus\bar{A}$ and $C(f)$, both being in $\mathcal{E}^w$. It follows that the symmetric spaces $(A,\alpha)\oplus\mathcal{H}(\bar{A}\oplus CX)$ and $\mathcal{H}(\bar{A}\oplus CX)$ lie in $\mathcal{E}^w$, where $\mathcal{H}E=(E\oplus E^*,\left(\begin{smallmatrix}0&1\\ \eta&0\end{smallmatrix}\right))$ is the hyperbolic space associated with E. This implies that (A,α) is trivial in $GW_0(\tilde{\mathcal{E}}^w)/GW_0(\mathcal{E}^w)$. □

For an exact category with weak equivalences and duality $(\mathcal{E},w,*,\eta)$, the category $\mathrm{Mor}\,\mathcal{E}=\mathrm{Fun}([1],\mathcal{E})$ of morphisms in $\mathcal{E}$ is an exact category with weak equivalences and duality such that the fully exact inclusion $\mathcal{E}\subset\mathrm{Mor}\,\mathcal{E}: E\mapsto id_E$, induced by the unique map $[1]\to[0]$, is duality preserving. The inclusion factors through the fully exact subcategory $\mathrm{Mor}_w\,\mathcal{E}=\mathrm{Fun}_w([1],\mathcal{E})\subset\mathrm{Mor}\,\mathcal{E}$ of weak equivalences in $\mathcal{E}$ and defines a duality preserving functor

$$I:\mathcal{E}\to\mathrm{Mor}_w\,\mathcal{E}.$$

Note that $(\mathrm{Mor}_w\,\mathcal{E})^w=\mathrm{Mor}_w(\mathcal{E}^w)=\mathrm{Mor}(\mathcal{E}^w)$.

The following proposition is the key to proving Theorem 6.

Proposition 4 *Let $(\mathcal{E},w,*,\eta)$ be an exact category with weak equivalences and strong duality. Assume that $(\mathcal{E},w,*,\eta)$ has a symmetric cone. Then the commutative diagram*

$$\begin{array}{ccc}\mathcal{E}^w & \longrightarrow & \mathrm{Mor}_w\,\mathcal{E}^w\\ \downarrow & & \downarrow\\ \mathcal{E} & \xrightarrow{I} & \mathrm{Mor}_w\,\mathcal{E}\end{array}\qquad(9)$$

of duality preserving inclusions of exact categories with duality (all weak equivalences being isomorphisms) induces a homotopy Cartesian square of associated Grothendieck-Witt spaces.

4.2 Cone exact categories

The proof of Proposition 4 uses the cone category construction of [25, Sect. 9]. We recall the relevant definitions and facts.

Let $\mathcal{A} \subset \mathcal{U}$ be a duality preserving fully exact inclusion of idempotent complete exact categories with duality $(*, \eta)$. In [25, Sect. 9], we constructed a duality preserving fully exact inclusion $\Gamma : \mathcal{U} \subset \mathcal{C}(\mathcal{U}, \mathcal{A})$ of exact categories with duality, depending functorially on the pair $\mathcal{A} \subset \mathcal{U}$, such that the duality preserving commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathcal{C}(\mathcal{A}, \mathcal{A}) \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{\Gamma} & \mathcal{C}(\mathcal{U}, \mathcal{A}) \end{array} \tag{10}$$

of exact categories with duality induces a homotopy Cartesian square of associated Grothendieck-Witt spaces, and the upper right corner $\mathcal{C}(\mathcal{A}, \mathcal{A})$ (also written as $\mathcal{C}(\mathcal{A})$) has contractible Grothendieck-Witt space [25, Theorem 9.7].

We recall the definition of the *cone category* $\mathcal{C}(\mathcal{U}, \mathcal{A})$, details can be found in [25, Sect. 9.1–9.3]. One first constructs a category $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$, a localization of which is $\mathcal{C}(\mathcal{U}, \mathcal{A})$. Objects of $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ are commutative diagrams in $\mathcal{U}$

$$\begin{array}{ccccccccc} U_0 & \overset{\sim}{\rightarrowtail} & U_1 & \overset{\sim}{\rightarrowtail} & U_2 & \overset{\sim}{\rightarrowtail} & U_3 & \overset{\sim}{\rightarrowtail} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ U^0 & \overset{\sim}{\twoheadleftarrow} & U^1 & \overset{\sim}{\twoheadleftarrow} & U^2 & \overset{\sim}{\twoheadleftarrow} & U^3 & \overset{\sim}{\twoheadleftarrow} & \cdots \end{array} \tag{11}$$

such that the maps $U_i \overset{\sim}{\rightarrowtail} U_{i+1}$ and $U^{i+1} \overset{\sim}{\twoheadrightarrow} U^i$, $i \in \mathbb{N}$ are inflations with cokernel in $\mathcal{A}$ and deflations with kernel in $\mathcal{A}$, respectively. Moreover, there has to be an integer d such that for every $i \geq j$, the map $U_j \to U^{i+d}$ is an inflation with cokernel in $\mathcal{A}$ and the map $U_{i+d} \to U^j$ is a deflation with kernel in $\mathcal{A}$. If the maps in diagram (11) are understood, we may abbreviate the diagram as $(U_\bullet \to U^\bullet)$. Maps in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ are natural transformations of diagrams. A sequence of diagrams in $\mathcal{C}_0(\mathcal{U}, \mathcal{A})$ is exact if at each U_i, U^j spot it is an exact sequence in $\mathcal{U}$. The dual of the diagram (11) is obtained by applying the duality to the diagram: $(U_\bullet \to U^\bullet)^* = ((U^\bullet)^* \to (U_\bullet)^*)$.

For each diagram (11), forgetting the upper left corner U_0 gives us a new object $(U_\bullet \to U^\bullet)_{[1]} = (U_{\bullet+1} \to U^\bullet)$ and a canonical map $(U_\bullet \to U^\bullet) \to (U_{\bullet+1} \to U^\bullet)$. Similarly, forgetting the lower left corner U^0 defines a new object $(U_\bullet \to U^\bullet)^{[1]} = (U_\bullet \to U^{\bullet+1})$ and a canonical map $(U_\bullet \to U^{\bullet+1}) \to$

$(U_\bullet \to U^\bullet)$. Finally, the category $\mathcal{C}(\mathcal{U},\mathcal{A})$ is the localization of $\mathcal{C}_0(\mathcal{U},\mathcal{A})$ with respect to the two types of canonical maps just defined. A sequence in $\mathcal{C}(\mathcal{U},\mathcal{A})$ is a conflation if and only if it is isomorphic in $\mathcal{C}(\mathcal{U},\mathcal{A})$ to the image under the localization functor $\mathcal{C}_0(\mathcal{U},\mathcal{A}) \to \mathcal{C}(\mathcal{U},\mathcal{A})$ of a conflation in $\mathcal{C}_0(\mathcal{U},\mathcal{A})$.

There is a fully exact duality preserving inclusion $\Gamma : \mathcal{U} \subset \mathcal{C}(\mathcal{U},\mathcal{A})$ which sends an object U of $\mathcal{U}$ to the constant diagram

$$\begin{array}{ccccccccc}
U & \overset{=}{\rightarrowtail} & U & \overset{=}{\rightarrowtail} & U & \overset{=}{\rightarrowtail} & U & \overset{=}{\rightarrowtail} & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \\
U & \overset{=}{\twoheadleftarrow} & U & \overset{=}{\twoheadleftarrow} & U & \overset{=}{\twoheadleftarrow} & U & \overset{=}{\twoheadleftarrow} & \cdots
\end{array}$$

Proof of Proposition 4 By Lemma 6, we can (and will) assume $\mathcal{E}$ to be idempotent complete. Then all categories in diagram (9) are idempotent complete.

We will write $\mathcal{C}(\mathcal{E},w)$ and $\mathcal{C}(\mathcal{E}^w)$ instead of the categories $\mathcal{C}(\mathcal{E},\mathcal{E}^w)$ and $\mathcal{C}(\mathcal{E}^w,\mathcal{E}^w)$ of Sect. 4.2, and we will write $(F,\varphi) \sim (G,\psi)$ if the two non-singular exact form functors (F,φ), (G,ψ) induce homotopic maps on Grothendieck-Witt spaces. The strategy of proof is as follows. We will extend diagram (9) to a commutative diagram of exact categories with duality and non-singular exact form functors

$$\begin{array}{ccccccc}
\mathcal{E}^w & \longrightarrow & \mathrm{Mor}_w\,\mathcal{E}^w & \longrightarrow & \mathcal{C}(\mathcal{E}^w) & \longrightarrow & \mathcal{C}(\mathrm{Mor}_w\,\mathcal{E}^w) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{I} & \mathrm{Mor}_w\,\mathcal{E} & \xrightarrow{(F,\varphi)} & \mathcal{C}(\mathcal{E},w) & \xrightarrow{\mathcal{C}(I)} & \mathcal{C}(\mathrm{Mor}_w\,\mathcal{E},w)
\end{array} \qquad (12)$$

where the right hand square is obtained from (9) by functoriality of the cone category construction in Sect. 4.2. We will show:

(†) (GW applied to) the compositions $(F,\varphi)\circ(I,id)$ and $(\mathcal{C}(I),id)\circ(F,\varphi)$ are homotopic to the constant diagram inclusions (Γ,id) of Sect. 4.2 in such a way that the homotopies restricted to $\mathcal{E}^w$ and $\mathrm{Mor}_w\,\mathcal{E}^w$ have images in the Grothendieck-Witt spaces of $\mathcal{C}(\mathcal{E}^w)$ and $\mathcal{C}(\mathrm{Mor}_w\,\mathcal{E}^w)$, respectively.

We will first derive Proposition 4 assuming (†). For that, call f_i the map on Grothendieck-Witt spaces of the i-th vertical map in diagram (12), call F_i the homotopy fibre of f_i, and A_i the cokernel of $\pi_0(f_i)$. (†) implies that the outer diagrams of the left two squares and of the right two squares of diagram (12) are homotopy Cartesian in hermitian K-theory. That is, the maps $F_1 \to F_3$ and $F_2 \to F_4$ are homotopy equivalences, and $A_1 \to A_3$ and $A_2 \to A_4$ are injective. The first homotopy equivalence implies that $F_2 \to F_3$ is surjective on homotopy groups. The second homotopy equivalence implies that $F_2 \to F_3$ is injective on homotopy groups, hence an isomorphism. The first homotopy equivalence then implies that $F_1 \to F_2$ is an isomorphism on homotopy groups. Moreover, injectivity of $A_1 \to A_3$ implies injectivity of $A_1 \to A_2$. By Remark 9, the left square in diagram (12) is homotopy Cartesian, hence the Proposition 4.

To construct diagram (12), we will define the non-singular exact form functor $(F,\varphi) : \mathrm{Mor}_w\,\mathcal{E} \to \mathcal{C}(\mathcal{E},w)$ as the composition of a non-singular exact form functor $(F_0,\varphi) : \mathrm{Mor}_w\,\mathcal{E} \to \mathcal{C}_0(\mathcal{E},w)$ and the localization functor $\mathcal{C}_0(\mathcal{E},w) \to \mathcal{C}(\mathcal{E},w)$ such that its restriction to $\mathrm{Mor}\,\mathcal{E}^w$ has image in $\mathcal{C}_0(\mathcal{E}^w)$. Recall that $(\mathcal{E},w,*,\eta)$ is assumed to have a symmetric cone (see Definition 4), where $i : id \rightarrowtail C$ and $p : P \twoheadrightarrow id$ denote the natural inflation and deflation which are part of the structure of a symmetric cone. The functor (F,φ) (or rather (F_0,φ)) sends an object $g : X \to Y$ of $\mathrm{Mor}_w\,\mathcal{E}$ to the object $F(g)$ given by the diagram

$$\begin{array}{ccccccccc}
X & \rightarrowtail & X\oplus PY & \rightarrowtail & X\oplus PY\oplus CX & \rightarrowtail & X\oplus PY\oplus CX\oplus PY & \rightarrowtail & \cdots\\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} g & p\\ i & 0\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} g & p & 0\\ i & 0 & 1\\ 0 & 1 & 0\end{smallmatrix}\right)} & & \downarrow & & \\
Y & \twoheadleftarrow & Y\oplus CX & \twoheadleftarrow & Y\oplus CX\oplus PY & \twoheadleftarrow & Y\oplus CX\oplus PY\oplus CX & \twoheadleftarrow & \cdots
\end{array} \tag{13}$$

of $\mathcal{C}_0(\mathcal{E},w)$. In the notation $(U_\bullet \to U^\bullet)$ of Sect. 4.2 corresponding to diagram (11), the object $F(g)$ is given by

$$U_n = X\oplus PY\oplus CX\oplus PY\oplus CX\oplus\cdots \quad (n+1 \text{ summands}),$$
$$U^n = Y\oplus CX\oplus PY\oplus CX\oplus PY\oplus\cdots \quad (n+1 \text{ summands}).$$

The maps $U_n \to U_{n+1}$ and $U^{n+1} \to U^n$ are the canonical inclusions into the first $n+1$ summands and the canonical projections onto the first $n+1$ factors. The maps $U_n \to U^n$ are given by the matrix

$$(u_{rs})_{0\le r,s\le n} = \begin{pmatrix} g & p & 0 & 0 & 0 & \cdots\\ i & 0 & 1 & 0 & & 0\\ 0 & 1 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 1 & \\ 0 & & 0 & 1 & & \\ \vdots & 0 & & & & \ddots \end{pmatrix}$$

with $u_{rs} = 0$ unless $r = s = 0$ or $|r-s| = 1$, and $u_{0,0} = g$, $u_{0,1} = p_Y$, $u_{1,0} = i_X$, $u_{r,r+1} = 1$, $u_{r+1,r} = 1$ for $r \ge 1$. The construction of diagram (13) is functorial in g, so that $F : \mathrm{Mor}_w\,\mathcal{E} \to \mathcal{C}(\mathcal{E},w)$ is indeed a functor. The duality compatibility map $\varphi_g : F(g^*) \to (Fg)^*$ for $g : X \to Y$ is the identity on

X^* and Y^*, it is γ_X on the summands $P(X^*)$ and $\bar{\gamma}_Y$ on $C(Y^*)$. It is clear that F sends the subcategory $\mathrm{Mor}_w\,\mathcal{E}^w$ to the full subcategory $\mathcal{C}(\mathcal{E}^w)$ of $\mathcal{C}(\mathcal{E},w)$. This defines diagram (12).

We are left with proving (†). Since $U_0 = X$ is the initial object of diagram (13), it defines a map $j = j_g : X = U_0 \to F(g)$, where U_0 (and X) is considered an object of $\mathcal{C}(\mathcal{E},w)$ via the constant diagram embedding $\Gamma : \mathcal{E} \to \mathcal{C}(\mathcal{E},w)$. The map $j : X \to F(g)$ is an inflation in $\mathcal{C}(\mathcal{E},w)$ with cokernel in $\mathcal{C}(\mathcal{E}^w)$ because $j : X \to (U_\bullet \to U^{\bullet+1})$ is an inflation in $\mathcal{C}_0(\mathcal{E},w)$ with cokernel in $\mathcal{C}_0(\mathcal{E}^w)$. Varying g, the map j_g defines a natural transformation $j : \Gamma \to F$. Similarly, U^0 is the final object of diagram (13) and thus defines a (functorial) map $q = q_g : F(g) \to U^0 = Y$ with kernel in $\mathcal{C}(\mathcal{E}^w)$. We have $g = qj$.

Let $\hat{\varphi} : F \to F^\sharp = *F*$ be the symmetric form on the functor F associated with the duality compatibility map φ (see Sect. 2.1). The form $\hat{\varphi} : F \to F^\sharp$ fits into a commutative diagram in $\mathcal{C}_0(\mathcal{E},w)$

$$\begin{array}{ccccc} X & \xrightarrow{j_g} & F(g) & \xrightarrow{q_g} & Y \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \hat{\varphi}_g} & & \downarrow{\scriptstyle \eta_Y} \\ X^{**} & \xrightarrow{q^*_{g^*}} & F(g^*)^* & \xrightarrow{j^*_{g^*}} & Y^{**}. \end{array} \tag{14}$$

Write (FI, φ_{FI}) for the composition $(F,\varphi)\circ(I, id)$ of form functors. The natural transformation j above makes the canonical inclusion $(\Gamma, id) : \mathcal{E} \to \mathcal{C}(\mathcal{E},w)$ into an admissible subfunctor $j : \Gamma \subset FI$ of FI. Commutativity of diagram (14) for $g = id_X$ implies that $\eta = j^\sharp \hat{\varphi}_{FI}\, j$, that is, j defines a map $(\Gamma,\eta) \to (FI, \hat{\varphi}_{FI})$ of symmetric spaces associated with the form functors (Γ, id) and (FI, φ_{FI}). Since the maps id and φ_{FI} are isomorphisms, the symmetric space $(FI, \hat{\varphi}_{FI})$ decomposes in $\mathrm{Fun}(\mathcal{E}, \mathcal{C}(\mathcal{E},w))$ as $(\Gamma,\eta) \perp (A, \hat{\varphi}_A)$ with $(A, \hat{\varphi}_A)$ the orthogonal complement of (Γ,η) in $(FI, \hat{\varphi}_{FI})$. As mentioned above, the cokernel of $j : X \to FI(X) = F(1_X)$ is in $\mathcal{C}(\mathcal{E}^w)$. Therefore, the form functor (A, φ_A) factors through the category $\mathcal{C}(\mathcal{E}^w)$ (whose hermitian K-theory space is contractible [25, Corollary 9.6]). So, $(A,\varphi_A) \sim 0$. Hence, $(FI, \varphi_{FI}) \cong (\Gamma, id) \perp (A, \varphi_A) \sim (\Gamma, id)$. By construction, the homotopy restricted to $\mathcal{E}^w$ has image in the Grothendieck-Witt space of $\mathcal{C}(\mathcal{E}^w)$. This shows the first half of the claim (†).

For the second half, write (IF, φ_{IF}) and (IF_0, φ_{IF_0}) for the compositions of form functors $(\mathcal{C}(I), id)\circ(F,\varphi)$ and $(\mathcal{C}(I), id)\circ(F_0,\varphi)$, and note that (IF, φ_{IF}) is just the composition of (IF_0, φ_{IF_0}) with the localization functor $\mathcal{C}_0(\mathrm{Mor}_w\,\mathcal{E}, w) \to \mathcal{C}(\mathrm{Mor}_w\,\mathcal{E}, w)$. There is an obvious isomorphism of exact categories with duality $\mathrm{Mor}_w\,\mathcal{C}_0(\mathcal{E},w) \cong \mathcal{C}_0(\mathrm{Mor}_w\,\mathcal{E}, w)$ such that the composition IF_0 sends the object $(g : X \to Y) \in \mathrm{Mor}_w\,\mathcal{E}$ to

$id_{F(g)} : F(g) \to F(g)$, and the duality compatibility morphism φ_{IF_0} becomes $(\varphi_g, \varphi_g) : id_{F(g^*)} \to id_{F(g)^*}$. Consider the functorial biCartesian square

$$\begin{array}{ccc} (X \xrightarrow{j} F(g)) & \xrightarrow{(j,1)} & (F(g) \xrightarrow{1} F(g)) \\ \downarrow{\scriptstyle (1,q)} & & \downarrow{\scriptstyle (1,q)} \\ (X \xrightarrow{g} Y) & \xrightarrow{(j,1)} & (F(g) \xrightarrow{q} Y) \end{array}$$

in $\mathrm{Mor}_w \mathcal{C}_0(\mathcal{E}, w)$. The total complex of the square (considered as a bicomplex) is a conflation in $\mathrm{Mor}_w \mathcal{C}_0(\mathcal{E}, w) = \mathcal{C}_0(\mathrm{Mor}_w, w)$. It is therefore also a conflation in $\mathcal{C}(\mathrm{Mor}_w, w)$, hence the square is also biCartesian in $\mathcal{C}(\mathrm{Mor}_w, w)$. In $\mathcal{C}(\mathrm{Mor}_w, w)$, the horizontal maps in the square are inflations with cokernel in $\mathcal{C}(\mathrm{Mor}_w \mathcal{E}^w)$ since $(j, 1)$ is isomorphic to the $\mathcal{C}_0(\mathrm{Mor}_w, w)$-inflation $(j, 1)^{[1]}$ which has cokernel in $\mathcal{C}_0(\mathrm{Mor}_w \mathcal{E}^w)$. Similarly, the vertical maps in the square are deflations in $\mathcal{C}(\mathrm{Mor}_w, w)$ with kernel in $\mathcal{C}(\mathrm{Mor}_w \mathcal{E}^w)$ since $(1, q)$ is isomorphic to the $\mathcal{C}_0(\mathrm{Mor}_w, w)$-deflation $(1, q)_{[1]}$ with kernel in $\mathcal{C}_0(\mathrm{Mor}_w \mathcal{E}^w)$.

Commutativity of diagram (14) implies that the form $\hat{\varphi}_{IF_0}$ on the upper right corner IF_0 of the square extends to a form on the whole biCartesian square such that its restriction to the lower left corner is the constant diagram inclusion $(\Gamma, \eta) : \mathrm{Mor}_w \mathcal{E} \to \mathcal{C}_0(\mathrm{Mor}_w \mathcal{E}, w)$. It follows that $\ker(1, q) \subset IF$ is a totally isotropic subfunctor of (IF, φ_{IF}) with induced form on $(X \xrightarrow{g} Y) = \ker(1, q)^{\perp}/\ker(1, q)$ isometric to the constant diagram inclusion (Γ, id) : $\mathrm{Mor}_w \mathcal{E} \to \mathcal{C}(\mathrm{Mor}_w \mathcal{E}, w)$. By the Additivity Theorem [25, Theorem 7.1] (or its generalization in Theorem 5), the form functors $(\Gamma, id) \perp \mathcal{H}\ker(1, q)$ and (IF, φ_{IF}) induce homotopic maps on Grothendieck-Witt spaces. Since $\mathcal{H}\ker(1, q)$ has image in $\mathcal{C}(\mathrm{Mor}_w \mathcal{E}^w)$ whose Grothendieck-Witt space is contractible, we have $(IF, \varphi_{IF}) \sim (\Gamma, id) \perp \mathcal{H}\ker(1, q) \sim (\Gamma, id)$. The homotopies restricted to $\mathrm{Mor}_w \mathcal{E}^w$ have image in the Grothendieck-Witt space of $\mathcal{C}(\mathrm{Mor}_w \mathcal{E}^w)$. This is clear for the second homotopy, and for the first, it follows from Remark 6. □

Next, we prove a variant of the Change-of-weak-equivalence Theorem.

Proposition 5 *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and strong duality which has a symmetric cone. Then the following commutative diagram of exact categories with weak equivalences and duality induces a homotopy Cartesian square of Grothendieck-Witt spaces with contractible* *upper right corner*

$$\begin{array}{ccc} (\mathcal{E}^w, i) & \longrightarrow & (\mathcal{E}^w, w) \\ \downarrow & & \downarrow \\ (\mathcal{E}, i) & \longrightarrow & (\mathcal{E}, w). \end{array} \tag{15}$$

Proof From Lemma 2 it is clear that $GW(\mathcal{E}^w, w)$ is contractible since $0 \to id$ is a natural weak equivalence in $(\mathcal{E}^w, w)$. Consider the commutative diagram of (simplicial) exact categories with dualities (all weak equivalences being isomorphisms)

$$\begin{array}{ccccc} \mathcal{E}^w & \longrightarrow & \mathrm{Fun}_w(\underline{0}, \mathcal{E}^w) & \longrightarrow & \left[n \mapsto \mathrm{Fun}_w(\underline{n}, \mathcal{E}^w) \right] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathrm{Fun}_w(\underline{0}, \mathcal{E}) & \longrightarrow & \left[n \mapsto \mathrm{Fun}_w(\underline{n}, \mathcal{E}) \right] \end{array}$$

in which the left hand square can be identified with the square of Proposition 4 and induces therefore a homotopy Cartesian square of Grothendieck-Witt spaces. On Grothendieck-Witt spaces, the right vertical map can be identified with the map $(\mathcal{E}^w, w) \to (\mathcal{E}, w)$ in view of the Simplicial Resolution Lemma 5. The right hand square is the inclusion of degree zero simplices. The proof of Proposition 5 is thus reduced to showing that the right hand square of the diagram induces a homotopy Cartesian square of Grothendieck-Witt spaces.

Let $\mathrm{Fun}^1_w(\underline{n}, \mathcal{E}) \subset \mathrm{Fun}_w(\underline{n}, \mathcal{E})$ be the full subcategory of those functors $A : \underline{n} \to \mathcal{E}$ for which $A_p \to A_q$ is an inflation, and $A_{q'} \to A_{p'}$ is a deflation, $0 \leq p \leqq q \leq n$. It inherits the structure of an exact category with duality from $\mathrm{Fun}_w(\underline{n}, \mathcal{E})$. Further, let $\mathrm{Fun}^0_w(\underline{n}, \mathcal{E})$ be the category which is equivalent to $\mathrm{Fun}^1_w(\underline{n}, \mathcal{E})$ but where an object is an object A of $\mathrm{Fun}^1_w(\underline{n}, \mathcal{E})$ together with a choice of subquotients $A_{p,q} = A_q / A_p = \mathrm{coker}(A_p \overset{\sim}{\rightarrowtail} A_q) \in \mathcal{E}^w$ and induced maps $A_{p,q} \to A_{p,q}$, and together with a choice of kernels $A_{q',p'} = \ker(A_{q'} \overset{\sim}{\twoheadrightarrow} A_{p'}) \in \mathcal{E}^w$ for $0 \leq p \leq q \leq n$. The category $\mathrm{Fun}^0_w(\underline{n}, \mathcal{E})$ is an exact category with duality such that the forgetful functor $\mathrm{Fun}^0_w(\underline{n}, \mathcal{E}) \to \mathrm{Fun}^1_w(\underline{n}, \mathcal{E})$ is an equivalence of exact categories with duality. We have an exact functor $\mathrm{Fun}^0_w(\underline{n}, \mathcal{E}) \to S_n\mathcal{E}^w : A \mapsto (A_{p,q})_{0 \leq p \leq q \leq n}$. By the Additivity Theorem [25, Theorem 7.1] (or Theorem 5), this functor induces a map which is part of a split homotopy fibration

$$GW(\mathrm{Fun}_w(\underline{0}, \mathcal{E})) \to GW(\mathrm{Fun}^0_w(\underline{n}, \mathcal{E})) \to K(S_n\mathcal{E}^w).$$

The same argument applies to $(\mathcal{E}^w, w)$ instead of $(\mathcal{E}, w)$. So, varying n, we obtain a map of homotopy fibrations after topological realization

$$\begin{array}{ccccc}
GW\mathrm{Fun}_w(\underline{0}, \mathcal{E}^w) & \longrightarrow & |n \mapsto GW\mathrm{Fun}^0_w(\underline{n}, \mathcal{E}^w)| & \longrightarrow & |n \mapsto K(S_n\mathcal{E}^w)| \\
\downarrow & & \downarrow & & \| \\
GW\mathrm{Fun}_w(\underline{0}, \mathcal{E}) & \longrightarrow & |n \mapsto GW\mathrm{Fun}^0_w(\underline{n}, \mathcal{E})| & \longrightarrow & |n \mapsto K(S_n\mathcal{E}^w)|
\end{array}$$

which shows that the left square is homotopy Cartesian.

Since $\mathrm{Fun}^0_w \to \mathrm{Fun}^1_w$ is an equivalence of exact categories with duality, the proposition follows once we show that the inclusion $I : \mathrm{Fun}^1_w(\underline{n}, \mathcal{A}) \subset \mathrm{Fun}_w(\underline{n}, \mathcal{A})$ induces a homotopy equivalence on Grothendieck-Witt spaces for $\mathcal{A} = \mathcal{E}, \mathcal{E}^w$. We illustrate the argument for $\mathcal{A} = \mathcal{E}$ and $n = 1$. The general case is mutatis mutandis the same.

We define two functors $F, G : \mathrm{Fun}_w(\underline{1}, \mathcal{E}) \to \mathrm{Fun}^1_w(\underline{1}, \mathcal{E})$. The functor F sends $E_{1'} \xrightarrow{\sim} E_{0'} \xrightarrow{\sim} E_0 \xrightarrow{\sim} E_1$ to $E_{1'} \oplus PE_{0'} \overset{\sim}{\twoheadrightarrow} E_{0'} \xrightarrow{\sim} E_0 \overset{\sim}{\rightarrowtail} E_1 \oplus CE_0$, the functor G sends the same object to $PE_{0'} \overset{\sim}{\twoheadrightarrow} 0 \xrightarrow{\sim} 0 \overset{\sim}{\rightarrowtail} CE_0$. Both functors F and G are equipped with canonical duality compatibility morphisms, induced by γ and $\bar{\gamma}$ from Definition 4, such that F and G are non-singular exact form functors. By the Additivity Theorem, we have $IF \sim id \perp IG$ and $FI \sim id \perp GI$. Therefore, $GW(F) - GW(G)$ defines an inverse of $GW(I)$, up to homotopy. $\square$

Proof of Theorem 6 By Lemma 2, $GW(\mathcal{E}^w, w)$ is contractible. Let $\mathcal{A} = \mathcal{E}^{\mathrm{str}}_w$ be the strictification of $\mathcal{E}$ from Lemma 4, and recall that it has a strong duality. Consider the commutative diagram of exact categories with weak equivalences and duality

$$\begin{array}{ccccc}
(\mathcal{A}^w, i) & \longrightarrow & (\mathcal{A}^w, w) & & \\
\downarrow & & \downarrow & & \\
(\mathcal{A}^v, i) & \longrightarrow & (\mathcal{A}^v, w) & \longrightarrow & (\mathcal{A}^v, v) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{A}, i) & \longrightarrow & (\mathcal{A}, w) & \longrightarrow & (\mathcal{A}, v).
\end{array}$$

By the strictification Lemma 4 and Lemma 2, the square in Theorem 6 is equivalent to the lower right square in the diagram. By Proposition 5, the upper square and the outer diagram of the left two squares are homotopy Cartesian in hermitian K-theory. Since the left vertical maps are surjective on GW_0 (because $\mathcal{A} = \mathcal{E}_w^{\mathrm{str}}$), it follows that the lower left square is homotopy Cartesian in hermitian K-theory, by Remark 9. Again, by Proposition 5, the outer diagram of the two lower squares induces a homotopy Cartesian square of Grothendieck-Witt spaces. Together with the facts that the lower left square is homotopy Cartesian in hermitian K-theory and that the lower left vertical map is surjective on GW_0, this implies that the lower right square induces a homotopy Cartesian square of Grothendieck-Witt spaces. □

Theorem 7 (Cofinality) *Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality which has a symmetric cone. Let $A \subset K_0(\mathcal{E}, w)$ be a subgroup closed under the duality action on $K_0(\mathcal{E}, w)$, and let $\mathcal{E}_A \subset \mathcal{E}$ be the full subcategory of those objects whose class in $K_0(\mathcal{E}, w)$ belongs to A. Then the category $\mathcal{E}_A$ inherits the structure of an exact category with weak equivalences and duality from $(\mathcal{E}, w, *, \eta)$, and the induced map on Grothendieck Witt spaces*

$$GW(\mathcal{E}_A, w, *, \eta) \longrightarrow GW(\mathcal{E}, w, *, \eta)$$

is an isomorphism on π_i, $i \geq 1$, and a monomorphism on π_0.

Proof Let $\mathcal{U} = \mathcal{E}_w^{\mathrm{str}}$, and consider the diagram of exact categories with weak equivalences and duality

$$\begin{array}{ccccc} (\mathcal{U}_A^w, i) & \longrightarrow & (\mathcal{U}_A, i) & \longrightarrow & (\mathcal{U}_A, w) \\ \| & & \downarrow & & \downarrow \\ (\mathcal{U}^w, i) & \longrightarrow & (\mathcal{U}, i) & \longrightarrow & (\mathcal{U}, w). \end{array}$$

On Grothendieck-Witt spaces, the right vertical map can be identified (up to homotopy) with the map in the theorem, by Lemmas 4 and 2. The rows are homotopy fibrations, by Proposition 5, and the right horizontal maps are surjective on GW_0 (as $\mathcal{U} = \mathcal{E}_w^{\mathrm{str}}$). It follows that the right square induces a homotopy Cartesian square of Grothendieck-Witt spaces. Since $\mathcal{U}_A \subset \mathcal{U}$ is a cofinal inclusion of exact categories with duality, the Cofinality Theorem of [25, Corollary 5.2] shows that the homotopy fibre of the Grothendieck-Witt spaces of the middle vertical map is contractible. As the right square is homotopy Cartesian in hermitian K-theory, the same is true for the right vertical map. □

5 Approximation, change of exact structure and resolution

In this section we prove in Theorems 8 and 9 variants of Waldhausen's Approximation Theorem [31, Theorem 1.6.7] which hold for higher Grothendieck-Witt groups. We explain two immediate consequences, one concerning the conditions under which a change of exact structure has no effect on Grothendieck-Witt groups (Lemma 7, compare [30, Theorem 1.9.2]) and the other concerning an analog of Quillen's Resolution Theorem (Lemma 9, compare [22, Sect. 4, Corollary 1]).

Theorem 8 (Approximation I) *Let $(F,\varphi):\mathcal{A}\to\mathcal{B}$ be a non-singular exact form functor between exact categories with weak equivalences and duality. Assume the following.*

(a) *Every map in $\mathcal{A}$ can be written as the composition of an inflation followed by a weak equivalence.*
(b) *A map in $\mathcal{A}$ is a weak equivalence iff its image in $\mathcal{B}$ is a weak equivalence.*
(c) *For every map $f:FA\to E$ in $\mathcal{B}$, there is a map $a:A\to B$ in $\mathcal{A}$ and a weak equivalence $g:FB\overset{\sim}{\to}E$ in $\mathcal{B}$ such that $f=g\circ Fa$:*

$$\begin{array}{ccc} FA & \overset{\forall f}{\dashrightarrow} & E \\ {\scriptstyle Fa}\downarrow & \overset{\sim}{\nearrow} & \\ \exists\ FB & & \end{array}$$

(d) *The duality compatibility morphism $\varphi_A:F(A^*)\to F(A)^*$ is an isomorphism for every A in $\mathcal{A}$.*
(e) *For every map $f:FA\to FB$ in $\mathcal{B}$, there is an exact functor $L:\mathcal{A}\to\mathcal{A}$, a natural weak equivalence $\lambda:L\overset{\sim}{\to}id_{\mathcal{A}}$ and a map $a:LA\to B$ in $\mathcal{A}$ such that $Fa=f\circ F\lambda_A$:*

$$\exists\ FLA \overset{\sim}{\longrightarrow} FA \overset{\forall f}{\dashrightarrow} FB, \qquad FLA \xrightarrow{Fa} FB.$$

(f) *For every map $a:A\to B$ in $\mathcal{A}$ such that $Fa=0$ in $\mathcal{B}$, there is an exact functor $L:\mathcal{A}\to\mathcal{A}$ and a natural weak equivalence $\lambda:L\overset{\sim}{\to}id_{\mathcal{A}}$ such that $a\circ\lambda_A=0$ in $\mathcal{A}$:*

$$\exists\ LA \overset{\sim}{\longrightarrow} A \overset{\forall a}{\longrightarrow} B,\quad Fa=0, \qquad LA \xrightarrow{0} B.$$

Then (F,φ) induces homotopy equivalences

$$(wS^e_\bullet\mathcal{A})_h \overset{\sim}{\longrightarrow} (wS^e_\bullet\mathcal{B})_h \quad \text{and} \quad GW(\mathcal{A},w,*,\eta) \overset{\sim}{\longrightarrow} GW(\mathcal{B},w,*,\eta).$$

Proof For the purpose of the proof, we call *lattice* a pair (L, λ) with $L : \mathcal{A} \to \mathcal{A}$ an exact functor and $\lambda : L \xrightarrow{\sim} id_{\mathcal{A}}$ a natural weak equivalence. Lattices form an associative monoid under composition

$$(L_2, \lambda_2) \circ (L_1, \lambda_1) := (L_2 L_1, \lambda_2 \circ L_2(\lambda_1)).$$

Since $\lambda_2 \circ L_2(\lambda_1) = \lambda_1 \circ \lambda_{2,L_1}$ (as λ_2 is a natural transformation), lattices behave like a multiplicative set in a commutative ring. More precisely, composition of lattices allows us to generalize properties (e) and (f) to finite families of maps:

(e′) for any finite set of maps $f_i : FA_i \to FB_i$ in $\mathcal{B}$, $i = 1, \dots, n$, there are a lattice (L, λ) and maps $a_i : LA_i \to B_i$ such that $Fa_i = f_i \circ F\lambda_{A_i}$, $i = 1, \dots, n$, and
(f′) for any finite set of maps $a_i : A_i \to B_i$ in $\mathcal{A}$ such that $Fa_i = 0$ in $\mathcal{B}$, $i = 1, \dots, n$, there is a lattice (L, λ) such that $a_i \lambda_{A_i} = 0$ in $\mathcal{A}$, $i = 1, \dots, n$.

We will refer to (e′) and (f′) as "clearing denominators", in analogy with the localization of a commutative ring with respect to a multiplicative subset. "Clearing denominators" together with (b) and (d) implies that we can lift non-degenerate symmetric forms from $\mathcal{B}$ to $\mathcal{A}$ in the following sense:

(†) For any non-degenerate symmetric form $(FA, \alpha) \in (w\mathcal{B})_h$ on the image FA of an object A of $\mathcal{A}$, there is a lattice (L, λ) and a non-degenerate symmetric form $(LA, \beta) \in (w\mathcal{A})_h$ on LA such that $F(\lambda_A)$ is a map of symmetric spaces $F(\lambda_A) : F(LA, \beta) \xrightarrow{\sim} (FA, \alpha)$.

The proof is the same as the classical proof which shows that a non-degenerate symmetric form over the fraction field of a Dedekind domain can be lifted to a (usual) lattice in the ring. In detail, the map $\varphi_A^{-1}\alpha : FA \to F(A^*)$ lifts to a map $a_1 : L_1 A \to A^*$ such that $\varphi_A F a_1 = \alpha F \lambda_{1,A}$ for some lattice (L_1, λ_1). The map $a : \lambda_{1,A}^* a_1 : L_1 A \to (L_1 A)^*$ is a weak equivalence but not necessarily symmetric. However, the difference $\delta = a - a^* \eta_{L_1 A}$ satisfies $F\delta = 0$. Therefore, there is a second lattice (L_2, λ_2) such that $\delta \lambda_{2,L_1 A} = 0$. Then $\beta = \lambda_{2,L_1 A}^* a \lambda_{2,L_1 A} : L_2 L_1 A \to (L_2 L_1 A)^*$ is a non-degenerate symmetric form on LA with $(L, \lambda) = (L_2, \lambda_2) \circ (L_1, \lambda_1)$, and $F(\lambda_A)$ defines a map of symmetric spaces $F(LA, \beta) = (FLA, \varphi_{LA} F\beta) \xrightarrow{\sim} (FA, \alpha)$.

Apart from "clearing denominators", the proof of the first homotopy equivalence in the theorem proceeds now as the proof of [24, Theorem 10] which was based on the proof of [31, Theorem 1.6.7]. We first note that under the assumptions of the theorem, the non-singular exact form functors $S_n(F, \varphi) : S_n\mathcal{A} \to S_n\mathcal{B}$ also satisfy (a)–(f), $n \in \mathbb{N}$. For (a)–(c), this is in [31, Lemma 1.6.6] (using the fact that in the presence of (a), the map $a : A \to B$ in (c) can be replaced by an inflation), (d) extends by functoriality, and the extension of (e), (f) to S_n easily follows by induction on n by successively clearing denominators.

In order to show that $(wS^e_\bullet\mathcal{A})_h \to (wS^e_\bullet\mathcal{B})_h$ is a homotopy equivalence, it suffices to prove that $(wS^e_n\mathcal{A})_h \to (wS^e_n\mathcal{B})_h$ is a homotopy equivalence for every $n \in \mathbb{N}$, which, by the argument of the previous paragraph, only needs to be checked for $n = 0$, that is, it is sufficient to prove that

$$F : (w\mathcal{A})_h \to (w\mathcal{B})_h$$

is a homotopy equivalence. The last claim will follow from Quillen's Theorem A once we show that for every object $X = (X, \psi)$ of $(w\mathcal{B})_h$, the comma category $(F \downarrow X)$ is non-empty and contractible.

By (a) with $A = 0$, there is an object B of $\mathcal{A}$ and a weak equivalence $FB \xrightarrow{\sim} X$, hence a map $(FB, \psi_{|FB}) \to (X, \psi)$ in $(w\mathcal{B})_h$. By (†) above, there is a symmetric space (C, γ) in $\mathcal{A}$ and a map $F(C, \gamma) \to (FB, \psi_{|FB})$ in $(w\mathcal{B})_h$. Hence, the category $(F \downarrow X)$ is non-empty.

In order to show that $(F \downarrow X)$ is contractible it suffices to show that every functor $\mathcal{P} \to (F \downarrow X)$ from a finite poset $\mathcal{P}$ to the comma category $(F \downarrow X)$ is homotopic to a constant map (see for instance [24, Lemma 14]). Such a functor is given by a triple $(\underline{A}, \alpha, f)$ where $\underline{A}$ is a functor $\underline{A} : \mathcal{P} \to w\mathcal{A} : i \mapsto A_i, (i \le j) \mapsto a_{j,i}$ together with a collection α of non-degenerate symmetric forms $\alpha_i : A_i \to A_i^*$ in $\mathcal{A}$, $i \in \mathcal{P}$, such that $\alpha_i = \alpha_{j|A_i}$ whenever $i \le j$ in $\mathcal{P}$, and f is a collection of compatible maps of symmetric spaces $f_i : F(A_i, \alpha_i) \to (X, \psi)$ in $(w\mathcal{B})_h$ such that $f_i = f_j F a_{j,i}$ whenever $i \le j \in \mathcal{P}$. It is convenient to consider f as a map $F(\underline{A}, \alpha) \to (X, \psi)$ of functors $\mathcal{P} \to (w\mathcal{B})_h$, where objects in $(w\mathcal{B})_h$ (or in $\mathcal{A}$, $\mathcal{B}$, $(w\mathcal{A})_h$) such as (X, ψ) are interpreted as constant $\mathcal{P}$-diagrams. By [24, Lemma 13], there is a map $b : \underline{B} \to \underline{A}$ of functors $\mathcal{P} \to w\mathcal{A}$ such that $b_i : B_i \to A_i$ is a weak equivalence, $i \in \mathcal{P}$, and the $\mathcal{P}$-diagram $\underline{B} : \mathcal{P} \to \mathcal{A}$ has a colimit in $\mathcal{A}$ such that $F(\underline{B}) \to F(\operatorname{colim}_{\mathcal{P}} \underline{B})$ represents the colimit of $F(\underline{B})$ in $\mathcal{B}$ (the diagram $\underline{B}$ is a cofibrant replacement of $\underline{A}$ in a suitable cofibration structure on the category of functors $\mathcal{P} \to \mathcal{A}$; see [24, Appendix A.2], cofibrant objects have colimits, and F, being an exact functor, preserves cofibrant objects and their colimits). By (c), the natural map $F(\operatorname{colim}_{\mathcal{P}} \underline{B}) = \operatorname{colim}_{\mathcal{P}} F(\underline{B}) \to X$ induced by $b \circ f$ factors as $F(\operatorname{colim}_{\mathcal{P}} \underline{B}) \to F(C) \xrightarrow{c} X$ where the first map is in the image of F and the second map is a weak equivalence in $\mathcal{B}$. Let $g : \underline{B} \to C$ be the composition $\underline{B} \to \operatorname{colim}_{\mathcal{P}} \underline{B} \to C$ which, by (b), is a weak equivalence since b, f, and c are. The null-homotopy to be constructed can be read off the following diagram

$$\begin{array}{ccccccc}
L'L\underline{B}, \beta & \xrightarrow{\lambda'} & L\underline{B} & \xrightarrow{\lambda} & \underline{B} & \xrightarrow{b} & \underline{A}, \alpha \\
{\scriptstyle \beta = \alpha_{|L'L B} = \gamma_{|L'LB}} & & \downarrow{\scriptstyle Lg} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
 & & LC, \gamma & \xrightarrow{\lambda} & C & \xrightarrow{c} & X, \psi
\end{array} \qquad (16)$$

of which we have constructed the right hand square, so far. In the diagram, a dashed arrow $A \dashrightarrow X$ stands for an arrow $FA \to X$ in $\mathcal{B}$, and solid arrows are arrows in $\mathcal{A}$. By (†), there is a lattice (L,λ) and a non-degenerate symmetric form $(LC,\gamma) \in (w\mathcal{A})_h$ such that $F\lambda_C : F(LC,\gamma) \to (FC,\psi_{|FC})$ defines a map in $(w\mathcal{B})_h$. The restrictions of γ and α_i to LB_i may not coincide, but their images under F coincide since in $\mathcal{B}$, both are (up to composition with the isomorphism φ_{B_i}) the restriction of ψ to FLB_i. Clearing denominators, we can find a lattice (L',λ') such that $\gamma_{|L'LB_i} = \alpha_{i|L'LB_i} =: \beta_i$ for all $i \in \mathcal{P}$ simultaneously. The outer part of the diagram involving $(L'L\underline{B},\beta)$, $(\underline{A},\alpha)$, (LC,γ), (X,ψ) and the maps between them, defines a homotopy between $(\underline{A},\alpha,f)$ and the constant functor $(LC,\gamma,cF\lambda) : \mathcal{P} \to (F\downarrow X)$ via the functor $(L'L\underline{B},\beta,fF(b\lambda\lambda')) : \mathcal{P} \to (F\downarrow X)$. This finishes the proof of the first homotopy equivalence in the theorem.

The same proof (forgetting forms), or an appeal to [31, 1.6.7], implies that the map $wS_\bullet\mathcal{A} \to wS_\bullet\mathcal{B}$ is a homotopy equivalence. Therefore, the map $GW(\mathcal{A},w,*,\eta) \to GW(\mathcal{B},w,*,\eta)$ is also a homotopy equivalence. □

5.1 Change of exact structure

Let $(\mathcal{A},w,*,\eta)$ be an additive category with weak equivalences and duality, and assume that $\mathcal{A}$ can be equipped with two exact structures $\mathfrak{E}_1$ and $\mathfrak{E}_2$ one smaller than the other, $\mathfrak{E}_1 \subset \mathfrak{E}_2$, so that the identity functor $(\mathcal{A},\mathfrak{E}_1) \to (\mathcal{A},\mathfrak{E}_2)$ is exact. Assume furthermore that the duality functor $* : \mathcal{A}^{op} \to \mathcal{A}$ is exact for both exact structures $\mathfrak{E}_2$ and $\mathfrak{E}_2$, so that the identity defines a duality preserving exact functor

$$(\mathcal{A},\mathfrak{E}_1,w,*,\eta) \to (\mathcal{A},\mathfrak{E}_2,w,*,\eta). \tag{17}$$

The following is an immediate consequence of Theorem 8.

Lemma 7 *In the situation of Sect. 5.1, if every map in $\mathcal{A}$ can be written as the composition of an inflation in $\mathfrak{E}_1$ followed by a weak equivalence, then the map (17) induces an equivalence of hermitian $S_\bullet$-constructions and of associated Grothendieck-Witt spaces.*

5.2 Approximation for categories of chain complexes and resolution

The purpose of the next lemma (Lemma 8 below) is to simplify some of the hypothesis of Theorem 8 provided the exact categories with weak equivalences in the theorem are "categories of complexes".

Definition 5 We call an exact category with weak equivalences $(\mathcal{C},w)$ a *category of complexes* (with underlying additive category $\mathcal{C}_0$) if $\mathcal{C} \subset \mathrm{Ch}\,\mathcal{C}_0$ is a full additive subcategory of the category $\mathrm{Ch}\,\mathcal{C}_0$ of chain complexes in $\mathcal{C}_0$ such that

- $\mathcal{C}$ is closed under degree-wise split extensions in $\mathrm{Ch}\,\mathcal{C}_0$,
- degree-wise split exact sequences are exact in $\mathcal{C}$,
- with a complex A in $\mathcal{C}$ its usual cone CA (that is, the cone on the identity map of A) and all its shifts $A[i]$, $i \in \mathbb{Z}$, are in $\mathcal{C}$, and
- the set of weak equivalences w contains at least all usual homotopy equivalences between complexes in $\mathcal{C}$.

Lemma 8 *Let* $(\mathcal{A}, w)$ *and* $(\mathcal{B}, w)$ *be categories of complexes with associated additive categories* $\mathcal{A}_0$ *and* $\mathcal{B}_0$, *and let* $F : (\mathcal{A}, w) \to (\mathcal{B}, w)$ *be an exact functor which is the induced functor on chain complexes of an additive functor* $\mathcal{A}_0 \to \mathcal{B}_0$. *Then condition* (a) *of Theorem 8 holds and condition* (c) *of Theorem 8 is implied by conditions* (b), (e) *of Theorem 8 and the following condition.*

(c′) *For every object* E *of* $\mathcal{B}$ *there is an object* A *of* $\mathcal{A}$ *and a weak equivalence*

$$FA \xrightarrow{\sim} E.$$

Proof Every map $f : A \to B$ in $\mathcal{A}$ can be written as the composition

$$A \xrightarrow{\binom{f}{i}} B \oplus CA \xrightarrow[\sim]{(1\ 0)} B,$$

where $i : A \rightarrowtail CA$ is the canonical inclusion of A into its cone CA. This shows that condition (a) of Theorem 8 holds.

To prove condition (c) of Theorem 8 assuming conditions (b), (e) of Theorem 8 and (c′) are satisfied, let $f : FA \to E$ be a map in $\mathcal{B}$ with A in $\mathcal{A}$. By (c′), there is a weak equivalence $s : F(B') \to E$ in $\mathcal{B}$ with B' in $\mathcal{A}$. Let M be the pull-back of $f : FA \to E$ along the degree-wise split surjection $(s, p) : F(B') \oplus PE \to E$ where $p : PE \to E$ is the canonical degree-wise split surjection from the contractible complex $PE = CE[-1]$ to E. Therefore, we have a homotopy commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & F(A) \\ \downarrow & & \downarrow f \\ F(B') & \xrightarrow[\sim]{s} & E. \end{array}$$

Since $F(B') \oplus PE \to E$ is a deflation and a weak equivalence, its pull-back, the map $M \to FA$, is a deflation and a weak equivalence, too. By condition (c′), there is a weak equivalence $F(A') \to M$ with A' in $\mathcal{A}$, and by condition (e) of Theorem 8 we can assume the compositions $F(A') \to FA$ and $F(A') \to F(B')$ to be the images $F\alpha$ and $F\beta$ of maps $\alpha : A' \to A$ and $\beta : A' \to B$ in $\mathcal{A}$. The resulting square involving $F(A')$, $F(B')$, FA and E homotopy commutes, and the map $\alpha : A' \to A$ is a weak equivalence, by condition (b) of Theorem 8. Replacing B' with $B' \oplus CA'$, we obtain a commutative diagram

$$\begin{array}{ccc} F(A') & \overset{F\alpha}{\underset{\sim}{\rightarrowtail}} & F(A) \\ {\scriptstyle \binom{F\beta}{Fi}} \downarrow & & \downarrow {\scriptstyle f} \\ F(B') \oplus F(CA') & \xrightarrow{(s\ g)} & E, \end{array}$$

where $i : A' \rightarrowtail CA'$ is the canonical inclusion of A' into its cone, $g : F(CA') = CF(A') \to E$ a map such that $g \circ Fi$ is the null-homotopic map $f \circ F\alpha - s \circ F\beta : F(A') \to E$. Let B be the push-out of $\alpha : A' \to A$ along the degree-wise split inclusion $A' \rightarrowtail B' \oplus CA'$ and call $a : A \to B$ and $b : B' \oplus CA' \to B$ the induced maps. Since $\alpha : A' \to A$ is a weak equivalence, so is b. The functor F preserves push-outs along inflations. Therefore, we obtain an induced map $t : F(B) \to E$ which is a weak equivalence since $F(b)$ and (s, g) are. By construction, we have $t \circ Fa = f$. □

Next, we prove an analog of Quillen's Resolution Theorem [22, Sect. 4].

Lemma 9 (Resolution) *Let $(\mathcal{B}, w, *, \eta)$ be an exact category with weak equivalences and duality such that $(\mathcal{B}, w)$ is a category of complexes. Let $\mathcal{A} \subset \mathcal{B}$ be a full subcategory closed under the duality, degree-wise split extensions and under taking cones and shifts in $\mathcal{B}$. Restricting $(w, *, \eta)$ to $\mathcal{A}$ makes $\mathcal{A}$ into an exact category with weak equivalences and duality. Assume that the following resolution condition holds*:

For every object E of $\mathcal{B}$, there is an object A of $\mathcal{A}$ and a weak equivalence

$$A \xrightarrow{\sim} E.$$

Then the duality preserving inclusion $\mathcal{A} \subset \mathcal{B}$ induces homotopy equivalences

$$(wS^e_\bullet \mathcal{A})_h \xrightarrow{\sim} (wS^e_\bullet \mathcal{B})_h \quad \textit{and} \quad GW(\mathcal{A}, w, *, \eta) \xrightarrow{\sim} GW(\mathcal{B}, w, *, \eta).$$

Proof This follows from Theorem 8 in view of Lemma 8 and the fact that conditions (e) and (g) of Theorem 8 trivially hold since $\mathcal{A} \subset \mathcal{B}$ is fully faithful, and condition (d) holds since $\mathcal{A} \to \mathcal{B}$ is duality preserving. □

5.3 Approximation and calculus of fractions

We finish the section with Theorem 9 below. It is a variant of Theorem 8 when conditions (e) and (f) of Theorem 8 can not be achieved in a functorial way but the form functor is a localization by a calculus of right fractions.

Definition 6 Call a functor $F : \mathcal{A} \to \mathcal{B}$ between small categories a *localization by a calculus of right fractions* if the following three conditions hold (compare Theorem 8(e), (f)).

(a) The functor $F : \mathcal{A} \to \mathcal{B}$ is essentially surjective.
(b) For every map $f : F(A) \to F(B)$ between the images of objects A, B of $\mathcal{A}$, there are maps $s : A' \to A$ and $g : A' \to B$ in $\mathcal{A}$ with $F(s)$ an isomorphism in $\mathcal{B}$ and $f \circ F(s) = F(g)$.
(c) For any two maps $a, b : A \to B$ in $\mathcal{A}$ such that $F(a) = F(b)$ there is a map $s : A' \to A$ such that $F(s)$ is an isomorphism in $\mathcal{B}$ and $as = bs$.

Remark 11 A functor $F : \mathcal{A} \to \mathcal{B}$ between small categories is a localization by a calculus of right fractions if and only if the set Σ of maps f in $\mathcal{A}$ such that $F(f)$ is an isomorphism in $\mathcal{B}$ satisfies a calculus of right fractions (the dual of [7, Sect. I.2.2]) and the induced functor $\mathcal{A}[\Sigma^{-1}] \to \mathcal{B}$ is an equivalence of categories.

Context for Theorem 9 Consider an additive functor $\mathcal{A} \to \mathcal{B}$ between additive categories. It induces an exact functor $F : \mathrm{Ch}^b \mathcal{A} \to \mathrm{Ch}^b \mathcal{B}$ between the associated exact categories of bounded chain complexes where we call a sequence of chain complexes exact if it is degree-wise split exact. We assume that F is part of an exact form functor

$$(F, \varphi) : (\mathrm{Ch}^b \mathcal{A}, w, *, \eta) \to (\mathrm{Ch}^b \mathcal{B}, w, *, \eta)$$

between exact categories with weak equivalences and duality such that the duality compatibility map $F* \to *F$ is a natural isomorphism.

Theorem 9 (Approximation II) *If in the context above, a map in* $\mathrm{Ch}^b \mathcal{A}$ *is a weak equivalence iff its image in* $\mathrm{Ch}^b \mathcal{B}$ *is a weak equivalence, and if the functor* $\mathcal{A} \to \mathcal{B}$ *is a localization by a calculus of right fractions, then* (F, φ) *induces homotopy equivalences*

$$\begin{aligned} (wS^e_\bullet \mathrm{Ch}^b \mathcal{A})_h &\xrightarrow{\sim} (wS^e_\bullet \mathrm{Ch}^b \mathcal{B})_h \quad \text{and} \\ GW(\mathrm{Ch}^b \mathcal{A}, w, *, \eta) &\xrightarrow{\sim} GW(\mathrm{Ch}^b \mathcal{B}, w, *, \eta). \end{aligned} \tag{18}$$

The proof of Theorem 9 is a consequence of the following two lemmas.

Lemma 10 *Let $F : \mathcal{A} \to \mathcal{B}$ be a localization by a calculus of right fractions. Then the following holds.*

(a) *For every integer $n \geq 0$, the induced functor* $\mathrm{Fun}([n], \mathcal{A}) \to \mathrm{Fun}([n], \mathcal{B})$ *on diagram categories is a localization by a calculus of right fractions.*
(b) *If $(F, \varphi) : (\mathcal{A}, *, \eta) \to (\mathcal{B}, *, \eta)$ is a form functor between categories with duality such that the duality compatibility map $\varphi : F* \to *F$ is a natural isomorphism, then the induced functor $\mathcal{A}_h \to \mathcal{B}_h$ on associated categories of symmetric forms is a localization by a calculus of right fractions.*
(c) *If F is an additive functor between additive categories, then the induced functors* $\mathrm{Ch}^b \mathcal{A} \to \mathrm{Ch}^b \mathcal{B}$ *and $S_n\mathcal{A} \to S_n\mathcal{B}$ are localizations by a calculus of right fractions.*

Proof The proof is an exercise in clearing denominators, and we omit the details. □

Lemma 11 *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between small categories $\mathcal{A}$ and $\mathcal{B}$ which is a localization by a calculus of right fractions. Then F induces a homotopy equivalence on classifying spaces*

$$|\mathcal{A}| \xrightarrow{\sim} |\mathcal{B}|.$$

Proof For a category $\mathcal{C}$ and a subcategory $w\mathcal{C}$, the category $\mathrm{Fun}_w([n], \mathcal{C})$ is the full subcategory of the category $\mathrm{Fun}([n], \mathcal{C})$ of functors $[n] \to \mathcal{C}$ which have image in $w\mathcal{C}$. Maps are natural transformations of functors $[n] \to \mathcal{C}$. There are homotopy equivalences of topological realizations of simplicial categories (a variant of which already appeared in the proof of Lemma 5)

$$|\mathcal{C}| \xrightarrow{\sim} |n \mapsto \mathrm{Fun}_w([n], \mathcal{C})| \simeq |n \mapsto w\mathrm{Fun}([n], \mathcal{C})| \tag{19}$$

which is functorial in the pair $(\mathcal{C}, w\mathcal{C})$. In the first map, the category $\mathcal{C}$ is considered as a constant simplicial category, and the functor $\mathcal{C} \to \mathrm{Fun}_w([n], \mathcal{C})$ sends an object $C \in \mathcal{C}$ to the string consisting of only identity maps on C. This functor is a homotopy equivalence with inverse the functor $\mathrm{Fun}_w([n], \mathcal{C}) \to \mathcal{C}$ which sends a string of maps $C_0 \to \cdots \to C_n$ to C_0. The composition of the two functors is the identity in one direction, and in the other, it is homotopic to the identity, where the homotopy is given by the natural transformation from $C_0 \xrightarrow{1} C_0 \xrightarrow{1} \cdots \to C_0$ to $C_0 \to C_1 \to \cdots \to C_n$ induced by the structure maps of the last string. Therefore, the first map in (19) is a homotopy equivalence. The second map in (19) is in fact a homeomorphism as it is the realization in two different orders of the same bisimplicial set.

In order to prove the lemma, let $\sigma\mathcal{A} \subset \mathcal{A}$ be the subcategory of $\mathcal{A}$ whose maps are the maps which are sent to isomorphisms in $\mathcal{B}$. By the natural homotopy equivalences in (19), the map $|\mathcal{A}| \to |\mathcal{B}|$ in the lemma is equivalent to the map $|n \mapsto \sigma\mathrm{Fun}([n],\mathcal{A})| \to |n \mapsto i\mathrm{Fun}([n],\mathcal{B})|$ which is the realization of a map of simplicial categories, so that it suffices to show that for each integer $n \geq 0$, the functor $\sigma\mathrm{Fun}([n],\mathcal{A}) \to i\mathrm{Fun}([n],\mathcal{B})$ is a homotopy equivalence. By part (a) of Lemma 10, this functor is a localization by a calculus of fractions. Therefore, we are reduced to proving that $G: \sigma\mathcal{A} \to i\mathcal{B}$ is a homotopy equivalence whenever $\mathcal{A} \to \mathcal{B}$ is a localization by a calculus of right fractions. In this case, the comma category $(G \downarrow B)$ is left filtering for every object B of $\mathcal{B}$, hence contractible. By Theorem A of Quillen, the functor $\sigma\mathcal{A} \to i\mathcal{B}$ is a homotopy equivalence. □

Proof of Theorem 9 We only prove the first homotopy equivalence in the theorem, the second homotopy equivalence follows from this and the homotopy equivalence $wS_\bullet\mathrm{Ch}^b\mathcal{A} \to wS_\bullet\mathrm{Ch}^b\mathcal{B}$ which is proved in the same way (forgetting forms).

A map of simplicial categories which is degree-wise a homotopy equivalence induces a homotopy equivalence after topological realization. Therefore, it suffices to show that for every $n \geq 0$, the form functor (F,φ) induces a homotopy equivalence

$$(wS^e_n\mathrm{Ch}^b\mathcal{A})_h \longrightarrow (wS^e_n\mathrm{Ch}^b\mathcal{B})_h. \tag{20}$$

By Lemma 10(c) and in view of the isomorphism $S_n\mathrm{Ch}^b\mathcal{E} = \mathrm{Ch}^b S_n\mathcal{E}$ of exact categories applied to $\mathcal{E} = \mathcal{A}, \mathcal{B}$, we are reduced to showing that the map (20) is a homotopy equivalence for $n = 0$. The functor $\mathrm{Ch}^b\mathcal{A} \to \mathrm{Ch}^b\mathcal{B}$ is a localization by a calculus of right fractions, by Lemma 10(c). The assumption that $\mathrm{Ch}^b\mathcal{A} \to \mathrm{Ch}^b\mathcal{B}$ preserves and detects weak equivalences, implies that, on subcategories of weak equivalences, the functor $w\mathrm{Ch}^b\mathcal{A} \to w\mathrm{Ch}^b\mathcal{B}$ is also a localization by a calculus of right fractions. By Lemma 10(b), the induced functor on categories of symmetric forms $(\mathrm{Ch}^b\mathcal{A})_h \to (\mathrm{Ch}^b\mathcal{B})_h$, which is the map (20) in degree $n = 0$, is a localization by a calculus of right fractions and therefore a homotopy equivalence, by Lemma 11. □

6 From exact categories to chain complexes

The purpose of this section is to prove Proposition 6 which allows us the replace the Grothendieck-Witt space of an exact category with duality by the Grothendieck-Witt space of the associated category of bounded chain complexes.

6.1 Chain complexes and dualities

Let $(\mathcal{E}, *, \eta)$ be an exact category with duality, that is, an exact category with weak equivalences and duality where all weak equivalences are isomorphisms. Let $\mathrm{Ch}^b(\mathcal{E})$ be the category of bounded chain complexes

$$(E, d): \qquad \cdots \to E_{n-1} \stackrel{d_{n-1}}{\to} E_n \stackrel{d_n}{\to} E_{n+1} \to \cdots, \quad d_n d_{n-1} = 0,$$

in $\mathcal{E}$. A sequence of chain complexes $(E', d) \to (E, d) \to (E'', d)$ is *exact* if it is degree-wise exact in $\mathcal{E}$, that is, if the sequence $E'_n \rightarrowtail E_n \twoheadrightarrow E''_n$ is exact for all n. Call a chain complex (E, d) in $\mathcal{E}$ *strictly acyclic* if every differential d_n is the composition $E_n \twoheadrightarrow \operatorname{im} d_n \rightarrowtail E_{n+1}$ of a deflation followed by an inflation, and the sequences $\operatorname{im} d_{n-1} \rightarrowtail E_n \twoheadrightarrow \operatorname{im} d_n$ are exact in $\mathcal{E}$. A chain complex is called *acyclic* if it is homotopy equivalent to a strictly acyclic chain complex. A map of chain complexes is a *quasi-isomorphism* if its cone is acyclic. Write quis for the set of quasi-isomorphisms, then the triple

$$(\mathrm{Ch}^b(\mathcal{E}), \text{quis})$$

is an exact category with weak equivalences.

For $n \in \mathbb{Z}$, the duality $(*, \eta)$ induces a (naive) duality $(*^n, \eta^n)$ on $\mathrm{Ch}^b \mathcal{E}$ which on objects (E, d) and on chain maps $f : (E, d) \to (E', d)$ is given by the formulas

$$\begin{aligned} (E^{*^n})_i &= (E_{-i-n})^*, \qquad & (f^{*^n})_i &= (f_{-i-n})^*, \\ (d^{*^n})_i &= (d_{-i-1-n})^*, & (\eta^n_E)_i &= (-1)^{\frac{n(n-1)}{2}} \eta_{E_i}. \end{aligned}$$

With these definitions, we have an exact category with weak equivalences and duality

$$(\mathrm{Ch}^b(\mathcal{E}), \text{quis}, *^n, \eta^n).$$

If $n = 0$ we may simply write $(*, \eta)$ for $(*^0, \eta^0)$.

Remark 12 The functor $T : \mathrm{Ch}^b \mathcal{E} \to \mathrm{Ch}^b \mathcal{E}$ given by the formula

$$(TE)_i = E_{i+1}, \qquad T(f)_i = f_{i+1}, \qquad (d_{TE})_i = d_{i+1}$$

defines a duality preserving isomorphism of exact categories with duality

$$T : (\mathrm{Ch}^b \mathcal{E}, *^n, \eta^n) \cong (\mathrm{Ch}^b \mathcal{E}, *^{n+2}, -\eta^{n+2}).$$

Remark 13 There is another (more natural) sign choice for defining induced dualities $(\sharp^n, \mathrm{can}^n)$ on $\mathrm{Ch}^b\,\mathcal{E}$ coming from the internal hom of chain complexes, compare Sect. 7.4. They are given by the formulas

$$\begin{aligned} (E^{\sharp^n})_i &= (E_{-i-n})^*, & (f^{\sharp^n})_i &= (f_{-i-n})^*, \\ (d^{\sharp^n})_i &= (-1)^{i+1}(d_{-i-1-n})^*, & (\mathrm{can}^n_E)_i &= (-1)^{i(i+n)}\eta_{E_i}. \end{aligned}$$

The identity functor on $\mathrm{Ch}^b\,\mathcal{E}$ together with the duality compatibility isomorphism $\varepsilon^n_E : E^{*^n} \to E^{\sharp^n}$ which in degree i is

$$(\varepsilon^n_E)_i = (-1)^{\frac{i(i+1)}{2}}\ id_{E^*_{-i-n}} : (E^{*^n})_i \to (E^{\sharp^n})_i$$

defines an isomorphism of exact categories with duality

$$(id, \varepsilon^n) : (\mathrm{Ch}^b\,\mathcal{E}, *^n, \eta^n) \xrightarrow{\cong} (\mathrm{Ch}^b\,\mathcal{E}, \sharp^n, \mathrm{can}^n).$$

For the purpose of proving Proposition 6 below, the duality $(*, \eta)$ is convenient. For most other purposes, the duality $(\sharp, \mathrm{can})$ is more natural. It is the latter duality, which we will use from Sect. 8 on. In any case, both give rise to isomorphic exact categories with duality and thus have isomorphic Grothendieck-Witt spaces.

Exercise. Show that the exact categories with weak equivalences and duality $(\mathrm{Ch}^b\,\mathcal{E}, *^n, \eta^n)$ and $(\mathrm{Ch}^b\,\mathcal{E}, \sharp^n, \mathrm{can}^n)$ have symmetric cones in the sense of Definition 4 (hint: see Sect. 7.5).

For an exact category with duality $(\mathcal{E}, *, \eta)$, inclusion as complexes concentrated in degree 0, defines a duality preserving exact functor

$$(\mathcal{E}, i, *, \eta) \to (\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis}, *, \eta). \tag{21}$$

The following proposition generalizes [30, Theorem 1.11.7]; see also Remark 14.

Proposition 6 *For an exact category with duality $(\mathcal{E}, *, \eta)$, the functor (21) induces a homotopy equivalence of Grothendieck-Witt spaces*

$$GW(\mathcal{E}, *, \eta) \xrightarrow{\sim} GW(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis}, *, \eta).$$

6.2 Semi-idempotent completions

We will reduce the proof of Proposition 6 to "semi-idempotent complete" exact categories. This has the advantage that for such categories, every acyclic complex is strictly acyclic. Here are the relevant definitions and facts.

Call an exact category $\mathcal{E}$ *semi-idempotent complete* if any map $p : A \to B$ which has a section $s : B \to A$, $pi = 1$, is a deflation in $\mathcal{E}$. A semi-idempotent complete exact category has the following property: any map $B \to C$ for which there is a map $A \to B$ such that the composition $A \to C$ is a deflation, is itself a deflation. This is because a semi-idempotent complete exact category satisfies Thomason's axiom [30, A.5.1]. Therefore, the standard embedding of $\mathcal{E}$ into the category of left exact functors $\mathcal{E}^{op} \to \langle ab \rangle$ into the category of Abelian groups is closed under kernels of surjections [30, Theorem A.7.1 and Proposition A.7.16 (b)]. For a semi-idempotent complete exact category, every acyclic complex is strictly acyclic.

The semi-idempotent completion of an exact category $\mathcal{E}$ is the full subcategory $\widetilde{\mathcal{E}}^{semi} \subset \widetilde{\mathcal{E}}$ of the idempotent completion $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ of those objects which are stably in $\mathcal{E}$. Clearly, $\widetilde{\mathcal{E}}^{semi}$ is semi-idempotent complete, and the map $K_0(\mathcal{E}) \to K_0(\widetilde{\mathcal{E}}^{semi})$ is an isomorphism. If $(\mathcal{E}, *, \eta)$ is an exact category with duality, then $(\widetilde{\mathcal{E}}^{semi}, *, \eta)$ is an exact category with duality such that the fully exact inclusion $\mathcal{E} \subset \widetilde{\mathcal{E}}^{semi}$ is duality preserving. If $(\mathcal{A}, w, *, \eta)$ is an exact category with weak equivalences and duality, then $(\widetilde{\mathcal{A}}^{semi}, w, *, \eta)$ inherits the structure of an exact category with weak equivalence and duality from $(\widetilde{\mathcal{A}}, w, *, \eta)$; see Sect. 4.1. Therefore, the inclusion $\mathcal{A} \subset \widetilde{\mathcal{A}}^{semi}$ is duality preserving.

Lemma 12 *Let $(\mathcal{A}, w, *, \eta)$ be an exact category with weak equivalences and strong duality, then the inclusion $\mathcal{A} \subset \widetilde{\mathcal{A}}^{semi}$ induces a homotopy equivalence of Grothendieck-Witt spaces*

$$GW(\mathcal{A}, w, *, \eta) \to GW(\widetilde{\mathcal{A}}^{semi}, w, *, \eta).$$

Proof By Cofinality [25], the map $GW_i(\mathcal{E}) \to GW_i(\widetilde{\mathcal{E}}^{semi})$ is an isomorphism for $i \geq 1$ and injective for $i = 0$ for any exact category with duality $(\mathcal{E}, *, \eta)$. The map $GW_0(\mathcal{E}) \to GW_0(\widetilde{\mathcal{E}}^{semi})$ is also surjective, hence an isomorphism, since for every symmetric space (X, φ) in $\widetilde{\mathcal{E}}^{semi}$, we can find an A in $\mathcal{E}$ such that $X \oplus A$ is in $\mathcal{E}$, and thus, $[X, \varphi] = [(X, \varphi) \perp \mathcal{H}A] - [\mathcal{H}A]$ is in the image of the map. Therefore, Lemma 12 holds when w is the set of isomorphisms. The lemma now follows from this case and the Simplicial Resolution Lemma 5 since $\mathrm{Fun}_w(\underline{n}, \widetilde{\mathcal{A}}^{semi})$ is the semi-idempotent completion of $\mathrm{Fun}_w(\underline{n}, \mathcal{A})$. (For a string $X_{n'} \to \cdots \to X_n$ of weak equivalences in $\widetilde{\mathcal{A}}^{semi}$, there is an object A of $\mathcal{A}$ such that $X_i \oplus A$ is in $\mathcal{A}$ for all $i \in \underline{n}$. Therefore, $(X_{n'} \to \cdots \to X_n) \oplus (A \xrightarrow{1} \cdots \xrightarrow{1} A)$ is a string of weak equivalences in $\mathcal{A}$.) □

6.3 Proof of Proposition 6

Proof of Proposition 6 In view of Lemma 12, we can assume $\mathcal{E}$ to be semi-idempotent complete. So, acyclic complexes are strictly acyclic. By the strictification Lemma 4, we can assume $\mathcal{E}$ to have a strict duality, since an exact category with duality is equivalent to its strictification. Let $\mathrm{Ac}^b\,\mathcal{E} \subset \mathrm{Ch}^b\,\mathcal{E}$ be the full subcategory of acyclic chain complexes. It inherits the structure of an exact category with weak equivalences and strict duality from $\mathrm{Ch}^b\,\mathcal{E}$. Consider the commutative diagram of exact categories with weak equivalences and strict duality

$$\begin{array}{ccccc}
0 & \longrightarrow & (\mathrm{Ac}^b\,\mathcal{E}, i) & \longrightarrow & (\mathrm{Ac}^b\,\mathcal{E}, \mathrm{quis}) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathcal{E}, i) & \longrightarrow & (\mathrm{Ch}^b\,\mathcal{E}, i) & \longrightarrow & (\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis}).
\end{array}$$

We will show that

(a) the left square induces a homotopy Cartesian square of Grothendieck-Witt spaces, and that
(b) the map $GW_0(\mathcal{E}, i) \to GW_0(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$ is surjective.

By Proposition 5, the right hand square induces a homotopy Cartesian square of Grothendieck-Witt spaces. By (a) the same is true for the outer square. Together with (b), this implies the proposition.

We prove (a). For $n \geq 0$, let $\mathrm{Ch}^b_{[-n,n]}\,\mathcal{E} \subset \mathrm{Ch}^b\,\mathcal{E}$ and $\mathrm{Ac}^b_{[-n,n]}\,\mathcal{E} \subset \mathrm{Ac}^b\,\mathcal{E}$ be the full subcategories of those chain complexes which are concentrated in degrees $[-n, n]$. They inherit a structure of exact categories with duality. Note that the inclusion $\mathrm{Ac}^b_{[-n,n]}\,\mathcal{E} \subset \mathrm{Ch}^b_{[-n,n]}\,\mathcal{E}$ is $0 \subset \mathcal{E}$ for $n = 0$. The natural inclusions induce a commutative diagram of exact categories with duality

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Ac}^b_{[-n,n]}\,\mathcal{E} & \longrightarrow & \mathrm{Ac}^b_{[-n-1,n+1]}\,\mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathrm{Ch}^b_{[-n,n]}\,\mathcal{E} & \longrightarrow & \mathrm{Ch}^b_{[-n-1,n+1]}\,\mathcal{E}.
\end{array} \tag{22}$$

We will show that the right-hand square induces a homotopy Cartesian square of Grothendieck-Witt spaces. Then, by induction, the outer square induces a homotopy Cartesian square, too. Taking the colimit over n of the Grothendieck-Witt spaces of the outer squares yields the desired homotopy Cartesian square, since the Grothendieck-Witt space functor GW commutes with filtered colimits.

Consider the following form functors between exact categories with duality (equipped with the obvious duality compatibility maps):

$$\begin{array}{ccl}
\mathcal{HE}\times \mathrm{Ac}^b_{[-n,n]}\mathcal{E} & \stackrel{s}{\to} & \mathrm{Ac}^b_{[-n-1,n+1]}\mathcal{E}:\\
(A,B),C & \mapsto & A \xrightarrow{(1\ 0)} A\oplus C_{-n}\to C_{-n+1}\to\cdots\to C_{n-1}\to C_n\oplus B^* \xrightarrow{(0\ 1)} B^*\\
\mathrm{Ch}^b_{[-n-1,n+1]}\mathcal{E} & \stackrel{\rho}{\to} & \mathcal{HE}\times \mathrm{Ch}^b_{[-n,n]}\mathcal{E}:\\
C & \mapsto & (C_{-n-1},(C_{n+1})^*), C_{-n}\to\cdots\to C_n\\
\mathcal{HE}\times \mathrm{Ac}^b_{[-n,n]}\mathcal{E} & \to & \mathcal{HE}\times \mathrm{Ch}^b_{[-n,n]}\mathcal{E}:\\
(A,B),C & \mapsto & (A,B),\ \ A\oplus C_{-n}\to C_{-n+1}\to\cdots\to C_{n-1}\to C_n\oplus B^*.
\end{array}$$

These functors, together with the natural inclusions, fit into a commutative diagram of exact categories with duality

$$\begin{array}{ccccccc}
\mathrm{Ac}^b_{[-n,n]}\mathcal{E} & & & \longrightarrow & & & \mathrm{Ch}^b_{[-n,n]}\mathcal{E}\\
\downarrow & \searrow^{\binom{0}{1}} & & & & \swarrow^{\binom{0}{1}} & \downarrow\\
 & & \mathcal{HE}\times \mathrm{Ac}^b_{[-n,n]}\mathcal{E} & \longrightarrow & \mathcal{HE}\times \mathrm{Ch}^b_{[-n,n]}\mathcal{E} & & \\
\downarrow & \swarrow_{s} & & & & \nwarrow_{\rho} & \downarrow\\
\mathrm{Ac}^b_{[-n-1,n+1]}\mathcal{E} & & & \longrightarrow & & & \mathrm{Ch}^b_{[-n-1,n+1]}\mathcal{E}.
\end{array}$$

The upper square induces a homotopy Cartesian square of Grothendieck-Witt spaces. By Additivity (Theorem 5 or [25, 7.1]), the diagonal maps s and ρ in the lower square induce homotopy equivalences of Grothendieck-Witt spaces (details below). Therefore, the outer square induces a homotopy Cartesian square as well. To see that the functors s and ρ induce homotopy equivalences, consider the following functors of categories with duality

$$\begin{array}{ccl}
\mathrm{Ac}^b_{[-n-1,n+1]}\mathcal{E} & \stackrel{r}{\to} & \mathcal{HE}\times \mathrm{Ac}^b_{[-n,n]}\mathcal{E}:\\
C & \mapsto & (C_{-n-1},(C_{n+1})^*),\\
 & & C_{-n}/C_{-n-1}\to C_{-n+1}\to\cdots\to C_{n-1}\to \ker(C_n\to C_{n+1})\\
\mathcal{HE}\times \mathrm{Ch}^b_{[-n,n]}\mathcal{E} & \stackrel{\sigma}{\to} & \mathrm{Ch}^b_{[-n-1,n+1]}\mathcal{E}:\\
(A,B),C & \mapsto & A\stackrel{0}{\to} C_{-n}\to C_{-n+1}\to\cdots\to C_{n-1}\to C_n\stackrel{0}{\to} B^*.
\end{array}$$

We have $\rho\sigma = id$. The identity functor id on $\mathrm{Ch}^b_{[-n-1,n+1]}$ has a totally isotropic subfunctor $G \subset id$ given by

$$\begin{array}{ccccccccc} G & & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & C_{n+1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow 1 \\ id & & C_{-n-1} & \longrightarrow & \cdots & \longrightarrow & C_n & \longrightarrow & C_{n+1}. \end{array}$$

By Additivity [25, 7.1], the identity functor id and $\sigma\rho = G^{\perp}/G \oplus \mathcal{H}G$ induce homotopic maps on Grothendieck-Witt spaces, thus ρ induces a homotopy equivalence. Similarly, we have $rs = id$, and the identity functor id on $\mathrm{Ac}^b_{[-n-1,n+1]}\mathcal{E}$ has a totally isotropic subfunctor $F \subset id$ given by

$$\begin{array}{ccccccccccc} F & & C_{-n-1} & \xrightarrow{1} & C_{-n-1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\ \downarrow & & \downarrow 1 & & \downarrow & & \downarrow & & & & \downarrow \\ id & & C_{-n-1} & \rightarrowtail & C_{-n} & \longrightarrow & C_{-n+1} & \longrightarrow & \cdots & \twoheadrightarrow & C_{n+1}. \end{array}$$

Again, by Additivity [25], the identity functor id on $\mathrm{Ac}^b_{[-n-1,n+1]}\mathcal{E}$ and $sr = F^{\perp}/F \oplus \mathcal{H}F$ induce homotopic maps on Grothendieck-Witt spaces. It follows that s induces a homotopy equivalence. This finishes the prove of (a).

We are left with proving (b). We will show that

(c) a symmetric space (A,α), where A is supported in $[-n,n]$, $n \geq 1$, equals $[A,\alpha] = [B,\beta] + [\mathcal{H}(C)]$ in the Grothendieck-Witt group $GW_0(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$, where B is supported in $[-n+1, n-1]$.

By induction, $[A,\alpha]$ is then a sum of hyperbolic objects plus a symmetric space supported in degree 0. Since the latter two kinds of symmetric spaces are obviously in the image of $GW_0(\mathcal{E}, i) \to GW_0(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$, this proves (b). To show (c), let $n \geq 1$ and let (A,α) be a symmetric space supported in $[-n,n]$. Since the cone of α is acyclic (hence strictly acyclic, by semi-idempotent completeness of $\mathcal{E}$), the map $\binom{d_{-n}}{\alpha_{-n}} : A_{-n} \to A_{-n+1} \oplus A^*_n$ is an inflation. Define a complex $\tilde{A}$, also supported in $[-n,n]$, by

$$A_{-n} \overset{\binom{d_{-n}}{\alpha_{-n}}}{\rightarrowtail} A_{-n+1} \oplus A^*_n \xrightarrow{(d\ 0)} A_{-n+2} \xrightarrow{d} \cdots \xrightarrow{d} A_{n-2} \xrightarrow{\binom{d}{0}} A_{n-1} \oplus A_n \overset{(d_{n-1}\ 1)}{\twoheadrightarrow} A_n.$$

The complex $\tilde{A}$ is equipped with a non-singular symmetric form $\tilde{\alpha}$ which is α_i in degree i except in degrees $i = -n+1, n-1$ where it is $\alpha_i \oplus 1$. We have a symmetric space in the category of admissible short complexes in $\mathrm{Ch}^b\,\mathcal{E}$

$$A_n^*[n-1] \rightarrowtail \tilde{A} \twoheadrightarrow A_n[-n+1]$$

with form $(1, \tilde{\alpha}, \eta)$, where for an object E of $\mathcal{E}$, we denote by $E[i]$ the complex which is E in degree $-i$ and 0 elsewhere. The maps $A_n^* \rightarrowtail \tilde{A}_{-n+1} = A_{-n+1} \oplus A_n^*$ and $A_{n-1} \oplus A_n = \tilde{A}_{n-1} \to A_n$ are the canonical inclusions and projections, respectively. Since (A, α) is the zero-th homology of this admissible short complex equipped with its form, we have $[\tilde{A}, \tilde{\alpha}] = [A, \alpha] + [\mathcal{H}(A_n[-n+1])]$ in $GW_0(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$. There is another symmetric space in the category of admissible short complexes in $\mathrm{Ch}^b\,\mathcal{E}$

$$CA_{-n}[n-1] \rightarrowtail \tilde{A} \twoheadrightarrow CA_n[n]$$

with non-singular from $(\alpha_{-n}, \tilde{\alpha}, \alpha_n)$, where for an object E of $\mathcal{E}$, we write $CE[i]$ for the complex $E \xrightarrow{1} E$ placed in degrees i and $i-1$. The maps $CA_{-n}[n-1] \rightarrowtail \tilde{A}$ and $\tilde{A} \twoheadrightarrow CA_n[n]$ are the unique maps which are the identity in degree $-n$ and n, respectively. Since $CA_{-n}[n-1]$ and $CA_n[n]$ are acyclic, the form on the admissible short complex is non-singular and its zero-th homology, which is concentrated in degrees $[-n+1, n-1]$, has the same class in $GW_0(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$ as $(\tilde{A}, \tilde{\alpha})$. □

Remark 14 We can equip $(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis}, *, \eta)$ with two exact structures, the one defined in Sect. 6.1, and the degree-wise split exact structure. By Lemma 7, the two yield homotopy equivalent Grothendieck-Witt spaces.

7 DG-Algebras on ringed spaces and dualities

In this section we recall basic definitions and facts about differential graded algebras and modules over them. Besides fixing terminology, the main point here is the construction of the canonical symmetric cone in Sect. 7.5, and the interpretation of certain form functors as symmetric forms in dg bimodule categories; see Sect. 7.7.

7.1 DG κ-modules

Let κ be a commutative ring. Unless otherwise indicated, modules will always mean left module, tensor product $\otimes$ will be tensor product $\otimes_\kappa$ over κ, and homomorphism sets $\mathrm{Hom}(M, N)$ between κ-modules means set of κ-linear homomorphisms, and is itself a κ-module. Recall that a differential graded κ-module M is a graded κ-module $\bigoplus_{n\in\mathbb{Z}} M_n$ together with a κ-linear map $d : M_n \to M_{n+1}$, $n \in \mathbb{Z}$, called differential of M, satisfying $d \circ d = 0$. In other words, M is a chain complex of κ-modules. A map of dg κ-modules is a map of graded κ-modules commuting with the differentials. For two dg κ-modules M, N, the tensor product dg κ-module $M \otimes N$ and the homomorphism dg κ-module $[M, N]$ are defined by the usual formulas

$$(M \otimes N)_n = \bigoplus_{i+j} M_i \otimes N_j \qquad d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$$
$$[M, N]_n = \prod_{j-i=n} \mathrm{Hom}(M_i, N_j) \quad d(f) = d_N \circ f - (-1)^{|f|} f \circ d_M.$$

7.2 DGAs and modules over them

A differential graded κ-algebra (dga) is a dg κ-module A equipped with dg κ-module maps $\cdot : A \otimes A \to A$ and $\kappa \to A$, called multiplication and unit, making the usual associativity and unit diagrams commute [18, diagrams (1), (2), p. 166]. In other words, A is an associative graded κ-algebra with multiplication satisfying $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$. For dg algebras A and B, we denote by A-Mod-B the category of left A and right B-modules. Its objects are the dg κ-modules M equipped with dg κ-maps $A \otimes M \to M$ and $M \otimes B \to M$, called multiplication, both of which are associative and unital, and furthermore, $(am)b = a(mb)$ for all $a \in A$, $m \in M$ and $b \in B$. We also denote by A-Mod $= A$-Mod-κ, Mod-$A = \kappa$-Mod-A, A-Bimod $= A$-Mod-A the categories of dg left A-modules, right A-modules, and of dg A-bimodules.

Let A, B, C be dg algebras. Recall that for a right B module M and left B-module N, tensor product $M \otimes_B N$ of M and N over B is the dg κ-module which is the co-equalizer

$$M \otimes B \otimes N \underset{\mu \otimes 1}{\overset{1 \otimes \mu}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_B N$$

in the category of dg κ-modules, where μ stands for the multiplications $M \otimes B \to M$ and $B \otimes N \to N$. For two dg right C-modules M and N, the dg κ-module $[M, N]_C$ of right C-module morphisms is the equalizer in the category of dg κ-modules

$$[M, N]_C \longrightarrow [M, N] \underset{[1,\mu]\circ(?\otimes 1_C)}{\overset{[\mu,1]}{\rightrightarrows}} [M \otimes C, N],$$

where $(? \otimes 1_C) : [M, N] \to [M \otimes C, N \otimes C]$ is the dg κ-module map $f \mapsto f \otimes 1_C$ defined by $(f \otimes 1_C)(x \otimes c) = f(x) \otimes c$ for $x \in M$ and $c \in C$. Similarly, one can define the dg κ-module of left C-module morphisms ${}_C[M, N]$ for two dg left C-modules.

Tensor product $\otimes_B$ and right C-module morphisms $[\,,\,]_C$ define functors

$$\otimes_B : \quad A\text{-Mod-}B \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C : \quad M, N \mapsto M \otimes_B N$$
$$[\,,\,]_C : (B\text{-Mod-}C)^{op} \times A\text{-Mod-}C \longrightarrow A\text{-Mod-}B : M, N \mapsto [M, N]_C.$$

7.3 DGAs with involution

For a dg κ-module M, we denote by M^{op} the opposite dg κ-module which, as a dg κ-module, is simply M itself. To avoid confusion we may sometimes write x^{op}, y^{op}, etc, for the elements in M^{op} corresponding to $x, y \in M$. For instance, $d(x^{op}) = (dx)^{op}$ denotes an equation in M^{op} as opposed to in M. For a dg κ-algebra A, its opposite dga A^{op} has underlying dg κ-module the opposite module of A and multiplication $x^{op} y^{op} = (-1)^{|x||y|}(yx)^{op}$. A *dg algebra with involution* is a dg κ-algebra A together with an isomorphism $A \to A^{op} : a \mapsto \bar{a}$ of dgas satisfying $\bar{\bar{a}} = a$ for all $a \in A$. The base commutative ring κ is always considered as a dga with trivial involution $\kappa \to \kappa^{op} : x \mapsto x$.

Let A, B be dgas with involution. For a left A, right B-module $M \in A$-Mod-B its opposite module M^{op} is a left B^{op} and right A^{op}-module which we consider as a left B and right A-module via the isomorphisms $A \to A^{op}$, $B \to B^{op}$, that is, $m^{op} \cdot a = (-1)^{|a||m|}(\bar{a} \cdot m)^{op}$, $b \cdot m^{op} = (-1)^{|b||m|}(m \cdot \bar{b})^{op}$ for $a \in A$, $b \in B$, $m \in M$. For dgas with involution A, B, C and $M \in A$-Mod-B, $N \in B$-Mod-C, the commutativity isomorphism of the tensor product defines an A-Mod-C-isomorphism

$$c : M \otimes_B N \xrightarrow{\cong} (N^{op} \otimes_B M^{op})^{op} : \;\; x \otimes y \mapsto (-1)^{|x||y|}(y^{op} \otimes x^{op})^{op}.$$

This can be iterated to obtain for rings with involution A, B, C, D and $M \in A$-Mod-B, $N \in B$-Mod-C, $P \in C$-Mod-D an isomorphism c_3 in A-Mod-D defined by

$$\begin{array}{ccc} (M \otimes_B N \otimes_C P) & \xrightarrow{c_3} & (P^{op} \otimes_C N^{op} \otimes_B M^{op})^{op} \\ x \otimes y \otimes z & \mapsto & (-1)^{|x||y|+|x||z|+|y||z|}(z^{op} \otimes y^{op} \otimes x^{op})^{op}. \end{array}$$

7.4 DG-modules and dualities

Let A be a dga with involution, and let I be an A-bimodule equipped with an A bimodule isomorphism $i : I \to I^{op}$ such that $i^{op} \circ i = id$, for instance A itself with $i(x) = \bar{x}$. We call the pair (I, i) a *duality coefficient* for the category A-Mod of dg A-modules, as it defines a duality $\sharp_{(I,i)} : (A\text{-Mod})^{op} \to A$-Mod by

$$M^{\sharp_{(I,i)}} = [M^{op}, I]_A.$$

The canonical double dual identification $\mathrm{can}_{(I,i),M} : M \to M^{\sharp_{(I,i)}\sharp_{(I,i)}}$ is the left A-module map given by the formula

$$\mathrm{can}_{(I,i),M}(x)(f^{op}) = (-1)^{|x||f|} i(f(x^{op}))$$

for $f \in [M^{op}, I]_A$ and $x \in M$. It is a straight forward to check the identity $\mathrm{can}^{\sharp}_{(I,i),M} \mathrm{can}_{(I,i),M^{\sharp}} = 1_{M^{\sharp}}$. Therefore, the triple $(A\text{-Mod}, \sharp_{(I,i)}, \mathrm{can}_{(I,i)})$ is a category with duality. In this paper, for the duality $\sharp_{(I,i)}$, the double dual identification will *always* be the natural map $\mathrm{can}_{(I,i)}$. With this in mind, we will write

$$(A\text{-Mod}, \sharp_{(I,i)})$$

for the category with duality $(A\text{-Mod}, \sharp_{(I,i)}, \mathrm{can}(I,i))$, the double dual identification being $\mathrm{can}_{(I,i)}$. If $i : I \to I^{op}$ is understood, we may write $\sharp_I$ instead of $\sharp_{(I,i)}$.

To give a symmetric form $\varphi : M \to [M^{op}, I]_A$ in $(A\text{-Mod}, \sharp_I)$ is the same as to give an A-bimodule map $\hat{\varphi} : M \otimes M^{op} \to I$ such that the diagram

$$\begin{array}{ccc} M \otimes M^{op} & \xrightarrow{\hat{\varphi}} & I \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle i} \\ (M \otimes M^{op})^{op} & \xrightarrow{\hat{\varphi}^{op}} & I^{op} \end{array}$$

commutes. The bijection is given by the identity $\hat{\varphi}(x \otimes y^{op}) = \varphi(x)(y^{op})$.

7.5 The canonical symmetric cone

Let $C = \kappa \cdot 1_C \oplus \kappa \cdot \epsilon$ be the dg κ-module whose underlying κ-module is free of rank 2 with basis 1_C and ϵ in degrees 0 and -1, respectively, and differential $d\epsilon = 1_C$. In fact, C is a commutative dga with unit 1_C and unique multiplication. Let A be a dga and M a (left) dg A-module. We write $C : A\text{-Mod} \to A\text{-Mod}$ for the functor $M \mapsto M \otimes C$, and i_M for the natural inclusion $M \to CM = M \otimes C : x \mapsto x \otimes 1_C$. Similarly, we write $P : A\text{-Mod} \to A\text{-Mod}$ for the functor $M \otimes [C^{op}, \kappa]$ and p_M for the natural surjection $PM = M \otimes [C^{op}, \kappa] \to M : m \otimes g \mapsto m \cdot g(1_C^{op})$. If A is a dga with involution, and (I, i) a duality coefficient for A-Mod, we define a natural transformation

$$\gamma_M : [M^{op}, I]_A \otimes [C^{op}, \kappa] \longrightarrow [(M \otimes C)^{op}, I]_A$$

by the formula $\gamma_M(f \otimes g)((x \otimes a)^{op}) = (-1)^{|a||x|} f(x^{op}) \cdot g(a^{op})$. One checks the equality $i_M^{\sharp_I} \circ \gamma_M = p_{M^{\sharp_I}}$.

Therefore, an exact category with weak equivalences and duality $(\mathcal{E}, w, *, \eta)$ which admits a fully faithful and duality preserving embedding into $(A\text{-Mod}, \sharp_I)$ has a symmetric cone in the sense of Definition 4 provided the functors C and P restrict to exact endofunctors of $\mathcal{E}$, the natural maps i_M and p_M are inflation and deflation, and the objects CM and PM are w-acyclic for all $M \in \mathcal{E}$.

7.6 Symmetric forms in bimodule categories and their tensor product

Let A and B be dgas with involution, and let (I, i), (J, j) be duality coefficients for A-Mod and B-Mod, respectively. A *symmetric form in* A-Mod-B*, with respect to the duality coefficients* (I, i) *and* (J, j), is a pair (M, φ) where $M \in A\text{-Mod-}B$ is a left A and right B-module, and $\varphi : M \otimes_B J \otimes_B M^{op} \to I$ is an A-bimodule map making the diagram of A-bimodule maps

$$\begin{array}{ccc} M \otimes_B J \otimes_B M^{op} & \xrightarrow{\varphi} & I \\ {\scriptstyle c_3\circ(1\otimes j\otimes 1)}\downarrow & & \downarrow{\scriptstyle i} \\ (M \otimes_B J \otimes_B M^{op})^{op} & \xrightarrow[\varphi^{op}]{} & I^{op} \end{array} \tag{23}$$

commute. Isometries and orthogonal sums of symmetric forms in A-Mod-B are defined in the obvious way.

Tensor product of symmetric forms is defined as follows. Let A, B, C be dg algebras with involution, and let (I, i), (J, j), (K, k) be duality coefficients for A-Mod, B-Mod and C-Mod, respectively. Further, let $M \in A\text{-Mod-}B$ and $N \in B\text{-Mod-}C$ be equipped with symmetric forms given by the A- and B-bimodule maps $\varphi : M \otimes_B J \otimes_B M^{op} \to I$ and $\psi : N \otimes_C K \otimes_C N^{op} \to J$ (making diagram (23) and its analog for ψ commute). The tensor product

$$(M, \varphi) \otimes_B (N, \psi)$$

of the symmetric forms (M, φ) and (N, ψ) has $M \otimes_B N$ as underlying left A and right C-module, and is equipped with the symmetric form which is the A-bimodule map

$$\begin{aligned} (M \otimes_B N) \otimes_C K \otimes_C (M \otimes_B N)^{op} &\xrightarrow{c} M \otimes_B N \otimes_C K \otimes_C N^{op} \otimes_B M^{op} \\ &\xrightarrow{\psi} \qquad M \otimes_B J \otimes_B M^{op} \qquad \xrightarrow{\varphi} I. \end{aligned}$$

7.7 Form functors as tensor product with symmetric forms

Let A and B be dgas with involution, and let (I, i), (J, j) be duality coefficients for A-Mod and B-Mod, respectively. We want to think of (certain) form functors $(B\text{-Mod}, \sharp_J) \to (A\text{-Mod}, \sharp_I)$ as tensor product with symmetric forms in A-Mod-B. For that, let (M, φ) be a symmetric form in A-Mod-B. It defines a form functor

$$(M, \varphi)\otimes_B ? : (B\text{-Mod}, \sharp_J) \xrightarrow{(F,\Phi)} (A\text{-Mod}, \sharp_I),$$

where $F(P) = M \otimes_B P$ and the duality compatibility map is the left A-module homomorphism

$$M \otimes_B [P^{op}, J]_B \xrightarrow{\Phi_P} [(M \otimes_B P)^{op}, I]_A$$

defined by

$$\Phi(x \otimes f)((y \otimes t)^{op}) = (-1)^{|y||t|}\varphi(x \otimes f(t^{op}) \otimes y^{op})$$

for $x, y \in M$, $f \in [P^{op}, J]_B$, and $t \in P$.

7.8 Basic properties of $(M, \varphi)\otimes_B ?$

Let A, B, C be dg algebras with involution, and let (I, i), (J, j), (K, k) be duality coefficients for A-Mod, B-Mod and C-Mod, respectively. Further, let (M, φ), (M', φ') be symmetric forms in A-Mod-B and (N, ψ) a symmetric form in B-Mod-C. Form functors induced by tensor product with symmetric forms have the following elementary properties.

(a) Tensor product $(A, \mu_I)\otimes_A ?$ with the symmetric form

$$\mu_I : A \otimes_A I \otimes_A A^{op} \to I : a \otimes t \otimes b \mapsto a \cdot t \cdot \bar{b}$$

on the A-bimodule A induces the identity form functor on $(A, \sharp_I)$.

(b) An isometry $(M, \varphi) \cong (M', \varphi')$ between symmetric forms in A-Mod-B defines an isometry of associated form functors

$$(M, \varphi)\otimes_B ? \cong (M', \varphi')\otimes_B ? : (B\text{-Mod}, \sharp_J) \to (A\text{-Mod}, \sharp_I).$$

(c) Orthogonal sum $(M, \varphi) \perp (M', \varphi')$ of symmetric forms in A-Mod-B corresponds to orthogonal sum of associated form functors:

$$(M, \varphi)\otimes_B ? \;\perp\; (M', \varphi')\otimes_B ? \;\cong\; [(M, \varphi) \perp (M', \varphi')]\otimes_B ?$$

(d) Tensor product of symmetric forms corresponds to composition of associated form functors:

$$[(M, \varphi) \otimes_B (N, \psi)]\otimes_C ? \cong [(M, \varphi)\otimes_B ?] \circ [(N, \psi)\otimes_C ?]$$

These properties follow directly from the definitions, and we omit the details.

7.9 Tensor product of dgas with involution

For two dgas A, V, the tensor product dg κ-module $AV = A \otimes_\kappa V$ is a dga with multiplication $(a \otimes v) \cdot (b \otimes w) = (-1)^{|b||v|}(a \cdot b) \otimes (v \cdot w)$. If A, V are dgas with involution, then the tensor product dga AV is a dga with involution $AV \to (AV)^{op} : a \otimes v \mapsto (\bar{a} \otimes \bar{v})^{op}$. Furthermore, if (I, i) and (U, u) are duality coefficients for A-Mod and V-Mod, then $IU = I \otimes U$ is an AV-bimodule with left multiplication $(a \otimes v) \cdot (t \otimes x) = (-1)^{|v||t|} a \cdot t \otimes v \cdot x$ and right multiplication $(t \otimes x) \cdot (a \otimes v) = (-1)^{|a||x|} t \cdot a \otimes x \cdot v$, for $a \in A$, $v \in V$, $t \in I$, $x \in U$, and the AV-bimodule map $iu = i \otimes u : I \otimes U \to (I \otimes U)^{op} : t \otimes x \mapsto (i(t) \otimes u(x))^{op}$ makes the pair (IU, iu) into a duality coefficient for AV-Mod.

If B is another dga with involution, and (J, j) is a duality coefficient for B-Mod, then, with the same formulas as in Sects. 7.6 and 7.7, any symmetric from (M, φ) in A-Mod-B with respect to the duality coefficients (I, i) and (J, j) defines a form functor

$$(M, \varphi) \otimes_B ? : (BV\text{-Mod}, \sharp_{(JU, ju)}) \to (AV\text{-Mod}, \sharp_{(IU, iu)})$$

satisfying the properties in Sect. 7.8.

7.10 Extension to ringed spaces

Let (X, O_X) be a ringed space with O_X a sheaf of commutative rings on a topological space X. Replacing in Sects. 7.1–7.9 the ground ring κ with the sheaf of commutative rings O_X, all definitions and properties from Sects. 7.1–7.9 extend to modules over differential graded sheaves of O_X-algebras. Definitions are extended by applying the definitions of Sects. 7.1–7.9 to sections over open subsets of X. For instance, let $\mathcal{A}$ be a sheaf of dg O_X-algebras with involution, (I, i) a duality coefficient for $\mathcal{A}$-Mod, and P a sheaf of left dg $\mathcal{A}$-modules. The canonical double dual identification $\mathrm{can} : P \to [[P^{op}, I]^{op}_{\mathcal{A}}, I]_{\mathcal{A}}$ is defined by sending a section $x \in P(U)$, $U \subset X$, to the map of sheaves of dg modules $\mathrm{can}(x) : ([P^{op}, I]^{op}_{\mathcal{A}})_{|U} \to I_{|U}$ defined on $V \subset U$ by $\mathrm{can}(x)(f^{op}) = (-1)^{|x||f|} i(f(x^{op}_{|V}))$ for $f \in [P^{op}, I]_{\mathcal{A}}(V)$.

8 Higher Grothendieck-Witt groups of schemes

Let X be a scheme, $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution, L a line bundle on X, $Z \subset X$ a closed subscheme and $n \in \mathbb{Z}$ an integer. The purpose of this section is to introduce the Grothendieck-Witt space $GW^n(\mathcal{A}_X \text{ on } Z, L)$ of symmetric spaces over $\mathcal{A}_X$ with coefficients in the n-th shifted line bundle $L[n]$ and support in Z. We work in this generality in order to be able to extent the localization and excision theorems of Sect. 9 to negative degrees.

Recall that, unless otherwise indicated, "module" will always mean "left module". In what follows, we will denote by $\otimes$ the tensor product $\otimes_{O_X}$ of O_X-modules.

8.1 Vector bundles and strictly perfect complexes

Let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution. The category of quasi-coherent left $\mathcal{A}_X$-modules (dg-modules concentrated in degree 0) is a fully exact Abelian subcategory of the Abelian category of left $\mathcal{A}_X$-modules. We denote by

$$\mathrm{Vect}(\mathcal{A}_X)$$

the full subcategory of $\mathcal{A}_X$ vector bundles, that is, of those quasi-coherent left $\mathcal{A}_X$-modules F for which $F(U)$ is a finitely generated projective $\mathcal{A}_X(U)$-module for every affine $U \subset X$. As usual, the last condition only needs to be checked for those U running through a choice of an affine open cover of X. The category of $\mathcal{A}_X$ vector bundles inherits the notion of exact sequences from the category of all (quasi-coherent) $\mathcal{A}_X$-modules. Note that $\mathcal{A}_X$ vector bundles need not be locally free, since $\mathcal{A}_{X,x}$ may not be commutative nor a local ring for $x \in X$. In case $\mathcal{A}_X = O_X$, the category $\mathrm{Vect}(X)$ is the usual exact category of vector-bundles on X.

A *strictly perfect* complex of $\mathcal{A}_X$-modules is a dg left $\mathcal{A}_X$-module M such that $M_n = 0$ for all but finitely many $n \in \mathbb{Z}$ and M_n is an $\mathcal{A}_X$ vector bundle for all $n \in \mathbb{Z}$. Denote by $\mathrm{sPerf}(\mathcal{A}_X)$ the category of strictly perfect complexes of $\mathcal{A}_X$-modules, in oder words, the category of bounded chain complexes of $\mathcal{A}_X$ vector bundles.

Let L be a line bundle on X. Then $\mathcal{A}_X L[n] = \mathcal{A}_X \otimes L[n]$ is a dg $\mathcal{A}_X$-bimodule via the multiplication defined on sections by $a(x \otimes l)b = (-1)^{|b||l|} axb \otimes l$ for $a, b, x \in \mathcal{A}_X$ and $l \in L[n]$. We equip $\mathcal{A}_X L[n]$ with a dg $\mathcal{A}_X$-bimodule isomorphism $i : \mathcal{A}_X L[n] \to (\mathcal{A}_X L[n])^{op} : a \otimes l \mapsto \bar{a} \otimes l$ satisfying $i^{op} \circ i = 1$. Therefore, $(\mathcal{A}_X L[n], i)$ is a duality coefficient for $\mathcal{A}_X$-Mod. If $\varepsilon \in \{+1, -1\}$, then $(\mathcal{A}_X L[n], \varepsilon i)$ is also a duality coefficient for $\mathcal{A}_X$-Mod. In the notation of Sect. 7.9, the duality coefficient $(\mathcal{A}_X L[n], \varepsilon i)$ is the tensor product of the duality coefficient $(\mathcal{A}_X, \mu)$ for $\mathcal{A}_X$-Mod and the duality coefficient $(L[n], \varepsilon)$ for O_X-Mod.

For a strictly perfect complex of $\mathcal{A}_X$-modules M, the left dg $\mathcal{A}_X$-module

$$M^{\sharp^n_{\varepsilon L}} = [\, M^{op}, \mathcal{A}_X L[n]\,]_{\mathcal{A}_X}$$

is also strictly perfect, the functor $M \mapsto M^{\sharp^n_{\varepsilon L}}$ is exact and preserves quasi-isomorphisms. Moreover, the double dual identification $\mathrm{can}_{(\mathcal{A}_X L[n], \varepsilon i)}$ defined in Sect. 7.4 is an isomorphism. Therefore, the triple

$$(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis}, \sharp^n_{\varepsilon L})$$

defines an exact category with weak equivalences and duality, the double dual identification being understood as $\mathrm{can}_{(\mathcal{A}_X L[n], \varepsilon i)}$. If $n = 0$ (or $\varepsilon = 1$, or $L = O_X$), we may omit the label corresponding to n (or ε, or L, respectively). For instance (A_X-Mod, $\sharp^n$) means (A_X-Mod, $\sharp^n_{1,O_X}$). By restriction of structure, we have an exact category with duality

$$(\mathrm{Vect}(\mathcal{A}_X), \sharp_{\varepsilon L}).$$

Let $Z \subset X$ be a closed subscheme with open complement $U = X - Z$. A strictly perfect complex M of $\mathcal{A}_X$-modules has *cohomological support in* Z if the restriction $M_{|U}$ of M to U is acyclic. We write sPerf($\mathcal{A}_X$ on Z) for the category of strictly perfect complexes of $\mathcal{A}_X$-modules which have cohomological support in Z. By restriction of structure, we have exact categories with weak equivalences and duality

$$(\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z), \mathrm{quis}, \sharp^n_{\varepsilon L}). \tag{24}$$

Definition 7 Let X be a scheme, $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution, L a line bundle on X, $Z \subset X$ a closed subscheme, $n \in \mathbb{Z}$ an integer, and $\varepsilon \in \{+1, -1\}$. The Grothendieck-Witt space

$${}_{\varepsilon}GW^n(\mathcal{A}_X \text{ on } Z, L)$$

of ε-symmetric spaces over $\mathcal{A}_X$ with coefficients in the n-th shifted line bundle $L[n]$ and (cohomological) support in Z is the Grothendieck-Witt space of the exact category with weak equivalences and duality (24). If $Z = X$ (or $\varepsilon = 1$, or $L = O_X$, or $n = 0$), we may omit the label corresponding to Z (ε, L, n, respectively). For instance, the space $GW(\mathcal{A}_X, L)$ denotes the Grothendieck-Witt space ${}_1GW^0(\mathcal{A}_X \text{ on } X, L)$.

Remark 15 By Sect. 7.5, the exact category with weak equivalences and duality (24) has a symmetric cone in the sense of Definition 4.

In the following proposition, we write $\mu : O_X \otimes O_X \to O_X$ for the multiplication in O_X.

Proposition 7 *Tensor product with the* (-1)*-symmetric space* $(O_X[1], \mu)$ *induces an equivalence of exact categories with weak equivalences and duality*

$$(\mathrm{sPerf}(\mathcal{A}_X \textit{ on } Z), \mathrm{quis}, \sharp^n_{\varepsilon L}) \cong (\mathrm{sPerf}(\mathcal{A}_X \textit{ on } Z), \mathrm{quis}, \sharp^{n+2}_{-\varepsilon L}).$$

In particular, we have homotopy equivalences of Grothendieck-Witt spaces

$$(O_X[1], \mu)\otimes ? : {}_{\varepsilon}GW^n(\mathcal{A}_X \text{ on } Z, L) \simeq {}_{-\varepsilon}GW^{n+2}(\mathcal{A}_X \text{ on } Z, L) \quad \text{and}$$
$$(O_X[2], \mu)\otimes ? : {}_{\varepsilon}GW^n(\mathcal{A}_X \text{ on } Z, L) \simeq {}_{\varepsilon}GW^{n+4}(\mathcal{A}_X \text{ on } Z, L).$$

Proof The pair $(O_X[1], \mu)$ defines a symmetric space in O_X-Mod with respect to the duality coefficient $(O_X[2], -1)$. Tensor product $(O_X[1], \mu)\otimes_{O_X} ?$ defines a form functor $(\mathcal{A}_X\text{-Mod}, \sharp^n_{\varepsilon L}) \to (\mathcal{A}_X\text{-Mod}, \sharp^{n+2}_{-\varepsilon L})$ as explained in Sects. 7.7–7.10. Since $(O_X[1], \mu) \otimes_{O_X} (O_X[-1], -\mu)$ and $(O_X[-1], -\mu) \otimes_{O_X} (O_X[1], \mu)$ are isometric to (O_X, μ) which induces the identity form functor, the equivalence of categories with duality and the first homotopy equivalence follow. The second map of spaces is a homotopy equivalence with inverse given by the tensor product with the symmetric space $(O_X[-2], \mu)$. □

Corollary 1 *For $n \in \mathbb{Z}$, there are functorial homotopy equivalences*

$$GW^{4n}(\mathcal{A}_X, L) \simeq GW(\mathrm{Vect}(\mathcal{A}_X), \sharp_L, \mathrm{can}_L), \quad \text{and}$$
$$GW^{4n+2}(\mathcal{A}_X, L) \simeq GW(\mathrm{Vect}(\mathcal{A}_X), \sharp_L, -\mathrm{can}_L).$$

where the Grothendieck-Witt spaces on the right hand side are the ones associated with the exact categories with duality $(\mathrm{Vect}(\mathcal{A}_X), \sharp_L, \pm\mathrm{can}_L)$ *as defined in* [25, *Definition* 4.4].

Proof The homotopy equivalences follow from Proposition 6, Remark 13, and Proposition 7. □

9 Localization and Zariski-excision in positive degrees

9.1 Schemes with an ample family of line bundles

A scheme X has an *ample family of line bundles* if there is a finite set $L_1, \ldots, L_n$ of line bundles with global sections $s_i \in \Gamma(X, L_i)$ such that the non-vanishing loci $X_{s_i} = \{x \in X \mid s_i(x) \neq 0\}$, $i = 1, \ldots, n$, form an open affine cover of X; see [30, Definition 2.1], [4, II 2.2.4].

Recall that if $f \in \Gamma(X, L)$ is a global section of a line bundle L on a scheme X, then the open inclusion $X_f \subset X$ is an affine map (as can be seen by choosing an open affine cover of X trivializing the line bundle L). As a special case, X_f is affine whenever X is affine. Thus, for the affine cover above $X = \bigcup X_{s_i}$, all finite intersections of the X_{s_i}'s are affine. In particular, a scheme with an ample family of line bundles is quasi-compact (as a finite union of affine, hence quasi-compact, subschemes) and it is quasi-separated. Recall that the latter means that the intersection of any two quasi-compact open subsets is quasi-compact (a condition which only needs to be checked for the pair-wise intersections $U_i \cap U_j$ of a cover of $X = \bigcup_i U_i$ by quasi-compact open subsets U_i; in our case, we can take $U_i = X_{s_i}$).

For a scheme X with an ample family of line bundles, there is a set L_i, $i \in I$, of line bundles on X with global sections $s_i \in \Gamma(X, L_i)$ such that the open subsets X_{s_i}, $i \in I$, form an open affine basis for the topology of X [30, 2.1.1(b)]. If X is affine, this follows from the definition of the Zariski topology. For a general X (with an ample family of line bundles), the sections which give rise to a basis of topology on an open affine X_s can be extended (up to a power of s) to global sections; see [9, Théorème 9.3.1]. Therefore, every open subset of a basis for X_s is also the non-vanishing locus of a global section of some line bundle on X.

For examples of schemes with an ample family of line bundles, see [30, 2.1.2]. Any quasi-compact open or closed subscheme of a scheme with an ample family of line bundles has itself an ample family of line bundles. Any scheme quasi-projective over an affine scheme, and any separated regular noetherian scheme has an ample family of line bundles.

The main purpose of this section is to prove the following two theorems.

Theorem 10 (Localization) *Let X be a scheme with an ample family of line-bundles, let $Z \subset X$ be a closed subscheme with quasi-compact open complement $j : U \subset X$, and let L be a line bundle on X. Let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution. Then for every $n \in \mathbb{Z}$ there is a homotopy fibration of Grothendieck-Witt spaces*

$$GW^n(\mathcal{A}_X \text{ on } Z, L) \longrightarrow GW^n(\mathcal{A}_X, L) \longrightarrow GW^n(\mathcal{A}_U, j^*L).$$

Theorem 11 (Zariski-excision) *Let X be a scheme with an ample family of line-bundles, let $Z \subset X$ be a closed subscheme with quasi-compact open complement, let L be a line bundle on X and let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution. Then for every $n \in \mathbb{Z}$ and every quasi-compact open subscheme $j : V \subset X$ containing Z, restriction of vector-bundles induces a homotopy equivalence*

$$GW^n(\mathcal{A}_X \text{ on } Z, L) \xrightarrow{\sim} GW^n(\mathcal{A}_V \text{ on } Z, j^*L).$$

The proofs of Theorems 10 and 11 will occupy the rest of this section. First, we introduce some terminology. For an open subscheme $U \subset X$, call a map $M \to N$ of left dg $\mathcal{A}_X$-modules a U-isomorphism (U-quasi-isomorphism) if its restriction $M_{|U} \to N_{|U}$ to U is an isomorphism (quasi-isomorphism). A left dg $\mathcal{A}_X$-module M is called U-acyclic if its restriction $M_{|U}$ to U is acyclic.

9.2 Notation for Sect. 9.3 and Lemmas 13 and 14

Below, we will consider the following objects:

- a scheme X which is quasi-compact and quasi-separated,
- a finite set of line bundles L_i, $i = 1, \dots, l$ together with global sections $s_i \in \Gamma(X, L_i)$,
- the union $U = \bigcup_{i=1}^{l} X_{s_i}$ of the non-vanishing loci X_{s_i} of the s_i's, denoting $j : U \subset X$ the corresponding open immersion, and
- a quasi-coherent sheaf of O_X-algebras $\mathcal{A}_X$.

9.3 Truncated Koszul complexes

In the situation of Sect. 9.2, the global sections s_i define maps $s_i : O_X \to L_i$ of line-bundles whose O_X-duals are denoted by $s_i^{-1} : L_i^{-1} \to O_X$. We consider the maps s_i^{-1} as (cohomologically graded) chain-complexes with O_X placed in degree 0. For an l-tuple $n = (n_1, \dots, n_l)$ of negative integers, the Koszul complex

$$\bigotimes_{i=1}^{l} (L_i^{n_i} \overset{s^{n_i}}{\to} O_X) \tag{25}$$

is acyclic over U. This is because the map $s^{n_i} = (s_i^{-1})^{\otimes |n_i|} : L_i^{n_i} \to O_X$ is an X_{s_i}-isomorphism. Therefore, the Koszul complex (25) is acyclic (even contractible) over each X_{s_i}. Let $K(s^n)$ denote the bounded complex whose nonzero part (which we place in degrees $-l + 1, \dots, 0$) is the Koszul complex (25) in degrees ≤ -1. The last differential d_{-1} of the Koszul complex defines a map

$$K(s^n) = \left[\bigotimes_{i=1}^{l} (L_i^{n_i} \overset{s^{n_i}}{\to} O_X) \right]_{\leq -1} [-1] \xrightarrow{\varepsilon} O_X$$

of strictly perfect complexes of O_X-modules which is a U-quasi-isomorphism since its cone, the Koszul complex, is U-acyclic. For a left dg $\mathcal{A}_X$-module M, we write ε_M for the tensor product map $\varepsilon_M = 1_M \otimes \varepsilon : M \otimes K(s^n) \to M \otimes O_X \cong M$.

The following lemma is a consequence of the well-known techniques of extending a section of a quasi-coherent sheaf from an open subset cut out by a divisor to the scheme itself [9, Théorème 9.3.1]. It is implicit in the proof of [30, Proposition 5.4.2].

Lemma 13 *In the situation Sect. 9.2, let M be a complex of quasi-coherent left $\mathcal{A}_X$-modules and let A be a strictly perfect complex of $\mathcal{A}_X$-modules. Then the following holds.*

(a) *For every map $f : j^*A \to j^*M$ of left dg $\mathcal{A}_U$-modules between the restrictions of A and M to U, there is an l-tuple of negative integers $n = (n_1, \dots, n_l)$, and a map $\tilde{f} : A \otimes K(s^n) \to M$ of left dg $\mathcal{A}_X$-modules such that $f \circ j^*(\varepsilon_A) = j^*(\tilde{f})$.*

(b) *For every map $f : A \to M$ of left dg $\mathcal{A}_X$-modules such that $j^*(f) = 0$, there is an l-tuple of negative integers $n = (n_1, \dots, n_l)$ such that $f \circ \varepsilon_A = 0$.*

Proof For any complex of O_X-modules K, to give a map $A \otimes K \to M$ of chain complexes of $\mathcal{A}_X$-modules is the same as to give a map $K \to {}_{\mathcal{A}_X}[A, M]$ of chain complexes of O_X-modules, by adjointness of the tensor product functor $A\otimes ? : O_X\text{-Mod} \to \mathcal{A}_X\text{-Mod}$ and the left $\mathcal{A}_X$-module map functor ${}_{\mathcal{A}_X}[A, \ ] : \mathcal{A}_X\text{-Mod} \to O_X\text{-Mod}$. If A is strictly perfect, the complex ${}_{\mathcal{A}_X}[A, M]$ is a complex of quasi-coherent O_X-modules and the natural map ${}_{\mathcal{A}_X}[A, j_*j^*M] \to j_*j^*\ {}_{\mathcal{A}_X}[A, M]$ is an isomorphism. The adjunction allows us to reduce the proof of the lemma to $\mathcal{A}_X = O_X$ and $A = O_X$ (concentrated in degree 0).

Every map $O_X \to M$ of chain complexes of O_X-modules factors through the subcomplex $Z_0M \subset M$ of M which is the complex $\ker(d_0 : M_0 \to M_1)$ concentrated in degree 0. By adjunction, every map $O_X \to j_*j^*M$ factors as $O_X \to j_*Z_0j^*M = j_*j^*Z_0M \to j_*j^*M$. This allows us to further reduce the proof to M a complex with $M_i = 0$, $i \neq 0$. In this case, the proof for $l = 1$ is classical; see [9, Théorème 9.3.1], [30, Lemma 5.4.1]. Hence, the lemma is proved in case $l = 1$.

Before we treat the case $l > 1$ (and $\mathcal{A}_X = O_X$; $A = O_X$, M concentrated in degree 0), we prove the following.

(†) For every map $A \to G$ of complexes of quasi-coherent O_X-modules with A strictly perfect and j^*G contractible, there is an l-tuple of negative integers $(n_1, \dots, n_l)$ and a commutative diagram of complexes of O_X-modules

$$\begin{array}{ccc} A & \xrightarrow{1_A \otimes \iota} & A \otimes \left[\bigotimes_{i}^{l} (L_i^{n_i} \stackrel{s^{n_i}}{\to} O_X) \right] \\ & \searrow & \downarrow \\ & & G \end{array}$$

where ι is the canonical embedding of O_X (concentrated in degree 0) into the Koszul complex.

It suffices to prove (†) for $l = 1$, since the general case is a repeated application of the case $l = 1$. For the proof of (†) with $l = 1$, we can assume

$A = O_X$, as above. The composition $O_X \to G \to j_* j^* G$ has contractible target and therefore factors through the cone $(O_X \to O_X)$ of O_X. By the case $l = 1$ of the lemma (proved above), there is an $n < 0$ such that the composition $(L^n \to L^n) \stackrel{s^n}{\to} (O_X \to O_X) \to j_* j^* G$ lifts to G. The two maps $(0 \to L^n) \stackrel{s^n}{\to} (0 \to O_X) \to G$ and $(0 \to L^n) \to (L^n \to L^n) \to G$ may not be the same, but their compositions with $G \to j_* j^* G$ are. Therefore, again by case $l = 1$ of the lemma, precomposing both maps with $L^{n+t} \stackrel{s^t}{\to} L^n$ makes the two maps with target G equal. Replacing n with $n + t$, we can assume that the two maps $(0 \to L^n) \to G$ above coincide. Then we obtain a map from the push-out $(L^n \stackrel{s^n}{\to} O_X)$ of $(L^n \to L^n) \leftarrow (0 \to L^n) \stackrel{s^n}{\to} (0 \to O_X)$ to G. This proves (†) for $l = 1$, hence for all l.

For part (a) of the general case of Lemma 13 (and $\mathcal{A}_X = O_X$; $A = O_X$, M concentrated in degree 0), we apply (†) to the map of chain-complexes of O_X-modules $(0 \to O_X) \to (M \to j_* j^* M)$, and obtain a factorization of that map through the Koszul complex $\bigotimes_i (L_i^{n_i} \stackrel{s^{n_i}}{\to} O_X)$ for some l-tuple of negative integers $(n_1, \ldots, n_l)$. The canonical map from the stupid truncation in degrees ≤ -1 (shifted by 1 degree) to its degree 0 part

$$\begin{array}{ccc}
K(s^n) = \left[\bigotimes_{i=1}^{l} (L_i^{n_i} \stackrel{s^{n_i}}{\to} O_X) \right]_{\leq -1} [-1] & \xrightarrow{d_{-1}} & O_X \\
\downarrow & & \downarrow \\
M = [M \to j_* j^* M]_{\leq -1} [-1] & \xrightarrow{d_{-1}} & j_* j^* M
\end{array}$$

yields (a). The general case of part (b) is a repeated application of the case $l = 1$. □

For an exact category with weak equivalences $(\mathcal{C}, w)$, we write $\mathcal{D}(\mathcal{C}, w)$ for its derived category, that is, the category $\mathcal{C}[w^{-1}]$ obtained from $\mathcal{C}$ by formally inverting the arrows in w. If $(\mathcal{C}, w)$ is a category of complexes in the sense of Definition 5, its derived category $\mathcal{D}(\mathcal{C}, w)$ is a triangulated category. In this case, it can also be obtained as the localization by a calculus of fractions of the homotopy category $\mathcal{K}(\mathcal{C})$ of $\mathcal{C}$ which is the factor category of $\mathcal{C}$ modulo the ideal of maps which are homotopic to zero.

Lemma 14 *In the situation Sect. 9.2, let $\mathcal{A} \subset \mathrm{sPerf}(\mathcal{A}_X)$ be a full subcategory of the category of strictly perfect $\mathcal{A}_X$-modules such that the inclusion $\mathcal{A} \subset \mathrm{sPerf}(\mathcal{A}_X)$ is closed under degree-wise split extensions, usual shifts and cones. Assume furthermore that for all $A \in \mathcal{A}$, $k \leq 0$ and $i = 1, \ldots, l$, we have $A \otimes L_i^k \in \mathcal{A}$.*

Then for every U*-quasi-isomorphism* $M \to A$ *of complexes of quasi-coherent* $\mathcal{A}_X$*-modules with* $A \in \mathcal{A}$*, there is a* U*-quasi-isomorphism* $B \to M$ *of complexes of* $\mathcal{A}_X$*-modules with* $B \in \mathcal{A}$:

$$B \xrightarrow[U\text{-quis}]{\exists} M \xrightarrow[U\text{-quis}]{\forall} A.$$

In particular, the inclusion $\mathcal{A} \subset \mathrm{sPerf}(\mathcal{A}_X)$ *induces a fully faithful triangle functor*

$$\mathcal{D}(\mathcal{A}, U\text{-quis}) \subset \mathcal{D}(\mathrm{sPerf}(\mathcal{A}_X), U\text{-quis}).$$

Proof We first prove the following statement.

(†) Let $s \in \Gamma(X, L)$ be a global section of a line-bundle L such that X_s is affine. Then for every X_s-quasi-isomorphism $N \to E$ of complexes of quasi-coherent $\mathcal{A}_X$-modules with E strictly perfect on X, there is an X_s-quasi-isomorphism $E \otimes L^{-k} \to N$ for some integer $k > 0$.

Write $j : X_s \subset X$ for the open inclusion. Since X_s is affine, we have an equivalence of categories between quasi-coherent $\mathcal{A}_{X_s}$-modules and $\mathcal{A}_X(X_s)$-modules under which the map $j^*N \to j^*E$ becomes a quasi-isomorphism of complexes of $\mathcal{A}_X(X_s)$-modules with j^*E a bounded complex of projectives. Such a map always has a section up to homotopy $f : j^*E \to j^*N$ which is then a quasi-isomorphism. By Lemma 13 with $l = 1$, there is a map of complexes $\tilde{f} : E \otimes L^k \to N$ such that $j^*\tilde{f} = f \cdot s^k$, for some $k < 0$. In particular, $\tilde{f}$ is a U-quasi-isomorphism.

Now we prove the lemma by induction on l. For $l = 1$, this is (†). Let $U_0 = \bigcup_{i=1}^{l-1} X_{s_i}$. By our induction hypothesis, there is a U_0-quasi-isomorphism $B_0 \to M$ with $B_0 \in \mathcal{A}$. Let M_0 and A_0 be the cones of the maps $B_0 \to M$ and $B_0 \to A$, that is, the push-out of these maps along the canonical (degree-wise split) injection of B_0 into its cone CB_0. We obtain a commutative (in fact bi-Cartesian) diagram involving M, M_0, A and A_0 with $M \to M_0$ and $A \to A_0$ degree-wise split injective. Factor the map $A \to A_0$ as in the diagram

$$\begin{array}{ccccc} M & \rightarrowtail & M_1 & \twoheadrightarrow & M_0 \\ \downarrow & & \downarrow & & \downarrow \\ A & \rightarrowtail & A \oplus PA_0 & \twoheadrightarrow & A_0 \end{array}$$

with $A \oplus PA_0 \to A_0$ degree-wise split surjective and $PA_0 = CA_0[-1] \in \mathrm{Ch}^b \mathcal{A}$ contractible. Then $M \to M_0$ factors through the pull-back M_1 of

$M_0 \to A_0$ along the surjection $A \oplus PA_0 \to A$. The map $M \to M_1$ is degree-wise split injective (as $M \to M_0$ is), and has cokernel the contractible complex PA_0. It follows that $M \to M_1$ is a homotopy equivalence, and we can choose a homotopy inverse $M_1 \to M$. By construction, A_0 and M_0 are acyclic over U_0 and $A_0 \in \mathcal{A}$. Moreover, the map $M_0 \to A_0$ is an X_{s_l}-quasi-isomorphism. By (†), there is an X_{s_l}-quasi-isomorphism $A_0 \otimes L^{-k} \to M_0$ for some $k > 0$. The complex $A_0 \otimes L^{-k}$ is U_0-acyclic since A_0 is Therefore, the map $A_0 \otimes L^{-k} \to M_0$ is in fact a U-quasi-isomorphism. Let B be the pull-back of $A_0 \otimes L^{-k} \to M_0$ along the surjection $M_1 \to M_0$. The resulting map $B \to M_1$ is a U-quasi-isomorphism. Moreover B is an object of $\mathcal{A}$ since B is also the pull-back of $A_0 \otimes L^{-k} \to A_0$ along the (degree-wise split) surjection $A \oplus PA_0 \to A_0$. Composing the map $B \to M_1$ with the homotopy inverse $M_1 \to M$ of $M \to M_1$ yields the desired U-quasi-isomorphism $B \to M$. □

9.4 The derived category of quasi-coherent $\mathcal{A}_X$-modules

Recall that a Grothendieck Abelian category is an Abelian category $\mathcal{A}$ in which all set-indexed direct sums exist, filtered colimits are exact, and $\mathcal{A}$ has a set of generators. We remind the reader that a set I of objects of $\mathcal{A}$ generates the Abelian category $\mathcal{A}$ if for every object $E \in \mathcal{A}$, there is a surjection $\bigoplus A_i \twoheadrightarrow E$ from a set indexed direct sum of (possibly repeated) objects $A_i \in I$ to E.

If X is a scheme with an ample family of line-bundles, and $\mathcal{A}_X$ a quasi-coherent O_X-algebra, then the category $\mathrm{Qcoh}(\mathcal{A}_X)$ of quasi-coherent $\mathcal{A}_X$-modules is an Abelian category with generating set the set $\mathcal{A}_X \otimes L_i^k$, $i = 1, \dots, n$, $k \leq 0$, where $L_1, \dots, L_n$ is a set of line bundles on X with global sections $s_i \in \Gamma(X, L_i)$ such that the non-vanishing loci X_{s_i} are affine and cover X.

For a Grothendieck Abelian category $\mathcal{A}$, write $\mathcal{D}(\mathcal{A})$ for the unbounded derived category of $\mathcal{A}$, that is, the triangulated category $\mathcal{D}(\mathrm{Ch}\,\mathcal{A}, \mathrm{quis})$. This category has small homomorphism sets, by [32, Remark 10.4.5]. Coproducts of complexes are also coproducts in $\mathcal{D}\mathcal{A}$. Therefore, the triangulated category $\mathcal{D}\mathcal{A}$ has all set-indexed coproducts. This applies in particular to the unbounded derived category $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ of quasi-coherent $\mathcal{A}_X$ modules. If $Z \subset X$ is a closed subset with quasi-compact open complement $U = X - Z$, we write $\mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X) \subset \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ for the full triangulated subcategory of those complexes E of quasi-coherent $\mathcal{A}_X$-modules whose restriction $E_{|U}$ to U are acyclic.

Lemma 15 *Let $\mathcal{A}$ be a Grothendieck Abelian category with generating set of objects I. Then, an object E of the triangulated category $\mathcal{D}\mathcal{A}$ is zero iff every map $A[j] \to E$ in $\mathcal{D}\mathcal{A}$ is the zero map for $A \in I$ and $j \in \mathbb{Z}$.*

Proof Let E be an object of the derived category $\mathcal{DA}$ of $\mathcal{A}$ such that every map $A[j] \to E$ in $\mathcal{DA}$ is the zero map for $A \in I$ and $j \in \mathbb{Z}$. We can choose a surjection $\bigoplus_J A_j \twoheadrightarrow \ker(d_0)$ in $\mathcal{A}$ with A_j objects in the generating set I. The inclusion of complexes $\ker(d_0) \to E$ yields a map of complexes $\bigoplus_J A_j \to \ker(d_k) \to E$ which induces a surjective map $\bigoplus_J A_j \twoheadrightarrow \ker(d_0) \twoheadrightarrow H^0 E$ on cohomology. Since every map $\bigoplus_J A_j \to E$ is zero in $\mathcal{DA}$, the induced surjective map $\bigoplus_J A_j \twoheadrightarrow H^0 E$ is the zero map, hence $H^0 E = 0$. The same argument applied to $E[k]$ instead of to E shows that $H^k E = 0$ for all $k \in \mathbb{Z}$. Therefore, E is quasi-isomorphic to the zero complex. □

Next, we recall the concept of a compactly generated triangulated category due to Neeman [20] in the form of [21].

9.5 Compactly generated triangulated categories

Let $\mathcal{T}$ be a triangulated category in which (all set-indexed) coproducts exist. An object A of $\mathcal{T}$ is called *compact* [21, Definition 1.6] if the natural map $\bigoplus_{j\in J} \mathrm{Hom}(A, M_j) \to \mathrm{Hom}(A, \bigoplus_{j\in J} M_j)$ is an isomorphism for any set M_j, $j \in J$, of objects in $\mathcal{T}$. The full subcategory $\mathcal{T}^c \subset \mathcal{T}$ of compact objects is closed under shifts and cones and thus is a triangulated subcategory.

A triangulated category $\mathcal{T}$ is *compactly generated* [21, Definition 1.7] if $\mathcal{T}$ has all set-indexed direct sums, and if there is a set I of compact objects in $\mathcal{T}$ such that an object M of $\mathcal{T}$ is the zero object iff all maps $A \to M$ are the zero map for $A \in I$.

A set I of compact objects in a compactly generated triangulated category $\mathcal{T}$ is called a *generating set* [21, Definition 1.7] if I is closed under shifts and if an object M of $\mathcal{T}$ is the zero object iff all maps $A \to M$ are the zero map for $A \in I$.

The following theorem is due to Neeman [21, Theorem 2.1].

Theorem 12 (Neeman)

(a) *Let $\mathcal{T}$ be a compactly generated triangulated category with generating set of objects I. Then the full triangulated subcategory $\mathcal{T}^c$ of compact objects in $\mathcal{T}$ is an essentially small category which coincides with the smallest idempotent complete triangulated subcategory of $\mathcal{T}$ containing I.*

(b) *Let $\mathcal{R}$ be a compactly generated triangulated category, $S_0 \subset \mathcal{R}^c$ a set of compact objects closed under taking shifts. Let $\mathcal{S} \subset \mathcal{R}$ be the smallest full triangulated subcategory closed under formation of coproducts in $\mathcal{R}$ which contains S_0. Then $\mathcal{S}$ and $\mathcal{R}/\mathcal{S}$ are compactly generated triangulated categories with generating sets S_0 and the image of* (*a set of representatives for the isomorphism classes of objects of*) *$\mathcal{R}^c$ in $\mathcal{R}/\mathcal{S}$. Moreover, the functor $\mathcal{R}^c/\mathcal{S}^c \to \mathcal{R}/\mathcal{S}$ induces an equivalence between the idempotent completion of $\mathcal{R}^c/\mathcal{S}^c$ and the category of compact objects in $\mathcal{R}/\mathcal{S}$.*

(c) *Let $\mathcal{S} \to \mathcal{R}$ be a triangle functor between compactly generated triangulated categories which preserves coproducts and compact objects. Then $\mathcal{S} \to \mathcal{R}$ is an equivalence iff the functor $\mathcal{S}^c \to \mathcal{R}^c$ on compact objects is an equivalence.*

The following two propositions are essentially due to Thomason [30]. We include the proofs here because only the commutative situation is considered in [30], and we need the explicit versions below.

For an exact category $\mathcal{E}$, we will write $\mathcal{D}^b(\mathcal{E})$ for the triangulated category $\mathcal{D}(\mathrm{Ch}^b\,\mathcal{E}, \mathrm{quis})$. Recall that a fully faithful functor $\mathcal{A} \to \mathcal{B}$ of additive categories is called *cofinal* if every object of B is a direct factor of an object of $\mathcal{A}$.

Proposition 8 *Let X be a quasi-compact and quasi-separated scheme which is the union $X = \bigcup_{i=1}^n X_{s_i}$ of open affine non-vanishing loci X_{s_i} of global sections $s_i \in \Gamma(X, L_i)$ of line-bundles L_i, $i = 1, \dots, n$. Let $\mathcal{A}_X$ be a quasi-coherent O_X-algebra. Then the triangulated category $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ is compactly generated by the set of objects $\mathcal{A}_X \otimes L_i^k[j]$ for $k \le 0$, $i = 1, \dots, n$ and $j \in \mathbb{Z}$.*

Moreover, the inclusion $\mathrm{Vect}(\mathcal{A}_X) \subset \mathrm{Qcoh}(\mathcal{A}_X)$ induces a fully faithful triangle functor $\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X) \subset \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ which identifies, up to equivalence, the category $\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X)$ with the full triangulated subcategory $\mathcal{D}^c\mathrm{Qcoh}(\mathcal{A}_X)$ of compact objects in $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$.

Proof In the triangulated category $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$, every strictly perfect complex of $\mathcal{A}_X$ modules is a compact object. To see this, note that for an $\mathcal{A}_X$ vector bundle A and a set M_j, $j \in J$, of quasi-coherent $\mathcal{A}_X$-modules the canonical map of sheaves of homomorphisms $\bigoplus_j \underline{\mathrm{Hom}}_{\mathcal{A}_X}(A, M_j) \to \underline{\mathrm{Hom}}_{\mathcal{A}_X}(A, \bigoplus_j M_j)$ is an isomorphism since this can be checked on an affine open cover of X where the statement is clear. Taking global sections, we obtain an isomorphism $\bigoplus_j \mathrm{Hom}_{\mathcal{A}_X}(A, M_j) \stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{A}_X}(A, \bigoplus M_j)$. This isomorphism extends to an isomorphism of homomorphism sets of complexes of $\mathcal{A}_X$-modules for A a strictly perfect complex and M an arbitrary complex of quasi-coherent $\mathcal{A}_X$-modules. For such complexes, the isomorphism induces an isomorphism

$$\bigoplus_j \mathrm{Hom}_{\mathcal{K}\mathrm{Qcoh}(\mathcal{A}_X)}(A, M_j) \stackrel{\cong}{\to} \mathrm{Hom}_{\mathcal{K}\mathrm{Qcoh}(\mathcal{A}_X)}\left(A, \bigoplus_j M_j\right) \tag{26}$$

of homomorphism sets in the homotopy category $\mathcal{K}\mathrm{Qcoh}(\mathcal{A}_X)$ of chain complexes of quasi-coherent $\mathcal{A}_X$-modules. It follows from Lemma 14 with $U = X$ and $\mathcal{A} = \mathrm{sPerf}(\mathcal{A}_X)$ that maps in $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ from a strictly perfect complex A to an arbitrary complex M of quasi-coherent $\mathcal{A}_X$-modules can be computed as the filtered colimit

$$\operatorname{colim}_{B \xrightarrow{\sim} A} \operatorname{Hom}_{\mathcal{K}\mathrm{Qcoh}(\mathcal{A}_X)}(B, M) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)}(A, M)$$

of homomorphism sets in $\mathcal{K}\mathrm{Qcoh}(\mathcal{A}_X)$ where the indexing category is the left filtering category of homotopy classes of quasi-isomorphisms $B \xrightarrow{\sim} A$ of strictly perfect complexes with target A. Taking the colimit over this filtering category of the isomorphism (26) yields the isomorphism which proves that A is compact in $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$.

Since the set $\mathcal{A}_X \otimes L_i^k$, $i = 1, \dots, n$, $k \leq 0$ is a set of generators for the Grothendieck Abelian category $\mathrm{Qcoh}(\mathcal{A}_X)$ all of which are compact in the derived category $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$, Lemma 15 shows that $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ is a compactly generated triangulated category with generating set $\mathcal{A}_X \otimes L_i^k[j]$, $i = 1, \dots, n$, $k \leq 0$, $j \in \mathbb{Z}$.

The inclusion $\mathrm{Vect}(\mathcal{A}_X) \subset \mathrm{Qcoh}(\mathcal{A}_X)$ of vector bundles into all quasi-coherent $\mathcal{A}_X$-modules induces a triangle functor $\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X) \to \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ which is fully faithful, by the existence of an ample family of line bundles and the criterion in [15, 12.1]. Since the exact category $\mathrm{Vect}(\mathcal{A}_X)$ is idempotent complete, its bounded derived category $\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X)$ is also idempotent complete [2, Theorem 2.8]. By Neeman's Theorem 12 (a), the inclusion $\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X) \to \mathcal{D}^c\mathrm{Qcoh}(\mathcal{A}_X)$ is an equivalence. □

9.6 Reminder on Rj_*

Let X be a scheme with an ample family of line bundles, and let $j : U \hookrightarrow X$ be an open immersion from a quasi-compact open subset U to X. We recall one possible construction of the right-derived functor $Rj_* : \mathcal{D}\mathrm{Qcoh}(U) \to \mathcal{D}\mathrm{Qcoh}(X)$ of $j_* : \mathrm{Qcoh}(U) \to \mathrm{Qcoh}(X)$. To that end, choose a finite cover $\mathcal{U} = \{U_0, \dots, U_n\}$ of U such that the inclusion of all finite intersections $U_{i_0} \cap \cdots \cap U_{i_k} \subset X$ into X are affine maps, $i_0, \dots, i_k \in \{0, \dots, n\}$. For instance, we can take as $\mathcal{U}$ an open cover of U by a finite number of non-vanishing loci X_{s_i} associated with a set of line bundles L_i on X and global sections $s_i \in \Gamma(X, L_i)$, $i = 0, \dots, n$. For a $k+1$-tuple $\underline{i} = (i_0, \dots, i_k)$, set $U_{\underline{i}} = U_{i_0} \cap \cdots \cap U_{i_k}$ and write $j_{\underline{i}} : U_{\underline{i}} \subset U$ for the corresponding open immersion.

For a quasi-coherent $\mathcal{A}_U$ module F, consider the sheafified Čech complex $\check{C}(\mathcal{U}, F)$ associated with this covering. In degree k it is the quasi-coherent $\mathcal{A}_U$-module

$$\check{C}(\mathcal{U}, F)_k = \bigoplus_{\underline{i}} j_{\underline{i},*} j_{\underline{i}}^* F$$

where the indexing set is taken over all $k+1$-tuples $\underline{i} = (i_0, \dots, i_k)$ such that $0 \leq i_0 < \cdots < i_k \leq n$. The differential $d_k : \check{C}(\mathcal{U}, F)_k \to \check{C}(\mathcal{U}, F)_{k+1}$ for the component $\underline{i} = (i_0, \ldots, i_{k+1})$ is given by the formula

$$(d_k(x))_{\underline{i}} = \sum_{l=0}^{k+1} (-1)^l \, j_{\underline{i},*} j_{\underline{i}}^* \, x_{(i_0,\ldots,\hat{i}_l,\ldots,i_{k+1})}.$$

Note that the complex $\check{C}(\mathcal{U}, F)$ is concentrated in degrees $0, \ldots, n$.

The units of adjunction $F \to j_{i*} j_i^* F$ define a map $F \to \check{C}(\mathcal{U}, F)_0 = \bigoplus_{i=0}^n j_{i*} j_i^* F$ into the degree zero part of the Čech complex with $d_0(F) = 0$, and thus a map of complexes of quasi-coherent $\mathcal{A}_U$-modules $\lambda_F : F \to \check{C}(\mathcal{U}, F)$. This map is a quasi-isomorphism for any quasi-coherent $\mathcal{A}_U$-module F as can be checked by restricting the map to the open subsets U_i of the cover of U. Since, by assumption, for every $k+1$-tuple, $\underline{i} = (i_0, \ldots, i_k)$, the open inclusion $j \circ j_{\underline{i}} : U_{\underline{i}} \subset X$ is an affine map, the functor

$$j_* \check{C}(\mathcal{U}) : \mathrm{Qcoh}(\mathcal{A}_U) \to \mathrm{ChQcoh}(\mathcal{A}_X) : F \mapsto j_* \check{C}(\mathcal{U}, F)$$

is exact. Taking total complexes, this functor extends to a functor on all complexes

$$\begin{aligned} j_* \mathrm{Tot} \check{C}(\mathcal{U}) : \quad & \mathrm{ChQcoh}(\mathcal{A}_U) \to \mathrm{ChQcoh}(\mathcal{A}_X) : \\ & F \qquad \mapsto j_* \mathrm{Tot} \check{C}(\mathcal{U}, F) = \mathrm{Tot}\, j_* \check{C}(\mathcal{U}, F). \end{aligned}$$

This functor preserves quasi-isomorphisms as it is exact and sends acyclics to acyclics. It is equipped with a natural quasi-isomorphism

$$\lambda_F : F \xrightarrow{\sim} \mathrm{Tot} \check{C}(\mathcal{U}, F). \tag{27}$$

Finally, the pair $(j_* \mathrm{Tot} \check{C}(\mathcal{U}), j_* \lambda)$ represents the right derived functor Rj_* of j_*, that is,

$$Rj_* = j_* \mathrm{Tot} \check{C}(\mathcal{U}) : \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_U) \to \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X).$$

Lemma 16 *Let X be a scheme with an ample family of line bundles, $j : U \subset X$ a quasi-compact open subscheme, $Z \subset X$ a closed subset with quasi-compact open complement $X - Z$ such that $Z \subset U$, then we have an equivalence of triangulated categories*

$$j^* : \mathcal{D}_Z \mathrm{Qcoh}(\mathcal{A}_X) \xrightarrow{\simeq} \mathcal{D}_Z \mathrm{Qcoh}(\mathcal{A}_U)$$

with inverse the functor Rj_*.

Proof We first check that Rj_* preserves cohomological support. Denote by $j_U : U - Z \subset U$, $j_X : X - Z \subset X$ and $j_Z : U - Z \subset X - Z$ the corresponding open immersions, and note that the canonical map $j_X^* j_* M \to j_{Z*} j_U^* M$ is an isomorphism for every quasi-coherent $\mathcal{A}_U$-module M. By the existence of an ample family of line bundles on X, we can choose a finite open cover $\mathcal{U} = \{U_0, \dots, U_n\}$ of U such that all inclusions $U_{i_0} \cap \cdots \cap U_{i_k} \subset X$ are affine maps. For a complex F of quasi-coherent $\mathcal{A}_U$-modules, we have

$$j_X^* Rj_* F = j_X^* j_* \mathrm{Tot}\check{C}(\mathcal{U}, F) = j_{Z*} j_U^* \mathrm{Tot}\check{C}(\mathcal{U}, F) = j_{Z*} \mathrm{Tot}\check{C}(\mathcal{U} - Z, j_U^* F)$$

where $\mathcal{U} - Z$ is the cover $\{U_0 - Z, \dots, U_n - Z\}$ of $U - Z$. As pull-backs of affine maps, all inclusions $(U_{i_0} - Z) \cap \cdots \cap (U_{i_k} - Z) \subset X - Z$ are also affine maps. Therefore, the functor $j_{Z*}\mathrm{Tot}\check{C}(\mathcal{U} - Z)$ represents Rj_{Z*}, and we obtain a natural isomorphism of functors

$$j_X^* \circ Rj_* \xrightarrow{\cong} Rj_{Z*} \circ j_U^*.$$

In particular, Rj_* sends $\mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_U)$ into $\mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X)$.

We prove the lemma. The functor $j^* Rj_* = j^* j_* \mathrm{Tot}\check{C}(\mathcal{U}) = \mathrm{Tot}\check{C}(\mathcal{U})$ is naturally quasi-isomorphic to the identity functor via the map (27). Furthermore, the unit of adjunction $F \to Rj_* \circ j^*(F)$, which is adjoint to the map (27) applied to j^*F, is a quasi-isomorphism for $F \in \mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X)$ since both complexes have cohomological support in Z. Therefore, we can check this property by restricting the map to U where it is the quasi-isomorphism (27). □

Proposition 9 *Let X be a scheme with an ample family of line-bundles. Let $Z \subset X$ be a closed subscheme with quasi-compact open complement $X - Z$. Let $j : U \subset X$ be a quasi-compact open subscheme. Let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras. Then the following hold.*

(a) *Restriction of vector bundles induces a fully faithful triangle functor*

$$j^* : \mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, U\text{-quis}) \hookrightarrow \mathcal{D}\mathrm{sPerf}(\mathcal{A}_U \text{ on } Z \cap U, U\text{-quis}).$$

(b) *The triangulated category $\mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X)$ is compactly generated and the triangle functor $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis}) \to \mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X)$ induces an equivalence of $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis})$ with the full triangulated subcategory of compact objects in $\mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_X)$.*

(c) *If $Z \subset U$, then restriction of vector bundles induces an equivalence of triangulated categories*

$$j^* : \mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis}) \xrightarrow{\simeq} \mathcal{D}\mathrm{sPerf}(\mathcal{A}_U \text{ on } Z, \mathrm{quis}).$$

(d) *The triangle functor in (a) is cofinal.*

Proof The functor in (a) is clearly conservative. It is full by the following argument. Let A and B be strictly perfect complexes of $\mathcal{A}_X$-modules with support in Z, and let $j^*A \stackrel{\sim}{\leftarrow} E \to j^*B$ be a diagram in $\mathrm{sPerf}(\mathcal{A}_U \text{ on } U \cap Z)$ representing a map $f : j^*A \to j^*B$ in $\mathcal{D}^c(\mathcal{A}_U \text{ on } Z \cap U, U\text{-quis})$ where $E \stackrel{\sim}{\to} j^*A$ is a U-quasi-isomorphism. Let M be the pull-back of $j_*E \to j_*j^*A$ and the U-isomorphism $A \to j_*j^*A$. The induced maps $M \to j_*E$ and $M \to A$ are U-isomorphism and U-quasi-isomorphism, respectively. By Lemma 14 with $\mathcal{A} = \mathrm{sPerf}(\mathcal{A}_X \text{ on } Z)$, there is a $A_0 \in \mathcal{A}$ and a U-quasi-isomorphism $A_0 \to M$. By Lemma 13, there is a map $A_0 \otimes K(s^n) \to B$ such that the two maps $A_0 \otimes K(s^n) \to A_0 \to M \to j_*E \to j_*j^*B$ and $A_0 \otimes K(s^n) \to B \to j_*j^*B$ coincide. If follows that the map $f : j^*A \to j^*B$ is the image of the map in $\mathcal{D}^c(\mathcal{A}_X \text{ on } Z, U\text{-quis})$ which is represented by the diagram $A \stackrel{\sim}{\leftarrow} A_0 \otimes K(s^n) \to B$. Therefore, the functor in (a) is full. Any conservative and full triangle functor is faithful, hence the triangle functor in (a) is fully faithful.

It follows from Proposition 8 and Neeman's Theorem 12(a) that the functor in (a) is cofinal for $Z = X$ since in this case both categories contain as a cofinal subcategory the triangulated category generated by $\mathcal{A}_X \otimes L$ where L runs through the line bundles on X. This shows part (d) when $Z = X$.

In order to prove (b), write $\mathcal{R}$ for the compactly generated triangulated category $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X)$ with category of compact objects $\mathcal{R}^c = \mathcal{D}\mathrm{sPerf}(\mathcal{A}_X, \mathrm{quis})$; see Proposition 8. Let $\mathcal{S} \subset \mathcal{R}$ be the full triangulated subcategory closed under the formation of coproducts in $\mathcal{R}$ which is generated by the set $S = \mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis}) \subset \mathcal{R}^c$ of compact objects. By part (d) for $Z = X$ proved above and Proposition 8, we have a cofinal inclusion $\mathcal{R}^c/\mathcal{S}^c \to \mathcal{D}^c\mathrm{Qcoh}(\mathcal{A}_U)$. By Neeman's Theorem 12(b) and (c), this implies that the functor $\mathcal{R}/\mathcal{S} \to \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_U)$ is an equivalence. In particular $\mathcal{S}$ is the kernel category of the functor $\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X) \to \mathcal{D}\mathrm{Qcoh}(\mathcal{A}_U)$. Therefore, $\mathcal{S} = \mathcal{D}_Z\mathrm{Qcoh}(\mathcal{A}_U)$ is compactly generated by $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis})$. Since the triangulated category $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X, \mathrm{quis}) = \mathcal{D}^c\mathrm{Qcoh}(\mathcal{A}_X)$ is idempotent complete, its epaisse subcategory $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis})$ is also idempotent complete. Therefore, we have the identification $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, \mathrm{quis}) = \mathcal{D}^c_Z\mathrm{Qcoh}(\mathcal{A}_X)$, by Neeman's Theorem 12(a).

In view of (b), the functor j^* in (c) is the restriction to compact objects of the equivalence of Lemma 16. It is therefore also an equivalence.

For the proof of (d), we simplify notation by writing $\mathcal{D}^c(\mathcal{A}_X \text{ on } Z, w)$ for the triangulated category $\mathcal{D}\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z, w)$. Let $V = U \cup (X - Z)$ and consider the commutative diagram of triangulated categories

$$\begin{array}{ccccc}
\mathcal{D}^c(\mathcal{A}_X \text{ on } Z, V\text{-quis}) & \longrightarrow & \mathcal{D}^c(\mathcal{A}_X, V\text{-quis}) & \longrightarrow & \mathcal{D}^c(\mathcal{A}_X, X - Z\text{-quis}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}^c(\mathcal{A}_V \text{ on } Z \cap U, V\text{-quis}) & \longrightarrow & \mathcal{D}^c(\mathcal{A}_V, V\text{-quis}) & \longrightarrow & \mathcal{D}^c(\mathcal{A}_V, X - Z\text{-quis})
\end{array}$$

in which all vertical functors are fully faithful, by (a). The middle and the right vertical functors are cofinal since all four categories have as cofinal subcategory the triangulated category generated by $\mathcal{A}_X \otimes L$ where L runs through the line-bundles on X, by Proposition 8. Therefore, the right two vertical functors are equivalences after idempotent completion. Since the two left triangulated categories are the "kernel categories" of the two right horizontal functors, and this property is preserved under idempotent completion, the left vertical functor is also an equivalence after idempotent completion. Thus, the left vertical functor is cofinal.

The functor $\mathcal{D}^c(\mathcal{A}_X \text{ on } Z, U\text{-quis}) \hookrightarrow \mathcal{D}^c(\mathcal{A}_U \text{ on } Z \cap U, U\text{-quis})$ in (a) can be identified with the left vertical functor in the diagram since U-quasi-isomorphisms are V-quasi-isomorphisms for complexes of A_X-modules cohomologically supported in Z, and since the functor $\mathcal{D}^c(\mathcal{A}_V \text{ on } Z \cap U, V\text{-quis}) \to \mathcal{D}^c(\mathcal{A}_U \text{ on } Z \cap U, U\text{-quis})$ is an equivalence, by (b). □

Corollary 2 *Let X be a scheme which has an ample family of line-bundles, let $Z \subset X$ be a closed subset with quasi-compact open complement $X - Z$, and let $j : U \subset X$ be a quasi-compact open subscheme. Let M be a complex of quasi-coherent $\mathcal{A}_X$-modules such that j^*M is strictly perfect on U and has cohomological support in $Z \cap U$. If the class $[j^*M] \in K_0(\mathcal{A}_U \text{ on } Z \cap U)$ is in the image of the map $K_0(\mathcal{A}_X \text{ on } Z) \to K_0(\mathcal{A}_U \text{ on } Z \cap U)$, then there is a U-quasi-isomorphism*

$$A \xrightarrow[U\text{-quis}]{\exists} M$$

with A a strictly perfect complex of $\mathcal{A}_X$-modules which has cohomological support in Z.

Proof We start with a standard fact about K_0 of triangulated categories. Let $\mathcal{T}_0 \subset \mathcal{T}_1$ be a (fully faithful and) cofinal functor between triangulated categories. Then an object T of $\mathcal{T}_1$ is isomorphic to an object of $\mathcal{T}_0$ if and only if its class $[T] \in K_0(\mathcal{T}_1)$ is in the image of $K_0(\mathcal{T}_0) \to K_0(\mathcal{T}_1)$. This is because the cokernel of $K_0(\mathcal{T}_0) \to K_0(\mathcal{T}_1)$ can be identified with the quotient monoid of the Abelian monoid of isomorphism classes of objects in $\mathcal{T}_1$ under direct sum modulo the submonoid of isomorphism classes of objects in $\mathcal{T}_0$. Therefore, an object of $\mathcal{T}_1$ defines the zero class in the cokernel if and only if it is stably in $\mathcal{T}_0$. But for triangulated categories, an object is stably in $\mathcal{T}_0$ iff it is isomorphic to an object in $\mathcal{T}_0$.

For the proof of the corollary, we apply this argument to the inclusion in Proposition 9(a) which is cofinal, by Proposition 9(d). We see that j^*M is isomorphic in $\mathcal{D}^c(\mathcal{A}_U \text{ on } Z \cap U, U\text{-quis})$ to an object j^*B, where B is a perfect complex of $\mathcal{A}_X$-modules with cohomological support in Z. It follows that there is a zig-zag of U-quasi-isomorphisms $j^*M \overset{\sim}{\leftarrow} F \overset{\sim}{\rightarrow} j^*B$. Let P be the pull-back of $j_*F \to j_*j^*B$ along the U-isomorphism $B \to j_*j^*B$. Then $P \to j_*F$ is a U-isomorphism, and it follows that $P \to B$ is a U-quasi-isomorphism. By Lemma 14 with $\mathcal{A}$ the subcategory of those strictly perfect complexes which are cohomologically supported in Z, there is a U-quasi-isomorphism $B' \to P$ with B' strictly perfect and cohomologically supported in Z. Since X has an ample family of line-bundles and U is quasi-compact, we can choose line-bundles L_i and global sections $s_i \in \Gamma(X, L_i)$, $i = 1, \dots, l$ such that the set of non-vanishing loci X_{s_i}, $i = 1, \dots, l$ is an affine open cover of U. By Lemma 13, we can find an l-tuple of negative integers n such that the composition of U-quasi-isomorphisms $A = B' \otimes K(s) \to B' \to j_*j^*M$ lifts to M. □

Proposition 10 *Let X be a scheme which has an ample family of line bundles, let $Z \subset X$ be a closed subscheme with quasi-compact open complement, and let $j : U \subset X$ be a quasi-compact open subscheme. Let $\mathcal{A}_X$ be a quasi-coherent O_X-algebra with involution. Then for any line-bundle L on X and any integer $n \in \mathbb{Z}$, restriction of $\mathcal{A}_X$ vector bundles to U defines non-singular exact form functors*

$$(\mathrm{sPerf}(\mathcal{A}_X \ on\ Z),\ U\text{-quis},\ \sharp^n_L)$$
$$\longrightarrow (\mathrm{sPerf}(\mathcal{A}_U \ on\ U \cap Z),\ U\text{-quis},\ \sharp^n_{j^*L})$$

which induce isomorphisms on higher Grothendieck-Witt groups GW_i for $i \geq 1$ and a monomorphism for GW_0.

If, moreover, we have $Z \subset U$, then the form functors induce isomorphisms for all higher Grothendieck-Witt groups GW_i where $i \geq 0$.

Proof Let $\mathrm{sPerf}_{K_0}(\mathcal{A}_U$ on $U \cap Z) \subset \mathrm{sPerf}(\mathcal{A}_U$ on $U \cap Z)$ be the full subcategory of those strictly perfect complexes of $\mathcal{A}_U$-modules with cohomological support in $U \cap Z$ which have class in the image of $K_0(\mathcal{A}_X$ on $Z) \to K_0(\mathcal{A}_U$ on $Z \cap U)$. By the Cofinality Theorem 7, the duality preserving inclusion

$$\mathrm{sPerf}_{K_0}(\mathcal{A}_U \text{ on } U \cap Z) \to \mathrm{sPerf}(\mathcal{A}_U \text{ on } U \cap Z) \tag{28}$$

of exact categories with weak equivalences the U-quasi-isomorphisms and duality $\sharp^n_{j^*L}$ induces maps on higher Grothendieck-Witt groups GW_i which are isomorphisms for $i \geq 1$ and a monomorphism for $i = 0$. Restriction of vector-bundles defines a non-singular exact form functor

$$(\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z),\ U\text{-quis}) \to (\mathrm{sPerf}_{K_0}(\mathcal{A}_U \text{ on } U \cap Z),\ U\text{-quis}) \tag{29}$$

which induces a homotopy equivalence of Grothendieck-Witt spaces by Theorem 8, where (c) of Theorem 8 follows from Corollary 2 and Lemma 8; parts 8(e) and (f) are proved in Lemma 13; the remaining hypothesis of Theorem 8 being trivially satisfied.

If $Z \subset U$, then $K_0(\mathcal{A}_X \text{ on } Z) = K_0(\mathcal{A}_U \text{ on } Z \cap U)$, by Proposition 9(c). Therefore, (28) is the identity inclusion, and the map (29), which induces a homotopy equivalence of Grothendieck-Witt spaces, is the map in the proposition. □

Proof of Theorem 10 By the Change-of-weak-equivalence Theorem 6, the sequence of exact categories with weak equivalences and duality

$$\begin{aligned}(\mathrm{sPerf}(\mathcal{A}_X \text{ on } Z), \text{quis}, \sharp^n_L) &\to (\mathrm{sPerf}(\mathcal{A}_X), \text{quis}, \sharp^n_L)\\ &\to (\mathrm{sPerf}(\mathcal{A}_X), U\text{-quis}, \sharp^n_L)\end{aligned}$$

induces a homotopy fibration of Grothendieck-Witt spaces. By Proposition 10, the form functor

$$(\mathrm{sPerf}(\mathcal{A}_X), U\text{-quis}, \sharp^n_L) \to (\mathrm{sPerf}(\mathcal{A}_U), \text{quis}, \sharp^n_{j^*L})$$

induces isomorphisms on GW_i for $i \geq 1$ and a monomorphism for $i = 0$. □

Proof of Theorem 11 The theorem is a special case of Proposition 10. □

10 Extension to negative Grothendieck-Witt groups

For an open subscheme $U \subset X$, the restriction map $GW_0(X) \to GW_0(U)$ is not surjective, in general, not even if X is regular. The purpose of this section is to extend the long exact sequence associated with the homotopy fibration of Theorem 10 to negative degrees. Theorems 10 and 11 will be extended to a fibration and a weak equivalence of non-connective spectra.

10.1 Cone and suspension of $\mathcal{A}_X$

The *cone ring* is the ring C of infinite matrices $(a_{i,j})_{i,j\in\mathbb{N}}$ with coefficients $a_{i,j}$ in $\mathbb{Z}$ for which each row and each column has only finitely many non-zero entries. Transposition of matrices ${}^t(a_{i,j}) = (a_{j,i})$ makes C into a ring with involution. As a $\mathbb{Z}$-module C is torsion free, hence flat.

The *suspension ring* S is the factor ring of C by the two sided ideal $M_\infty \subset C$ of those matrices which have only finitely many non-zero entries. Transposition also makes S into a ring with involution such that the quotient map $C \twoheadrightarrow S$ is a map of rings with involution. For another description of the suspension ring S, consider the matrices $e_n \in C$, $n \in \mathbb{N}$, with entries $(e_n)_{i,j} = 1$ for $i = j \geq n$ and zero otherwise. They are symmetric idempotents, i.e., ${}^t e_n = e_n = e_n^2$, and they form a multiplicative subset of C which satisfies the Øre condition, that is, the multiplicative subset satisfies the axioms for a calculus of fractions. One checks that the quotient map $C \twoheadrightarrow S$ identifies the suspension ring S with the localization of the cone ring C with respect to the elements $e_n \in C$, $n \in \mathbb{N}$. In particular, the suspension ring S is also a flat $\mathbb{Z}$-module.

Let X be a quasi-compact and quasi-separated scheme. For a quasi-coherent sheaf $\mathcal{A}_X$ of O_X-algebras, write $C\mathcal{A}_X$ and $S\mathcal{A}_X$ for the quasi-coherent sheaves of O_X-algebras associated with the presheaves $C \otimes_{\mathbb{Z}} \mathcal{A}_X$ and $S \otimes_Z \mathcal{A}_X$. On quasi-compact open subsets $U \subset X$, we have $(C\mathcal{A}_X)(U) = C \otimes_{\mathbb{Z}} A_X(U)$ and $S\mathcal{A}_X = S \otimes_{\mathbb{Z}} A_X(U)$, by flatness of C and S. If $\mathcal{A}_X$ is a sheaf of algebras with involutions, then the involutions on C and on S make $C\mathcal{A}_X$ and $S\mathcal{A}_X$ into sheaves of O_X-algebras with involution.

Let $\epsilon = 1 - e_1 \in C$ be the symmetric idempotent with entries 1 at (0, 0) and zero otherwise. The image $C\epsilon$ of the right multiplication map $\times\epsilon : C \to C$ is a finitely generated projective left C-module. It is equipped with a symmetric form $\varphi : C\epsilon \otimes_{\mathbb{Z}} (C\epsilon)^{op} \to C : x \otimes y^{op} \mapsto x \cdot {}^t y$. The idempotent ϵ makes $(C\epsilon, \varphi)$ into a direct factor of the unit symmetric form (C, μ); see Sect. 7.8(a). Therefore, tensor product $(C\epsilon, \varphi) \otimes_{\mathbb{Z}} ?$ defines a non-singular exact form functor

$$\iota : (\mathrm{sPerf}(\mathcal{A}_X), \sharp^n_L) \to (\mathrm{sPerf}(C\mathcal{A}_X), \sharp^n_L) : V \mapsto C\epsilon \otimes_{\mathbb{Z}} V.$$

Since S is a flat C-algebra, the quotient map $C \to S$ induces an exact functor $\rho : C\mathcal{A}_X\text{-Mod} \to S\mathcal{A}_X\text{-Mod} : M \mapsto S \otimes_C M$ on categories of modules which sends vector bundles to vector bundles. The two functors ι and ρ yield a sequence of non-singular exact form functors

$$(\mathrm{sPerf}(\mathcal{A}_X), \sharp^n_L) \xrightarrow{\iota} (\mathrm{sPerf}(C\mathcal{A}_X), \sharp^n_L) \xrightarrow{\rho} (\mathrm{sPerf}(S\mathcal{A}_X), \sharp^n_L). \tag{30}$$

The functors satisfy $\rho \circ \iota = 0$ because $S \otimes_C C\epsilon = \mathrm{im}(\times\epsilon : S \to S) = 0$ as $0 = \epsilon \in S$.

The following theorem will allow us to extend the results of Sect. 9 to negative Grothendieck-Witt groups.

Theorem 13 *Let X be a scheme with an ample family of line-bundles, let $Z \subset X$ be a closed subset with quasi-compact open complement $X - Z$, and let $\mathcal{A}_X$ be a quasi-coherent O_X-algebra with involution. Then for any line bundle L on X, and any integer $n \in \mathbb{Z}$, the sequence (30) induces a homotopy fibration of Grothendieck-Witt spaces with contractible total space*

$$GW^n(\mathcal{A}_X \text{ on } Z, L) \to GW^n(C\mathcal{A}_X \text{ on } Z, L) \to GW^n(S\mathcal{A}_X \text{ on } Z, L).$$

The proof of Theorem 13 will occupy us until Definition 8.

Lemma 17 *The functor ι in (30) is fully faithful.*

Proof The image ϵC of the left multiplication map $\epsilon\times : C \to C$ is a right C-module. We have a $\mathbb{Z}$-bimodule isomorphism $\eta : \mathbb{Z} \to \epsilon C \otimes_C C\epsilon : 1 \mapsto \epsilon \otimes_C \epsilon = 1 \otimes \epsilon = \epsilon \otimes 1$ and a C-bimodule map $\mu : C\epsilon \otimes_{\mathbb{Z}} \epsilon C \to C : A\epsilon \otimes \epsilon B \mapsto A\epsilon B$ such that the compositions

$$C\epsilon \cong C\epsilon \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{id\otimes\eta} C\epsilon \otimes_{\mathbb{Z}} \epsilon C \otimes_C C\epsilon \xrightarrow{\mu\otimes id} C \otimes_C C\epsilon \cong C\epsilon \quad \text{and}$$

$$\epsilon C \cong \mathbb{Z} \otimes_{\mathbb{Z}} \epsilon C \xrightarrow{\eta\otimes id} \epsilon C \otimes_C C\epsilon \otimes_{\mathbb{Z}} \epsilon C \xrightarrow{id\otimes\mu} \epsilon C \otimes_C C \cong \epsilon C$$

are the identity maps. It follows that η and μ define unit and counit of an adjunction between the functors $\mathcal{A}_X$-Mod $\to C\mathcal{A}_X$-Mod : $M \mapsto C\epsilon \otimes_{\mathbb{Z}} M$ and $C\mathcal{A}_X$-Mod $\to \mathcal{A}_X$-Mod : $N \mapsto \epsilon C \otimes_C N$. Since the unit η is an isomorphism, the first functor is fully faithful. In particular, ι is fully faithful. □

Proposition 11 *The sequence of triangulated categories*

$$\mathcal{D}^b\mathrm{Vect}(\mathcal{A}_X) \xrightarrow{\iota} \mathcal{D}^b\mathrm{Vect}(C\mathcal{A}_X) \xrightarrow{\rho} \mathcal{D}^b\mathrm{Vect}(S\mathcal{A}_X)$$

is exact up to direct factors.

Proof The multiplication map $\mu : C\epsilon \otimes_{\mathbb{Z}} \epsilon C \to C$ factors through $M_\infty \subset C$ and induces an isomorphism $\mu : C\epsilon \otimes_{\mathbb{Z}} \epsilon C \to M_\infty$ (it is a filtered colimit of isomorphisms of finitely generated free $\mathbb{Z}$-modules). The exact functors

$$\mathrm{Qcoh}(\mathcal{A}_X) \xrightarrow{\iota} \mathrm{Qcoh}(C\mathcal{A}_X) \xrightarrow{\rho} \mathrm{Qcoh}(S\mathcal{A}_X)$$

have exact right adjoints $\kappa : \mathrm{Qcoh}(C\mathcal{A}_X) \to \mathrm{Qcoh}(\mathcal{A}_X) : M \mapsto \epsilon C \otimes_C M$ and $\sigma : \mathrm{Qcoh}(S\mathcal{A}_X) \to \mathrm{Qcoh}(C\mathcal{A}_X) : N \mapsto N$ such that for a left $C\mathcal{A}_X$-module M the adjuntion maps $\iota\kappa \to id$ and $id \to \sigma\rho$ are part of a functorial exact sequence

$$0 \to \iota\kappa M \to M \to \sigma\rho M \to 0 \tag{31}$$

which is the tensor product (over C) of M with the exact sequence of flat C-modules $0 \to M_\infty \to C \to S \to 0$. It follows that the sequence of triangulated categories

$$\mathcal{D}\mathrm{Qcoh}(\mathcal{A}_X) \xrightarrow{\iota} \mathcal{D}\mathrm{Qcoh}(C\mathcal{A}_X) \xrightarrow{\rho} \mathcal{D}\mathrm{Qcoh}(S\mathcal{A}_X)$$

is exact as κ and ρ induce right adjoint functors on derived categories, and (31) induces a functorial distinguished triangle for every object of

$\mathcal{D}\mathrm{Qcoh}(C\mathcal{A}_X)$. By Proposition 8, these triangulated categories are compactly generated. Since ι and ρ preserve compact objects, the associated sequence of compact objects—which is the sequence in Proposition 11—is exact up to factors, by Theorem 12. □

Let $\mathrm{sPerf}_S(C\mathcal{A}_X) \subset \mathrm{sPerf}(C\mathcal{A}_X)$ be the full subcategory of those complexes V for which $S \otimes_C V$ is acyclic. This subcategory is closed under the involution $\sharp^n_L$. Therefore, $\mathrm{sPerf}_S(C\mathcal{A}_X)$ inherits the structure of an exact category with weak equivalences and duality from $\mathrm{sPerf}(C\mathcal{A}_X)$. Since $\rho\iota = 0$, ι induces a non-singular exact form functor $\iota : \mathrm{sPerf}(\mathcal{A}_X) \to \mathrm{sPerf}_S(C\mathcal{A}_X)$.

Proposition 12 *For any line bundle L on X, and any $n \in \mathbb{Z}$, the functor ι induces a homotopy equivalence*

$$GW^n(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis}, L) \xrightarrow{\sim} GW^n(\mathrm{sPerf}_S(C\mathcal{A}_X), \mathrm{quis}, L).$$

Proof The proof is a consequence of Theorem 8 (or of Lemma 9). Since ι is fully faithful, conditions (e) and (f) are satisfied. Since ι induces a fully faithful functor on derived categories, by Proposition 11, condition (b) is also satisfied. The only non-trivial condition to check is (c). By Lemma 8, we only need to show that for every $M \in \mathrm{sPerf}_S(C\mathcal{A}_X)$ there is an $A \in \mathrm{sPerf}(\mathcal{A}_X)$ and a quasi-isomorphism $C\epsilon \otimes_{\mathbb{Z}} A \to M$. Let M be a strictly perfect complex of $C\mathcal{A}_X$ modules with $S \otimes_C M$ acyclic. By Proposition 11, there is a zigzag of quasi-isomorphisms $C\epsilon \otimes_{\mathbb{Z}} B \leftarrow N \to M$ in $\mathrm{sPerf}_S(C\mathcal{A}_X)$ with $B \in \mathrm{sPerf}(\mathcal{A}_X)$. Since $N \in \mathrm{sPerf}_S(C\mathcal{A}_X)$, Proposition 11 implies that the counit of adjunction $C\epsilon \otimes_{\mathbb{Z}} \epsilon C \otimes_C N \to N$ is a quasi-isomorphism. We apply Lemma 14 with $C\mathcal{A}_X$ in place of $\mathcal{A}_X$ and $U = X$, $\mathcal{A} = \mathrm{sPerf}(\mathcal{A}_X)$ to the quasi-isomorphism $\epsilon C \otimes_C N \to \epsilon C \otimes_C C\epsilon \otimes_{\mathbb{Z}} B \cong B$, and obtain a strictly perfect complex A of $\mathcal{A}_X$-modules and a quasi-isomorphism $A \to \epsilon C \otimes_C N$. Finally, the composition $C\epsilon \otimes_{\mathbb{Z}} A \to C\epsilon \otimes_{\mathbb{Z}} \epsilon C \otimes_C N = M_\infty \otimes_C N \to N \to M$ is a quasi-isomorphism. □

For a quasi-coherent O_X-algebra $\mathcal{A}_X$, call an $\mathcal{A}_X$-module M *quasi-free* if it is isomorphic to a finite direct sum $\bigoplus_i \mathcal{A}_X \otimes L_i$ of A_X-modules of the form $\mathcal{A}_X \otimes L_i$ for some line bundles L_i on X. Note that a quasi-free $\mathcal{A}_X$-module is a vector bundle.

Lemma 18 *Let X be a quasi-compact and quasi-separated scheme, and let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras. Let A, M be quasi-coherent $C\mathcal{A}_X$-modules with A quasi-free. Then the following hold.*

(a) *For every map $f : \rho A \to \rho M$ of $S\mathcal{A}_X$-modules, there are maps $s : B \to A$ and $g : B \to M$ of $C\mathcal{A}_X$-modules with B quasi-free such that $f \circ \rho(s) = \rho(g)$ and $\rho(s)$ an isomorphism.*

(b) *For any two maps* $f, g : A \to M$ *of* $C\mathcal{A}_X$*-modules such that* $\rho(f) = \rho(g)$ *there is a map* $s : B \to A$ *of quasi-free* $C\mathcal{A}_X$*-modules such that* $f \circ s = g \circ s$ *and* $\rho(s)$ *is an isomorphism.*

Proof The proof reduces to $A = C\mathcal{A}_X \otimes L$ with L a line-bundle on X. For such an A, the map $\mathrm{Hom}_{C\mathcal{A}_X}(A, M) \to \mathrm{Hom}_{S\mathcal{A}_X}(\rho S, \rho M)$ can be identified with the map on global sections $\Gamma(X, M \otimes L^{-1}) \to \Gamma(X, S \otimes_C M \otimes L^{-1}) = S \otimes_C \Gamma(X, M \otimes L^{-1})$. This map is surjective since $C \to S$ is, proving (a). The map is also a localization by a calculus of fractions with respect to the set of elements $e_n \in C$, $n \in \mathbb{N}$, of Sect. 10.1. This shows that (b) also holds. □

Lemma 19 *Let* X *be a quasi-compact and quasi-separated scheme*, L *a line bundle on* X, *and let* $\mathcal{A}_X$ *be a quasi-coherent sheaf of* O_X*-algebras with involution. Then for every* $n \in \mathbb{Z}$*, the Grothendieck-Witt space*

$$GW^n(C\mathcal{A}_X, L)$$

is contractible.

Proof (Compare [13]) We will define a C-bimodule M, which is finitely generated projective as left C-module, together with a symmetric form $\varphi : M \otimes_C M^{op} \to C$ in C-Bimod whose adjoint $M \to [M^{op}, C]_C$ is an isomorphism. Furthermore, we will construct an isometry $(C, \mu) \perp (M, \varphi) \cong (M, \varphi)$ of symmetric forms in C-Bimod. Therefore, tensor product $(M, \varphi) \otimes_C ?$ defines a non-singular exact form functor $(F, \varphi) : (\mathrm{sPerf}(C\mathcal{A}_X), \mathrm{quis}, \sharp^n_L) \to (\mathrm{sPerf}(C\mathcal{A}_X), \mathrm{quis}, \sharp^n_L)$ which satisfies $id \perp (F, \varphi) \cong (F, \varphi)$. Therefore, on higher Grothendieck-Witt groups we have $GW^n_i(id) + GW^n_i(F, \varphi) = GW^n_i(F, \varphi)$ which implies $GW^n_i(id) = 0$, that is, $GW^n_i(C\mathcal{A}_X, L) = 0$, hence $GW^n(C\mathcal{A}_X, L)$ is contractible.

To construct (M, φ) and the bimodule isometry $(C, \mu) \perp (M, \varphi) \cong (M, \varphi)$ we choose a bijection $\sigma : \mathbb{N} \xrightarrow{\cong} \mathbb{N} \times \mathbb{N} : n \mapsto (\sigma_1(n), \sigma_2(n))$ and define a homomorphism of rings with involutions

$$I : C \to C : a \mapsto I(a) \quad \text{with } I(a)_{ij} = \begin{cases} a_{\sigma_1(i), \sigma_1(j)} & \text{if } \sigma_2(i) = \sigma_2(j) \\ 0 & \text{otherwise.} \end{cases}$$

The C-bimodule M is C as a left module, and has right multiplication defined by $M \times C \to M : (x, a) \mapsto x \cdot I(a)$. The symmetric form φ is the C-bimodule map $M \otimes_C M^{op} \to C : x \otimes y^{op} \mapsto x \cdot {}^t y$. Since, as a left C-module, (M, φ) is just the unit symmetric form (C, μ) (see Sect. 7.8 (a)), the adjoint $M \to [M^{op}, C]_C$ of φ is an isomorphism.

In order to define the bimodule isometry $(C, \mu) \perp (M, \varphi) \cong (M, \varphi)$, consider the elements $\gamma, \delta \in C$ defined by

$$\gamma_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = (i, 0) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \delta_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = \sigma(i) + (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

The homomorphism I and the elements $\gamma, \delta \in C$ are related by the following identities

$$\delta \cdot {}^t\gamma = 0, \qquad \gamma \cdot {}^t\gamma = \delta \cdot {}^t\delta = 1, \qquad {}^t\gamma \cdot \gamma + {}^t\delta \cdot \delta = 1,$$
$$a \cdot \gamma = \gamma \cdot I(a), \qquad I(a) \cdot \delta = \delta \cdot I(a)$$

for all $a \in C$. Therefore, the map $C \oplus M \to M : (a, x) \mapsto a \cdot \gamma + x \cdot \delta$ is a C-bimodule isomorphism with inverse the map $M \to C \oplus M : x \mapsto (x \cdot {}^t\gamma, x \cdot {}^t\delta)$. It preserves forms because $(a\gamma + x\delta) \cdot {}^t(b\gamma + y\delta) = a \cdot {}^tb + x \cdot {}^ty$. □

Write $\mathrm{sPerf}^0(\mathcal{A}_X) \subset \mathrm{sPerf}(\mathcal{A}_X)$ for the full subcategory of those strictly perfect complexes of $\mathcal{A}_X$-modules which are degree-wise quasi-free. Note that this category is closed under the duality $\sharp^n_L$. We equip the category $\mathrm{sPerf}^0(\mathcal{A}_X)$ with the degree-wise split exact structure. Together with the set of quasi-isomorphisms of complexes of $\mathcal{A}_X$ vector bundles, it becomes a category of complexes in the sense of Definition 5.

Lemma 20 *Let X be a scheme with an ample family of line bundles. The inclusion of quasi-free modules into the category of vector bundles induces a (fully faithful) cofinal triangle functor*

$$\mathcal{D}(\mathrm{sPerf}^0(\mathcal{A}_X), \mathrm{quis}) \subset \mathcal{D}(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis}).$$

Moreover, for every strictly perfect complex M of $\mathcal{A}_X$-modules with class $[M]$ in the image of the map $K_0(\mathrm{sPerf}^0(\mathcal{A}_X), \mathrm{quis}) \to K_0(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis})$ there is a quasi-isomorphism $A \to M$ of complexes of $\mathcal{A}_X$-modules with A a bounded complex of quasi-free modules.

Proof The triangle functor in the lemma is fully faithful, by Lemma 14 with $U = X$ and $\mathcal{A} = \mathrm{sPerf}^0(\mathcal{A}_X)$. It is cofinal, by Neeman's Theorem 12(a) and Proposition 8. Let $\mathrm{sPerf}_{K_0}(\mathcal{A}_X) \subset \mathrm{sPerf}(\mathcal{A}_X)$ be the full subcategory of those strictly perfect complexes of $\mathcal{A}_X$-modules M whose class $[M]$ is in the image of the map $K_0(\mathrm{sPerf}^0(\mathcal{A}_X), \mathrm{quis}) \to K_0(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis})$. Then the inclusion $(\mathrm{sPerf}^0(\mathcal{A}_X), \mathrm{quis}) \subset (\mathrm{sPerf}_{K_0}(\mathcal{A}_X), \mathrm{quis})$ of exact categories with weak equivalences induces an equivalence of derived categories, so that another application of Lemma 14 with $U = X$ and $\mathcal{A} = \mathrm{sPerf}^0(\mathcal{A}_X)$ finishes the proof of the claim. □

Proof of Theorem 13 By Theorem 10, we only need to treat the case $Z = X$. In this case, the total spaces are contractible, by Lemma 19.

Let $\mathrm{sPerf}_{K_0}(S\mathcal{A}_X) \subset \mathrm{sPerf}(S\mathcal{A}_X)$ be the full subcategory of those strictly perfect complexes of $S\mathcal{A}_X$-modules E whose class $[E]$ is zero in the Grothendieck group $K_0(\mathrm{sPerf}(S\mathcal{A}_X), \mathrm{quis})$ of $S\mathcal{A}_X$-vector bundles. Furthermore, call a map f of strictly perfect complexes of $C\mathcal{A}_X$-modules an S-quasi-isomorphism if $\rho(f)$ is a quasi-isomorphism of complexes of $S\mathcal{A}_X$-modules. The set of S-quasi-isomorphisms is denoted by S-quis. Consider the commutative diagram of exact categories with weak equivalences and duality $\sharp^n_L$ induced by inclusions and the map of rings with involution $C \to S$

$$\begin{array}{ccccc}
(\mathrm{sPerf}^0(C\mathcal{A}_X), \mathrm{quis}) & \longrightarrow & (\mathrm{sPerf}^0(C\mathcal{A}_X), S\text{-quis}) & \xrightarrow{\rho} & (\mathrm{sPerf}^0(S\mathcal{A}_X), \mathrm{quis}) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{sPerf}(C\mathcal{A}_X), \mathrm{quis}) & \longrightarrow & (\mathrm{sPerf}(C\mathcal{A}_X), S\text{-quis}) & \xrightarrow{\rho} & (\mathrm{sPerf}_{K_0}(S\mathcal{A}_X), \mathrm{quis}).
\end{array}$$

Note that K_0 of all categories with weak equivalences in the diagram is 0. For the two left hand categories, this follows from Lemmas 19 and 20, since for cofinal triangle functors $\mathcal{T}^0 \subset \mathcal{T}$, the map $K_0(\mathcal{T}^0) \to K_0(\mathcal{T})$ is injective. Since the left horizontal maps are surjective on K_0, the middle two categories with weak equivalences have trivial Grothendieck group K_0. For the upper right corner, vanishing of K_0 follows moreover from the fact that its K_0 is generated by classes of complexes concentrated in degree 0 and the fact that every quasi-free $S\mathcal{A}_X$-module is the image of a quasi-free $C\mathcal{A}_X$-module. Therefore, the upper horizontal map is surjective on K_0. Hence, the right vertical and the lower right horizontal functors, which—a priori—have images in $\mathrm{sPerf}(S\mathcal{A}_X)$, have indeed image in $\mathrm{sPerf}_{K_0}(S\mathcal{A}_X)$. We will show that the upper right horizontal and middle and right vertical functors induce equivalences of Grothendieck-Witt spaces (for any duality $\sharp^n_L$). So, the lower right horizontal functor will induce an equivalence, too.

The upper right horizontal functor is a localization by a calculus of right fractions, by Lemmas 18 and 10(c). Therefore, Theorem 9 shows that it induces a homotopy equivalence of Grothendieck-Witt spaces. For the right vertical functor, the Resolution Lemma 9 (which we can apply because of Lemma 20) shows that it induces an equivalence of Grothendieck-Witt spaces. Similarly, by Lemma 20, for every strictly perfect complex of $C\mathcal{A}_X$-modules M, there is a bounded complex A of quasi-free $C\mathcal{A}_X$-modules and a quasi-isomorphism $A \to M$. A quasi-isomorphism of complexes of $C\mathcal{A}_X$-modules is, a fortiori, an S-quasi-isomorphism. Therefore, the Resolution Lemma applies to show that the middle vertical functor induces an equivalence of Grothendieck-Witt spaces. Summarizing, we have shown that the lower right horizontal functor induces an equivalence of Grothendieck-Witt spaces.

By the Change-of-weak-equivalence Theorem (Theorem 6), the sequence of exact categories with weak equivalences and duality $\sharp^n_L$

$$(\mathrm{sPerf}_S(C\mathcal{A}_X),\ \mathrm{quis}) \to (\mathrm{sPerf}(C\mathcal{A}_X),\ \mathrm{quis}) \to (\mathrm{sPerf}(C\mathcal{A}_X),\ S\text{-quis})$$

induces a homotopy fibration of Grothendieck-Witt spaces. Using Proposition 12 we can replace the left hand term with $(\mathrm{sPerf}(\mathcal{A}_X), \mathrm{quis})$. Using the equivalence of Grothendieck-Witt spaces of the lower right horizontal functor above and Cofinality (Theorem 7) applied to the inclusion of exact categories with weak equivalences and duality $(\mathrm{sPerf}_{K_0}(S\mathcal{A}_X), \mathrm{quis}) \subset (\mathrm{sPerf}(S\mathcal{A}_X), \mathrm{quis})$, we can replace the right hand term in the sequence by $(\mathrm{sPerf}(S\mathcal{A}_X), \mathrm{quis})$. □

Since the total space in the fibration of Theorem 13 is contractible, we obtain a homotopy equivalence of spaces

$$GW^n(\mathcal{A}_X \text{ on } Z, L) \xrightarrow{\simeq} \Omega\, GW^n(S\mathcal{A}_X \text{ on } Z, L). \tag{32}$$

Definition 8 Let X be a scheme with an ample family of line-bundles, $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution, L a line bundle on X, $Z \subset X$ a closed subscheme with quasi-compact open complement $X - Z$ and $n \in \mathbb{Z}$ an integer. The *Grothendieck-Witt spectrum*

$$\mathbb{G}W^n(\mathcal{A}_X \text{ on } Z, L)$$

of symmetric spaces over $\mathcal{A}_X$ with coefficients in the n-th shifted line bundle $L[n]$ and support in Z is the sequence

$$GW^n(S^k\mathcal{A}_X \text{ on } Z, L), \quad k \in \mathbb{N},$$

of Grothendieck-Witt spaces together with the bonding maps given by the homotopy equivalence (32). As usual, if $Z = X$, $n = 0$, $\mathcal{A}_X = O_X$ or $L = O_X$, we omit the label corresponding to Z, n, $\mathcal{A}$, or L, respectively.

By construction, we have

$$\pi_i \mathbb{G}W^n(\mathcal{A}_X \text{ on } Z, L) = \begin{cases} GW^n_i(\mathcal{A}_X \text{ on } Z, L) & \text{for } i \geq 0 \\ GW^n_0(S^{-i}\mathcal{A}_X \text{ on } Z, L) & \text{for } i \leq 0. \end{cases}$$

Remark 16 By Proposition 7, there are natural homotopy equivalences of spectra $\mathbb{G}W^n(\mathcal{A}_X \text{ on } Z, L) \simeq \mathbb{G}W^{n+4}(\mathcal{A}_X \text{ on } Z, L)$.

Finally, we are in position to prove the main theorems of this article.

Theorem 14 (Localization) *Let X be a scheme with an ample family of line-bundles, let $Z \subset X$ be a closed subscheme with quasi-compact open complement $j : U \subset X$, and let L be a line bundle on X. Let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution. Then for every $n \in \mathbb{Z}$, the following sequence is a homotopy fibration of Grothendieck-Witt spectra*

$$\mathbb{G}W^n(\mathcal{A}_X \text{ on } Z, L) \longrightarrow \mathbb{G}W^n(\mathcal{A}_X, L) \longrightarrow \mathbb{G}W^n(\mathcal{A}_U, \ j^*L).$$

Proof This is because the sequences

$$GW^n(S^i\mathcal{A}_X \text{ on } Z, L) \longrightarrow GW^n(S^i\mathcal{A}_X, L) \longrightarrow GW^n(S^i\mathcal{A}_U, \ j^*L)$$

are homotopy fibrations for $i \in \mathbb{N}$, by Theorem 10. □

Theorem 15 (Zariski-excision) *Let X be a scheme with an ample family of line-bundles, let $Z \subset X$ be a closed subscheme with quasi-compact open complement, let L be a line bundle on X and let $\mathcal{A}_X$ be a quasi-coherent sheaf of O_X-algebras with involution. Then for every $n \in \mathbb{Z}$ and every quasi-compact open subscheme $j : V \subset X$ containing Z, restriction of vector-bundles induces a homotopy equivalence of Grothendieck-Witt spectra*

$$\mathbb{G}W^n(\mathcal{A}_X \text{ on } Z, L) \xrightarrow{\sim} \mathbb{G}W^n(\mathcal{A}_V \text{ on } Z, \ j^*L).$$

Proof This is because the maps

$$GW^n(S^i\mathcal{A}_X \text{ on } Z, L) \longrightarrow GW^n(S^i\mathcal{A}_V \text{ on } Z, \ j^*L).$$

are homotopy equivalences for $i \in \mathbb{N}$, by Theorem 11. □

Theorem 16 (Mayer-Vietoris for open covers) *Let $X = U \cup V$ be a scheme with an ample family of line-bundles which is covered by two open quasi-compact subschemes $U, V \subset X$. Let $\mathcal{A}_X$ be a quasi-coherent O_X-module with involution. Let L be a line-bundle on X, and $n \in \mathbb{Z}$. Then restriction of vector bundles induces a homotopy Cartesian square of Grothendieck-Witt spectra*

$$\begin{array}{ccc} \mathbb{G}W^n(\mathcal{A}_X, L) & \longrightarrow & \mathbb{G}W^n(\mathcal{A}_U, L) \\ \downarrow & & \downarrow \\ \mathbb{G}W^n(\mathcal{A}_V, L) & \longrightarrow & GW^n(\mathcal{A}_{U\cap V}, L). \end{array}$$

Proof The map on vertical homotopy fibres is an equivalence, by Theorems 14 and 15. □

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