Cyclic Foam Topological Field Theories

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Abstract

This paper proposes an axiomatic for Cyclic Foam Topological Field Theories. That is Topological Field Theories, corresponding to String Theories, where particles are arbitrary graphs. World surfaces in this case are two-manifolds with one-dimensional singularities. I prove that Cyclic Foam Topological Field Theories one-to-one correspond to graph-Cardy-Frobenius algebras, that are families $(A, B_\star, \phi)$, where $A = \{A^s | s \in S\}$ are families of commutative associative Frobenius algebras, $B_\star = \bigoplus_{\sigma\in\Sigma} B_\sigma$ is an graduated by graphes, associative algebras of Frobenius type and $\phi = \{\phi^s_\sigma : A^s \to \mathsf{End}(B_\sigma) | s \in S, \sigma \in \Sigma\}$ is a family of special representations. There are constructed examples of Cyclic Foam Topological Field Theories and its graph-Cardy-Frobenius algebras

Contents

1. Introduction

Two-dimensional Topological Field Theories were introduced by Segal [23], Atiyah [5] and Witten [25]. An example of them is a topological approach of the String Theory. The String Theory is a modern variant of uniform Field Theory. It treads particles as one-dimensional objects. A path of a particle is represented by a world surface, that is,

[1]Supported by grants RFBR-07-01-00593, NWO 047.011.2004.026 (05-02-89000-NWO-a), NSh-4719.2006.1, INTAS 05-7805

a two-dimensional space. The Topological String Theory assumes that a probability of a world surface depends only on the state of the particle at the moments of birth/death and on the topological type of the world surface. The standard properties of the measure on world lines extend to properties of Topological String Theory [10].

The Topological Field Theory is a direct axiomatization of the Topological String Theory. The Topological Field Theory is a function on the set of two-dimensional spaces (world surface in String Theory) endowed with marked points (points of birth/death in String Theory) and also with vectors in the marked points (vectors of states in String Theory). The function depends from the vectors linearly. The properties of Topological String Theory can be reformulated as properties of the function by a surgery of surfaces.

The simplest model treads particles as closed contours. Thus its world surface is a closed surface, that is, two-dimensional topological manifold without boundary. The corresponding Topological Field Theory was constructed in [5], [8] for orientable surfaces and in [1] for arbitrary (orientable and non-orientable) surfaces.

In this case, the values of a Topological Field Theory on spheres with one, two and three marked points determine the values of the Topological Field Theory on all oriented surfaces. As well, the values of a Topological Field Theory on spheres with one, two and three marked points are structure constants for some associative, commutative Frobenius algebra A with a unit. Moreover, this construction gives a one-to-one correspondence between Topological Field Theories on closed orientable surfaces and associative, commutative Frobenius algebras with unit [8]. To extend a Topological Field Theory to non-orientable surfaces one has to add new structures to A, namely, an involution of A and an element $U \in A$ which defines the value of the Topological Field Theory on a projective sphere with a marked point [1].

A String Theory where particles are closed contours and segments is called an Open-Closed Theory. In this case world surfaces are two-dimensional topological manifolds without boundary or with a boundary consisting of closed contours. For orientable surfaces of this type Topological Field Theory was constructed in [17], [18], where it is determined a pair of associative Frobenius algebras with unit, connected by a special homomorphism ϕ. The first algebra is A, corresponding to closed surfaces. The second algebra B, which in general is non-commutative, corresponds to disks with marked points at the boundary. The one-to-one correspondent between Open-Closed Topological Field Theories and the families (A, B, ϕ) was proved in [1] and later, independently, in [16], [19].

For orientable and non-orientable surfaces with boundary a Topological Field Theory was constructed in [1]. We call it the Klein Topological Field Theory. Klein Topological Field Theories are in one-to-one correspondence with the Cardy-Frobenius algebras which are the tuples (A, B, ϕ) with equipments [1].

In the present paper I construct Cyclic Foam Topological Field Theories that correspond to String Theories where particles are arbitrary graphs (for physical motivation see [6], and also [20] and references there in). In this case the world surface is a CW-complex glued from finitely many surfaces ("patches") with boundaries by gluing some segments of the boundaries. The boundaries of surfaces form the singular part of the complex is called "seamed graph". Complexes of this type are called "world-sheet foam" or "seamed surfaces". They appear also in A-models [22], Landau-Ginsburg models [15] and 3-dimensional topology [13] [14].

In this paper I consider a special class of seamed surfaces which I call cyclic foam. They satisfy the following conditions: (1) glued boundary contours of patches have compatible orientation; (2) different boundary contours a patch included to different connected components of the seamed graph. I assume also that any patch has a "color" from a set S and the closures of two patches have no intersections if they have the same color.

We start with the definition of Cyclic Foam Topological Field Theories on cyclic foams where all patches are disks (Section 2). We prove that such Topological Field Theories are in one-to-one correspondence with graph-Frobenius algebras which we define in Section 3. A graph-Frobenius algebra is an associative algebra that satisfy all the properties of a Frobenius algebra except for the finite dimensionality property. Instead, it is presented as a sum of finite dimensional vector spaces $B_\star = \bigoplus_{\sigma\in\Sigma} B_\sigma$ where Σ is the set of oriented colored graphs. The algebra B from Cardy-Frobenius algebras is the subalgebra of $B_\star$ that corresponds to the segment.

In Section 4 we define Topological Field Theories for arbitrary cyclic foams. Later (Section 5) we prove that Cyclic Foam Topological Field Theories are in one-to-one correspondence with families $(A, B_\star, \phi)$, where $A = \{A^s | s \in S\}$ is a family of commutative associative Frobenius algebras with units and $\phi = \{\phi^s_\sigma : A^s \to \mathsf{End}(B_\sigma) | s \in S, \sigma \in \Sigma\}$ is a special family of representations in B_σ.

Some classical topological objects satisfy the axioms of Topological Field Theory. The algebraic description of Topological Field Theories makes it possible to reduce some topological problems to algebraic one. Such applications of Topological Field Theory appear in the Theory of Links [24], [21] and in the Theory of Hurwitz Numbers.

The classical Hurwitz numbers are weighted numbers of meromorphic functions with prescribed topological types of critical values [12]. These numbers depend on topological types of surfaces and critical values only. The classical Hurwitz numbers and Hurwitz numbers of regular coverings generate Topological Field Theories on closed surfaces [9], [7]. Hurwitz numbers for surfaces with boundary are defined in [1] for coverings by surfaces with boundary, in [2], [3] for coverings by seamed surfaces, and in [4] for regular coverings by seamed surfaces. In these papers we proved that each type of these Hurwitz numbers forms a Klein Topological Field Theory and we described their Cardy-Frobenius algebras.

In Section 6 of preset paper it is constructed examples of Cyclic Foam Topological Field Theories and corresponding graph-Cardy-Frobenius algebras. These examples extend to cyclic foams the Klein Topological Field Theories of regular Hurwitz numbers from [4].

I thank A.Alekseevskii, S.Lando, and L.Rozansky for useful discussions.

2. Film Topological Field Theories

2.1. **Film surfaces.** In this paper a *graph* is a compact simplicial complex that consist of simplexes of dimension 1 (*edges*) and dimension 0 (*vertices*). An edge is either a segment or a loop depending on the topological type of its closure. A graph is said to be *regular* if all its edges are segments.

A compact CW-complex that consists of oriented cells of dimension 2 (*disks*), cells of dimension 1 (*edges*) and cells of dimension 0 (*vertices*) is said to be *regular* if its edges form a regular graph. Thus regular CW-complex Ω is defined by a set $(\check{\Omega}, \Delta, \varphi)$, where $\check{\Omega} = \check{\Omega}(\Omega)$ is a set of closed oriented disks , $\Delta = \Delta(\Omega)$ is a regular graph and $\varphi : \partial\check{\Omega} \to \Delta$ is a gluing map, that is, a homeomorphism on any connected component of $\partial\Omega$ and $\varphi(\partial\check{\Omega}) = \Delta$.

A system of cyclic orders on vertexes of connected components of a regular CW-complex Ω is called a *cyclic order on* Ω, if the cyclic orders on vertexes are compatible with the orientations of the disks $\check{\Omega}(\Omega)$. A regular CW-complex with a cyclic order is called *almost cyclic complex.*

A connected regular graph $\gamma \subset \Omega$ on a connected almost cyclic complex Ω is called a *graph-cut* if:

- the restriction of γ to any disk $\omega \in \check{\Omega}$ either is empty, or forms one of edge of γ;
- γ divides Ω into two connected components that splits vertices of Ω into two nonempty groups, compatible with the cyclic order on Ω.

A almost cyclic complex Ω is called *cyclic complex* if for any compatible with the cyclic order division of vertexes Ω there exists a graph-cut that realize it. A small neighborhood of a vertex q of a CW-complex is a cone over a regular *vertex graph* σ_q, with orientation of edges generated by the orientation of the disks outside the neighborhood. It is obviously that vertex graphs of cyclic complexes are connected.

Fix a set S of *colors*. A graph (respectively CW-complex) is called *colored* if a color $s(l) \in S$ is assigned to each of its edges (respectively disks) l and all the colors are pairwise different for any connected component. A colored cyclic complex is called a *film surface.* Vertex the graph σ_q of a vertices q of a film surface is a colored graph, where colors of edges generated by colors of the disks. Denote by Ω_b the set of vertices of a film surface Ω.

2.2. **Topological Field Theory.** Below we assume that all vector spaces are defined over a field $\mathbb{K} \supset \mathbb{Q}$. Let $\{X_m | m \in M\}$ be a finite set of $n = |M|$ vector spaces X_m over the field of complex numbers $\mathbb{C}$. The action of the symmetric group S_n on $\{1, \dots, n\}$ induces its action on the sum of the vector spaces $\left(\oplus_\sigma X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}\right)$ where σ runs over the bijections $\{1, \dots, n\} \to M$, an element $s \in S_n$ takes $X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}$ to $X_{\sigma(s(1))} \otimes \dots \otimes X_{\sigma(s(n))}$. Denote by $\otimes_{m \in M} X_m$ the subspace of all invariants of this action. The vector space $\otimes_{m \in M} X_m$ is canonically isomorphic to the tensor product of all X_m in any fixed order; the isomorphism is the projection of $\otimes_{m \in M} X_m$ to the summand that is equal to the tensor product of X_m in that order.

Two regular oriented colored graphs said to be *isomorphic* if there exists a homeomorphism that maps one to the other preserving the colors and the orientations. Denote by $\Sigma = \Sigma(S)$ the set of all isomorphism classes of connected oriented colored graphs. The inversions of the orientations generate the involution $* : \Sigma \to \Sigma$. Denote it by $\sigma \mapsto \sigma^*$.

Consider a family of finite-dimensional vector spaces $\{B_\sigma | \sigma \in \Sigma\}$ and a family of tensors $\{K_\sigma^\otimes \in B_\sigma \otimes B_{\sigma^*} | \sigma \in \Sigma\}$. Using these data, we define now a functor $\mathcal{V}$ from the category of film surfaces to the category of vector spaces. This functor assigns the vector space $V_\Omega = (\otimes_{q \in \Omega_b} B_q)$ to any film surface Ω. Here B_q is the copy of B_{σ_q} that is a vector space with a fixed isomorphism $B_q \to B_{\sigma_q}$.

We are going to describe all morphisms of the monoidal category $\mathcal{S}$ of film surfaces and morphisms of the category of vector spaces that correspond to it.

(1) *Isomorphism.* Let $\phi : \Omega \to \Omega'$ be a homeomorphism of film surfaces, preserving the cyclic orders, orientations of disks and its colors. Define $\mathcal{V}(\phi) = \phi_* : V_\Omega \to V_{\Omega'}$ as linear operator generated by the bijections $\phi|_{\Omega_b} : \Omega_b \to \Omega'_b$.

(2) *Cut.* Let Ω be a connected film surface and $\gamma \subset \Omega$ be a graph-cut. The graph γ is represented by two graphs γ_+ and γ_- on the closure $\overline{\Omega \setminus \gamma}$ of $\Omega \setminus \gamma$. Contract these graphs to points $q_+ = q_+[\gamma]$ and $q_- = q_-[\gamma]$, respectively. The contraction produces a film surface $\Omega' = \Omega[\gamma]$. Its vertices $\Omega' = \Omega[\gamma]$ are the vertices of Ω and the points q_+, q_-. The cyclic order, orientation and the coloring of Ω induces an orientation and a coloring of

Ω'. Thus we can assume that Ω' is a film surface and $V_{\Omega'} = V_\Omega \otimes B_{q+} \otimes B_{q-}$. The functor takes the morphism $\mathcal{V}(\eta)(x) = \eta_*(x) = x \otimes K_\sigma^\otimes$, where $\sigma = \sigma_{q+} = \sigma_{q-}^*$, to the morphism $\eta : \Omega \to \Omega'$.

(3) The tensor product in $\mathcal{S}$ defined by the disjoint union of surfaces $\Omega' \otimes \Omega'' \to \Omega' \coprod \Omega''$ induces the tensor product of vector spaces $\theta_* : V_{\Omega'} \otimes V_{\Omega''} \to V_{\Omega' \sqcup \Omega''}$.

The functorial properties of $\mathcal{V}$ can be easily verified.

Fix a tuple of vector spaces and vectors $\{B_\sigma, K_\sigma^\otimes \in B_\sigma \otimes B_{\sigma^*} | \sigma \in \Sigma\}$, defining the functor $\mathcal{V}$. A family of linear forms $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ defined for all film surfaces $\Omega \in \mathcal{S}$ is called a *Film Topological Field Theory* if it satisfies the following axioms:

1° *Topological invariance.*

$$\Phi_{\Omega'}(\phi_*(x)) = \Phi_\Omega(x)$$

for any isomorphism $\phi : \Omega \to \Omega'$ of film surfaces.

2° *Non-degeneracy.*

Let Ω be a film surface with only two vertices q_1, q_2. Then $\sigma_{q_2} = \sigma^*$ if $\sigma_{q_1} = \sigma$. Denote by $(.,.)_\sigma$ the bilinear form $(.,.)_\sigma : B_\sigma \times B_{\sigma^*} \to \mathbb{K}$, where $(x', x'')_\sigma = \Phi_\Omega(x'_{q_1} \otimes x''_{q_2})$. Axiom 2° asserts that the forms $(.,.)_\sigma$ are non-degenerated for all $\sigma \in \Sigma$.

3° *Cut invariance.*

$$\Phi_{\Omega'}(\eta_*(x)) = \Phi_\Omega(x)$$

for any cut morphism $\eta : \Omega \to \Omega'$ of film surfaces.

4° *Multiplicativity.*

$$\Phi_\Omega(\theta_*(x' \otimes x')) = \Phi_{\Omega'}(x')\Phi_{\Omega''}(x'')$$

for $\Omega = \Omega' \coprod \Omega''$, $x' \in V_{\Omega'}$, $x'' \in V_{\Omega''}$.

Note that a Topological Field Theory defines the tensors $\{K_\sigma^\otimes \in B_\sigma \otimes B_{\sigma^*} | \sigma \in \Sigma\}$, since it is not difficult to prove:

Lemma 2.1. *Let $\{\Phi_\Omega\}$ be a Film Topological Field Theory. Then $(K_\sigma^\otimes, x_1 \otimes x_2)_\sigma = (x_1, x_2)_\sigma$, for all $x_1 \in B_\sigma$, $x_2 \in B_{\sigma^*}$.*

3. Graph-Frobenius algebras

3.1. **Definitions.** We say that a connected film surface Ω is a *compatible surface* for colored graphes $\sigma_1, \sigma_2, ..., \sigma_n$ if these graphes are vertex graphs of Ω and the numeration the graphs σ_i generates the cyclic order of vertexes of film surface Ω. Denote by $\Omega(\sigma_1, \sigma_2, ..., \sigma_n)$ the set of all isomorphism classes of compatible surfaces for $\sigma_1, \sigma_2, ..., \sigma_n$. Then $\Omega(\sigma_1, \sigma_2, ..., \sigma_n)$ is either empty or consists of a single element.

Let $\Omega(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \neq \emptyset$. Then there exist unique classes of graph-cuts $\sigma_{(1,2|3,4)}, \sigma_{(4,1|2,3)} \in \Sigma$ such that $\Omega(\sigma_1, \sigma_2, \sigma_{(1,2|3,4)}) \neq \emptyset$, $\Omega(\sigma_{(3,4|1,2)}, \sigma_3, \sigma_4) \neq \emptyset$, $\Omega(\sigma_4, \sigma_1, \sigma_{(4,1|2,3)}) \neq \emptyset$, $\Omega(\sigma_{(4,1|2,3)}, \sigma_2, \sigma_3) \neq \emptyset$ and $\sigma_{(3,4|1,2)} = \sigma_{(1,2|3,4)}^*$, $\sigma_{(2,3|4,1)} = \sigma_{(4,1|2,3)}^*$.

Consider a tuple of finite dimensional vector spaces $\{B_\sigma | \sigma \in \Sigma\}$. Its direct sum $B_\star = \bigoplus_{\sigma \in \Sigma} B_\sigma$ is called a *colored graph-graded vector space.*

A colored graph-graded vector space with a bilinear form $(.,.) : B_\star \times B_\star \to \mathbb{K}$ and a tree-linear form $(.,.,.) : B_\star \times B_\star \times B_\star \to \mathbb{K}$ is called a *graph-Frobenius algebra* if

- $(B_{\sigma_1}, B_{\sigma_2}) = 0$ for $\sigma_1 \neq \sigma_2^*$;
- the form $(.,.)$ is not-degenerate;
- $(B_{\sigma_1}, B_{\sigma_2}, B_{\sigma_3}) = 0$ for $\Omega(\sigma_1, \sigma_2, \sigma_3) = \emptyset$
- $\sum_{i,j}(x_1, x_2, b_i^{(1,2|3,4)})F^{ij}_{(1,2|3,4)}(b_j^{(3,4|1,2)}, x_3, x_4) = \sum_{i,j}(x_4, x_1, b_i^{(4,1|2,3)})F^{ij}_{(4,1|2,3)}(b_j^{(2,3|4,1)}, x_2, x_3)$.

Here $x_k \in B_{\sigma_k}$, $\{b_i^{(s,t|k,r)}\}$ is a basis of $B_{(s,t|k,r)}$ and $F^{ij}_{(s,t|k,r)}$ is the inverse matrix for $(b_i^{(s,t|k,r)}, b_j^{(k,r|s,t)})$.

We will consider B_* as an algebra with the multiplication $(x_1x_2, x_3) = (x_1, x_2, x_3)$, for $x_k \in B_{\sigma_k}$. The axiom $\sum_{i,j}(x_1, x_2, b_i^{(1,2|3,4)})F^{ij}_{(1,2|3,4)}(b_j^{(3,4|1,2)}, x_3, x_4) = \sum_{i,j}(x_4, x_1, b_i^{(4,1|2,3)})F^{ij}_{(4,1|2,3)}(b_j^{(2,3|4,1)}, x_2, x_3)$ is equivalent to associativity for the algebra B_*. Moreover it is a Frobenius algebra in the sense of [11] if its dimension is finite.

3.2. **One-to one correspondence.**

Theorem 3.1. *Let $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ be a Film Topological Field Theory on a tuple of finite-dimensional vector spaces $\{B_\sigma | \sigma \in \Sigma\}$. Then the poli-linear forms*

- $(x', x'') = \Phi_{\Omega(\sigma_1,\sigma_2)}(x'_{q_1} \otimes x''_{q_2})$, *где $x' \in B_{\sigma_1}$, $x'' \in B_{\sigma_2}$*
- $(x', x'', x''') = \Phi_{\Omega(\sigma_1,\sigma_2,\sigma_3)}(x'_{q_1} \otimes x''_{q_2} \otimes x'''_{q_3})$, *где $x' \in B_{\sigma_1}$, $x'' \in B_{\sigma_2}$, $x''' \in B_{\sigma_3}$.*

generate a structure of graph-Frobenius algebra on $B_\star = \bigoplus_{\sigma\in\Sigma} B_\sigma$.

Proof. Only the last axiom is not obvious. Let us consider a film surface $\Omega \in \Omega(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and a graph-cut between the pairs of vertexes σ_1, σ_2 and σ_3, σ_4. Then the cut-invariant axiom and lemma 2.1 give $\sum_{i,j}(x_1, x_2, b_i^{(1,2|3,4)})F^{ij}_{(1,2|3,4)}(b_j^{(3,4|1,2)}, x_3, x_4) = \Phi_\Omega(x_1, x_2, x_3, x_4)$. Similarly, $\sum_{i,j}(x_4, x_1, b_i^{(4,1|2,3)})F^{ij}_{(4,1|2,3)}(b_j^{(2,3|4,1)}, x_2, x_3) = \Phi_\Omega(x_1, x_2, x_3, x_4)$
□

Theorem 3.2. *Let $B_\star = \bigoplus_{\sigma\in\Sigma} B_\sigma$ be a graph-Frobenius algebra with poli-linear forms $(.,.)$ and $(.,.,.)$. Then it generates a Film Topological Field Theory on $\{B_\sigma|\sigma \in \Sigma\}$ by means of following construction. Fix a basis $\{b_i^\sigma\}$ of any vector space B_σ, $\sigma \in \Sigma$. Consider the matrix F^{ij}_σ that is the inverse matrix for $F^\sigma_{ij} = (b_i^\sigma, b_j^{\sigma^*})$. Define the linear functionals on connected film surfaces by*

- $\Phi_{\Omega(\sigma_1,\sigma_2\dots,\sigma_n)}(x^1_{q_1} \otimes x^2_{q_2} \otimes \dots \otimes x^n_{q_n}) = \sum_{\varsigma_1,\varsigma_2,\dots,\varsigma_{n-3}\in\Sigma}(x^1, x^2, b^{\varsigma_1}_{i_1})F^{\varsigma_1}_{i_1j_1}(b^{\varsigma_1^*}_{j_1}, x^3_{q_3}, b^{\varsigma_2}_{i_2})$ $F^{\varsigma_2}_{i_2j_2}(b^{\varsigma_2^*}_{j_2}, x^4_{q_4}, b^{\varsigma_3}_{i_3})\dots\dots F^{\varsigma_{n-4}}_{i_{n-4}j_{n-4}}(b^{\varsigma^*_{n-4}}_{j_{n-4}}, x^{n-2}_{q_{n-2}}, b^{\varsigma_{n-3}}_{i_{n-3}})F^{\varsigma_{n-3}}_{i_{n-3}j_{n-3}}(b^{\varsigma^*_{n-3}}_{j_{n-3}}, x^{n-1}_{q_{n-1}}, x^n_{q_n})$, *where $x^i_{q_i} \in B_{\sigma_i}$.*

Define the linear functionals on non-connected film surfaces by multiplicativity axiom.

Proof. The topological invariance follows from the invariance under cyclic renumbering the vertices of Ω. The invariance under the renumbering $q_i \mapsto q_j$ $j \equiv i + 1(\text{mod}2)$ follows from the last axiom for the tree-linear form. The cut invariance follows directly from the definition of Φ if we renumber the vertices marking the cut divide the vertices $q_1, q_2, \dots q_k$ and $q_{k+1}, q_{k+2}, \dots q_n$.
□

These two theorems determine the one-to-one correspondence between Film Topological Field Theories and isomorphic classes of graph-Frobenius algebras.

4. Cyclic Foam Topological Field Theories

4.1. **Cyclic foams.** *Cyclic foams* Ω is defined by a 4-tuple $(\check{\Omega}, \overrightarrow{\partial\Omega}, \Delta, \varphi)$, where

- $\check{\Omega} = \check{\Omega}(\Omega)$ is a a compact 2-manifold with a boundary $\partial\check{\Omega}$ that consists of pairwise non-intersecting circles; moreover, some of these circles are oriented;
- $\overrightarrow{\partial\Omega} \subset \partial\check{\Omega}$ is the subset of all oriented circles. The rest circles are called *free circles*;
- $\Delta = \Delta(\Omega)$ is a regular graph
- $\varphi : \overrightarrow{\partial\Omega} \to \Delta$ is a *gluing map*, that is, a homeomorphism on any circle and $\varphi(\overrightarrow{\partial\Omega}) = \Delta$.

Here:

- the cyclic foam Ω has a cyclic order; this means that the vertices of any connected component of Δ have cyclic order that is agreed with the orientation of $\varphi(\overrightarrow{\partial\Omega})$;
- the cyclic foam Ω is colored; this means that a color $s(\omega) \in S$ corresponds to each connected component of $\omega \in \check{\Omega}$ and the colors $s(\omega)$ are pairwise different for any connected component of Ω;
- for any connected component $\omega \in \check{\Omega}$ different connected components of $\partial\omega \cap \overrightarrow{\partial\Omega}$ are mapped by φ to different connected components of Δ;
- consider a tuple disks $\tilde{\Omega}$ with $\partial\tilde{\Omega} = \overrightarrow{\partial\Omega}$ and colors and orientation generated by colors and orientation of $\overrightarrow{\partial\Omega}$, then the cluing map φ generate a film surface $\dot{\Omega}$;
- a finite set of *marked points* is fixed on Ω:
  (a) marked points from $\check{\Omega} \setminus \partial\check{\Omega}$ are said to be *interior* and form a set Ω_a; a point $a \in \Omega_a$ is equipped with a local orientation and the color $s(a) = s(\omega)$ for $a \in \omega \in \check{\Omega}$;
  (b) remaining marked points form a set Ω_b of *vertices* of Ω; they are all the vertices of Δ and the marked points on the free circles; a vertex graph σ_q for the vertex $q \in \Delta$ is defined as vertex graph for $q \in \dot{\Omega}$; we assume that each free circle contains a vertex; the graph of this vertex q is a segment with an orientation (local orientation of the marked point) and the color $s(q) = s(\omega)$ for $q \in \omega \in \check{\Omega}$.

Thus the family of cyclic foams contains the family of film surfaces defined in section 2 and marked compact 2-manifolds with boundary considered in [1].

We assume 3 types of cuts:

- a contour-cut, that is a simple closed contour $\gamma \in (\check{\Omega} \setminus \partial\check{\Omega})$;
- a segment-cut, a connects without self intersections segment $\gamma \in \check{\Omega}$ with ends on free contours and without other intersection with $\partial\check{\Omega}$, that divide the set of marked points if it divide Ω;
- a regular graph $\gamma \in \Omega$ that generate a graph-cut on $\hat{\Omega}$.

Denote by $I_s \in \Sigma$ the isomorphism class of oriented segments of color s. Then $I_s^* = I_s$. Consider families of finite dimensional vector spaces $\{A_s | s \in S\}$ and $\{B_\sigma | \sigma \in \Sigma\}$. Fix families of tensors $\{K_s^\otimes \in A_s \otimes A_s | s \in S\}$ and $\{K_\sigma^\otimes \in B_\sigma \otimes B_{\sigma^*} | \sigma \in \Sigma\}$. Fix families of elements $\{1_{A_s}, U_s \in A_s | s \in S\}$ and $\{1_{B_{I_s}} \in B_{I_s} | s \in S\}$. Fix families of involutions $\{*_s : A_s \to A_s | s \in S\}$ and $\{*_{I_s} : B_{I_s} \to B_{I_s} | s \in S\}$.

Define a functor $\mathcal{V}$ from the category of cyclic foams to the category of vector spaces. This functor extends the functor on film surfaces form section 2, and the functor on marked compact 2-manifolds with boundary considered in [1].

The functor $\mathcal{V}$ associates the vector space $V_\Omega = (\otimes_{p\in\Omega_a} A_{s_p}) \otimes (\otimes_{q\in\Omega_b} B_q)$ to any cyclic foam Ω. Here A_p is a copy of $A_{s(p)}$, and B_q is a copy of B_{σ_q}. We are going to describe all morphisms of a monoidal category of cyclic foams and morphisms of the category of cyclic foams that correspond to it.

(1) *Isomorphism.* Let $\phi : \Omega \to \Omega'$ be a homeomorphism of cyclic foams preserving colors, orientations and other structures. Define $\mathcal{V}(\phi) = \phi_* : V_\Omega \to V_{\Omega'}$ as a linear operator generated by the bijections $\phi|_{\Omega_a} : \Omega_b \to \Omega'_a$ and $\phi|_{\Omega_b} : \Omega_b \to \Omega'_b$.

(2) *Cut.* Let Ω be a connected film surface and $\gamma \subset \Omega$ be a cut.

a) Let $\gamma \subset \omega \in \check{\Omega}$ be a non-coorientable contour-cut. It is presented by a simple coorientable contour γ' on the closure $\overline{\Omega \setminus \gamma}$ of $\Omega \setminus \gamma$. Contracting γ' to a point p' with arbitrary local orientation gives the cyclic foam Ω', where $V_{\Omega'} = V_\Omega \otimes A_{s(\omega)}$. We associate the morphism $\mathcal{V}(\eta)(x) = \eta_*(x) = x \otimes U_{s(\omega)}$ to the morphism $\eta : \Omega \to \Omega'$

(b) Let $\gamma \subset \omega \in \check{\Omega}$ be a coorientable contour-cut. It is presented by simple contours γ_+ and γ_- on the closure $\overline{\Omega \setminus \gamma}$ of $\Omega \setminus \gamma$. Contracting γ_+ and γ_- gives points $p_+ =$ and p_-. We assume that their local orientation are not generated by an orientations of γ. Thus we have a cyclic foam Ω' and $V_{\Omega'} = V_\Omega \otimes A_{s(\omega)} \otimes A_{s(\omega)}$. We assume the morphism $\mathcal{V}(\eta)(x) = \eta_*(x) = x \otimes K^{\otimes}_{s(\omega)}$ to the morphism $\eta : \Omega \to \Omega'$.

(c) If $\gamma \subset \omega \in \check{\Omega}$ be a segment-cut, then we define the result of the cutting by γ and the value of the functor on it by analogy with case b) changing K_s by K_{I_s}.

(d) If $\gamma \subset \omega \in \check{\Omega}$ be a graph-cut, then we define the result of the cutting by γ and the value of the functor on it by analogy with section 2.

(3) *Addition of marked point.*

(a)Let us add a not marked point $p \in \check{\Omega} \setminus \partial\check{\Omega}$ with a local orientation to the set Ω_a. This operation generates a morphism $\xi : \Omega \to \Omega'$, where $V_{\Omega'} = V_\Omega \otimes A_{s(\omega)}$ and $p \in \omega \in \check{\Omega}$. Associate the morphism $\mathcal{V}(\xi)(x) = \xi_*(x) = x \otimes 1_{s(\omega)}$ to it.

(b) Similarly, let us add a not marked point $q \in \overrightarrow{\partial\Omega} \setminus \partial\check{\Omega}$ with a local orientation to the set Ω_b. This operation generates a morphism $\xi : \Omega \to \Omega'$, where $V_{\Omega'} = V_\Omega \otimes B_{I_s(\omega)}$and $p \in \omega \in \check{\Omega}$. Associate the morphism $\mathcal{V}(\xi)(x) = \xi_*(x) = x \otimes 1_{B_{Is(\omega)}}$ to it.

(4) *Change of local orientations of marked points.*

Let $\psi : \Omega \to \Omega'$ be a morphism of change of a local orientation of a marked point $p \in \Omega_a$ or $q \in \Omega_b$. It generates an involution $*_{s(p)} : A_p \to A_p$ or $*_{I_{s(q)}} : B_q \to B_q$ and thus the homomorphism $\mathcal{V}(\phi) = \psi_* : V_\Omega \to V_{\Omega'}$.

(5) The tensor product in $\mathcal{S}$ defined by the disjoint union of surfaces $\Omega' \otimes \Omega'' \to \Omega' \coprod \Omega''$ induces the tensor product of vector spaces $\theta_* : V_{\Omega'} \otimes V_{\Omega''} \to V_{\Omega' \sqcup \Omega''}$.

The functorial properties of $\mathcal{V}$ can be easily verified.

4.2. **Topological Field Theory.** Fix families of vector spaces $\{A_s | s \in S\}$ and $\{B_\sigma | \sigma \in \Sigma\}$ and families of tensors, elements and involutions defining the functor $\mathcal{V}$.

A family of linear forms $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$, defined for all cyclic foams $\Omega \in \mathcal{S}$, is called a *Cyclic Foam Topological Field Theory* if it satisfies the following axioms:

1° *Topological invariance.*

$$\Phi_{\Omega'}(\phi_*(x)) = \Phi_\Omega(x)$$

for any isomorphism $\phi : \Omega \to \Omega'$ of cyclic foams.

2° *Non-degeneracy.*

Let Ω (respectively Ω^*) be a sphere with exactly two marked locally oriented points, where their orientations are induced by an orientation of the sphere (respectively, there does not exist an orientation that induces the orientations of the points). Define the bilinear forms $(.,.)_s, (.,.)^*_s : A_s \times A_s \to \mathbb{K}$ by $(x', x'')_s = \Phi_\Omega(x'_{p_1} \otimes x''_{p_2})$ and $(x', x'')^*_s = \Phi^*_\Omega(x'_{p_1} \otimes x''_{p_2})$.

Define also the bilinear forms $(.,.)_{I_s}, (.,.)^*_{I_s} : B_{I_s} \times B_{I_s} \to \mathbb{K}$. Their definitions are similar to the definition of $(.,.)_s, (.,.)^*_s$ after the change of the sphere by the disk and interior marked points by vertices. Axiom 2° says, that the forms $(.,.)_s, (.,.)^*_s$, $(.,.)_{I_s}, (.,.)^*_{I_s}$ and the forms $(.,.)_\sigma$ subsection 2.2 are non-degenerate.

Let Ω be a film surface with only two vertices q_1, q_2. Then $\sigma_{q_2} = \sigma^*$ if $\sigma_{q_1} = \sigma$. Denote by $(.,.)_\sigma$ the bilinear form $(.,.)_\sigma : B_\sigma \times B_{\sigma^*} \to \mathbb{K}$, where $(x', x'')_\sigma = \Phi_\Omega(x'_{q_1} \otimes x''_{q_2})$. Axiom 2° asserts that the forms $(.,.)_\sigma$ are non-degenerated for all $\sigma \in \Sigma$.

3° *Cut invariance.*

$$\Phi_{\Omega'}(\eta_*(x)) = \Phi_\Omega(x)$$

for any cut morphism $\eta : \Omega \to \Omega'$ of film surfaces.

4° *Invariance under addition of a marked point.*

$$\Phi_{\Omega'}(\xi_*(x)) = \Phi_\Omega(x)$$

for any morphism of addition of a marked point $\xi : \Omega \to \Omega'$ of film surfaces.

5° *Invariance under a change of local orientations.*

$$\Phi_{\Omega'}(\psi_*(x)) = \Phi_\Omega(x)$$

for any morphism of change of the local orientation of a marked point $\psi : \Omega \to \Omega'$.

6° *Multiplicativity.*

$$\Phi_\Omega(\theta^*(x' \otimes x')) = \Phi_{\Omega'}(x')\Phi_{\Omega''}(x'')$$

for $\Omega = \Omega' \cup \Omega''$, $x' \in V_{\Omega'}$, $x'' \in V_{\Omega''}$ and the morphism of tensor product $\theta : \Omega' \times \Omega'' \to \Omega$.

Note that a Topological Field Theory defines the families of tensors, elements and involutions, defining the functor $\mathcal{V}$. This follows from lemma 2.1 and [[3] lemma 3.1.]

5. Graph-Cardy-Frobenius algebras

5.1. **Definitions.** A 3-tuple $(D, l_D, *_D)$ is called an *equipped Frobenius algebra*(see [4]) if D is an associative Frobenius algebra with unit 1_D, $l_D : D \to \mathbb{K}$ is a linear functional such that the bilinear form$(x_1, x_2)_D = l_D(x_1x_2)$ is non-degenerate and $*_D : D \to D$ is an involution such that $l_D(x^*) = l_D(x)$ and $(x_1x_2)^* = x_2^*x_1^*$ (here and below $x^* = *(x)$).

Consider a basis $\{d_i | i = 1, ..., n\} \subset D$, the matrix $F^D_{ij} = (d_i, d_j)_D$ and the matrix F^{ij}_D inverse to F^D_{ij}. The elements $K_D = F^{ij}_D d_i d_j$ and $K^*_D = F^{ij}_D d_i d^*_j$ are called the *Casimir* and the *twisted Casimir* elements, respectively. They don't depend on the choice of the basis.

We say that a pair of equipped Frobenius algebras $((A, l_A, *_A), (B, l_B, *_B))$, a homomorphism $\phi : A \to B$ and an element $U \in A$ form a *Cardy-Frobenius algebra* if

- A is commutative and the image $\phi(A)$ belongs to the centre of B;
- $\phi(x^*) = (\phi(x))^*$;
- $(\phi^*(x), \phi^*(y))_A = \mathsf{tr}\, W_{x,y}$, where $x, y \in B$, $(a, \phi^*(b)) = (\phi(a), b)_B$, $W \in \mathsf{End}(B)$ and $W(z) = xzy$;
- $U^2 = K^*_A$ and $\phi(U) = K_B$.

It is proved in [1] that Cardy-Frobenius algebras are in one-to-one correspondence with Klein Topological Field Theories that are Topological Field Theories on 2-dimensional manifolds with boundary. The paper [1] also contains a complete classification of semi-simple Cardy-Frobenius algebras.

Define now a *graph-Cardy-Frobenius algebra* as a family that consists of:

- a family of Cardy-Frobenius algebras $\{(A^s, l^s_A, *^s_A), (B^s, l^s_B, *^s_B), \phi^s, U^s | s \in S\}$
- a graph-Frobenius algebra $B_\star$ with a bilinear form $(.,.)_B : B_\star \times B_\star \to \mathbb{K}$, and a three-linear form $(.,.,.)_B : B_\star \times B_\star \times B_\star \to \mathbb{K}$
- a family of homomorphisms $\{\phi^s_\sigma : A^s \to \mathsf{End}(B_\sigma) | s \in S, \sigma \in \Sigma\}$, where $\phi^s_\sigma = 0$ if s is not the color of an edge of σ.

Here:

- B^s coincides with $B_{I_s} \subset B_\star$ and $\phi^s_{I^s}(a)(b) = \phi^s(a)b$ for $a \in A^s$, $b \in B^s$;
- $(\phi^s_{\sigma_1}(a)(x_1), x_2)_B = (x_1, \phi^s_{\sigma_2}(a)(x_2))_B$, where $a \in A^s$, $x_i \in B_{\sigma_i}$;
- $(\phi^s_{\sigma_1}(a)(x_1), x_2, x_3)_B = (x_1, \phi^s_{\sigma_2}(a)(x_2), x_3)_B = (x_1, x_2, \phi^s_{\sigma_3}(a)(x_3))_B$, where $a \in A^s$, $x_i \in B_{\sigma_i}$.

5.2. **One-to-one correspondence.** Let $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ be a Cyclic Foam Topological Field Theory. Then its restriction to 2-dimensional manifolds with boundary forms a Klein Topological Field Theory $\mathcal{F}_K$ and, therefore, a family of Cardy-Frobenius algebras $\{((A^s, l^s_A, *_{A^s}), (B^s, l^s_B, *_{B^s}), \phi^s, U^s) | s \in S\}_\mathcal{F}$. The restriction to film surfaces forms Film Topological Field Theory $\mathcal{F}_N$ and, therefore, a graph-Frobenius algebra $(B_\star, (.,.)_B, (.,.,.)_B)_\mathcal{F}$.

Let us define the homomorphisms $\{\phi^s_\sigma : A^s \to \mathsf{End}(B_\sigma) | s \in S, \sigma \in \Sigma\}_\mathcal{F}$. Let Ω be a cyclic foam with two vertices q_1, q_2 and one interior marked point p. Put $\sigma = \sigma_{q_1}$. The functional Φ_Ω generates the homomorphism ϕ^s_σ of A_s to the space E of linear functionals on $B_\sigma \otimes B_{\sigma^*}$. The bilinear form $(.,.)_\sigma$ generates the isomorphism between B_{σ^*} and the space of linear functionals on B_σ. Thus we can identify E with $\mathsf{Hom}(B_\sigma, B_\sigma)$.

Theorem 5.1. *Let $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ be a cyclic Foam Topological Field Theory on families of vector spaces $\{A_s | s \in S\}$, $\{B_\sigma | \sigma \in \Sigma\}$. Then the Cardy-Frobenius algebras $\{((A^s, l^s_A, *_{A^s}), (B^s, l^s_B, *_{B^s}), \phi^s, U^s) | s \in S\}_\mathcal{F}$, the graph-Frobenius algebra $(B_\star, (.,.)_B, (.,.,.)_B)_\mathcal{F}$ and the homomorphisms $\{\phi^s_\sigma : A^s \to \mathsf{End}(B_\sigma) | s \in S, \sigma \in \Sigma\}_\mathcal{F}$ form a graph-Cardy-Frobenius algebra.*

Proof. The properties $B^s = B_{I_s}$, and $\phi^s_{I^s}(a)(b) = \phi^s(a)b$ follow from the corresponding axiom. Let us prove that $(\phi^s_{\sigma_1}(a)(x_1), x_2)_B = (x_1, \phi^s_{\sigma_2}(a)(x_2))_B$. Consider a cyclic foam Ω that is the film surface $\Omega(\sigma_1, \sigma_2)$ with an interior marked point p of color s. Then the cut axiom gives $(\phi^s_{\sigma_1}(a)(x_1),x_2)_B=\Phi_\Omega(a\otimes x_1\otimes x_2)$ and $(x_1,\phi^s_{\sigma_2}(a)(x_2))_B=\Phi_\Omega(a\otimes x_1\otimes x_2)$. A prove of the identities $(\phi^s_{\sigma_1}(a)(x_1),x_2,x_3)_B=(x_1,\phi^s_{\sigma_2}(a)(x_2),x_3)_B=(x_1,x_2,\phi^s_{\sigma_3}(a)(x_3))_B$ is similar.

□

Theorem 5.2. *The correspondence from theorem 5.1 generates a one-to-one correspondence between Cyclic Foam Topological Field Theories and isomorphism classes of graph-Cardy-Frobenius algebras.*

Proof. Let $(\{((A^s,l^s_A,*_{A^s}),(B^s,l^s_B,*_{B^s}),\phi^s,U^s)|s\in S\},\ (B_\star,(.,.)_B,(.,.,.)_B),\ \{\phi^s_\sigma:A^s\to\mathsf{End}(B_\sigma)|s\in S,\sigma\in\Sigma\})$ be a graph-Cardy-Frobenius algebra. Let us construct a Cyclic Foam Topological Field Theory $\mathcal{F}=\{\Phi_\Omega:V_\Omega\to\mathbb{K}\}$ that generates it. According to the cut axiom and the axiom $\phi^s_{I^s}(a)(b)=\phi^s(a)b$, the Theory $\mathcal{F}$ is defined by its restrictions to:

- 2-dimensional manifolds with boundary and arbitrary number of marked points;
- film surfaces without interior marked points;
- film surfaces with two vertices and one interior marked point.

According to [1], Topological Field Theories on 2-dimensional manifolds with boundary and arbitrary number of marked points are in one-to-one correspondence with isomorphism classes of Cardy-Frobenius algebras $(\{((A^s,l^s_A,*_{A^s}),(B^s,l^s_B,*_{B^s}),\phi^s,U^s)|s\in S\}$. According to 3.2, Topological Field Theories on film surfaces without interior marked points are in one-to-one correspondence with isomorphism classes of graph-Frobenius algebras $(B_\star,(.,.)_B,(.,.,.)_B)$. Define the value of $\mathcal{F}$ on surfaces Ω with two vertices and one interior marked point by $\Phi_\Omega(a\otimes x_1\otimes x_2)=(\phi^s_\sigma(a)(x_1),x_2)_B$. The properties $\phi^s_{I^s}(a)(b)=\phi^s(a)b$, $(\phi^s_{\sigma_1}(x_1),x_2)_B=(x_1,\phi^s_{\sigma_2}(x_2))_B$ and $(\phi^s_{\sigma_1}(a)(x_1),x_2,x_3)_B=(x_1,\phi^s_{\sigma_2}(a)(x_2),x_3)_B=(x_1,x_2,\phi^s_{\sigma_3}(a)(x_3))_B$ guarantee that the values of $\mathcal{F}$ satisfy the axiom of Cyclic Foam Topological Field Theory.

□

Remark 5.1. *The category of cyclic foams contain the subcategory of* oriented foams $\Omega=(\check{\Omega},\overrightarrow{\partial\Omega},\Delta,\varphi)$, *where the orientation of* $\overrightarrow{\partial\Omega}$ *generate orientations of edges of* Δ*. Our constructions make possible to define the Topological Field Theories for oriented foams and to prove that these Topological Field Theories one-to-one correspond to the analog of graph-Cardy-Frobenius algebras where arbitrary colored graphes change to bipartite colored graphs.*

6. Examples of Cyclic Foam Topological Field Theories

In this section we construct an example of Cyclic Foam Topological Field Theory. Its restriction to 2-dimensional manifolds with boundary is the Klein Topological Field Theory of regular covering, constructed in [4].

Associate a group G_s of a set X_s and an action of G_s on X_s to any color. Consider the vector space A_s which is the center of the group algebra of G_s. Associate $X_{\tilde{S}}=\times_{s\in S}X_s$ to any finite subset $\tilde{S}\subset S$. The actions of G_s on X_s generate the action of $G=\bigoplus_{s\in S}G_s$ on $X_{\tilde{S}}$.

Let $L=L(\tilde{\sigma})$ be the set of edges of a colored graph $\tilde{\sigma}$. Let $s(l)$ be the color of $l\in L$. Denote by $\tilde{\sigma}^X$ the set of all maps $\psi:L\to X_{s(L)}\times X_{s(L)}$, where $\psi(l)\in X_{s(l)}\times X_{s(l)}$. Define the action of G on $\tilde{\sigma}^X$ by $g(\psi(l))=g(x')\times g(x'')$ for $\psi(l)=x'\times x''$. Let $\tilde{\sigma}^{X_G}$ be the set of orbits of this action.

A pair $(\tilde{\sigma}, \psi_G)$, where $\tilde{\sigma}$ is a colored graph and $\psi_G \in \tilde{\sigma}^{X_G}$, is called an *equipped colored graph*, or a colored graph with *equipment* ψ_G. An isomorphism $\varphi : \tilde{\sigma}_1 \to \tilde{\sigma}_2$ of colored graphs is called an *isomorphism* of the equipped colored graphs $(\tilde{\sigma}_i, \psi^i_G)$ if it takes ψ^2_G to ψ^1_G. Denote by $|\operatorname{Aut}(\sigma, \psi_G)|$ the order of the group $\{g \in G | g\psi = \psi\}$, where $\psi \in \psi_G \in \tilde{\sigma}^{X_G}$ and $\tilde{\sigma} \in \sigma \in \Sigma$.

Consider the set $E_{\tilde{\sigma}}$ of equipments of a colored graph $\tilde{\sigma}$. Isomorphisms of colored graphs generate the canonical bijections between the corresponding sets $E_{\tilde{\sigma}}$. Thus we can associate the set E_σ to any $\sigma \in \Sigma$. Let B_σ be a vector space generated by E_σ. Denote by $* : B_\sigma \to B_{\sigma^*}$ the involution generated by changing orientation of σ and changing of the components of $X_s \times X_s$.

Construct a Cyclic Foam Topological Field Theory with families of vector spaces $\{A_s | s \in S\}$, $\{B_\sigma | \sigma \in \Sigma\}$, by its restriction to

- 2-dimensional manifolds with boundary and arbitrary number of marked points;
- film surfaces without interior marked points;
- film surfaces with two vertices and one interior marked point.

We start with its description of $\mathcal{F}$ on film surfaces. Consider an additional structure on film surfaces. Let $V = V(\Omega)$ be the set of edges of a film surface Ω. Denote by $S(v)$ the set of the colors of the disks that are incident to $v \in V$. Consider the set Ω^X of maps $\psi : V \to \bigcup_{v \in V} X_{S(v)}$, where $\psi(v) \in X_{S(v)}$. Denote by Ω^X the set of orbits for the action of G on Ω^X.

A pair (Ω, ψ_G), where Ω is a colored graph and $\psi_G \in \Omega^{X_G}$, is called an *equipped film surface* or film surface with *equipment* ψ_G. An isomorphism $\varphi : \Omega_1 \to \Omega_2$ of film surfaces is called an *equivalence* of the equipment film surfaces (Ω_i, ψ^i_G) if it takes ψ^2_G to ψ^1_G. Denote by $|\operatorname{Aut}(\Omega, \psi_G)|$ the order of the group $\{g \in G | g\psi = \psi\}$, where $\psi \in \psi_G \in \Omega^{X_G}$. An equipment ψ of the film surface Ω generates an equipment ψ_σ for the graph σ of any vertex of Ω. We assume that $\psi_\sigma(l) = (x^1_{s(l)}, x^2_{s(l)})$, where $l \in L(\sigma)$ is an oriented edge from $v^1 \in V(\Omega)$ to $v^2 \in V(\Omega)$ and $\psi(v^i) = \times_{s \in S(v^i)} x^i_s$.

We say that a connected equipped film surface (Ω, ψ_G) is a *compatible surface* for equipped colored graphes $\varsigma_1, \varsigma_2, ..., \varsigma_n$ if these graphes are the vertex graphs of Ω and the numeration the graphs σ_i generates the cyclic order of vertexes of film surface Ω. Denote by $\Psi(\varsigma_1, \varsigma_2, ..., \varsigma_n)$ the set of all isomorphism classes of compatible surfaces for $\varsigma_1, \varsigma_2, ..., \varsigma_n$.

Define a set of linear functionals $\mathcal{F}_N = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ on connected equipped film surfaces by $\Phi_\Omega(\varsigma_1 \otimes \varsigma_2 \otimes ... \otimes \varsigma_n) = \sum_{\Psi \in \Psi(\varsigma_1, \varsigma_2, ..., \varsigma_n)} \frac{1}{|\operatorname{Aut}(\Psi)|}$.

Lemma 6.1. *The set $\mathcal{F}_N = \{\Phi_\Omega : V_\Omega \to \mathbb{K}\}$ generates a Film Topological Field Theory.*

Proof It follows from our definition that $(\varsigma_1, \varsigma_2^*)_\sigma = \frac{\delta_{\varsigma_1, \varsigma_2^*}}{|\operatorname{Aut}(\varsigma_1)|}$ for $\varsigma_1, \varsigma_2 \in E_\sigma$, and thus $K_\sigma = \sum_{\varsigma \in E_\sigma} |\operatorname{Aut}(\varsigma)| \varsigma \otimes \varsigma^*$. Let us prove the cut invariance.

Consider an equipment film surface $\Psi \in \Psi(\varsigma_1, \varsigma_2, ..., \varsigma_n) = (\Omega(\sigma_1, \sigma_2, ..., \sigma_n), \psi_G)$. Let η be the cut morphism by graph-cut $\gamma \subset \Omega(\sigma_1, \sigma_2, ..., \sigma_n)$. It associates the pair of film surfaces $\Omega(\sigma_1, \sigma_2, ..., \sigma_k, \sigma')$, $\Omega(\sigma'', \sigma_{k+1}, \sigma_2, ..., \sigma_n)$ to the film surface $\Omega(\sigma_1, \sigma_2, ..., \sigma_n)$. Any equipment of $\Omega(\sigma_1, \sigma_2, ..., \sigma_n)$ generates equipments of $\Omega(\sigma_1, \sigma_2, ..., \sigma_k, \sigma')$, $\Omega(\sigma'', \sigma_{k+1}, \sigma_2, ..., \sigma_n)$. Thus we receive equipment film surfaces Ψ' и Ψ'' and an equipment of σ'.

Fix an equipped colored graph $\varsigma' = (\sigma', \psi'_G)$ and consider the set of equipped film surfaces $\Psi_{\varsigma'}(\varsigma_1, \varsigma_2, ..., \varsigma_n) \subset \Psi(\varsigma_1, \varsigma_2, ..., \varsigma_n)$ that generate the equipment ψ'_G on σ'.

Then $\sum_{\Psi\in\Psi_{\varsigma'}(\varsigma_1,\varsigma_2,...,\varsigma_n)}\frac{1}{|\mathsf{Aut}(\Psi)|} = \frac{|\mathsf{Aut}(\varsigma)|}{|\mathsf{Aut}(\Psi')||\mathsf{Aut}(\Psi'')|}$. Summations over all equipments ψ'_G of σ' gives $\Phi_\Omega(\varsigma_1\otimes\varsigma_2\otimes...\otimes\varsigma_n) = \sum_{\Psi\in\Psi_\sigma(\varsigma_1,\varsigma_2,...,\varsigma_n)}\frac{1}{|\mathsf{Aut}(\Psi)|} = \sum_{\varsigma\in E_\sigma}\frac{|\mathsf{Aut}(\varsigma)|}{|\mathsf{Aut}(\Psi_+)||\mathsf{Aut}(\Psi_-)|} = \Phi_{\Omega'}\eta_*(\varsigma_1\otimes\varsigma_2\otimes...\otimes\varsigma_n)$.
□

It follows from the previous section that $\mathcal{F}_N$ is generated and is defined by the graph-Frobenius algebra B_*. This algebra has the basis $E=\bigcup_{\sigma\in\Sigma}E_\sigma$ and is defined by polilinear forms

- $(\varsigma_1,\varsigma_2)_B = \sum_{\Psi\in\Psi(\varsigma_1,\varsigma_2)}\frac{1}{|\mathsf{Aut}(\Psi)|} = \frac{\delta_{\varsigma_1,\varsigma_2^*}}{|\mathsf{Aut}(\varsigma_1)|}$ for $\varsigma_1,\varsigma_2\in E$;
- $(\varsigma_1,\varsigma_2,\varsigma_3)_B = \sum_{\Psi\in\Psi(\varsigma_1,\varsigma_2,\varsigma_3)}\frac{1}{|\mathsf{Aut}(\Psi)|}$ for $\varsigma_1,\varsigma_2,\varsigma_3\in E$.

Define now the action of the group algebra A_s on B_σ. It is identical if there are no edges of color s between the edges of σ. Let $s(l)=s$, $\psi\in\psi_G\in\sigma^{X_G}$, $\psi(l)=(x',x'')$ and $a=\Sigma_{g\in G}\lambda_g g$. Then we assume that $\phi^s_\sigma(a)(\psi)=(\Sigma_{g\in G}\lambda_g gx',x'')$ on l and $\phi^s_\sigma(a)(\psi)=\psi$ on the other edges of σ. The function $\phi^s_\sigma(a)(\psi)$ depends only on its orbit ψ_G, and thus generate the linear operator $\phi^s_\sigma(a): B_\sigma\to B_\sigma$.

Define now the system of linear operators $\mathcal{F}_C=\{\Phi_\Omega: V_\Omega\to\mathbb{K}\}$ on cyclic foams with two vertices by $\Phi_\Omega(a^1\otimes...\otimes a^r\otimes x_1\otimes x_2)=(\phi^{s(a^1)}_\sigma(a^1)...\phi^{s(a^r)}_\sigma(a^r)(x_1),x_2)_B$.

Define a system of linear operators $\mathcal{F}_s=\{\Phi_\Omega: V_\Omega\to\mathbb{K}\}$ on 2-dimensional manifolds with boundary of color s. We set it to be the Klein Topological Field Theory of G_s-regular covering with trivial stationary subgroup from [4].

Theorem 6.1. *There exists a unique Cyclic Foam Topological Field Theory $\mathcal{F}$ with the restrictions $\mathcal{F}_N$, $\mathcal{F}_C$ u $\mathcal{F}_s$.*

Proof It follows from lemma 6.1 and [4] that the families $\mathcal{F}_N$ and $\mathcal{F}_s$ satisfy the axioms of Cyclic Foam Topological Field Theory. By our definitions, the value of $\mathcal{F}_C$ on Ω is equal to the product of the values $\mathcal{F}_s$ on the disks that form Ω. Thus $\mathcal{F}_C$ also satisfies the axioms of Cyclic Foam Topological Field Theory. Moreover, the families $\mathcal{F}_N$, $\mathcal{F}_C$, and $\mathcal{F}_s$ coincide on common areas of definition.

To define $\mathcal{F}$ on an arbitrary cyclic foam one can use the cut axiom and cut the surface into 2-dimensional manifolds with boundary, film surfaces without interior marked points and surfaces with two vertices. The result does not depend on the cut system because any two such systems are different only on 2-dimensional manifolds with boundary, film surfaces without interior marked points or surfaces with two vertices.
□

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