ONE CLASS OF WILD BUT BRICK-TAME MATRIX PROBLEMS

LESYA BODNARCHUK AND YURIY DROZD

To the memory of A.V. Roiter

Abstract. We present a class of wild matrix problems (representations of boxes), which are "*brick-tame*," i.e. only have one-parameter families of *bricks* (representations with trivial endomorphism algebra). This class includes several boxes that arise in study of simple vector bundles on degenerations of elliptic curves, as well as those arising from the coadjoint action of some linear groups.

Contents

1. Introduction

Tame–wild dichotomy theorem asserts that any finitely dimensional algebra or a Roiter box is either tame or wild, i.e. either indecomposable representations of any fixed vector dimension form at most finitely

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many one-parameter families or their description contains that of representations of any finitely generated algebra [D79, CB, D01]. In the latter ("wild") case there is no chance to get a more or less reasonable classification of *all* representations. Nevertheless, there are some wild algebras and boxes, where one can get a good description of the so called "*bricks*", i.e. representations with only scalar endomorphisms. Such algebras and boxes appear, for instance, in the theory of unitary representations of Lie groups [D92, BDF] and in the study of vector bundles on degenerations of elliptic curves [BD, Bod].

In this paper we consider a rather wide class of boxes (called BT-boxes), which, though being wild, behave well under the so called "small reduction" in the sense of [D92]. It implies that the set of bricks of any fixed vector dimension is either empty or form one one-parameter family. This class of boxes includes, in particular, the boxes that have appeared in our study of vector bundles on Kodaira fibers in [BD, Bod], so that in these cases the bricks correspond to *simple vector bundles*. Thus BT-boxes play the key role in the classification of simple vector bundles on Kodaira fibers in the same way as the "*bunches of chains*" do in the description of *all* vector bundles on Kodaira cycles [DG, BBDG], and this paper gives the representation-theoretic background for such applications.

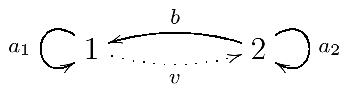
The following conjecture (due to Claus Ringel) provides another motivation for studying such sorts of boxes:

Conjecture. *Let $\mathfrak{A}$ be a finite dimensional algebra or a Roiter box. Then either the bricks over $\mathfrak{A}$ form at most one-parameter families in every fixed vector dimension, or there is a fully faithful exact functor $\Lambda\text{-}\mathsf{mod} \to \mathfrak{A}\text{-}\mathsf{mod}$ for every finitely generated $\Bbbk$-algebra Λ.* (In this case they say that $\mathfrak{A}$ is *fully wild.*)

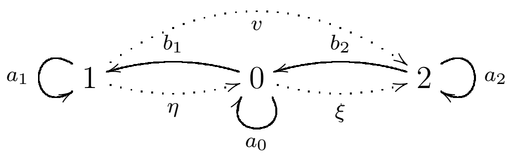
Recall the general method to study representations of boxes, especially effective for tame ones. The idea can be explained as follows. For a given class of representations $\mathcal{C}$ one constructs a reduction morphism $\mathfrak{f} : \mathfrak{A} \to \widetilde{\mathfrak{A}}$ replacing the box $\mathfrak{A}$ by a new one $\widetilde{\mathfrak{A}}$ such that the induced functor $\mathfrak{f}^* : \widetilde{\mathfrak{A}}\text{-}\mathsf{mod} \to \mathfrak{A}\text{-}\mathsf{mod}$ is fully faithful and its image contains all representations from $\mathcal{C}$. Moreover, for representations $M \in \mathcal{C}$ and $\widetilde{M} \in \widetilde{\mathfrak{A}}\text{-}\mathsf{mod}$ such that $M = \mathfrak{f}^*(\widetilde{M})$ one has $\|\widetilde{M}\| < \|M\|$, where $\|M\|$ is the *norm* of M defined in Subsection 2.3. Proceeding this way, we construct a morphism of boxes $\mathfrak{f} = \mathfrak{f}_m\mathfrak{f}_{m-1}\dots\mathfrak{f}_1 : \mathfrak{A} = \mathfrak{A}_1 \to \mathfrak{A}_2 \to \dots \to \mathfrak{A}_m$, such that $\mathcal{C}$ is contained in the image of $\mathfrak{f}^*$ and $\mathfrak{A}_m = (A, V)$ is a *minimal box*, i.e. such that the category A is a direct product of several copies of the field $\Bbbk$ and *rational algebras* $\mathbf{R}_i$, i.e. localizations $\mathbf{R}_i = \Bbbk[t, f_i^{-1}]$ of the polynomial algebra by nonzero polynomials f_i. Indecomposable $\mathbf{R}_i$-modules are *Jordan cells* $J_r(\lambda) = \mathbf{R}_i/(t-\lambda)^r$, where $\lambda \in \Bbbk \setminus \{\text{roots of } f_i\}$. Thus, all indecomposable modules $M \in \mathcal{C}$ are of the form $M = \mathfrak{f}^*(J_k(\lambda))$. For instance, the proof of the tame–wild dichotomy is just constructing, for any non-wild box $\mathfrak{A}$ and any vector dimension $\mathbb{d}$, a reduction $\mathfrak{f}$ like above, where $\mathfrak{C}$ is the set of all representations of vector dimensions $\mathbb{d}' \le \mathbb{d}$.

The following example shows how this method can be applied for bricks.

Example 1.1. Consider the box $\mathfrak{A}$ given, as it is explained below, by the differential biquiver

(1.1)  $\partial(a_1) = bv,\ \partial(a_2) = -vb.$

(We do not write the zero differentials.) It is known to be wild (see the proof of Theorem 4.2). One of the standard reduction steps, the reduction of the minimal edge b (see Subsection 2.5 or [D01]) induces a morphism of boxes $\mathfrak{f} : \mathfrak{A} \to \widetilde{\mathfrak{A}}$, where the box $\widetilde{\mathfrak{A}}$ is given by the differential biquiver

(1.2) 

$$\partial(a_1) = b_1\eta,\ \partial(a_2) = -\xi b_2,\ \partial(a_0) = -\eta b_1 + b_2\xi.$$

In this case $\mathfrak{f}^*$ is an equivalence, hence, $\widetilde{\mathfrak{A}}$ is also wild. Nevertheless, since v does not occur in any differential, if a representation M of $\widetilde{\mathfrak{A}}$ is a brick, either $M_1 = 0$ or $M_2 = 0$. But if we cut from $\widetilde{\mathfrak{A}}$ one of the vertices 1 or 2, we obtain the same box $\mathfrak{A}$. Thus, if we only consider bricks, the box $\mathfrak{A}$ is "*self-reproducing.*" It easily implies a description of bricks [BD]. In particular, if there are bricks of some vector dimension $\mathbb{d}$, they form one family parameterized by the elements $\lambda \in \Bbbk$.

Actually, this procedure is a partial case of "small reduction".

The paper is organized as follows. In Section 2 we fix notations and recall some results concerning representations of boxes and reduction algorithm. In Section 3 we consider bricks and define *brick-tame* boxes. In Section 4 we generalize Example 1.1 introducing BT-boxes and prove their main property, Theorem 4.2, which claims that the boxes of this class are always brick-tame despite being wild (except some trivial cases). Section 5 is devoted to a special class of BT-boxes arising from the adjoint action of linear groups over finite dimensional algebras on the dual spaces of their Lie algebras.

2. Preliminaries

2.1. **Boxes.** Let $\Bbbk$ be an algebraically closed field. Recall that a $\Bbbk$-*category* is a category A such that all morphism sets $A(X, Y)$ are vector spaces over $\Bbbk$, while the multiplication of morphisms is $\Bbbk$-bilinear. In what follows we only consider $\Bbbk$-categories and identify $\Bbbk$-algebras with $\Bbbk$-categories with a unique object. A tuple $\mathfrak{A} = (A, V, \varepsilon, \mu)$ is called a *box* if A is a $\Bbbk$-category and V is a *coalgebra* over A, that is an A-A-bimodule with A-homomorphisms $\varepsilon : V \to A$ (*counit*) and $\mu : V \to V \otimes_A V$ (*comultiplication*) such that

$$(id_V \otimes \mu) \circ \mu = (\mu \otimes id_V) \circ \mu \quad \text{and} \quad (id_V \otimes \varepsilon) \circ \mu = (\varepsilon \otimes id_V) \circ \mu = id_V$$

(under the natural identification of $A \otimes_A V$ and $V \otimes_A A$ with V). Note that any $\Bbbk$-category A can be considered as a box if we set $V = A$ as a bimodule over itself, $\varepsilon = id_A$ and μ being the identification $A \otimes_A A = A$. This box is called *principal* over the category A and is usually identified with this category.

A box $\mathfrak{A} = (A, V)$ is called *free* if A is the path category of a quiver (oriented graph) and the *kernel* of the box $\overline{V} = \ker(\varepsilon)$ is a *free A-bimodule*, i.e. a direct sum of bimodules of the type $A_{ij} = A1_i \otimes 1_j A$, where 1_i denotes the empty path at the vertex i (it is a primitive idempotent of A). We always suppose that the set of vertices of the quiver is $I = \{1, 2, \dots, n\}$ and denote by Q_0 its set of arrows, which we call the *solid arrows* of the box $\mathfrak{A}$. Moreover, we also consider the set of *dotted arrows* Q_1, where the number of dotted arrows from j to i (denoted as $v : j \dashrightarrow i$) equals the number of summands isomorphic to A_{ij} in the kernel $\overline{V}$. In other words, the arrows of Q_1 are in one-to-one correspondence with the *free generators* of the kernel $\overline{V}$, i.e. those coming from the natural generators $1_i \otimes 1_j$ of A_{ij}, and we usually identify them. Thus we obtain a *biquiver* $Q = Q_{\mathfrak{A}} = (I, Q_0, Q_1)$ of the box $\mathfrak{A}$. If p is a path in the biquiver Q, its *degree* $|\,p\,|$ is defined as the number of dotted arrows occurring in p. Thus the path category $\Bbbk Q$ becomes a graded category. We call a free normal box *solid-connected* if the solid part (I, Q_0) of its biquiver is connected (as a graph).

The box $\mathfrak{A} = (A, V)$ is called *normal* (or *group-like*) if there are elements $\omega_i \in V(i, i)$ such that $\varepsilon(\omega_i) = 1_i$ and $\mu(\omega_i) = \omega_i \otimes \omega_i$ for every $i \in \operatorname{Ob} A$. The set $\boldsymbol{\omega} = \{\, \omega_i \mid i \in \operatorname{Ob} A \,\}$ is called a *normal section* of the box $\mathfrak{A}$. Given a normal section, the *differential* ∂ of the box $\mathfrak{A}$ is defined for a solid arrow $a : j \to i$ as $\omega_i a - a\omega_j$ (it belongs to $\overline{V}$) and for a dotted arrow $v : j \dashrightarrow i$ as $\mu(v) - v \otimes \omega_j - \omega_i \otimes v$ (it belongs to $\overline{V} \otimes_A \overline{V}$). This differential extends to a derivation of the graded category $\Bbbk Q$, i.e. to a linear map $\partial : \Bbbk Q \to \Bbbk Q$ of degree 1 such that $\partial^2 = 0$ and the *Leibniz rule* holds:

$$\partial(xy) = \partial(x)y + (-1)^{|x|} x \partial(y).$$

The pair (Q, ∂) is called the *differential biquiver* of the box $\mathfrak{A}$. It completely determines the free normal box $\mathfrak{A}$.

A differential biquiver (Q, ∂) is called *solid-triangular* (respectively, triangular) if there is a map $h : Q_0 \to \mathbb{N}$ (respectively $h : Q_0 \cup Q_1 \to \mathbb{N}$)

such that, for every arrow $a \in Q_0$, (respectively, $a \in Q_0 \cup Q_1$) its differential $\partial(a)$ only contains solid arrows (respectively, arrows) b with $h(b) < h(a)$ (for instance, it never contains a itself). We call the function h the *triangulation for the differential biquiver* (Q, ∂). Certainly, the existence of triangulation can depend on the choice of free generators. A free normal box $\mathfrak{A} = (A, V)$ is called *solid-triangular* (respectively, *triangular*, or a *Roiter box*) if there is a set of free generators for A and V such that the resulting differential biquiver is solid-triangular (respectively, triangular).

2.2. **Representations of boxes.** Let $\mathfrak{A} = (A, V)$ be a box. The category $\mathfrak{A}$-Mod of $\mathfrak{A}$-*modules*, or *representations of* $\mathfrak{A}$, is defined as follows.

- Its *objects* are just A-modules.
- A *morphism* $S : M \to N$ between two representations M and N is a homomorphism of A-modules $V \otimes_A M \to N$.
- The product $S' \circ S$ of two morphisms $S : M \to N$ and $S' : N \to L$ is defined as the composition

$$S'(1 \otimes S)(\mu \otimes 1) : V \otimes_A M \to V \otimes_A V \otimes_A M \to V \otimes_A N \to L.$$

One easily sees that if $\mathfrak{A}$ is the principal box over an algebra A, the category of $\mathfrak{A}$-modules can be identified with that of A-modules, and we always do so.

If $\mathfrak{A}$ is a normal free box, the category of $\mathfrak{A}$-modules can be described in terms of its differential biquiver (Q, ∂). Namely:

- A representation M of $\mathfrak{A}$ is given by two sets:

  $$\{ M_i \mid i \in I \} \quad \text{and} \quad \{ M(a) : M_i \to M_j \mid a \in Q_0,\ a : i \to j \},$$

  where M_i are vector spaces and $M(a)$ are linear maps.
- A morphism $M \to N$ is given by the set of linear maps

  $$\{ S_i : M_i \to N_i \mid i \in I \} \cup \{ S(v) : M_i \to N_j \mid v \in Q_1,\ v : i \dashrightarrow j \},$$

  where $S_i(x) = S(\omega_i \otimes x)$ and $S_v(x) = S(v \otimes x)$ for $x \in M_i$, such that for any solid arrow $a : i \to j$ the following relation holds:

  $$S_j M(a) - N(a) S_i = S(\partial(a)) = \sum \lambda N(p') S(u) M(p),$$

  if $\partial(a) = \sum \lambda p' u p$, where $\lambda \in \Bbbk$, $u \in Q_1$ and p, p' are some solid paths in Q.
- The components of the product $T = S' \circ S$ are defined as follows:

  $$T_i = S'_i S_i,$$
  $$T(v) = S'_j S(v) + S'(v) S_j + \sum \lambda L(p_1) S'(u') N(p_2) S(u) M(p_3),$$

  if $v : i \dashrightarrow j$, $\partial(v) = \sum \lambda p_1 u' p_2 u p_3$, where $\lambda \in \Bbbk$, $u, u' \in Q_1$ and p_1, p_2, p_3 are some solid paths.

The following lemma expresses the main properties of Roiter boxes.

Lemma 2.1. [KR, Ro] *Let $\mathfrak{A}$ be a Roiter box and $M, N \in \mathfrak{A}$-Mod.*

(1) *A morphism $S : M \to N$ is an isomorphism if and only if so are all maps S_i.*
(2) *If $S : M \to M$ is an idempotent, there is a representation N such that S factors as $S = S_1 S_2$, where $S_1 : N \to M,\ S_2 : M \to N$ and $S_2 S_1 = \mathbb{I}_N$ (the identity map of N).*

In other words, all idempotents in the category $\mathfrak{A}$-Mod split, i.e. it is *fully additive* (or *Karoubian*). Note that this lemma does not hold for arbitrary solid-triangular boxes. We call a free normal solid-triangular box $\mathfrak{A}$ *layered* (by Crawley-Boevey [CB]) if the statement (1) of Lemma 2.1 holds for its representations. (In fact, it is a specification of [CB, Definition 3.6] for the case of free boxes.)

From now on we only consider normal free boxes $\mathfrak{A} = (A, V)$ such that in the corresponding biquiver $Q = (I, Q_0, Q_1)$ all sets I, Q_0, Q_1 are *finite*. We call a module $M \in \mathfrak{A}$-Mod *finite dimensional* if all spaces $M(i)$ $(i \in I)$ are finite dimensional, and denote by $\mathfrak{A}$-mod the full subcategory of $\mathfrak{A}$-Mod consisting of finite dimensional modules. Then all spaces $\mathrm{Hom}_{\mathfrak{A}}(M, N)$ are also finite dimensional, therefore, if $\mathfrak{A}$ is a Roiter box, $\mathfrak{A}$-mod is a *Krull–Schmidt* category, i.e. a fully additive category with unique decomposition of objects into direct sums of indecomposable ones.

2.3. **Base change Lemma.** Recall that the *vector dimension* of a representation $M \in \mathfrak{A}$-mod is a tuple $\mathbb{d}(M) = (d_1, \dots, d_n) \in \mathbb{N}^n$, where $d_i = \dim_{\mathbb{k}}(M_i)$. The *norm* of M is defined as $\|M\| = \sum_{i,j} q_{ij} d_i d_j$, where q_{ij} is the number of solid arrows $i \to j$. If we choose bases in all spaces M_i, then $\|M\|$ is just the numbers of coefficients in all matrices defining the maps $M(a)$, where a runs through Q_0. Note that it coincides with the negative part of the *Tits form* of the box $\mathfrak{A}$ as defined, for instance, in [D01].

Now we explain the usual procedures that are the base of the reduction algorithm mentioned in Introduction. The proofs of the statements can be found, for example, in [D01] .

Let $\mathfrak{A} = (A, V)$ and $\mathfrak{B} = (B, W)$ be some boxes. A *morphism* $\mathfrak{f} = (f_0, f_1) : \mathfrak{A} \to \mathfrak{B}$ consists of a functor $f_0 : A \to B$ and a morphism of A-bimodules $f_1 : V \to W$ such that

$$\varepsilon(f_1(v)) = f_0(\varepsilon(v)) \quad \text{and} \quad \mu(f_1(v)) = f_2(\mu(v)),$$

where W is considered as an A-bimodule using the functor f_0[1], and $f_2 : V \otimes_A V \to W \otimes_B W$ is the composition

$$V \otimes_A V \xrightarrow{f_1 \otimes f_1} W \otimes_A W \xrightarrow{\nu} W \otimes_B W,$$

[1] It means that $W(i,j) = W(f_0(i), f_0(j))$ for $i, j \in \mathrm{Ob}\, A$ and $a'xa = f_0(a')xf(a)$ for $x \in W(i,j),\ a : i' \to i,\ a' : j \to j'$,

ν being the natural surjection. Here (and later on) we denote by ε and μ the counit and comultiplication in all boxes that we consider (if it cannot lead to misunderstanding). Such a morphism $\mathfrak{f}$ induces a functor $\mathfrak{f}^* : \mathfrak{B}\text{-mod} \to \mathfrak{A}\text{-mod}$, where $\mathfrak{f}^*M$ is just the composition $M \circ f_0$ (or, the same, we consider the B-module M as A-module using f_0) and, for $S \in \mathrm{Hom}_{\mathfrak{B}}(M,N)$, $\mathfrak{f}^*S$ is the composition

$$V \otimes_A M \xrightarrow{f_1 \otimes 1} W \otimes_A M \xrightarrow{\nu} W \otimes_B M \xrightarrow{S} N,$$

ν being again the natural surjection.

This functor is especially useful in the following situation. Let $\mathfrak{A} = (A,V)$ be a box, $f : A \to B$ be a functor. Set $\mathfrak{A}^f = (B,W)$, where $W = B \otimes_A V \otimes_A B$. It becomes a box under naturally defined counit and comultiplication, and the pair $\mathfrak{f} = (f, f_1)$, where $f_1 : V \to W$ is the natural map, is a morphism of boxes. The following "Base Change Lemma" is the most important tool in constructing reduction algorithms.

Lemma 2.2 (Base Change)**.** *If $\mathfrak{B} = \mathfrak{A}^f$ for a functor $f : A \to B$, then the functor $\mathfrak{f}^* : \mathfrak{A}\text{-mod} \to \mathfrak{B}\text{-mod}$ defined above is fully faithful.*

This lemma is mostly used in the following situation. Let $\mathfrak{A}' = (A',V')$ be a *subbox* of the box $\mathfrak{A} = (A,V)$. It means that A' is a subcategory of A and V' is an A'-subbimodule of V such that $\varepsilon(a) \in V'$ for all $a \in A'$ and $\mu(v) \in \nu(V' \otimes_{A'} V')$ for all $v \in V'$, where again ν is the natural surjection $V \otimes_{A'} V \to V \otimes_A V$. If $\mathfrak{A}$ is a free normal box with the differential biquiver (Q,∂) and Q' is a sub-biquiver of Q such that, for every arrow (solid or dotted) $a \in Q'$, its differential $\partial(a)$ only contains arrows from Q', the box $\mathfrak{A}'$ defined by the biquiver Q' and the differential $\partial|_{Q'}$ is a subbox of $\mathfrak{A}$. In this case we say that $\mathfrak{A}'$ is a *Roiter subbox* of $\mathfrak{A}$. Lemma 2.2, together with the universal property of push-down (amalgamation), imply the following fact.

Corollary 2.3. *Suppose that $\mathfrak{A}' = (A',V')$ is a subbox of the box $\mathfrak{A} = (A,V)$ and a functor $f' : A' \to B'$ is given. Let B be the* amalgamation *of the categories A and B' over A', i.e. the push-down*

$$\begin{array}{ccc} A' & \xrightarrow{\iota} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \longrightarrow & B\,, \end{array}$$

where ι denotes the embedding $A' \hookrightarrow A$. Then the image of the functor $\mathfrak{f}^* : \mathfrak{A}^f\text{-mod} \to \mathfrak{A}\text{-mod}$ consists of the modules M whose restrictions $M|_{A'}$ factor through $\mathfrak{f}'$. In particular, if every $\mathfrak{A}'$-module M' is isomorphic (in $\mathfrak{A}'$-mod) to a module that factors through f', the image of $\mathfrak{f}^*$ is dense, so $\mathfrak{f}^*$ is an equivalence of categories.

Note that, if the subbox $\mathfrak{A}'$ is representation-finite, there is a natural functor $f' : A' \to \mathsf{add}\, B'$, the additive hull of a discrete category B' [2] whose objects are isomorphism classes of indecomposable $\mathfrak{A}'$-modules, and any representation of $\mathfrak{A}'$ is isomorphic to one that factors through f'. This morphism is called the *semisimple approximation* and was introduces in [Aus] for algebras of finite type. (Its extension to representation-finite boxes is obvious.)

We shall use it in the following two situations: *regularization* and *minimal edge reduction*, first introduced in [KR]. We recall some details (see, for instance, [D01]). In both cases $\mathfrak{A}$ is a free normal box with the differential biquiver (I, Q_0, Q_1, ∂) and $\mathfrak{A}'$ is a Roiter subbox of $\mathfrak{A}$.

2.4. **Regularization.** The subbox $\mathfrak{A}'$ has the differential biquiver

$$2 \overset{v}{\underset{b}{\rightrightarrows}} 1, \qquad \partial(b) = v.$$

(In this case they say that the solid arrow b is *superfluous*). Then B' is the discrete category with two objects, which we also denote by 1 and 2, $f'(1) = 1$, $f'(2) = 2$, $f'(b) = 0$. The box $\mathfrak{A}^f$ is obtained from $\mathfrak{A}$ just by deleting the arrows b and v from the biquiver as well as omitting all terms containing these arrows in all formulae for the differential. Note that the case when the vertices 1 and 2 coincide is also possible. We denote the box $\mathfrak{A}^f$ by $\mathfrak{A}^b$. Evidently, this box is solid-triangular (respectively, layered or a Roiter box) if so is $\mathfrak{A}$.

2.5. **Minimal edge reduction.** The subbox $\mathfrak{A}'$ has the differential biquiver $2 \xrightarrow{b} 1$, $\partial(b) = 0$, where 1 and 2 are different vertices. (In this case they say that b is a *minimal edge*.) Take for B' the additive hull of the trivial category with 3 objects $0, 1, 2$, where 0 is a new symbol, and set $f'(1) = 1 \oplus 0$, $f'(2) = 2 \oplus 0$, $f'(b) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then every A'-module factors through f', so Corollary 2.3 can be applied and $\mathfrak{f}^*$ is an equivalence of categories. One can check (cf. [KR] or [D01]) that the box $\mathfrak{A}^f$ can be identified with the additive hull of the box $\mathfrak{B}$ with the differential biquiver $(\widetilde{Q}, \tilde{\partial})$ defined as follows:

- The set of vertices of $\widetilde{Q}$ is $I \cup \{0\}$.
- The set of arrows of $\widetilde{Q}$ consists of:
  - The arrows $x : i \to j$, where $\{\, i, j \,\} \cap \{\, 1, 2 \,\} = \emptyset$.
  - For each arrow $x : i \to j$ (or $j \to i$), where $i \in \{\, 1, 2 \,\}$ and $j \notin \{\, 1, 2 \,\}$, we have two arrows $x_i : i \to j$ and $x_0 : 0 \to j$ (respectively, $x_i : j \to i$ and $x_0 : j \to 0$). Then we set $\mathfrak{f}(x) = \begin{pmatrix} x_i & x_0 \end{pmatrix}$ (respectively, $\mathfrak{f}(x) = \begin{pmatrix} x_i \\ x_0 \end{pmatrix}$).

[2] The category B' is called *discrete* if for $i, j \in \mathrm{Ob}(B')$ we have $B'(i, i) = \Bbbk$ and $B'(i, j) = 0$ if $i \neq j$.

– For each arrow $x : j \to i$, where both $i, j \in \{1, 2\}$ and $x \neq b$, we have four arrows $x_{kl} : l \to k$, where $k \in \{i, 0\}$, $l \in \{j, 0\}$. Then we set $\mathfrak{f}(x) = \begin{pmatrix} x_{ij} & x_{i0} \\ x_{0j} & x_{00} \end{pmatrix}$.

– Two new dotted arrows $\xi : 0 \dashrightarrow 1$ and $\eta : 2 \dashrightarrow 0$.

Certainly, the arrows arising from x are solid or dotted respectively to the sort of x. We also set $\mathfrak{f}(b) = f'(b)$, $\mathfrak{f}(\omega_i) = \omega_i$ if $i \notin \{1, 2\}$, $\mathfrak{f}(\omega_1) = \begin{pmatrix} \omega_1 & 0 \\ \eta & \omega_0 \end{pmatrix}$, $\mathfrak{f}(\omega_2) = \begin{pmatrix} \omega_2 & \xi \\ 0 & \omega_0 \end{pmatrix}$ and extend the map $\mathfrak{f}$ naturally to all elements from A and V.

- The differential $\tilde{\partial}$ is obtained from the rules

$$\mathfrak{f}(\omega_j)\mathfrak{f}(a) - \mathfrak{f}(a)\mathfrak{f}(\omega_i) = \tilde{\partial}(\mathfrak{f}(x)) \text{ for } a : i \to j,$$

$$\mu(\mathfrak{f}(v)) - \mathfrak{f}(v) \otimes \mathfrak{f}(\omega_i) - \mathfrak{f}(\omega_j) \otimes \mathfrak{f}(v) = \tilde{\partial}(\mathfrak{f}(v)) \text{ for } v : i \dashrightarrow j,$$

where all products, as well as tensor products, are calculated by usual matrix rules, while $\tilde{\partial}$ and μ are applied to matrices component-wise.

Therefore, $\mathfrak{A}\text{-mod} \simeq \mathfrak{A}^{\mathfrak{f}}\text{-mod} \simeq \mathfrak{B}\text{-mod}$. We denote the box $\mathfrak{B}$ by $\mathfrak{A}^b$. Again this new box is solid-triangular (respectively, layered or a Roiter box) if so is $\mathfrak{A}$.

The following theorem summarizes the above considerations.

Theorem 2.4 (Kleiner–Roiter)**.** *Let $\mathfrak{A}$ be a free normal box, $b : 2 \to 1$ be either a superfluous arrow or a minimal edge of its differential biquiver. Then there is a free normal box $\mathfrak{A}^b$ and an equivalence of module categories $\mathfrak{f}^b : \mathfrak{A}^b\text{-mod} \to \mathfrak{A}\text{-mod}$ such that $\|\mathfrak{f}^b(M)\| < \|M\|$ whenever $M \simeq \mathfrak{f}^b(N)$ is such that both $M(1) \neq 0$ and $M(2) \neq 0$. Moreover, the box $\mathfrak{A}^b$ is solid-triangular (respectively, layered or a Roiter box) if so is $\mathfrak{A}$.*

We also often need to delete vertices from a free normal box $\mathfrak{A}$. If i is a vertex of the biquiver of $\mathfrak{A}$, we denote by $\mathfrak{A}^i$ the box that is obtained from $\mathfrak{A}$ by deleting the vertex i from its biquiver and omitting all terms in differentials containing arrows starting or ending at i. Obviously, $\mathfrak{A}^i\text{-mod}$ is identified with the full subcategory of $\mathfrak{A}\text{-mod}$ consisting of all modules M with $M_i = 0$.

3. Bricks

Definition 3.1. A representation of a box (in particular, of an algebra) is called a *brick* if it admits no non-scalar endomorphisms. The full subcategory of bricks of $\mathfrak{A}\text{-mod}$ is denoted by $\mathsf{Br}(\mathfrak{A})$. We also denote by $\mathsf{Br}(\mathbb{d}, \mathfrak{A})$ the set of isomorphism classes of bricks of vector dimension $\mathbb{d}$.

Lemma 3.2. *Let $\mathfrak{A}$ be a normal free box with the differential biquiver (Q, ∂) containing a dotted arrow $u : i \dashrightarrow j$ that does not occur in the* *differential of any solid arrow. If $M \in \mathsf{Br}(\mathfrak{A})$, then either $M_i = 0$ or $M_j = 0$. Thus $\mathsf{Br}(\mathfrak{A}) = \mathsf{Br}(\mathfrak{A}^i) \cup \mathsf{Br}(\mathfrak{A}^j)$.*

Proof. If both $M_i \neq 0$ and $M_j \neq 0$, we construct a non-scalar endomorphism S of M setting $S_k = 0$ for all vertices k, $S(v) = 0$ for all dotted arrows $v \neq u$ and taking for $S(u)$ any nonzero linear map $M_i \to M_j$. □

We have actually applied this lemma to the box (1.2) in Example 1.1 of the Introduction.

Since we are going to study bricks instead of indecomposable representations, we have to adapt the classical definition of tameness for our purposes. Recall [D01] that a *rational family* of representations of a box $\mathfrak{A} = (A, V)$ is defined as a functor $\mathcal{F} : A \to \mathsf{add}\,\mathbf{R}$, where $\mathbf{R} = \Bbbk[t, f(t)^{-1}]$ is a rational algebra. Note that $\mathsf{add}\,\mathbf{R}$ can be identified with the category of finitely generated projective $\mathbf{R}$-modules. The $\mathbf{R}$-bricks (or, the same, $\mathsf{add}\,\mathbf{R}$-bricks) are just one-dimensional representations of $\mathbf{R}$, which we identify with the elements $\lambda \in \Bbbk$ such that $f(\lambda) \neq 0$. If for every such λ the $\mathfrak{A}$-module $\mathcal{F}^*(\lambda)$ is a brick and, moreover, $\mathcal{F}^*(\lambda) \not\simeq \mathcal{F}^*(\lambda')$ for all $\lambda \neq \lambda'$, we say that $\mathcal{F}$ is a *rational family of bricks*. We also say that the bricks isomorphic to $\mathcal{F}^*(\lambda)$ *belong to the family* $\mathcal{F}$.

Definition 3.3. A box $\mathfrak{A}$ is called *brick-tame* if for any vector dimension $\mathbb{d}$ there is a finite set Σ of rational families of bricks such that all $\mathfrak{A}$-bricks of vector dimension $\mathbb{d}$, except, possibly, finitely many of them, belong to one of the families from Σ. (Note that we allow the case when there are only finitely many bricks of vector dimension $\mathbb{d}$.)

Obviously, every tame box is brick-tame, but not vice versa: the box (1.1) from the Introduction is wild, but brick-tame.

4. BT-boxes

In this section we introduce a special class of brick-tame boxes that generalizes the boxes from Example 1.1.

Definition 4.1. A solid-triangular box $\mathfrak{A}$ with the differential biquiver (Q, ∂) is said to be of BT-*type*, or a BT-*box*, if Q_0 contains a set of loops (called *distinguished loops*) $\mathfrak{a} = \{\, a_i : i \to i \mid i \in I \,\}$ and there is an injective map $\tilde{}\, : \mathfrak{b} = Q_0 \setminus \mathfrak{a} \to Q_1$, $x \mapsto \tilde{x}$, such that $\tilde{x} : j \dashrightarrow i$ if $x : i \to j$ and

$$\partial(a_i) = \sum_{x \in \hat{\mathfrak{b}}(\cdot, i)} (-1)^{|x|} x\tilde{x} \tag{4.1}$$

for each distinguished loop $a_i \in \mathfrak{a}$, where we set $\tilde{\mathfrak{b}} = \{\tilde{x} \mid x \in \mathfrak{b}\}$, $\hat{\mathfrak{b}} = \mathfrak{b} \cup \tilde{\mathfrak{b}}$ and $\tilde{\tilde{x}} = x$ for each $b \in \mathfrak{b}$.

Both boxes (1.1) and (1.2) from Example 1.1 are BT-boxes. The polynomial algebra $\Bbbk[t]$ is also a BT-box (having only one vertex and one solid arrow, which is automatically a distinguished loop). The following theorem asserts that the BT-boxes are brick-tame despite being wild in general.

Theorem 4.2. *Let $\mathfrak{A}$ be a BT-box.*

(1) *$\mathfrak{A}$ is brick-tame. Moreover, if $\mathsf{Br}(\mathbb{d},\mathfrak{A}) \neq \emptyset$, all bricks of dimension $\mathbb{d}$ belong to a unique rational family $\mathfrak{F}_{\mathbb{d}} : A \to \mathsf{add}\,\Bbbk[t]$.*
(2) *If $\mathfrak{A}$ is solid-connected, has no superfluous arrows and does not coincide with $\Bbbk[t]$, it is wild.*

The claims of the theorem are trivial if $\mathfrak{A} = \Bbbk[t]$. Moreover, we may suppose that $\mathfrak{A}$ is solid-connected. The proof of the theorem is based on several lemmas. *In all of them $\mathfrak{A} = (A, V)$ denotes a solid-connected BT-box that does not coincide with $\Bbbk[t]$, (Q, ∂) is its differential biquiver and $h : Q_0 \to \mathbb{N}$ is a triangularity for this biquiver.* Note that if a is a distinguished loop from $\mathfrak{a}$, then $\partial(a)$ contains some solid arrows b with $h(b) < h(a)$. Hence, if b is an arrow with the minimal value of $h(b)$, it belongs to $\mathfrak{b}$. We then call b an *h-minimal arrow.*

Lemma 4.3. *Suppose that b is an h-minimal arrow in Q_0 and $\partial(b) \neq 0$. Then one can choose free generators of the category A and of the kernel $\overline{V}$ in such a way that*

(1) *$\partial(b) = \tilde{c}$ for some solid arrow c and $\partial(c) = -\tilde{b} + \theta$, where θ does not contain the arrow $\tilde{b}$.*
(2) *$\tilde{b}$ does not occur in $\partial(x)$ for any arrow (solid or dotted) $x \notin \{c, \tilde{b}\}$.*

Moreover, with respect to the new generators $\mathfrak{A}$ remains solid-triangular. We call c the partner *of b.*

Proof. Let $b : i \to j$ (possibly $i = j$). Since $\partial(b)$ cannot contain any solid arrow, we may suppose that $\partial(b) = u + \sigma$, where $u \in Q_1$ and σ is a sum of dotted arrows other than u. Then

$$\partial(a_i) = -\tilde{b}b + \sum_{\substack{x \in \tilde{\mathfrak{b}}(\cdot, i) \\ x \neq \tilde{b}}} (-1)^{|x|} x\tilde{x},$$

$$\partial^2(a_i) = -\partial(\tilde{b})b + \underline{\tilde{b}u} + \tilde{b}\sigma + \sum_{x \neq \tilde{b}} \big((-1)^{|x|}\partial(x)\tilde{x} + x\partial(\tilde{x})\big) = 0.$$

Since the underlined term must vanish, there must be $x \neq \tilde{b}$ such that $u = \tilde{x}$ and $\partial(x) = -\tilde{b} + \theta$, where θ does not contain the monomial $\tilde{b}$. Therefore, $\partial(b) = \sum_k \tilde{x}_k$ for some $x_k : j \to i$. Let $h(x_1) \leq h(x_k)$ for all k. Set $c = x_1$, $x'_k = x_k - x_1$ for $k \neq 1$; $\tilde{c} = \partial(b)$, $\tilde{x}'_k = \tilde{x}_k$ for $k \neq 1$, $h(x'_k) = h(x_k)$. One easily sees that after this change of generators the BT-condition (4.1) as well as the triangularity condition hold, but now $\partial(b) = \tilde{c}$ and $\partial(c) = -\tilde{b} + \theta$, as stated in (1). Note also that $\partial(\tilde{c}) = \partial^2(b) = 0$.

To prove (2) we have to show that it is impossible that

$$\partial(y) = q\tilde{b}p + \phi \tag{4.2}$$

for some arrow $y \neq c$ and some paths p, q, where ϕ does not contain the monomial $q\tilde{b}p$. We prove this claim using induction on the length $l(q)$ of the path q. If $l(q) = 0$, $\partial(y) = \tilde{b}p + \phi$. We first suppose that $y \neq \tilde{b}$. Then

$$\begin{aligned} \partial(a_i) &= -\tilde{b}b + c\tilde{c} + (-1)^{|y|} y\tilde{y} + \sum_{\substack{x \in \hat{\mathfrak{b}}(\cdot, i) \\ x \notin \{\tilde{b}, c, y\}}} (-1)^{|x|} x\tilde{x}, \\ \partial^2(a_i) &= -\partial(\tilde{b})b + \theta c + (-1)^{|y|}(\underline{\tilde{b}p\tilde{y}} + \phi\tilde{y}) + y\partial(\tilde{y}) \\ &\quad + \sum_{x \notin \{\tilde{b}, c, y\}} \big((-1)^{|x|}\partial(x)\tilde{x} + x\partial(\tilde{x})\big) = 0. \end{aligned}$$

But the underlined term cannot vanish, since no other term both starts with $\tilde{b}$ and ends with $\tilde{y}$. If $y = \tilde{b}$, we must omit in these equalities all terms with y and $\tilde{y}$ and replace $\partial(\tilde{b})$ by $\tilde{b}p + \phi$. Then the term $-\tilde{b}pb$ cannot vanish. (This case can happen if $\mathfrak{A}$ is only solid-triangular, but not a Roiter box).

Suppose now that (4.2) is impossible if $l(q) = l - 1$, but holds for an arrow $y \neq \tilde{b}$ and a path q of length l. (The case $y = \tilde{b}$ is handled in the same way, as above.) Then

$$\begin{aligned} \partial^2(a_i) &= -\partial(\tilde{b})b + \theta c + (-1)^{|y|}(\underline{q\tilde{b}p\tilde{y}} + \phi\tilde{y}) + y\partial(\tilde{y}) \\ &\quad + \sum_{x \notin \{\tilde{b}, c, y\}} \big((-1)^{|x|}\partial(x)\tilde{x} + x\partial(\tilde{x})\big) = 0. \end{aligned}$$

Therefore, the underlined term must vanish. It is only possible if there is an arrow x such that $q = xq'$ and $\partial(\tilde{x})$ contains the term $q'\tilde{b}p\tilde{y}$. Since $l(q') = l - 1$, it is impossible, which accomplishes the proof.

Just in the same way one proves that the term θ in $\partial(c)$ cannot contain monomials $q\tilde{b}p$ other than $\tilde{b}$. □

Analogous observations can be applied to the case when $\partial(b) = 0$.

Lemma 4.4. *If $\partial(b) = 0$ for some solid arrow b, the arrow $\tilde{b}$ does not occur in the differential of any arrow x (solid or dotted).*

Proof. It practically coincides with the proof of Lemma 4.3, so we omit the details. □

These lemmas imply several nice properties.

Corollary 4.5. *Let $b : i \to j$ be an h-minimal arrow with $\partial(b) \neq 0$, c be its partner as defined in Lemma 4.3, $B = A/\langle b, c\rangle$, $f : A \to B$ be the natural surjection and $\mathfrak{B} = \mathfrak{A}^f$. Then $\mathfrak{B}$ is also of* BT*-type, the* *functor* $\mathfrak{f}^* : \mathfrak{B}\text{-mod} \to \mathfrak{A}\text{-mod}$ *is an equivalence and, if* $M_1 \neq 0$, $M_2 \neq 0$ *and* $M \simeq \mathfrak{f}^*(N)$*, then* $\|N\| < \|M\|$.

Proof. Set $v = -\partial(c)$. Obviously, we can replace $\tilde{b}$ by v in the set of free generators of $\overline{V}$. Then both b and c become superfluous, so we can use the regularization procedure of Subsection 2.4 for both of them obtaining just the box $\mathfrak{B}$. The images of $b, c, \tilde{c}$ and $v = \phi - \tilde{b}$ become zero in $\mathfrak{B}$. Since $\tilde{b}$ only occurs in differentials of distinguished loops, always in terms $b\tilde{b}$ and $\tilde{b}b$, which disappear in $\mathfrak{B}$, the box $\mathfrak{B}$ is also solid-triangular (with the same triangulation) and of BT-type. The statement now follows from Theorem 2.4. □

Let now $\partial(b) = 0$. First we show that the case when b is a loop actually cannot occur. Recall that we suppose that $\mathfrak{A}$ is solid-connected and does not coincide with $\mathbb{k}[t]$, hence, b cannot be distinguished.

Corollary 4.6. *If* $b : 1 \to 1$ *is a solid loop with* $\partial(b) = 0$ *and* $M \in \mathsf{Br}(\mathfrak{A})$*, then* $M_1 = 0$.

Proof. Suppose that $M_1 \neq 0$. Then

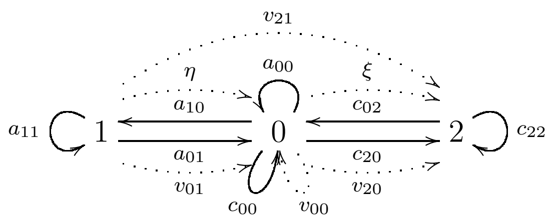
$$\partial(a_1) = b\tilde{b} - \tilde{b}b + \sum_{x \notin \{b, \tilde{b}\}} (-1)^{|x|} x\tilde{x}.$$

Set $S_i = 0$ for all i, $S(v) = 0$ for all $v \in Q_1 \setminus \{\tilde{b}\}$ and $S(\tilde{b}) = \mathbb{I}_{M_1}$. Since $\tilde{b}$ does not occur in any differential of a solid arrow other than a_1, S is a non-scalar endomorphism of M, so M is not am brick. □

The minimal edge reduction described in Subsection 2.4 usually does not give a BT-box if the original one was so. Nevertheless, the following result holds.

Lemma 4.7 (Self-Reproduction)**.** *Let* $\mathfrak{A}$ *be a* BT*-box,* $b : 2 \to 1$ *be a minimal edge in* $\mathfrak{A}$*. Then there is a morphism of boxes* $\mathfrak{A}^b \to \mathfrak{B}$*, which is actually a composition of regularizations, such that* $\mathfrak{B}$ *is also a* BT *-box and* $\mathsf{Br}(\mathfrak{B}) = \mathsf{Br}(\mathfrak{B}^1) \cup \mathsf{Br}(\mathfrak{B}^2)$*. Moreover, if* $M_1 \neq 0$, $M_2 \neq 0$ *and* $M \simeq \mathfrak{f}^*(N)$*, where* $\mathfrak{f}$ *is the composition* $\mathfrak{A} \to \mathfrak{A}^b \to \mathfrak{B}$*, then* $\|N\| < \|M\|$.

Proof. We denote $a = a_1$, $c = a_2$, $v = \tilde{b}$. Then the rules for the minimal edge reduction (see page 8) result in the biquiver



with the differential

$$
\begin{aligned}
\partial(a_{11}) &= a_{10}\eta + \alpha_{11},\\
\partial(a_{10}) &= \alpha_{10},\\
\partial(a_{01}) &= v_{01} + a_{00}\eta - \eta a_{11} + \alpha_{01},\\
\partial(a_{00}) &= v_{00} - \eta a_{10} + \alpha_{00},\\
\partial(c_{22}) &= -\xi c_{02} + \beta_{22},\\
\partial(c_{20}) &= -v_{20} + c_{22}\xi - \xi c_{00} + \beta_{20},\\
\partial(c_{02}) &= \beta_{02},\\
\partial(c_{00}) &= c_{02}\xi - v_{00} + \beta_{00},
\end{aligned} \tag{4.3}
$$

where α_{kl} and β_{kl} are collections of the other terms, which do not contain the arrows a_{kl}, c_{kl}, v_{kl}. Moreover, the terms α_{kk} and β_{kk} are just of the form $\sum_x (-1)^{|x|} x\tilde{x}$, as in (4.1), with x and $\tilde{x}$ different from a_{kl} and c_{kl}. Set $\partial(a_{01}) = u_{01}$, $\partial(c_{20}) = u_{20}$, $\partial(a_{00}) = u_{00}$. One easily sees that we can replace the generators v_{kl} of the kernel of the box $\mathfrak{A}^b$ by u_{kl} so that the resulting set of generators remains solid-triangular. Then the arrows a_{01}, a_{00}, c_{20} become superfluous. After regularization they disappear, as well as the dotted arrows u_{01}, u_{00}, u_{20}, and the formulae (4.3) change to:

$$
\begin{aligned}
\partial(a_{11}) &= a_{10}\eta + \alpha_{11},\\
\partial(a_{10}) &= \alpha_{10},\\
\partial(c_{22}) &= -\xi c_{02} + \beta_{22},\\
\partial(c_{02}) &= \beta_{02},\\
\partial(c_{00}) &= c_{02}\xi - \eta a_{10} + \alpha_{00} + \beta_{00}.
\end{aligned} \tag{4.4}
$$

Therefore, if we set $a_1 = a_{11}$, $a_2 = c_{22}$, $a_0 = c_{00}$, $\tilde{a}_{10} = \eta$ and $\tilde{c}_{02} = \xi$, we see that the resulting box $\mathfrak{B}$ is indeed a BT-box.

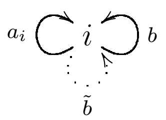
Moreover, the dotted arrow v_{21} does not occur in the differential of any solid arrow (since, by Lemma 4.4, the arrow v was not involved in the differentials of arrows from $\mathfrak{b}$). Therefore, by Lemma 3.2, $\mathsf{Br}(\mathfrak{B}) = \mathsf{Br}(\mathfrak{B}^1) \cup \mathsf{Br}(\mathfrak{B}^2)$. The other statements follow from Theorem 2.4. □

Remark 4.8. The boxes $\mathfrak{B}^1$ and $\mathfrak{B}^2$ are actually obtained from $\mathfrak{A}$ by the *small reduction* of the minimal edge b as defined in [D92].

Having these Lemmas, the proof of the theorem is quite obvious.

Proof of Theorem 4.2. (1) Let $\mathfrak{A}$ be a BT-box with the differential biquiver (Q, ∂) and $\mathbb{d}$ be a vector dimension such that the set $\mathsf{Br}(\mathbb{d}, \mathfrak{A})$ of bricks of vector dimension $\mathbb{d}$ is non-empty. Without loss of generality, we may suppose that $\mathfrak{A}$ is solid-connected and $d_i \neq 0$ for all i. Then Q contains no non-distinguished loops with zero differential by Corollary 4.6. Thus, if Q only has one vertex, $\mathfrak{A} = \mathbb{k}[t]$ and the statement is trivial. Let $b : 2 \to 1$ be an h-minimal arrow. Without loss of generality assume that $d_2 \le d_1$. Consider the box $\mathfrak{B}$ and the morphism $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ constructed in Corollary 4.5 or Lemma 4.7. The box $\mathfrak{B}$ is also of BT-type, $\mathfrak{f}^* : \mathfrak{B}\text{-mod} \to \mathfrak{A}\text{-mod}$ is an equivalence of categories, and $\|N\| < \|M\|$ for any $M \in \mathfrak{A}\text{-mod}$ such that $\mathbb{d}(M) = \mathbb{d}$ and $\mathfrak{f}^*(N) \simeq M$. Especially, this inequality holds if $M \in \mathsf{Br}(\mathbb{d}, \mathfrak{A})$. Moreover, then N is also a brick. If b was superfluous, $\mathbb{d}(N) = \mathbb{d}(M)$. If b was a minimal edge, then either $N_1 = 0$ or $N_2 = 0$, while $\dim M_1 = \dim N_1 + \dim N_0$ and $\dim M_2 = \dim N_2 + \dim N_0$. Since $d_2 \le d_1$, it implies that $N_2 = 0$, $\dim N_0 = d_2$ and $\dim N_1 = d_1 - d_2$. Hence, the vector dimension of N is uniquely determined. So we can proceed by induction on $\|M\|$, since, if $\|M\| = 1$ and $\mathfrak{A}$ is solid-connected, it only contains one vertex.

(2) Let $b : i \to j$ be an h-minimal arrow of $\mathfrak{A}$. Then $\partial(b) = 0$, so the differential biquiver (Q, ∂) of $\mathfrak{A}$ has a fragment (Q', ∂') that coincides either with the biquiver (1.1) from the Introduction or with the biquiver

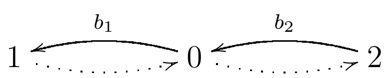
(4.5) 

$\partial(a_i) = b\tilde{b} - \tilde{b}b,\ \partial(b) = 0.$

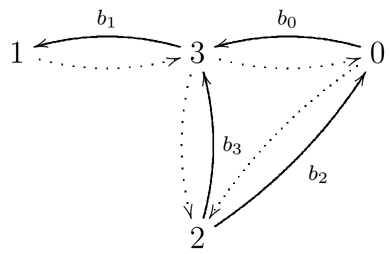
Let $\overline{\mathfrak{A}}$ be the box with the biquiver (Q', ∂'), $\overline{M}$ be an $\overline{\mathfrak{A}}$-module and M be the $\mathfrak{A}$-module that coincide with $\overline{M}$ on Q' and is zero outside it. One easily checks that for two such $\mathfrak{A}$-modules M and M' we have $M \simeq M'$ if and only if $\overline{M} \simeq \overline{M}'$. Hence, if $\overline{\mathfrak{A}}$ is wild, so is also $\mathfrak{A}$. The box (4.5) is well-known to be wild [D79, CB]. So we only have to prove that the box (1.1) (or, equivalently, (1.2)) is also wild. To do it, we use the reduction procedure described above.

In what follows, when drawing the biquiver of a BT-box, we omit the distinguished loops and their differentials and do not precise the names of the dotted arrows $\tilde{b}$, since these data can be uniquely restored. We also do not mention in the list of differentials the arrows b with $\partial(b) = 0$. Moreover, we usually omit the dotted arrows that do not occur in the differentials. Obviously, omitting such dotted arrows does not affect the representation type. In particular the differential biquiver (1.1) will be presented as

and the differential biquiver (1.2) as

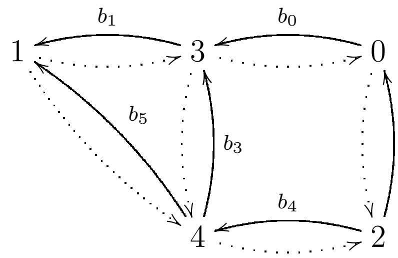


(we omit v). After the reduction of the minimal edge b_1 and regularization, we get the BT-box with the biquiver



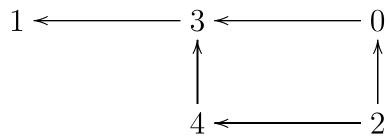
$$\partial(b_2) = -\tilde{b}_0 b_3,\ \partial(b_0) = b_3\tilde{b}_2.$$

Now we reduce the minimal edge b_3. After regularization, we get the BT-box with the biquiver



$$\partial(b_1) = b_5\tilde{b}_3,\ \partial(b_3) = -\tilde{b}_1 b_5.$$

If we factor out the arrow b_5, the remaining non-distinguished arrows become minimal and form the quiver



which is neither Dynkin nor Euclidean, hence, wild. Therefore, so is also the box (1.1). It accomplishes the proof. □

5. BT-boxes and coadjoint action

A natural class of BT-boxes arises from linear groups over algebras. Recall [D92, BDF] that a *linear group over an algebra* Λ is, by definition, the group $\mathsf{GL}(P,\Lambda)$ of automorphisms of a finitely generated projective Λ-module P. If Λ is finite dimensional over a field $\Bbbk$, $\mathsf{GL}(P,\Lambda)$ is a linear group over $\Bbbk$ (Lie group if $\Bbbk$ is the field of complex numbers). Its Lie algebra $\mathfrak{gl}(P,\Lambda)$ is just the commutator algebra of the endomorphism algebra of P. In the representation theory of linear and Lie groups the coadjoint action of a group on the dual space of its Lie algebra, especially its orbit space, plays an important role. Note that the dual space of $\mathfrak{gl}(P,\Lambda)$ is

$$\begin{aligned}\mathfrak{gl}^*(P,\Lambda) &= \mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_\Lambda(P,P),\Bbbk) \simeq \mathrm{Hom}_{\Bbbk}(P^\vee \otimes_\Lambda P,\Bbbk)\\ &\simeq \mathrm{Hom}_\Lambda(P,\mathrm{Hom}_{\Bbbk}(P^\vee,\Bbbk)) \simeq \mathrm{Hom}_\Lambda(P,\mathrm{Hom}_{\Bbbk}(P^\vee\otimes_\Lambda \Lambda,\Bbbk))\\ &\simeq \mathrm{Hom}_\Lambda(P,\mathrm{Hom}_\Lambda(P^\vee,\Lambda^*)) \simeq \mathrm{Hom}_\Lambda(P,\Lambda^*\otimes_\Lambda P),\end{aligned}$$

where $P^\vee = \mathrm{Hom}_\Lambda(P,\Lambda)$ and $\Lambda^* = \mathrm{Hom}_{\Bbbk}(\Lambda,\Bbbk)$.

From now on suppose that the algebra Λ is *basic*, i.e. if $1=\sum_{i=1}^n e_i$, where e_i are pairwise orthogonal primitive idempotents, $\Lambda e_i \not\simeq \Lambda e_j$ as Λ-modules for $i\neq j$. Since every finite dimensional algebra is Morita-equivalent to a basic one, every linear group over a finite dimensional algebra is isomorphic to a linear group over a basic algebra. If we fix the algebra Λ and consider all linear groups $\mathsf{GL}(P,\Lambda)$, the description of orbits in all dual spaces $\mathfrak{gl}(P,\Lambda)$ coincides with the "*bimodule problem*," namely, the description of isomorphism classes in the *bimodule category*, or the category of *elements of the bimodule* $\mathsf{El}(\Lambda^*)$. Recall [D01, BDF] that

- the *objects* of $\mathsf{El}(\Lambda^*)$ are just the elements of $\mathrm{Hom}_\Lambda(P,\Lambda^*\otimes_\Lambda P)$, where P runs through projective Λ-modules;
- if $u\in \mathrm{Hom}_\Lambda(P,\Lambda^*\otimes_\Lambda P)$, $v\in\mathrm{Hom}_\Lambda(P',\Lambda^*\otimes_\Lambda P')$, *morphisms* $u\to v$ are homomorphisms $\alpha: P\to P'$ such that $v\alpha=(1\otimes\alpha)u$.

Thus isomorphisms $u\to v$, where $u,v\in\mathrm{Hom}_\Lambda(P,\Lambda^*\otimes_\Lambda P)$, are the elements $g\in\mathsf{GL}(P,\Lambda)$ such that $v=(1\otimes g)ug^{-1}$, which coincides with the adjoint action of $\mathsf{GL}(P,\Lambda)$ on $\mathfrak{gl}^*(P,\Lambda)$.

Recall [D01] that the category $\mathsf{El}(\Lambda^*)$ can be identified with the category of representations of a Roiter box $\mathfrak{L}_\Lambda=(A,V)$. Namely, let $\mathfrak{R}$ be the radical of Λ, $\Lambda_{ij}=e_i\Lambda e_j$, $\mathfrak{R}_{ij}=e_i\mathfrak{R}e_j$. Note that $\Lambda_{ij}=\mathfrak{R}_{ij}$ if $i\neq j$, while $\Lambda_{ii}=\mathfrak{R}_{ii}\oplus\Bbbk e_i$ (since Λ is basic and $\Bbbk$ is algebraically closed). Choose a basis $\mathrm{B}_0(j,i)$ of $\mathfrak{R}_{ij}$ and set

$$\mathrm{B}(j,i)=\begin{cases}\mathrm{B}_0(j,i) & \text{if } i\neq j,\\ \mathrm{B}_0(i,i)\cup\{e_i\} & \text{if } i=j.\end{cases}$$

(It is a basis of Λ_{ij}.) Let $\mathrm{D}(j,i)$ be the basis of $(\mathfrak{R}_{ij})^*$ dual to $\mathrm{B}_0(j,i)$, $\mathrm{B}=\bigcup_{i,j}\mathrm{B}(j,i)$, $\mathrm{D}=\bigcup_{i,j}\mathrm{D}(j,i)$, and $\gamma(x,y,b)$ are the structure constants of the algebra Λ, i.e. $xy=\sum_b\gamma(x,y,b)b$ for $x,y,b\in\mathrm{B}$. It implies that

$$\begin{aligned}&x^*y=\sum_b\gamma(y,b,x)b^* \quad\text{and}\quad xy^*=\sum_b\gamma(b,x,y)b^*,\\ &\gamma(x,e_j,b)=\gamma(e_j,x,b)=\delta_{xb}\quad\text{for}\quad x\in\mathrm{B}(j,i).\end{aligned}$$

Then the set of solid arrows $j \to i$ in $\mathfrak{L}_\Lambda$ is $\mathrm{B}(j,i)$, while the set of dotted arrows $j \dashrightarrow i$ is $\mathrm{D}(j,i)$. The differential is defined by the rules

$$\partial(b) = \sum_{x,y} (\gamma(b,x,y)xy^* - \gamma(y,b,x)x^*y),$$
$$\partial(b^*) = \sum_{x,y} \gamma(x,y,b)x^* \otimes y^*,$$

Especially,

$$\partial(e_i) = \sum_{x \in \mathrm{B}(\cdot,i)} xx^* - \sum_{y \in \mathrm{B}(i,\cdot)} y^*y.$$

Hence, setting $\mathfrak{a} = \{ e_i \}$ and $\tilde{b} = b^*$ for $b \in \mathrm{B}_0 = \mathrm{B} \setminus \mathfrak{a}$, we get the following statement.

Proposition 5.1. *The box $\mathfrak{L}_\Lambda$ is of* BT*-type.*

By the way, it implies, due to Theorem 4.2 (2), that the problem of description of orbits of the coadjoint action of $\mathsf{GL}(P,\Lambda)$ for all P is wild, whenever the algebra Λ is not semisimple.

Obviously, an element $\xi \in \mathfrak{gl}^*(P,\Lambda)$ is a brick if and only if it has the trivial stabilizer: $\mathrm{Stab}_{\mathsf{GL}(P,\Lambda)}(\xi) = \Bbbk^\times$ (the multiplicative group of the field $\Bbbk$).

Corollary 5.2. *If the set $S(P,\Lambda)$ of bricks in $\mathfrak{gl}^*(P,\Lambda)$ is non-empty, all elements from this set belong to a single rational family. In particular, $S(P,\Lambda)/\,\mathsf{GL}(P,\Lambda) \simeq \Bbbk$.*

Remark 5.3. One easily sees that if $S(P,\Lambda) \neq \emptyset$, it is open and dense in $\mathfrak{gl}^*(P,\Lambda)$. Unfortunately, in most cases it is empty. Nevertheless, there is at least one case, when it is big indeed (see [D92, BDF]). It happens, when Λ is a *Dynkinian algebra*, that is an algebra derived equivalent to the path algebra of a Dynkin quiver. Namely, if Λ is Dynkinian, the space $\mathfrak{gl}^*(P,\Lambda)$ always contains an open dense subset U such that all elements $\xi \in U$ are *semisimple*, i.e. direct sums of bricks that are *mutually orthogonal*, that is every morphism between them is either zero or isomorphism. Therefore, the stabilizer of a semisimple element is a product of full linear groups over $\Bbbk$.

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Institut des Hautes Études Scientifiques
E-mail address: lesya_bod@ihes.fr
URL: http://www.mathematik.uni-kl.de/~bodnarchuk

Institute of Mathematics, National Academy of Sciences of Ukraine
E-mail address: drozd@imath.kiev.ua
URL: http://www.imath.kiev.ua/~drozd