

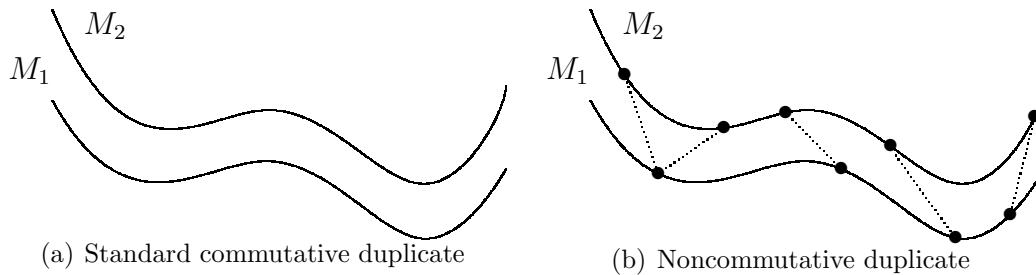
FACTORIZATION STRUCTURES WITH A 2-DIMENSIONAL FACTOR

ÓSCAR CORTADELLAS IZQUIERDO, JAVIER LÓPEZ PEÑA, AND GABRIEL NAVARRO

ABSTRACT. We introduce the notion of quantum duplicates of an (associative, unital) algebra, motivated by the problem of constructing toy-models for quantizations of certain configuration spaces in quantum mechanics. The proposed (algebraic) model relies on the classification of factorization structures with a two-dimensional factor. In the present paper, main properties of this particular kind of structures are determined, and we present a complete description of quantum duplicates of finite set algebras. As an application, we obtain a classification (up to isomorphism) of all the algebras of dimension 4 (over an arbitrary field) that can be factorized as a product of two factors.

INTRODUCTION

Consider a manifold M representing some physical system. From a dual point of view, this manifold can also be represented by some algebra of functions A (that could be taken, for instance, to be $A = C^\infty(M)$, the algebra of smooth functions on M) over some base field k (usually $k = \mathbb{R}$ or $k = \mathbb{C}$ when dealing with actual physical systems). Now, if we want to describe a physical system consisting on two “parallel” sheets M_1 and M_2 , both equal to M , the natural algebra to consider from a classical point of view is the direct product $A \times A$, which is isomorphic to the algebra $A \otimes k^2$.



However, assume that we keep studying this system while making the two sheets come closer and closer. After we reach certain critical distance, one could expect that measurements on M_1 would start to interfere with measurements on M_2 , so that they do not commute anymore. Under these premises, the algebra $A \otimes k^2$ can no longer be assumed (since it is commutative) to represent the measurements on the “parallel, but very close” sheets M_1 and M_2 . In the present work, our aim is to define and study, inside an algebraic framework, a suitable replacement for the algebra $A \otimes k^2$ in this quantized situation, i.e.

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a (noncommutative) algebra that may be regarded as a deformation of $A \otimes k^2$ and retains similar structural properties.

In particular, we would like to keep the linear dimension of the algebra $A \otimes k^2$, since the fact that a system starts showing up quantum effects should not alter the number of (linearly independent) quantities that we can measure on it. Thus, we are led to finding an algebra X , a “*quantum duplicate*” of A , which is isomorphic, as a vector space, to $A \otimes k^2$. In particular, if A is finite dimensional, the dimension of a quantum duplicate of A should be twice the dimension of A .

The method we propose in order to build these so-called quantum duplicates is the use of a **factorization structure**, or **twisted tensor product** involving the algebra A plus a (necessarily commutative) 2-dimensional factor. The number of ways in which we can choose this two dimensional factor depends on the field k . More concretely, if k admits a degree 2 field extension \bar{k} , (as happens, for instance, with the real numbers), then we have three kinds of non-isomorphic algebras of dimension 2 (over k), namely:

- (1) The trivial direct product k^2 ,
- (2) Quadratic field extensions of k ,
- (3) The ring of dual numbers $k[\xi] \cong k[x]/(x^2)$.

On the other hand, if k does not admit nontrivial extensions (for instance, if $k = \mathbb{C}$), there are only two possible algebras: the direct product k^2 and the dual numbers.

From a purely algebraic point of view, the notion of twisted tensor product comes directly from the **factorization problem**:

Given some kind of (algebraic) object, is it possible to find to suitable subobjects, having minimal intersection and such that they generate our original object?

The factorization problem has been intensively studied in the case of groups, coalgebras and Hopf algebras, and algebras (cf. for instance [19], [4], [5], [1]). In the particular case of algebras, a well known result (independently proven many times) establishes a one-to-one correspondence between the set of factorization structures admitting two given algebras A and B as factors and the set of so-called **twisting maps**, which are linear maps $\tau : B \otimes A \rightarrow A \otimes B$ satisfying certain compatibility conditions with respect with the units and products of A and B .

Henceforth, the problem of constructing factorization structures with given factors boils down to the problem of finding all the existing twisting maps for those factors. Under suitable, very mild, conditions (for instance, whenever A and B are affine algebras), the set $\mathcal{T}(A, B)$ of all the twisting maps $\tau : B \otimes A \rightarrow A \otimes B$ is an algebraic variety, and two interesting problems arise:

Problem 0.1. *Is it possible to describe explicitly the variety $\mathcal{T}(A, B)$?*

Problem 0.2. *Once the variety $\mathcal{T}(A, B)$ is known, is it possible to determine which points of the variety give rise to isomorphic algebras?*

These two problems, even in the simplest cases, turn out to be very difficult. Though there are many different methods that produce twisted tensor products of two given algebras, not a single one that produces all the existing ones is known, let alone describing the properties of the algebraic variety. Even harder is the problem of the determination

of the isomorphism classes of algebras obtained from the same factors through different tensor products, or finding any description of these isomorphism classes in terms of the variety $\mathcal{T}(A, B)$.

Recently, Cibils showed in [7] that the set $\mathcal{T}(k^2, A)$ of twisted tensor products between any algebra A and the commutative, semisimple algebra k^2 (also called the set of **2-interlacings**) is in one to one correspondence with couples of linear endomorphisms of the algebra A satisfying certain conditions. If we take $A = k^n$, these couples of linear maps can be described by combinatorial means using certain families of colored quivers, and this description gives a simple way to describe all the twisted tensor products $k^n \otimes_{\tau} k^2$, up to isomorphism (cf. [7, 15]). Some other partial steps in the classification problem for factorization structures have been undertaken in [3] and the final sections of [11].

Beyond the physical motivation originating the notion of noncommutative duplicate (defined by Cibils), the remaining choices of two-dimensional factors have their own source of interest. In particular, the building of duplicates by means of dual numbers has yet another physical interpretation. Being the algebra of dual numbers one of the simplest examples of non-trivial superspaces, where the ξ -direction can be reinterpreted as the fermionic direction, and the scalar component as the bosonic direction, the procedure of duplicating a manifold using dual numbers can be regarded as a simple way of adding a superstructure to the physical system described by the manifold. Moreover, quantizations of the tensor product $A \otimes k[\xi]$ admit an interpretation as **infinitesimal deformations** (for a *central* formal parameter) in the framework of formal deformation theory (cf. [9]), so twisted tensor products $A \otimes_{\tau} k[\xi]$ may be regarded as infinitesimal deformations with respect to a non-central parameter, with the added advantage of the existence of such a kind of deformations for algebras that are rigid from the formal point of view (like separable algebras). Finally, the remaining case of quantum duplicates obtained using a quadratic field extension have similar properties to complexifications of real algebras, hinting the possibility of thinking about them as noncommutative scalar extensions.

In Section 1 we introduce the definition of **quantum duplicates** of an algebra A as twisted tensor products $A \otimes_{\tau} B$ where B is any two-dimensional algebra, characterize the set of twisting maps as a set of couples (f, δ) where f is an algebra endomorphism of A and δ is an f -derivation, satisfying certain compatibility conditions. We also show how to lift certain classes of endomorphisms and involutions to such kind of twisted tensor products, with special attention to the case in which the algebra B is a quadratic field extension of k , obtaining a simple criterion (similar to the one existing for telling whether a vector space is a complexification of a real one) for determining whether or not certain k -algebras factorize as quantum duplicates with a quadratic field extension.

Sections 2 and 3 deal with the classification problem of isomorphism classes of the resulting twisted tensor products. More concretely, in Section 2 we classify (up to isomorphism) all quantum duplicates of the finite set algebras k^n by means of combinatorial techniques, obtaining results similar to the ones contained in [7].

The paper concludes with the complete classification, in Section 3, of all the algebras of dimension 4 that can be obtained as a twisted tensor product, that should necessarily be of two factors of dimension 2. For the case of an algebraically closed field, the resulting algebras are displayed inside the diagram of all the four dimensional algebras, showing that no apparent pattern relating those *factorizable* algebras appears.

Along this paper, k will denote a field. All the algebras will be unital, associative k -algebras. The tensor product will be taken over k and all maps will be k -linear. An algebra X is a **factorization structure** of the algebras A and B if there exist two injective algebra maps $i_A : A \hookrightarrow X$ and $i_B : B \hookrightarrow X$ and the map $\varphi : A \otimes B \rightarrow X$ defined by $\varphi(a \otimes b) = i_A(a) \cdot i_B(b)$ is a linear isomorphism.

A k -linear map $\tau : B \otimes A \rightarrow A \otimes B$ is said to be a **(unital) twisting map** if

$$(0.1) \quad \tau \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes \tau) \circ (\tau \otimes A)$$

$$(0.2) \quad \tau \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (\tau \otimes B) \circ (B \otimes \tau),$$

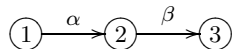
$$(0.3) \quad \tau(1 \otimes a) = a \otimes 1, \quad \tau(b \otimes 1) = 1 \otimes b \quad \text{for all } a \in A, b \in B.$$

where μ_A and μ_B stand for the product of A and B respectively and A and B stand for the identity maps on each algebra.

The map $\mu_\tau := (\mu_A \otimes \mu_B) \circ (A \otimes \tau \otimes B)$ defines an associative product over $A \otimes B$ if, and only if, τ is a twisting map. So we can endow the vectorial space $A \otimes B$ with this product and get a new algebra $(A \otimes B, \mu_\tau)$, which will be denoted by $A \otimes_\tau B$.

Proposition 0.3 ([6], [16]). *Let (C, i_A, i_B) a factorization structure of C with factors A and B . Then there exists a unique twisting map $\tau : B \otimes A \rightarrow A \otimes B$ such that C is isomorphic to $A \otimes_\tau B$ as a twisted tensor product.*

When working with path algebras kQ of quivers Q (cf. for instance [2]) we shall assume that arrows are multiplied as if they were maps, for instance, in the quiver



the length-two path from $\textcircled{1}$ to $\textcircled{3}$ will be written as $\beta\alpha$. All along the work, the ideal $(Q_{\geq 2})$ of kQ generated by all paths in Q of length greater than one will play a fundamental rôle. The quotient $kQ/(Q_{\geq 2})$ will be denoted by $kQ_{<2}$. We shall denote by Q^{op} the opposite quiver of Q , that is, the quiver that has the same set of vertices while arrows are reversed.

1. GENERALITIES ABOUT QUANTUM DUPLICATES

Let A and B be two (unitary) k -algebras, with $\dim_k B = 2$, so that we may consider it given as a quotient $B = k[x]/(p(x))$, where $p(x)$ is a polynomial of degree two. All along this work we write it as $p(x) = x^2 - \alpha x + \beta$ where $\alpha, \beta \in k$. We also denote by q the polynomial $q(x) = x^2 + \alpha x + \beta$, that is, the polynomial whose roots (in the algebraic closure of k) are the opposite to the ones of p .

1.1. Basic definitions and properties. Our purpose is to describe the twisting maps between A and B , that is, the k -linear maps

$$(1.4) \quad \tau : k[x]/(p(x)) \otimes A \longrightarrow A \otimes k[x]/(p(x))$$

verifying the twisting conditions (0.1) and (0.2). Following the method developed in [7], it is worth noting that $A \otimes k[x]/(p(x)) \cong A[x]/(p(x))$ and then a twisting map as in (1.4)

is determined by the values $\tau(x \otimes a)$ corresponding to the product xa in $A[x]/(p(x))$. For any $a \in A$, we put $\tau(x \otimes a) = xa = \delta(a) + f(a)x$. Then,

$$(1.5) \quad x^2a = (\alpha x - \beta)a = \alpha(\delta(a) + f(a)x) - \beta a = \alpha\delta(a) - \beta a + \alpha f(a)x$$

$$(1.6) \quad x(xa) = x(\delta(a) + f(a)x) = \delta^2(a) - \beta f^2(a) + f\delta(a)x + \delta f(a)x + \alpha f^2(a)x$$

for any $a \in A$. Thus the associativity $x^2a = x(xa)$ produces the equalities

$$(1.7) \quad p(\delta) = \delta^2 - \alpha\delta + \beta id_A = \beta f^2$$

$$(1.8) \quad f\delta + \delta f = \alpha(f - f^2)$$

As in the proof of [7, Proposition 2.10], the condition $x(ab) = (xa)b$ for any $a, b \in A$, produces that $f : A \rightarrow A$ is a morphism of algebras and $\delta : A \rightarrow A$ is a left f -derivation. It is also clear that if we have such an f and δ verifying (1.7) and (1.8), the linear map defined by $\tau(x \otimes a) = \delta(a) \otimes 1 + f(a) \otimes x$ is a twisting map between A and B . In other words, we have proven the following result.

Lemma 1.1. *The set of twisting maps $\tau : k[x]/(p(x)) \otimes A \rightarrow A \otimes k[x]/(p(x))$ is in one-to-one correspondence with the set of pairs (f, δ) , where $f : A \rightarrow A$ is an endomorphism of algebras and $\delta : A \rightarrow A$ is a left f -derivation, verifying the conditions*

$$p(\delta) = \beta f^2 \text{ and } f\delta + \delta f = \alpha(f - f^2).$$

A twisting tensor product of the form $A \otimes_\tau B$, where B is an algebra of dimension two, will be referred by a **quantum duplicate** of A .

Example 1.2. *If $\alpha = 1$ and $\beta = 0$, then $B \cong k^2$ and we recover the approach given by Cibils [7], since, in that case, the twisting maps are in one-to-one correspondence with the pairs (f, δ) verifying that $p(\delta) = \delta^2 - \delta = 0$ and $\delta f + f\delta = f - f^2$.*

Remark 1.3. *The previous lemma admits a refinement when the characteristic of k is different from two. Taking the linear transformation $\phi(x) = (\alpha/2)x + 1$, we obtain that $p(\phi(x)) = x^2 + \gamma$ and $k[x]/(p(x)) \cong k[x]/(x^2 + \gamma)$. Thus equations (1.7), (1.8) are rewritten as follows:*

$$(1.9) \quad \delta^2 = \gamma(f^2 - id_A)$$

$$(1.10) \quad f\delta + \delta f = 0$$

1.2. Characterization of certain quantum duplicates. When we have a real vector space V , we can construct the **complexification** of V , called $V^{\mathbb{C}}$, as the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$. The original vector space V remains a *real* vector subspace of $V^{\mathbb{C}}$, and can be recovered if we take advantage of the canonical conjugation map $\chi : V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ given by $\chi(v \otimes z) := v \otimes \bar{z}$. When the two dimensional algebra $l = k[x]/(p(x)) = k(\eta)$ is a quadratic (Galois) field extension, twisted tensor products $A \otimes_\tau l$ can be regarded as "noncommutative scalar extensions" to l . Our goal in this subsection is to establish a result that guarantees us that a given k -algebra B can be factorized as $A \otimes_\tau l$ for some twisted tensor product τ . In the forecoming, l will be assumed to be some fixed (Galois) quadratic field extension of k , with k a field of characteristic different from 2.

Lemma 1.4. *Let B a k -algebra endowed with a right l -module structure $B \otimes l \rightarrow B$. Then the map $i : l \rightarrow B$ given by $z \mapsto 1_B \cdot z$ is an injective algebra map.*

Proof. It is a straightforward calculation. \square

Assume now that B is endowed with a conjugation map $\sigma : B \rightarrow B$ satisfying the following conditions:

- (1) $\sigma^2 = \text{Id}_B$,
- (2) $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in B$.
- (3) $\sigma(a\lambda) = \sigma(a)\bar{\lambda}$ for all $\lambda \in l, a \in B$

and let

$$A := B^\sigma = \{a \in B \mid \sigma(a) = a\}.$$

We have an obvious algebra map (the inclusion map) $i_A : A \rightarrow B$.

Lemma 1.5. *The mapping*

$$\begin{aligned} \varphi : A \otimes l &\longrightarrow B \\ a \otimes z &\longmapsto a \cdot z \end{aligned}$$

is a linear isomorphism.

Proof. Let $b \in B$, consider the elements

$$a_1 := \frac{b + \sigma(b)}{2} \quad \text{and} \quad a_2 := \frac{b - \sigma(b)}{2\eta}$$

then, obviously, $b = a_1 + a_2 \cdot \eta$. Now,

$$\sigma(a_1) = \frac{\sigma(b) + \sigma(\sigma(b))}{2} = \frac{\sigma(b) + b}{2} = a_1 \quad \text{and} \quad \sigma(a_2) = \frac{\sigma(b) - b}{-2\eta} = a_2$$

and therefore $a_1, a_2 \in A$. Thus $b = \varphi(a_1 \otimes 1 + a_2 \otimes \eta)$ and φ is surjective.

Let $\alpha \in B \otimes C$, we may write $\alpha = a \otimes 1 + b \otimes \eta$ with $a, b \in A$, and then $\varphi(\alpha) = \varphi(a \otimes 1) + \varphi(b \otimes \eta) = a + b \cdot \eta$. Assume $\varphi(\alpha) = 0$, that is, $a + b \cdot \eta = 0$. Applying σ , $a - b \cdot \eta = 0$. As a consequence, $a = b = 0$ and thus φ is injective. \square

As a consequence of the previous lemmas, by using [6, Proposition 2.7] (see also [20] or [17]), we obtain the following corollary:

Corollary 1.6. *An algebra B factorizes as a twisted tensor product $A \otimes_\tau l$ for some twisting map τ if, and only if, B is endowed with a right l -module structure and a conjugation map satisfying (1), (2) and (3).*

1.3. Lifting of endomorphisms and involutions. Let A be an algebra, $\varphi : A \rightarrow A$ an algebra map, and $A \otimes_\tau B$ a quantum duplicate of A , with $B = k[x]/(p(x))$, induced by the couple (f, δ) . The map φ admits a natural lifting $\tilde{\varphi} : A \otimes_\tau B \rightarrow A \otimes_\tau B$ defined by $\tilde{\varphi}(a \otimes b) := \varphi(a) \otimes b$.

Assume that the lifting $\tilde{\varphi}$ is an algebra map. Then on the one hand we have

$$\tilde{\varphi}(xa) = \tilde{\varphi}(f(a)x + \delta(a)) = \varphi(f(a))x + \varphi(\delta(a)),$$

while, in the other one,

$$\tilde{\varphi}(xa) = \tilde{\varphi}(x)\tilde{\varphi}(a) = x\varphi(a) = f(\varphi(a))x + \delta(\varphi(a)).$$

Thus, we have established the following result:

Theorem 1.7. *An endomorphism φ of A can be lifted to an endomorphism of the twisted tensor product $A \otimes_{(f,\delta)} B$ if, and only if*

$$f\varphi = \varphi f, \quad \text{and} \quad \delta\varphi = \varphi\delta.$$

In other words, the set of endomorphisms of A admitting a lifting to $A \otimes_{(f,\delta)} B$ coincides with the set of morphisms that simultaneously commute with f and δ .

Assume now that A is an algebra endowed with an involution $a \mapsto a^*$, and that $l = k(\eta)$ is a quadratic field extension of k , with $\text{char}(k) \neq 2$, so that the Galois automorphism $\eta \mapsto -\eta$ gives us an involutive k -automorphism of l . Assume also that we are given a twisting map τ induced from a couple (f, δ) . Our purpose is to determine conditions in f and δ in such a way that we can ensure the existence of an involution j in $A \otimes_{(f,\delta)} l$.

First, recall that for any linear endomorphism φ of A we can define its conjugate $\bar{\varphi}$ by $\bar{\varphi}(a) := \varphi(a^*)^*$. An easy computation shows that if f is an algebra map, then so is \bar{f} , and if δ is a left f -derivation, then the conjugate $\bar{\delta}$ is a right \bar{f} -derivation.

Any involution j defined on a twisted tensor product $A \otimes_{\tau} B$ which is compatible with the ones existing in A and B must satisfy

$$\begin{aligned} j(a \otimes b) &= j((a \otimes 1)(1 \otimes b)) = j(1 \otimes b)j(a \otimes 1) = \\ &= (1 \otimes j_B(b))(j_A(a) \otimes 1) = \tau(j_B(b) \otimes j_A(a)), \end{aligned}$$

i.e., we necessarily have $j = \tau \circ (j_B \otimes j_A) \circ \tau_{BA}$, where τ_{BA} is the flip map. Henceforth, the involutions can be combined in a unique compatible way, provided that the twisting map τ satisfies the involutive condition $(\tau \circ (j_B \otimes j_A) \circ \tau_{BA})^2 = \text{Id}_{A \otimes B}$ (cf. [21]). In our particular situation, where the involution in B is given by $\eta \mapsto -\eta$ and the twisting map is giving in terms of f and δ , the lifting of the involution of A becomes

$$(a\eta)^* = \bar{\eta}a^* = -\eta a^* = -f(a^*)\eta - \delta(a^*)$$

so the involutive condition is

$$\begin{aligned} a\eta &= ((a\eta)^*)^* = (-f(a^*)\eta - \delta(a^*))^* = \\ &= \eta f(a^*)^* + \delta(a^*)^* = \\ &= f(f(a^*)^*)\eta + \delta(f(a^*)^*) - \delta(a^*)^*, \end{aligned}$$

for all $a \in A$. We have proved the following result:

Proposition 1.8. *If $(A, *)$ is an algebra with involution, $l = k(\eta)$ is a quadratic field extension (with Galois automorphism $\eta \mapsto -\eta$), and we have a twisting map induced by a couple (f, δ) , then the involution of A lifts in a compatible way to an involution of $A \otimes_{(f,\delta)} l$ if, and only if, the following conditions are satisfied:*

$$\begin{aligned} f \circ \bar{f} &= \text{Id}_A, \\ \delta \circ \bar{f} &= \bar{\delta}. \end{aligned}$$

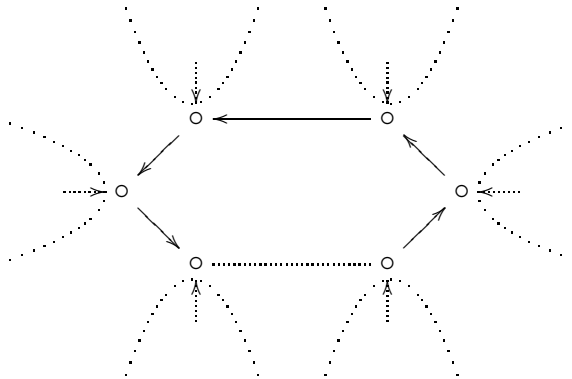
2. QUANTUM DUPLICATES OF k^n

In this section we describe and classify all quantum duplicates of k^n for some natural number $n \geq 2$. Denote by $\{e_1, \dots, e_n\}$ the canonical basis of k^n . Following Cibils' procedure [7], the set of algebra morphisms $f : k^n \rightarrow k^n$ is in one-to-one correspondence with the set of set maps $\varphi : \{e_1, \dots, e_n\} \rightarrow \{e_1, \dots, e_n\}$, where, to each set map φ , we associate the algebra morphism f defined as

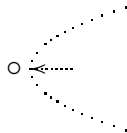
$$(2.11) \quad f(e_i) = \sum_{\{e_j | \varphi(e_j) = e_i\}} e_j, \quad \text{for any } i = 1, \dots, n.$$

This is also in one-to-one correspondence with the set of quivers with n vertices verifying that from each vertex starts precisely one arrow. From now on we denote by Q_f the quiver associated to the endomorphism f (by meanings of φ). We recall that Q_f has $\{e_1, \dots, e_n\}$ as set of vertices, and there is an arrow $e_i \rightarrow e_j$ in Q_f if, and only if, $\varphi(e_i) = e_j$. For the convenience of the reader we introduce the following notation:

Definition 2.1. *For any positive integer s , we say that a connected component of Q_f is an s -cycle if the component has the following shape:*

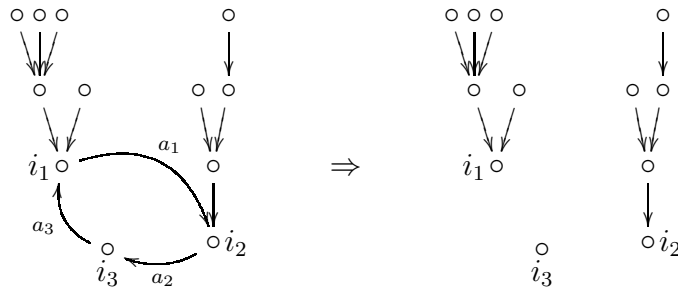


where the central cycle has s vertices and, by a diagram

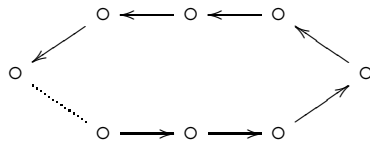


we mean a tree with ascendent orientation. Observe that, for any s -cycle, if we remove the arrows of the central cycle, we obtain s disjoint trees with ascendent orientation. For

instance, in the figure below we have a 3-cycle.



We say that a connected component of Q_f is an **strict s -cycle** if such component is the quiver $\tilde{\mathbb{A}}_s$, i.e., the quiver



with s vertices. Observe that, according to this nomenclature, an strict 1-cycle component is nothing more than a single vertex with a loop whilst an strict 2-cycle is the round-trip quiver.

Therefore we have the following result, see [12] or [10]:

Lemma 2.2. *For any algebra map $f : k^n \rightarrow k^n$, each connected component of the quiver Q_f is a s -cycle for certain integer s .*

With respect to the f -derivations, since k^n is separable, all of them are inner, so each $\delta \in \text{Der}(k^n, {}^f(k^n))$ is determined by certain element $a = (a_1, \dots, a_n) \in k^n$ such that

$$(2.12) \quad \delta(e_i) = (f(e_i) - e_i)a = \sum_{\varphi(e_j)=e_i} e_j a_j - e_i a_i, \quad \text{for any } i = 1, \dots, n.$$

Observe that if e_i is a loop vertex, then $\delta(e_i)$ does not depend of the value of a_i . For that reason, we normalize the element a by taking $a_i = 0$ for any loop vertex e_i . Then each inner derivation is given by a unique normalized element of k^n (that, following the nomenclature of [7], we shall call a **coloration** of Q_f)

In order to characterize the derivations (=colorations) verifying (1.7) and (1.8), we will need the following formulae:

$$(2.13) \quad f^2(e_i) = \sum_{\varphi^2(e_j)=e_i} e_j$$

$$(2.14) \quad \delta^2(e_i) = \sum_{\varphi^2(e_k)=e_i} a_k a_{\varphi(e_k)} e_k - \sum_{\varphi(e_j)=e_i} a_j (a_j + a_i) e_j + a_i^2 e_i$$

for any vertex e_i of Q_f . Therefore, we rewrite (1.7) and (1.8) as follows:

$$(2.15) \quad \sum_{\varphi^2(e_k)=e_i} (a_k a_{\varphi(e_k)} - \beta) e_k - \sum_{\varphi(e_j)=e_i} (a_j(a_j + a_i) + \alpha a_j) e_j + (a_i^2 + \alpha a_i + \beta) e_i = 0$$

$$(2.16) \quad \sum_{\varphi^2(e_k)=e_i} (a_k + a_{\varphi(e_k)} + \alpha) e_k - \sum_{\varphi(e_j)=e_i} (a_j + a_i + \alpha) e_j = 0$$

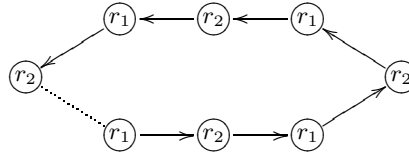
for any vertex e_i of Q_f . It is easy to see that the study of the possible colorations of the quiver Q_f can be reduced to the study of each connected component separately. Let us start by showing up the case of an strict cycle.

Proposition 2.3. *Let $p(x) = x^2 - \alpha x + \beta \in k[x]$ be a polynomial of degree two. The set of twisting maps $(f, \delta) : k^n \otimes k[x]/(p(x)) \rightarrow k[x]/(p(x)) \otimes k^n$, where Q_f is a strict s -cycle for some $s > 1$, is the following:*

- (1) *If Q_f is a strict 2-cycle, it is a one-parameter family of twisting maps, indexed in the field, given by the colorations*

$$\textcircled{a} \rightleftarrows \textcircled{b} \quad \text{where} \quad a + b = -\alpha.$$

- (2) *If Q_f is a strict s -cycle with $s > 2$, each twisting map is given by coloring each vertex of the cycle with a root of q satisfying that, if a vertex is colored by a root r_1 , the immediate predecessor and the immediate successor of such vertex are colored by $r_2 = -\alpha - r_1$.*



As a consequence, if s is even, there are as many colorations as different roots of p ; and, if s is odd, there exists a coloration (which would then be unique) if, and only if, p has a unique (double) root. In particular, there are at most two twisting maps.

Proof. Let us suppose that we treat with a strict 2-cycle, i.e., a connected component with shape

$$\textcircled{} \rightleftarrows \textcircled{} .$$

We also denote by e_i and e_j the vertices of this component. Then (2.15) and (2.16) applied to the vertex e_i reduces to

$$(2.17) \quad (a_i a_j - \beta) e_i - a_j(a_j + a_i + \alpha) e_j + (a_i^2 + \alpha a_i + \beta) e_i = 0$$

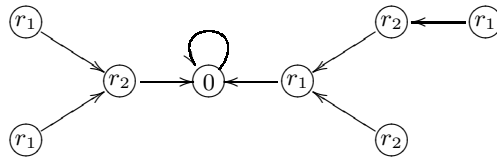
$$(2.18) \quad (a_i + a_j + \alpha) e_i - (a_i + a_j + \alpha) e_j = 0$$

Clearly, these equations hold if and only if $a_i + a_j = -\alpha$.

Let us now consider an strict s -cycle with $s > 2$. Then, for any vertex e_i of Q_f , the component i of (2.15) provide us the equation $q(a_i) = 0$. Therefore (2.15) holds if, and only if, each vertex is colored with a root of q . Now, denote by e_j the immediate predecessor of e_i . Then the component i of (2.16) provides us the equation $a_i + a_j + \alpha = 0$. Similarly, we may prove that the immediate successor of e_i must be colored by $-\alpha - a_i$. \square

Theorem 2.4. *Let $p(x) = x^2 - \alpha x + \beta \in k[x]$ be a polynomial of degree two. The set of twisting maps $\tau : k^n \otimes k[x]/(p(x)) \rightarrow k[x]/(p(x)) \otimes k^n$ is in one-to-one correspondence with the set of colored quivers Q_f , where each connected component Q_i of Q_f is colored according to the following rules:*

- (1) *If Q_i is a 1-cycle, any immediate predecessor of the loop vertex must be colored by a root of q . Given an immediate predecessor of the loop vertex e colored by r_1 , other vertex d in the same tree as e must be colored by r_1 if t is even, and by $r_2 = -\alpha - r_1$ if t is odd, where t is the length of the path from d to e .*



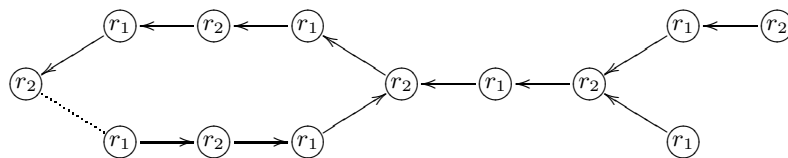
As a consequence, each connected component of this kind increases the number of twisting maps by at most a factor of 2^c , where c is the number of immediate predecessors of the loop vertex. Of course, if q has no roots in k , the only connected components of this kind that may show up are strict 1-cycles.

- (2) *If Q_i is a strict 2-cycle, any valid coloration must satisfy*

$$\textcircled{a} \rightleftarrows \textcircled{b} \quad \text{where} \quad a + b = -\alpha.$$

Connected components of this kind give rise to a one-parameter family of twisting maps, the parameter ranging in the field,

- (3) *If Q_i is either a non-strict 2-cycle or a s -cycle with $s > 2$, any valid coloration is determined by choosing a coloration of the central cycle. Each vertex of the central cycle must be colored by roots of q in alternating order, meaning that, if a vertex is colored by a root r_1 , any immediate predecessor and the immediate successor of such vertex are colored by $r_2 = -\alpha - r_1$. Moreover, for any vertex e in the cycle, the tree attached to e is also colored by alternating roots of q ; i.e. if e is colored by r_1 , any other vertex d in the same tree as e must be colored by r_1 if t is even, and by $r_2 = -\alpha - r_1$ if t is odd, where t is the length of the path from d to e .*



As a consequence, if s is even, there are as many colorations as different roots of p ; and, if s is odd, there exists a (unique) coloration if, and only if, p has a unique root. In particular, this kind of components increase the number of twisting maps by at most a factor of 2.

Proof. The second case is proven in Proposition 2.3. Let us suppose that Q_f is a 1-cycle. The loop vertex is colored by 0 by the arguments given above. Since any of the other vertices e_i is not inside a cycle, by applying the same arguments as in Proposition 2.3, we obtain that $q(a_i) = 0$ and thus all the vertices remaining are colored by a root of q . Now, let us consider an arrow $\textcircled{e_i} \longrightarrow \textcircled{e_j}$ inside one of the trees engaged to the loop vertex. Then (2.16) in the component i provides us the equation $a_i + a_j + \alpha = 0$. That is, for one

of these trees, if one level is colored by a root r_1 , the next level is colored by $-\alpha - r_1$, see Figure 1.

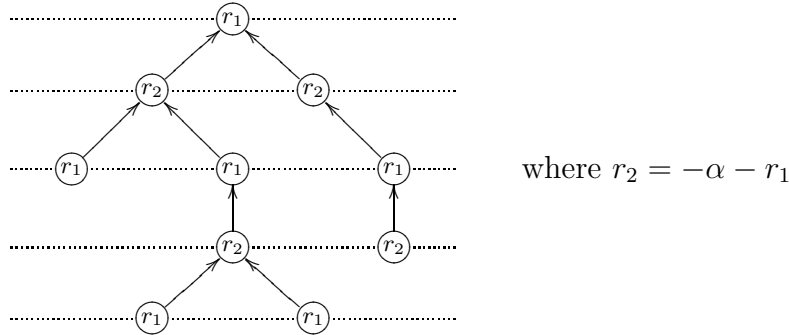


FIGURE 1. Coloration of a tree engaged to the cycle

Finally, suppose that Q_f is as stated in (3). Following the same proof as in Proposition 2.3, we deduce that the central cycle is colored by choosing to each vertex a root of q and, if a vertex is colored by a root r_1 , the immediate predecessor and the immediate successor of such vertex are colored by $-\alpha - r_1$. Now, applying the above reasoning to any of the trees engaged to a vertex of the cycle, we get the statement of (3). It is clear that these conditions are sufficient to have a coloration, thus the proof is completed. \square

Corollary 2.5. *Let $p \in k[x]$ be an irreducible polynomial of degree two. The set of twisting maps $\tau : k^n \otimes k[x]/(p(x)) \rightarrow k[x]/(p(x)) \otimes k^n$ is in one-to-one correspondence with the set of colored quivers Q_f such that each connected component is either a single loop vertex, or a round-trip quiver $\textcircled{a} \rightleftarrows \textcircled{b}$ where $a + b = -\alpha$.*

Proof. By Theorem 2.4, if a connected component of Q_f is neither a strict 1-cycle nor a strict 2-cycle, the vertices are labeled by a root of q in k . Therefore there is no derivations for an algebra map f producing such components. Thus all components of f must be either a single loop vertex or a round-trip quiver. Now, apply Theorem 2.4 for such cases. \square

Corollary 2.6. *Let $p \in k[x]$ be a polynomial of degree two such that it has a unique root s in k . The set of twisting maps $\tau : k^n \otimes k[x]/(p(x)) \rightarrow k[x]/(p(x)) \otimes k^n$ is in one-to-one correspondence with the set of colored quivers Q_f such that:*

- (1) Q_f is the quiver associated to a set map $\{1, \dots, n\} \rightarrow \{1, \dots, n\}$.
- (2) The coloration of a connected component different of a round-trip quiver is given by putting 0 in the loop vertex and $-s$ in the others.
- (3) A round-trip connected component is colored by $\textcircled{a} \rightleftarrows \textcircled{b}$, where $a + b = -2s$.

Now we are going to describe the isomorphism classes of the algebras that we have obtained. There is no loss of generality on assuming that the quiver Q_f is connected; otherwise we reason over every connected component, and the resulting algebra will be the direct product of the algebras associated to the distinct components (cf. [7]). As the reader should note, the case of a round-trip quiver becomes very special. For that reason we study this case separately.

Proposition 2.7. *Let $p(x) = x^2 - \alpha x + \beta \in k[x]$ be a polynomial of degree two. The twisted tensor product $k^2 \otimes_{(f,\delta)} k[x]/(p(x))$, where (f, δ) is the twisting map associated to the quiver*

$$\textcircled{1} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \textcircled{2} \quad \text{colored by} \quad \textcircled{a} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \textcircled{b}, \quad \text{where } a + b = -\alpha,$$

is isomorphic to one of the following algebras:

- (1) *The matrix ring $\mathcal{M}_2(k)$ if a and b are not roots of q .*
- (2) *The quotient algebra $kQ_{<2}$, otherwise, where Q is the round-trip quiver.*

Proof. Let us assume that a and b are not roots of q . In order to show the isomorphism, it is enough to give the image of $e_1 \otimes 1$, $e_2 \otimes 1$ and $1 \otimes x$ which, for brevity, we denote simply by e_1 , e_2 and x , respectively. Since $e_1 + e_2 = 1$, we map

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad x \mapsto \begin{pmatrix} -a & 1 \\ -q(a) & -b \end{pmatrix}$$

Since $q(a) \neq 0$, this provides us an isomorphism of algebras.

Now, suppose that a and b are roots of q . Let us denote the elements of Q as follows:

$$\textcircled{u} \begin{array}{c} \xrightarrow{R} \\ \xleftarrow{S} \end{array} \textcircled{v}$$

Then we consider the isomorphism of k -algebras $\Phi : kQ_{<2} \rightarrow k^2 \otimes_{(f,\delta)} k[x]/(p(x))$ given by:

$$u \mapsto e_1 \quad v \mapsto e_2 \quad R \mapsto e_2(x + b) = (x + a)e_1 \quad \text{and} \quad S \mapsto e_1(x + a) = (x + b)e_2$$

This is clearly well-defined. For instance, observe that

$$\Phi(R)\Phi(S) = (e_2x + be_2)(e_1x + ae_1) = -q(a)e_2 = 0,$$

as a is a root of q . This completes the proof. \square

For the convenience of the reader, in order to state the main result of this section, we remind the following quiver-transformation introduced by Cibils [7]. Let us consider an 1-cycle Q colored by means of the procedure explained above. Then we construct the forest \widehat{Q} obtained as follows:

- Remove the loop vertex and all the arrows ending at it. This produces a finite number of trees.
- Insert two isolated vertices, numbered by 1 and 2.
- For each tree of the ones obtained above, insert an arrow from the root node to 1 if that vertex is colored by r_1 , or to 2 if it is colored by r_2 .

Theorem 2.8. *Let $p(x) = x^2 - \alpha x + \beta \in k[x]$ be a polynomial of degree two. The twisted tensor product $R = k^n \otimes_{(f,\delta)} k[x]/(p(x))$, where (f, δ) is the twisting map associated to a connected colored quiver Q , is isomorphic to the following algebra:*

- (a) *When Q is an strict 2-cycle,*
 - (i) *If Q is colored by roots of q , then $R \cong kQ_{<2}$.*
 - (ii) *Otherwise $R \cong \mathcal{M}_2(k)$.*
- (b) *When Q is a 1-cycle,*
 - (i) *If p has no root, $n = 1$ and therefore $R \cong k[x]/(p(x))$.*

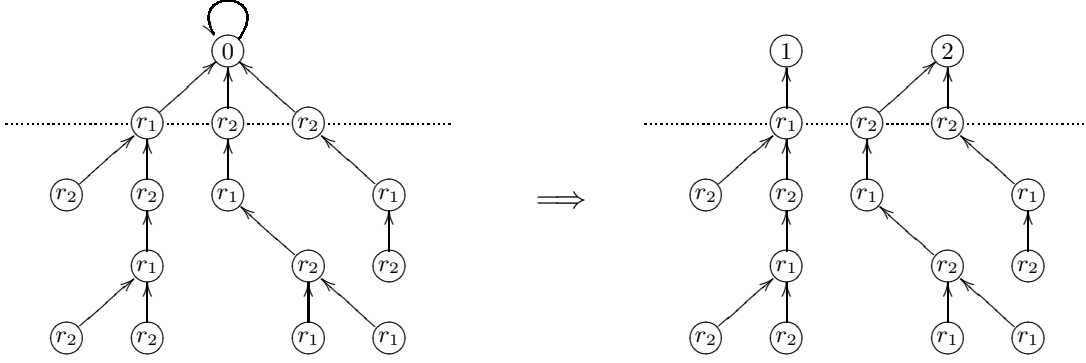


FIGURE 2. Cibils' transformation

(ii) If p has a single root, then $R \cong k(Q^{op})_{<2}$.

(iii) If p has two different roots, then $R \cong k(\widehat{Q})^{op}_{<2}$.

(c) Otherwise, $R \cong k(Q^{op})_{<2}$.

Proof. (a) is Proposition 2.7. (b) – (i) is given by Corollary 2.5 since Q must be a single loop vertex. Then the statement follows trivially.

Let us prove (b) – (ii) and (c). We remind that each vertex in Q^{op} is the target of a unique arrow. For any vertex e_i in Q^{op} , the twisting map given by (f, δ) , say τ , verifies that

$$(2.19) \quad \tau(x \otimes e_i) = \delta(e_i) \otimes 1 + f(e_i) \otimes x = \sum_{e_j} \epsilon_j(e_j \otimes 1) - \epsilon_i(e_i \otimes 1) + \sum_{e_j} e_j \otimes x,$$

where the e_j 's are the target of the arrows starting at e_i in Q^{op} and, ϵ_i and ϵ_j are the labels of e_i and e_j , respectively. Then we define a morphism $\Phi : kQ^{op} \rightarrow k^n \otimes_{(f,\delta)} k[x]/(p(x))$ as follows:

- For any vertex e_i in Q^{op} , we set $\Phi(e_i) = e_i \otimes 1$.
- For any arrow α_i in Q^{op} , we set $\Phi(\alpha_i) = e_i \otimes x + \epsilon_i(e_i \otimes 1)$, where e_i is the target of α_i in Q^{op} , and ϵ_i is its color.

It is a straightforward calculation to prove that Φ is a surjective algebra map such that $(Q_{\geq 2}^{op}) = \text{Ker } \Phi$, and it is left to the reader. For instance, by using (2.19),

$$\Phi(\alpha_i)\Phi(e_i) = (e_i \otimes x + \epsilon_i(e_i \otimes 1))(e_i \otimes 1) = -\epsilon_i(e_i \otimes 1) + \epsilon_i(e_i \otimes 1) = 0$$

if e_i is not a loop vertex, whilst

$$\Phi(e_i)\Phi(\alpha_i) = (e_i \otimes 1)(e_i \otimes x + \epsilon_i(e_i \otimes 1)) = (e_i \otimes x + \epsilon_i(e_i \otimes 1)) = \Phi(\alpha_i).$$

In order to prove (b) – (iii) we use a reformulation of the morphism Φ . Then we define $\widehat{\Phi} : K(\widehat{Q})^{op} \rightarrow k^n \otimes_{(f,\delta)} k[x]/(p(x))$ as follows:

- For any vertex $e_i \neq \textcircled{1}, \textcircled{2}$; we set $\widehat{\Phi}(e_i) = e_i \otimes 1$.
- $\widehat{\Phi}(\textcircled{1}) = e_0 \otimes \left(\frac{x-r_1}{r_2-r_1}\right)$ and $\widehat{\Phi}(\textcircled{2}) = e_0 \otimes \left(\frac{x-r_2}{r_1-r_2}\right)$, where e_0 is the loop vertex in Q .
- For any arrow α_i that does not start neither at $\textcircled{1}$ nor $\textcircled{2}$, we set $\widehat{\Phi}(\alpha_i) = e_i \otimes x + \epsilon_i(e_i \otimes 1)$, where e_i is the target of α_i and ϵ_i is its color.
- For any arrow $\alpha_k : \textcircled{1} \rightarrow e_k$, we set $\widehat{\Phi}(\alpha_k) = \alpha \left(e_k \otimes \frac{x-r_1}{r_2-r_1} \right)$

- For any arrow $\alpha_k : \textcircled{2} \rightarrow e_k$, we set $\widehat{\Phi}(\alpha_k) = \alpha \left(e_k \otimes \frac{x-r_2}{r_1-r_2} \right)$

We leave to the reader the details of the proof, see [7, Theorem 4.2] □

Corollary 2.9. *Let $p \in k[x]$ be an irreducible polynomial of degree two. Any twisted tensor product*

$$k^n \otimes_{(f,\delta)} \bar{k} \cong \mathcal{M}_2(k)^t \times (\bar{k})^r,$$

where $\bar{k} = k[x]/(p)$ and t and r are the number of strict 2-cycles and strict 1-cycles in Q_f , respectively.

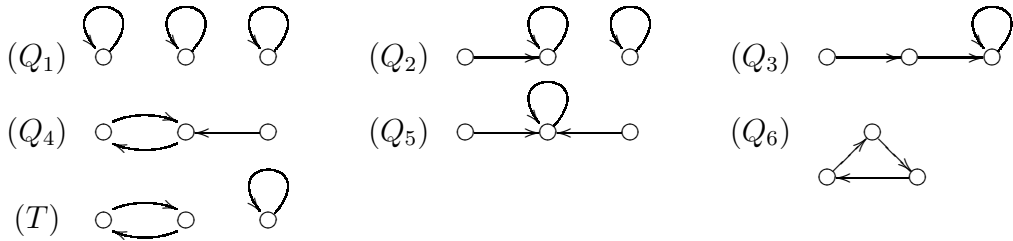
Theorem 2.10. *Any quantum duplicate of k^n , $k^n \otimes_{\tau} k[k]/(p(x))$, where $p \in k[x]$ is a reducible polynomial, is isomorphic to an algebra of type*

$$(\mathcal{M}_2(k))^t \times kQ_{<2},$$

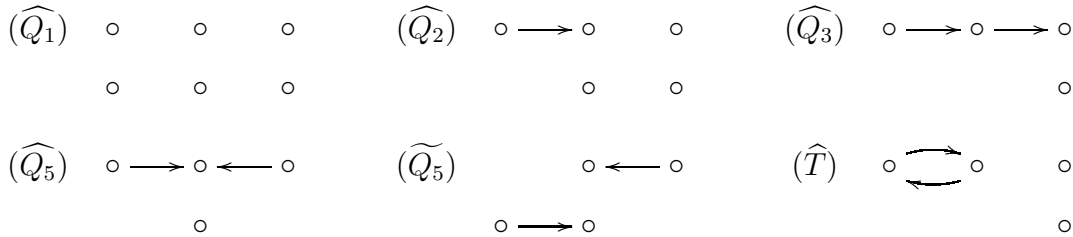
where t is a natural number, and Q an appropriate quiver.

Corollary 2.11. *There is a finite number of quantum duplicates of k^n for any fixed $n > 2$.*

Example 2.12. *Let us describe all quantum duplicates of k^3 , $R = k^3 \otimes_{\tau} k[x]/(p(x))$. If τ is given by a pair (f, δ) , the only possibilities for the quiver Q_f are:*



We should also consider the Cibil's transformation of the quivers with a 1-cycle component



where \widehat{Q}_5 and \widetilde{Q}_5 are the two possibilities depending on the coloration of Q_5 .

By Corollary 2.9, if p is irreducible, Q_f must be Q_1 or T . Then $R \cong (k[x]/(p(x)))^3$ or $R \cong \mathcal{M}_2(k) \times k[x]/(p(x))$. Otherwise, if $Q_f \neq T$, R is isomorphic to one of the (truncated) path algebras of the opposite quivers of $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, \widehat{Q}_1, \widehat{Q}_2, \widehat{Q}_3, \widehat{Q}_5$ or \widetilde{Q}_5 . In case of $Q_f = T$, R depends on the coloration of the round-trip component of T . Then we obtain that R is the truncated path algebra of T or \widehat{T} if it is colored by roots of q ; or $R \cong \mathcal{M}_2(k) \times k[x]/(x^2)$, or $R \cong \mathcal{M}_2(k) \times k^2$ if not.

3. FACTORIZATION STRUCTURES OF DIMENSION 4

The simplest nontrivial algebras that can be factorized as twisted tensor product ought to have factors of dimension at least 2, and thus the dimension of the product has to be greater or equal than 4. In the present Section, our purpose is to classify, up to isomorphism, all the algebras of dimension 4 that can be factorized. This problem turns out to have a strong dependence on the base field k . Basically we face two different situations:

- (1) The field k does not admit a field extension of degree 2 (for instance, if k is algebraically closed). In this case, there are only two different algebras of dimension 2 that can appear as factors, the semisimple algebra k^2 , and the algebra of dual numbers $k[\xi]$. In the realization of these algebras as quotients $k[x]/(p(x))$, the algebra k^2 corresponds to the cases in which p has two distinct roots in k , and $k[\xi]$ to those for which p has a double root.
- (2) If k admits quadratic extensions, or equivalently, there are polynomials of degree 2 with no roots in k , we have to take into account all these extensions as possible factors. The number of non-isomorphic quadratic extensions may range among one (for instance, if $k = \mathbb{R}$ the field of real numbers) to an infinite family of them (as happens for $k = \mathbb{Q}$ the field of rational numbers).

Some of the possible combinations of the previous kinds of algebras to form 4 dimensional factorization structures have already been classified, concretely all the algebras of the form $k^2 \otimes_\tau k^2$ (classified in [7] and [15]), and all the factors of the form $k^2 \otimes_\tau A$, with $A = k[x]/(p(x))$ 2-dimensional, are described as a particular case of Theorem 2.8. For the sake of completeness, we relist the resulting algebras (without any proof).

3.1. Twisted tensor products of the form $k^2 \otimes_\tau k^2$. Every twisted tensor product of the form $k^2 \otimes_\tau k^2$ is isomorphic to one of the following algebras:

- (1) The commutative algebra k^4 .
- (2) The algebra of matrices $\mathcal{M}_2(k)$.
- (3) The quotient $kQ_{<2}$ of the path algebra kQ of the round-trip quiver

$$Q = \circ \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \circ$$

- (4) The path algebra $k\tilde{Q}$ of the quiver

$$\tilde{Q} = \begin{array}{ccc} & \circ & \\ & \longrightarrow & \\ \circ & & \circ \end{array}$$

3.2. Twisted tensor products of the form $k^2 \otimes_\tau k[\xi]$. Every twisted tensor product of the form $k^2 \otimes_\tau k[\xi]$ is isomorphic to one of the following algebras:

- (1) The commutative algebra $k[\xi] \times k[\xi]$.
- (2) The algebra of matrices $\mathcal{M}_2(k)$.
- (3) The quotient $kQ_{<2}$ of the path algebra kQ of the round-trip quiver

$$Q = \circ \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \circ$$

(4) The quotient $kQ'_{<2}$ of the path algebra kQ' of the quiver

$$Q' = \circ \longrightarrow \circ \overset{\curvearrowright}{\circ}$$

3.3. Twisted tensor products of the form $k[\xi] \otimes_{\tau} k[\xi]$. Since the algebra $k[\xi]$ of dual numbers is not separable, the description of twisting maps by means of endomorphisms and derivations is not as simple as it is when we use k^2 . However, for this particular situation, the equations derived from the twisting conditions are simple enough to solve by a direct approach. If we consider the two copies of $k[\xi]$ respectively generated by x and y with $x^2 = y^2 = 0$, and identify x and y with their images in the twisted tensor product $k[\xi] \otimes_{\tau} k[\xi]$, the twisting map is given by $yx = a + bx + cy + dxy$, and imposing the twisting conditions we obtain a system of equations that can be reduced to

$$(3.20) \quad \begin{cases} a(1+d) = 0 \\ b = 0 \\ c = 0 \end{cases}$$

and thus the variety $\mathcal{T}(k[\xi], k[\xi])$ of twisting maps has two lines as irreducible components, giving rise to two one-parameter families of algebras, corresponding respectively to the solutions $a = 0$ and $d = -1$ of the previous system:

$$(3.21) \quad A_q := k\langle x, y \mid x^2 = y^2 = 0, yx = qxy \rangle$$

$$(3.22) \quad X_t := k\langle x, y \mid x^2 = y^2 = 0, yx + xy = t \rangle$$

both families intersect at the algebra $A_{-1} = X_0$ (which is the exterior algebra $\bigwedge k^2$ of k^2).

Lemma 3.1. *For any $t \neq 0$, the algebra X_t is isomorphic to the matrix ring $\mathcal{M}_2(k)$.*

Proof. Just take the isomorphism given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

□

The other irreducible component of the twisting variety actually gives rise to a complete family of non-isomorphic algebras. More concretely, we have the following result (whose proof is a straightforward computation):

Lemma 3.2. *The algebras A_q and A_h are isomorphic if, and only if, $q = h$ or $q = h^{-1}$.*

Remark 3.3. *Remarkable algebras contained in the family A_q are $A_{-1} = \bigwedge k^2$, the exterior algebra of k^2 , and $A_1 \cong k[x, y]/(x^2, y^2)$, the only commutative algebra in the family, corresponding to the classical tensor product $k[\xi] \otimes k[\xi]$. It is also worth noticing that, unlike in the previous situations, the twisting variety $\mathcal{T}(k[\xi], k[\xi])$ is connected.*

3.4. Twisted tensor products of the form $k^2 \otimes_{\tau} l$. Twisted tensor products of the form $k^2 \otimes_{\tau} l$, with $l = k[x]/(p(x))$ a quadratic field extension of k , are again classified using Theorem 2.8, or more precisely its Corollary 2.9. According to it, all the resulting algebras are of the following form:

- (1) The commutative algebra $k^2 \otimes l \cong l \times l$.
- (2) The matrix algebra $\mathcal{M}_2(k)$.

3.5. Twisted tensor products of the form $k[\xi] \otimes_{\tau} l$. For l a quadratic field extension of k , by Lemma 1.1, twisted tensor products of the form $k[\xi] \otimes_{\tau} l$ are given by couples (f, δ) , being f an algebra endomorphism of l , and δ and f -derivation such that

$$(3.23) \quad \delta^2 = 0$$

$$(3.24) \quad f\delta + \delta f = 0$$

Now, if l is a Galois extension of k (which is always the case if $\text{char } k \neq 2$), its k -linear endomorphisms are in one-to-one correspondence with elements of the Galois group $\text{Gal}(l/k) \cong \mathbb{Z}_2$, which in this case correspond to the identity map and the nontrivial morphism σ that exchanges the roots of $p(x)$. Moreover, since every Galois extension is separable, all derivations in l are inner. If f is the identity map on l , there are no nontrivial inner derivations, so we only get one couple $(\text{Id}_l, 0)$, that trivially satisfies (3.23) and (3.24). The twisting map associated to the couple $(\text{Id}_l, 0)$ is the classical tensor product $k[\xi] \otimes l \cong l[\xi]$.

Let us describe the σ -derivations for the nontrivial morphism σ . Assume that l is generated over k by η , a root of the polynomial $p(x) = x^2 - \alpha x + \beta$. As any derivation is inner, there exists $\theta \in l$ such that $\delta(x) = (\sigma(x) - x)\theta$, so in particular we get

$$\delta(\eta) = (\sigma(\eta) - \eta)\theta = (\alpha - 2\eta)\theta.$$

For the study of this example it is useful to take into account the characteristic of the field. If $\text{char } k \neq 2$, as stated in Remark 1.3, the change of variables $x \mapsto (\alpha/2)x + 1$ takes the polynomial p into $p'(x) = x^2 + \beta$, and p' generates the same field extension l as p , so we may just assume that $\alpha = 0$ and obtain $\delta(\eta) = -2\eta\theta$. Writing $\theta = a + b\eta$, and taking into account that $\eta^2 = -\beta$, we get

$$\delta(\eta) = -2a\eta - 2b\eta^2 = 2b\beta - 2a\eta.$$

Plugin this expression into equations (3.23) and (3.24) leads to conditions

$$(3.25) \quad 4a\eta = 0$$

$$(3.26) \quad 4a^2\eta - 4ab\beta = 0$$

that are satisfied if, and only if, $a = 0$. Thus, the only σ -derivations providing valid twisting maps are of the form $\delta_q(\eta) = q$ for some $q \in k$, and we get a 1-parameter family of twisting maps given by the couples (σ, δ_q) , leading to the family of algebras

$$B_q := k\langle x, y \mid x^2 = 0, y^2 = \gamma, xy + yx = q \rangle$$

where in order to simplify notation we are writing γ in the place of $-\beta$.

Lemma 3.4. *For all $q \neq 0$, the algebra B_q is isomorphic to the matrix algebra $\mathcal{M}_2(k)$.*

Proof. Just take the isomorphism given by

$$x \mapsto \begin{pmatrix} 0 & 0 \\ q/\gamma & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & \gamma \\ 1 & 0 \end{pmatrix}.$$

□

Lemma 3.5. *The algebra B_0 is isomorphic to the invariant ring $(lQ_{<2})^G$, where $Q = \circ \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \circ$ is the round-trip quiver, and G denotes the group generated by the non-trivial*

automorphism that exchanges the vertices and the arrows of Q and conjugate the scalar elements of l with respect to the nontrivial element of the Galois group $\text{Gal}(l/k)$.

Proof. Let us denote by u, v the vertices of Q , and by R, S its arrows. First, realize that the l -algebra $B_0 \otimes l = l\langle x, y \mid x^2 = 0, y^2 = \gamma, xy + yx = q \rangle$ is isomorphic to the algebra $lQ_{<2}$ via the automorphism defined through

$$x \longmapsto R + S, \quad y \longmapsto \sqrt{\gamma}u - \sqrt{\gamma}v.$$

The automorphism σ of the Galois group lifts in a trivial way to $B_0 \otimes l$, and obviously $B_0 \cong (B_0 \otimes l)^\sigma$; now, it is straightforward to check that, under the above mapping, the image of B_0 is invariant under the action of G , resulting in the desired isomorphism $B_0 \cong (lQ_{<2})^G$. □

Let us study now the case for k a field with $\text{char } k = 2$. In this case, for the twisted derivation we get

$$\delta(\eta) = \alpha a + \alpha b \eta,$$

which leads to equations

$$(3.27) \quad \alpha^2 b = 0$$

$$(3.28) \quad \alpha^2 ab + \alpha^2 b^2 \eta = 0,$$

that are satisfied if, and only if, $b = 0$ or $\alpha = 0$. Assume that $\alpha \neq 0$, that is, that the polynomial p has two distinct roots on l ; this is always the case if k is a perfect field, otherwise the situation is very different, and will be treated separately. Then from the above equations we obtain $b = 0$, yielding $\delta(\eta) = q \in k$, which gives us exactly the same family of algebras

$$B_q := k\langle x, y \mid x^2 = 0, y^2 = \gamma, xy + yx = q \rangle$$

previously mentioned. Same proof as in Lemma 3.4 tells us that $B_q \cong \mathcal{M}_2(k)$ whenever $q \neq 0$, but in this case for $q = 0$ the algebra B_0 is commutative and thus $B_0 \cong k[\xi] \otimes l \cong l[\xi]$.

The remaining case, namely when $\text{char } k = 2$ and $p(x) = x^2 + \beta$, is a special one. Realize that in this case the irreducible polynomial of η over k has η as its unique (double) root. Under these premises, l is not a Galois extension of k , since it fails to be separable, and the former reasoning cannot be applied. In particular, as there is only one root in p , there are no nontrivial automorphisms of l , so the only possible choice for f is the identity; however, as the extension is not separable anymore, now we must take into account that there can be derivations that are not inner. In the present situation, with $f = \text{Id}$, equations (3.23) and (3.24) become simply

$$(3.29) \quad \delta^2 = 0,$$

$$(3.30) \quad 2\delta = 0,$$

of which the second one is trivially satisfied. Thus, we only need to classify derivations of square 0.

For any derivation δ we have

$$(3.31) \quad \delta(1) = \delta(1^2) = 2\delta(1) = 0,$$

$$(3.32) \quad \delta(\eta) = a\eta + b$$

for some $a, b \in k$. It is straightforward to check that any choice of a and b leads to a valid derivation in $k(\eta)$, and we have that $\delta^2 = 0$ if, and only if,

$$0 = \delta^2(\eta) = \delta(a\eta + b) = a^2\eta + ab,$$

and this equation is satisfied if, and only if, $a = 0$. Henceforth, we have a one-parameter family of square 0 derivations that give us twisting maps, and these derivations are given by $\delta(\eta) = b$. Once again, the resulting twisted tensor products are the family of algebras

$$B_q := k\langle x, y \mid x^2 = 0, y^2 = \gamma, xy + yx = q \rangle,$$

so we obtain the same twisted tensor products regardless of the characteristic of the field.

3.6. Twisted tensor products of the form $l \otimes_\tau l'$. In order to classify twisted tensor products of two field extensions, we shall initially assume that the characteristic of the base field is different from 2. As it happened in the last case, for fields of characteristic 2 weird phenomena may show up, so we will study them separately.

So, take k such that $\text{char } k \neq 2$, and assume (without loss of generality) that the field extensions l and l' are given as splitting fields of the polynomials $x^2 - \alpha$ and $x^2 - \beta$, respectively, and let us denote by η, ζ their respective generators over k , so that $l = k\langle \eta \mid \eta^2 = \alpha \rangle$, and $l' = k\langle \zeta \mid \zeta^2 = \beta \rangle$. Using again Lemma 1.1, twisting maps $\tau : l' \otimes l \rightarrow l \otimes l'$ are in one-to-one correspondence with couples (f, δ) where f is a k -linear endomorphism of l , satisfying the corresponding compatibility conditions. Since l is a Galois extension of k of degree 2, there are only two k -linear endomorphisms of l , namely the identity and the map σ given by $\sigma(\eta) = -\eta$. Since l is separable, all derivations are inner, and thus there are no nontrivial derivations associated to the identity map. For the map σ , we need to find all the σ -derivations satisfying

$$(3.33) \quad \delta^2 = 0$$

$$(3.34) \quad \delta\sigma + \sigma\delta = 0,$$

where we are using the fact that $\sigma^2 = \text{Id}_l$. Realize that the above conditions are trivially satisfied for the identity morphism and the trivial derivation (yielding the usual tensor product $l \otimes l'$). If the derivation δ , which is inner, is induced by an element $\theta = a + b\eta$, this leads us again to the same equations that showed up in the former paragraph:

$$(3.35) \quad 4a\eta = 0$$

$$(3.36) \quad 4a^2\eta + 4ab\alpha = 0$$

with solutions given by $\delta(\eta) = q \in k$, and once again we obtain the 1-parameter family of algebras leading to the family of algebras

$$C_q := k\langle x, y \mid x^2 = \alpha, y^2 = \beta, xy + yx = q \rangle.$$

However, the classification of these algebras is not as easy as in the former situations, and depends strongly on the ground field k . As a first approximation for the classification of the algebras C_q , we have the following result:

Lemma 3.6. *The algebra C_q is isomorphic to the generalized quaternion algebra ${}^\alpha k^t$, with $t = \frac{q^2 - 4\alpha\beta}{4\alpha^2}$. In particular, the algebras C_q form a family of linked quaternion algebras.*

Proof. Take the isomorphism $C_q \rightarrow {}^\alpha k^t$ given by

$$x \mapsto i, \quad y \mapsto \frac{q}{2\alpha}i + ij,$$

where i and j are the generators of ${}^\alpha k^t = k\langle i, j \mid i^2 = \alpha, j^2 = t, ij + ji = 0 \rangle$. \square

We can draw some immediate consequences from the previous lemma. Firstly, it is a well known fact that the quaternion algebra ${}^\alpha k^t$ is a central simple algebra whenever $\alpha, t \neq 0$, and since in our present situation $\alpha \neq 0$, and $t = 0$ if, and only if, $q^2 = 4\alpha\beta$; which might happen for at most two values of q . Henceforth, all the algebras C_q (except maybe two of them) are central simple. In particular, since $\text{char } k \neq 2$, the field k must have at least three elements, and we can assure that there always exists a twisted tensor product $l \otimes_\tau l'$ which is simple, giving some supporting evidence to the following conjecture, due to J. Gómez-Torrecillas and F. van Oystaeyen:

Conjecture 3.7. *For any algebra A , there exists a twisting map $\tau : A \otimes A \rightarrow A \otimes A$ such that $A \otimes_\tau A$ is simple.*

The solution to the isomorphism problem for quaternion algebras over a generic field is not explicitly known; however, we have the following result establishing necessary and sufficient conditions for two linked quaternion algebras to be isomorphic:

Lemma 3.8. *Two linked quaternion algebras ${}^a k^b$ and ${}^a k^c$ are isomorphic if, and only if, $b/c \in N_{l/k}(l^\times)$, being $l = k(\sqrt{a})$, and $N_{l/k} : l \rightarrow k$ the norm map of the extension l/k . As a consequence, ${}^a k^b$ is a matrix ring if, and only if, $b \in N_{l/k}(l^\times)$.*

This result, as well as some others dealing with the problem of classifying quaternion algebras, can be found in [18, Section 1.7] (cf. also [13, Chapter III] for a more recent revision). Applied to our concrete situation, and taking into account that for a field extension $l = k(\sqrt{\alpha})$ the norm map is given by $N_{l/k}(x + y\sqrt{\alpha}) = x^2 - \alpha y^2$, we obtain the following result:

Theorem 3.9. *Let $q, h \in k$ such that $4\alpha\beta - q \neq 0$, $4\alpha\beta - h \neq 0$.*

(1) *The algebras C_q and C_h are isomorphic if, and only if, there exist $x, y \in k$ such that*

$$(3.37) \quad x^2 - \alpha y^2 = \frac{q^2 - 4\alpha\beta}{h^2 - 4\alpha\beta}$$

(2) *C_q is isomorphic to the matrix ring $\mathcal{M}_2(k)$ if, and only if, there exist $x, y \in k$ such that*

$$(3.38) \quad x^2 - \alpha y^2 = q^2 - 4\alpha\beta$$

In other words, the isomorphism classes of twisted tensor products of the form $l \otimes_\tau l'$ are given by:

- (1) The orbits of the action of $N_{l/k}(l^\times)$, seen as a multiplicative subgroup of k^\times , that intersect the image of the map $q \mapsto q^2 - 4\alpha\beta$.
- (2) The algebra $C_q \cong C_{-q}$, provided that $q = 2\sqrt{\alpha\beta}$ belongs to k .
- (3) The commutative algebra $l \otimes l'$.

The degenerate case (2) only shows up under very special conditions. Assume that $\sqrt{\alpha\beta} \in k$. Since obviously $\sqrt{\alpha} \in l = k(\sqrt{\alpha})$, this means that also $\sqrt{\beta} \in l$, and henceforth $l = l'$. Conversely, if $l = l' = k(\sqrt{\alpha})$, we may take $q = 2\alpha$ and obtain the exceptional algebra.

Proposition 3.10. *Let $l = l' = k(\sqrt{\alpha})$; the algebra*

$$\overline{C}_{2\alpha} := l \otimes C_{2\alpha} = l\langle x, y \mid x^2 = y^2 = \alpha, xy + yx = 2\alpha \rangle.$$

is isomorphic to the truncated path algebra $lQ_{<2}$ of the round-trip quiver $Q = \circ \overset{\curvearrowright}{\rightleftarrows} \circ$.

As a consequence, we have an isomorphism between $C_{2\alpha}$ and the algebra of invariants $(lQ_{<2})^G$, being G the group generated by the nontrivial automorphism of kQ that conjugates scalars while exchanging vertices and arrows of Q .

Proof. The proof of this result follows the same lines as the one of lemma 3.5. In this case, we use the isomorphism $C_{2\alpha} \otimes l \rightarrow lQ_{<2}$ given by

$$x \mapsto \sqrt{\alpha}u - \sqrt{\alpha}v + R + S, \quad y \mapsto \sqrt{\alpha}u - \sqrt{\alpha}v.$$

□

For the family of central simple algebras (1), as previously mentioned, the number of isomorphism classes (or orbits of the group action) depends strongly on the ground field k . A nice recent survey on quaternion algebras over different ground fields can be found in [14] (cf. also [18] and [13]). Recall that any quaternion algebra must be isomorphic to either a division ring over k , or to the matrix ring $\mathcal{M}_2(k)$ (this can easily be proven by using Artin-Wedderburn structure theorem). For some familiar fields, a more concrete description can be given:

- (1) If k is the field \mathbb{R} of real numbers, $\alpha, \beta < 0$ then the isomorphism class of $C_q \cong {}^\alpha\mathbb{R}^b$, with $b = (4\alpha\beta - q^2)/4\alpha$, depends on the sign of b , which is the sign of $q^2 - 4\alpha\beta$. More concretely, if $|q| > 2\sqrt{\alpha\beta}$, then $C_q \cong \mathcal{M}_2(\mathbb{R})$. If, on the other hand, $|q| < 2\sqrt{\alpha\beta}$, then $C_q \cong \mathbb{H}$, the usual quaternion algebra.
- (2) If $k = \mathbb{F}_n$ is a finite field, a well-known theorem by Wedderburn states that any division ring over k is commutative. Since the quaternion algebras are noncommutative, they must all be isomorphic to the matrix ring, and thus $C_q \cong \mathcal{M}_2(k)$ for all values of q .
- (3) If k is an algebraic number field, i.e. a finite extension field of the rational numbers \mathbb{Q} , then there exist an infinite number of nonisomorphic quaternion algebras over k . Though there is no easy way to list the isomorphism classes, given any concrete couple of algebras of type C_q , it is possible to tell whether they are or not isomorphic in a finite number of steps by studying the existence of rational points in the conic given by equation 3.10. The existence (or not) of such solutions is obtained as a consequence of the Hasse-Minkowski principle and Hensel's lemma.
- (4) If $k = \mathbb{Q}_p$ is a field of p -adic numbers, there is only one isomorphism class of quaternion algebras, which is never isomorphic to the matrix ring. This follows from the theory of quadratic forms over the p -adic numbers, see [13, Chapter VI] for details.

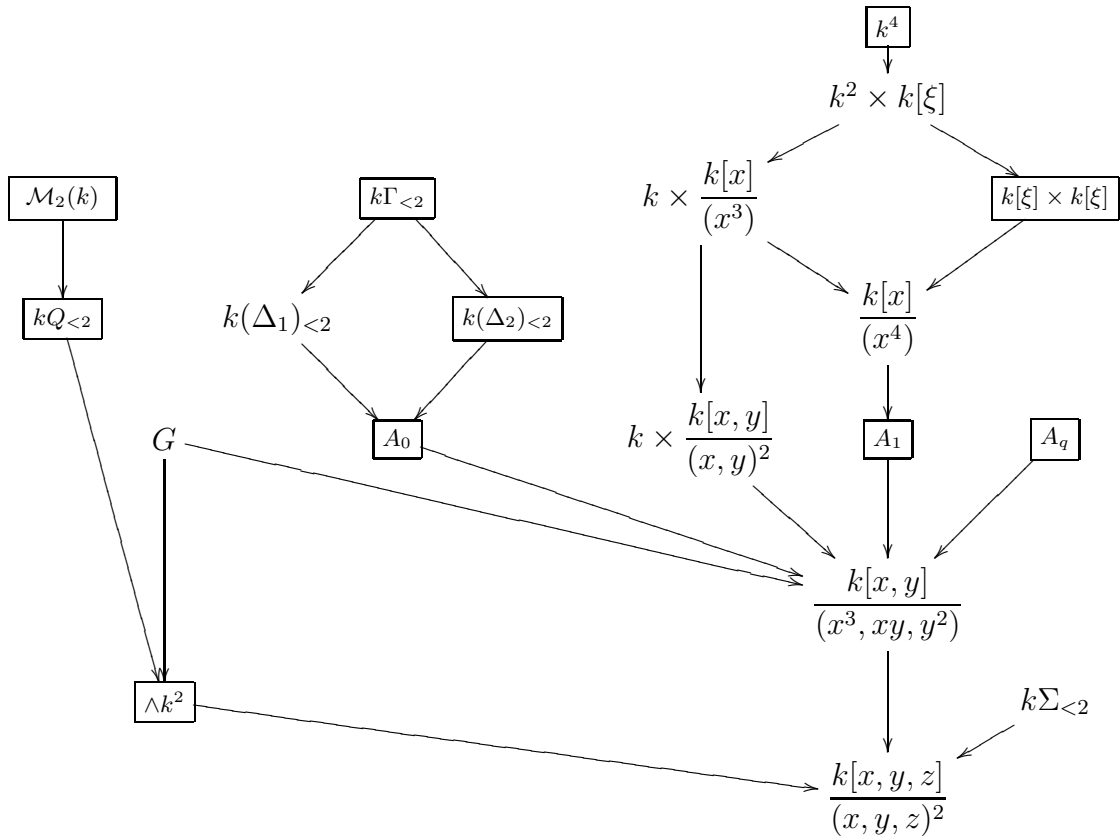
The remaining case, of field extensions of characteristic 2, cannot be described in such a fancy way using quaternion algebras. However, doing some computations (left to the reader) similar to the ones at the end of section 3.5, we obtain the following result:

Theorem 3.11. *Let k be a field with $\text{char } k = 2$, and let l and l' be quadratic field extensions of k generated by polynomials $p(x) = x^2 + \alpha x + \beta$ and $p'(x) = x^2 + \alpha'x + \beta'$. Then, all twisted tensor products $l \otimes_{\tau} l'$ of l and l' are described as*

$$D_q = k\langle x, y \mid x^2 = \alpha x + \beta, y^2 = \alpha'y + \beta', xy + yx = q \rangle,$$

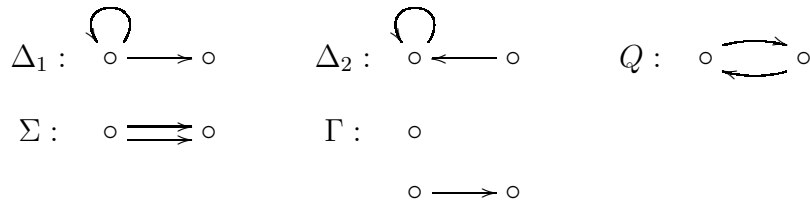
where the algebra D_0 corresponds to the usual tensor product $l \otimes l'$.

Let us finish the paper writing down the “indecomposable” (i.e., non-decomposable as a non-trivial twisted tensor product) algebras of dimension four over an algebraically closed field. A complete list of all algebras of dimension four can be found in [8]. We reproduce the scheme given there, highlighting the “decomposable” algebras putting them into a box.



where:

- (1) The scalar $q \neq 1, -1, 0$.
- (2) $G = \frac{k\langle x, y \rangle}{(x^2, y^2 + xy, xy + yx)}$
- (3) The quivers appearing above are the following:



- (4) The arrow $A \rightarrow B$ means that the algebra B can be obtained by a degeneration of the structure of A .

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