arXiv:0902.4542v1 [math.GR] 26 Feb 2009

ON ABSTRACT COMMENSURATORS OF GROUPS

L. BARTHOLDI AND O. BOGOPOLSKI

Abstract. We prove that the abstract commensurator of a nonabelian free group, an infinite surface group, or more generally of a group that splits appropriately over a cyclic subgroup, is not finitely generated.

This applies in particular to all torsion-free word-hyperbolic groups with infinite outer automorphism group and abelianization of rank at least 2.

We also construct a finitely generated, torsion-free group which can be mapped onto $\mathbb{Z}$ and which has a finitely generated commensurator.

1. Introduction

Let G be a group. Consider the set $\Omega(G)$ of all isomorphisms between subgroups of finite index of G. Two such isomorphisms $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ are called *equivalent*, written $\varphi_1 \sim \varphi_2$, if there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1\!\downharpoonright_H = \varphi_2\!\downharpoonright_H$.

For any two isomorphisms $\alpha : G_1 \to G_1'$ and $\beta : G_2 \to G_2'$ in $\Omega(G)$, we define their product $\alpha\beta : \alpha^{-1}(G_1' \cap G_2) \to \beta(G_1' \cap G_2)$ in $\Omega(G)$. The factor-set $\Omega(G)/\!\sim$ inherits the multiplication $[\alpha][\beta] = [\alpha\beta]$ and is a group, called the *abstract commensurator* of G and denoted $\mathrm{Comm}(G)$.

$\mathrm{Comm}(G)$ is in general much larger than $\mathrm{Aut}(G)$. For example $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n,\mathbb{Z})$ whereas $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}(n,\mathbb{Q})$. Margulis proved that an irreducible lattice Λ in a semisimple Lie group G is arithmetic if and only if it has infinite index in its *relative commensurator in* G,

$$\mathrm{Comm}_G(\Lambda) := \{g \in G : g\Lambda g^{-1} \cap \Lambda \text{ has finite index in both } \Lambda \text{ and } g\Lambda g^{-1}\}.$$

'Mostow-Prasad-Margulis strong rigidity' for irreducible lattices Λ in $G \neq \mathrm{SL}(2,\mathbb{R})$ implies that the abstract commensurator $\mathrm{Comm}(\Lambda)$ is isomorphic to the commensurator of Λ in G, which in turn is computed concretely by Margulis and Borel-Harish-Chandra; see e.g. [7, 14]. Analogously, for many groups acting on rooted trees, their abstract commensurator equals their relative commensurator in the automorphism group of the tree [11].

Few abstract commensurators were explicitly computed. The group $\mathrm{Comm}(\mathrm{MCG}_g)$ was computed for surface mapping class groups MCG_g by Ivanov [4]. Farb and Handel proved in [3] that $\mathrm{Comm}(\mathrm{Out}(F_n)) \cong \mathrm{Out}(F_n)$ for $n \geq 4$. Leininger and Margalit [5] computed the abstract commensurator of the braid group B_n on $n \geq 4$ strings: $\mathrm{Comm}(B_n) \cong (\mathbb{Q}^\infty \rtimes \mathbb{Q}^*) \rtimes \mathrm{MCG}_{0,n+1}$, where $\mathrm{MCG}_{0,n+1}$ is the mapping class group of the sphere with $n+1$ punctures.

Clearly, if G is finitely generated, then $\mathrm{Comm}(G)$ is countable. We show that, in many cases, it may be 'large' in the sense that it is not finitely generated. The cases we consider are groups G which split into an amalgamated product or an HNN extension over 1 or $\mathbb{Z}$, and satisfy some technical assumptions (see Theorems 3.2, 4.2 and 4.4). We deduce for example

Date: February 25, 2009.

Corollary A. *Let G be either a non-Abelian free group, or a surface group $\pi_1(S)$ where S is a closed surface of negative Euler characteristic. Then* $\mathrm{Comm}(G)$ *is not finitely generated.*

Using a result by Paulin [10], we deduce the more general

Corollary B. *Let G be a torsion-free word-hyperbolic group with infinite* $\mathrm{Out}(G)$*; suppose that G can be homomorphically mapped onto $\mathbb{Z} \times \mathbb{Z}$. Then* $\mathrm{Comm}(G)$ *is not finitely generated.*

We then consider some possible relaxations of the hypotheses (in particular (2) in Theorem 4.4), and show that in each case there are G with finitely generated commensurator:

Theorem C. *(1) There exist word-hyperbolic groups with finitely generated commensurator.*

(2) There exists a finitely generated group which has the unique root property (so in particular is torsion-free), can be mapped onto $\mathbb{Z}$, and whose commensurator is finitely generated.

It is a fundamental open question as whether all word-hyperbolic groups are residually finite; this is actually equivalent to asking whether all word-hyperbolic groups have arbitrarily large quotients [9, Theorem 2]. Of course, a group with no non-trivial finite quotient has identical automorphism group and abstract commensurator.

We start, in the next section, by a sufficient condition to ensure that an abstract commensurator cannot be finitely generated.

2. Infinitely generated abstract commensurators

Two groups G, H are *abstractly commensurable* if there exist finite index subgroups $G_1 \leqslant G$ and $H_1 \leqslant H$, such that $G_1 \cong H_1$. The following useful lemma is well-known; for completeness we give its proof.

Lemma 2.1. *If G and H are abstractly commensurable groups, then* $\mathrm{Comm}(G) \cong \mathrm{Comm}(H)$.

Proof. Without loss of generality we can assume that H is a subgroup of finite index in G. The embedding of H in G induces a canonical map $\Psi : \mathrm{Comm}(H) \to \mathrm{Comm}(G)$. Now we define a map $\Phi : \mathrm{Comm}(G) \to \mathrm{Comm}(H)$ by the rule: for $\alpha : G_1 \to G_2$ from $\mathrm{Comm}(G)$ we set $\Phi(\alpha) = \alpha\mid_{H_1} : H_1 \to H_2$, where $H_1 = \alpha^{-1}(G_2 \cap H) \cap H$ and $H_2 = \alpha(G_1 \cap H) \cap H$. Clearly $\Phi(\alpha)$ belongs to $\mathrm{Comm}(H)$. We leave it to the reader to check that Ψ and Φ are homomorphisms, and that both compositions $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity. □

A group G has the *unique root property* if for any $x, y \in G$ and any positive integer n, the equality $x^n = y^n$ implies $x = y$. Groups with the unique root property are torsion-free. It is well known that, in torsion-free word-hyperbolic groups, nontrivial elements have cyclic centralizers [1, pages 462–463]; so they have the unique root property, by the following standard

Lemma 2.2. *Let G be a torsion-free group with cyclic centralizers of nontrivial elements. Then G has the unique root property.*

Proof. Let x, y be nontrivial elements of G. If $x^n = y^n$, then $Z(x^n) \geqslant \langle x, y\rangle$. But $Z(x^n) = \langle z\rangle$ for some z, so there are $p, q \in \mathbb{Z}$ with $x = z^p$ and $y = z^q$. Then $x^n = y^n$ gives $z^{pn} = z^{qn}$, so $p = q$ and $x = y$. □

The usefulness of the unique root property can be seen immediately in the following two lemmas.

Lemma 2.3. *Let G be a group with the unique root property. Then* $\mathrm{Aut}(G)$ *naturally embeds in* $\mathrm{Comm}(G)$.

Proof. There is a natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Comm}(G)$. Suppose that some $\alpha \in \mathrm{Aut}(G)$ lies in its kernel. Then $\alpha\downharpoonright_H = \mathrm{id}$ for some subgroup H of finite index in G. If m is this index, then $g^{m!} \in H$ for every $g \in G$. Then $\alpha(g^{m!}) = g^{m!}$. Extracting roots, we get $\alpha(g) = g$, that is $\alpha = \mathrm{id}$. □

Lemma 2.4. *Let G be a group with the unique root property. Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in G. Suppose that $[\varphi_1] = [\varphi_2]$ in* $\mathrm{Comm}(G)$. *Then $\varphi_1\downharpoonright_{H_1 \cap H_2} = \varphi_2\downharpoonright_{H_1 \cap H_2}$.*

Proof. The equality $[\varphi_1] = [\varphi_2]$ means that there exists a subgroup H of finite index in G such that both φ_1 and φ_2 are defined on H and $\varphi_1\downharpoonright_H = \varphi_2\downharpoonright_H$. Clearly $H \leqslant H_1 \cap H_2$. Denote $m = |(H_1 \cap H_2) : H|$. Let h be an arbitrary element of $H_1 \cap H_2$. Then $h^{m!} \in H$ and so $\varphi_1(h^{m!}) = \varphi_2(h^{m!})$. Since G is a group with the unique root property, we get $\varphi_1(h) = \varphi_2(h)$. □

Let us call the *subindex* of a finite-index subgroup $H \leqslant G$ the minimal n, denoted $|G :: H|$, such that there exists a sequence of subgroups $H = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_k = G$ with $|G_i : G_{i-1}| \leq n$ for all $i \in \{1, \ldots, k\}$. Observe that given $F \leqslant H \leqslant G$, we have $|G :: F| \leq \max\{|G :: H|, |H :: F|\}$.

Lemma 2.5. *Let G be a group and let $\alpha_i : H_i \to H_i'$, for $i = 1, \ldots, r$ be isomorphisms between subgroups of finite index of G. Assume that $|G :: H_i| \leq n$ and $|G :: H_i'| \leq n$ for all i. Then any finite product of $[\alpha_i]$'s can be realized by an isomorphism $\beta : H \to H'$, where H, H' are subgroups of finite index and subindex at most n.*

Proof. By induction, it suffices to consider $\alpha_1 : H_1 \to H_1'$ and $\alpha_2 : H_2 \to H_2'$, and their product $\beta = \alpha_1\alpha_2$. Set $K = H_1' \cap H_2$, $H = \alpha_1^{-1}(K)$ and $H' = \alpha_2(K)$, so that $\beta : H \to H'$. Let $H_2 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_k = G$ be a sequence of subgroups with $|G_i :: G_{i-1}| \leq n$. The sequence $K = G_0 \cap H_1' \leqslant G_1 \cap H_1' \leqslant \cdots \leqslant G_k \cap H_1' = H_1'$ shows that $|H_1' :: K| \leq n$. Then

$$|G :: H| \leq \max\{|G :: H_1|, |H_1 :: H|\} = \max\{|G :: H_1|, |H_1' :: K|\} \leq n;$$

and similarly $|G :: H'| \leq n$. □

Lemma 2.6. *Let G be a group with the unique root property. Let $\varphi_1 : H_1 \to H_1'$ and $\varphi_2 : H_2 \to H_2'$ be two isomorphisms between subgroups of finite index in G. Suppose that*

(1) *H_2 is a normal subgroup of G;*
(2) $\varphi_1\downharpoonright_{H_1 \cap H_2} = \varphi_2\downharpoonright_{H_1 \cap H_2}$.

Then φ_1, φ_2 have a common extension, that is there exists an isomorphism $\varphi : H_1H_2 \to H_1'H_2'$, such that $\varphi\downharpoonright_{H_i} = \varphi_i$ for $i = 1, 2$.

Proof. We define $\varphi : H_1H_2 \to H_1'H_2'$ by $\varphi(h_1h_2) = \varphi_1(h_1)\varphi_2(h_2)$ for any $h_1 \in H_1$ and $h_2 \in H_2$. This definition is unambiguous because of Property (2). We prove first that φ is a homomorphism.

Take $x \in H_1H_2$ and $y \in H_1H_2$. Then $x = g_1g_2$ and $y = h_1h_2$ for some $g_1, h_1 \in H_1$ and $g_2, h_2 \in H_2$. Since $xy = g_1h_1 \cdot h_1^{-1}g_2h_1h_2$, where $h_1^{-1}g_2h_1 \in H_2$ by Property (1), we have

$$\varphi(xy) = \varphi_1(g_1)\varphi_1(h_1) \cdot \varphi_2(h_1^{-1}g_2h_1)\varphi_2(h_2).$$

On the other hand we have

$$\varphi(x)\varphi(y) = \varphi_1(g_1)\varphi_2(g_2)\varphi_1(h_1)\varphi_2(h_2).$$

Thus it is enough to verify that

$$(*)\qquad \varphi_2(h_1^{-1}g_2h_1)=\varphi_1(h_1)^{-1}\varphi_2(g_2)\varphi_1(h_1).$$

Since $H_1\cap H_2$ has finite index in H_2, we have $g_2^m\in H_1\cap H_2$ for some positive integer m. Then $h_1^{-1}g_2^m h_1\in H_1\cap H_2$ and so

$$\varphi_2(h_1^{-1}g_2^m h_1)=\varphi_1(h_1^{-1}g_2^m h_1)=\varphi_1(h_1^{-1})\varphi_1(g_2^m)\varphi_1(h_1)=\varphi_1(h_1)^{-1}\varphi_2(g_2)^m\varphi_1(h_1).$$

Since G is a group with the unique root property, we can extract m-th roots from both sides of the last equation and get (*).

Clearly φ maps onto $H_1'H_2'$. Assume for contradiction that φ is not injective; then, since G is torsion-free, $\ker\varphi$ is infinite. Since H_1 has finite index, $\ker\varphi\cap H_1$ is non-trivial, so φ_1 is not injective, a contradiction. □

Theorem 2.7. *Let G be a group with the unique root property. Suppose that, for infinitely many primes p, there exists a normal subgroup H of index p in G and an automorphism of H that cannot be extended to an automorphism of G.*

Then the commensurator of G is not finitely generated.

Proof. Suppose that $\mathrm{Comm}(G)$ is generated by a finite number of classes of isomorphisms $\alpha_i : H_i\to H_i'$, for $i=1,\dots,k$, where H_i, H_i' are subgroups of finite index in G. Set $n=\max\{|G::H_i|,|G::H_i'| : i=1,\dots,k\}$.

Now take a prime number $p>n$. By assumption, there exists a normal subgroup H of index p in G and an automorphism β of H, which cannot be extended to an automorphism of G.

Clearly $[\beta]\in\mathrm{Comm}(G)$. By Lemma 2.5, the class $[\beta]$ can be realized by an isomorphism $\alpha : A\to B$, where A, B are subgroups of finite index in G and subindex at most n. By Lemma 2.4, the automorphisms β and α coincide on the subgroup $H\cap A$.

By Lemma 2.6, the automorphism β can be extended to an isomorphism $\varphi : AH\to BH$. Note that $AH=BH=G$ because the indices of A and H are coprime and the indices of B and H are coprime. We have reached a contradiction. □

We shall also need a variant of the previous result:

Proposition 2.8. *Let G be a group with the unique root property. Suppose that, for infinitely many primes p, there exists a subgroup H of index p which is isomorphic to G.*

Then the commensurator of G is not finitely generated.

Proof. Suppose as above that $\mathrm{Comm}(G)$ is generated by a finite number of classes of isomorphisms $\alpha_i : H_i\to H_i'$, for $i=1,\dots,k$, where H_i, H_i' are subgroups of finite index in G. Set $n=\max\{|G::H_i|,|G::H_i'| : i=1,\dots,k\}$.

Now take a prime number $p>n$. By assumption, there exists a subgroup H of index p in G and an isomorphism $\beta : G\to H$.

Clearly $[\beta]\in\mathrm{Comm}(G)$. By Lemma 2.5, the class $[\beta]$ can be realized by an isomorphism $\alpha : A\to B$, where A, B are subgroups of finite index in G and subindex at most n. By Lemma 2.4, the automorphisms β and α coincide on A.

By Lemma 2.6, the automorphism β can be extended to an isomorphism $\varphi : G\to BH$. Note that $BH=G$ because the indices of B and H are coprime. We have reached a contradiction. □

Proof of Corollary A. It is well known that G has the unique root property (e.g. because G is a torsion-free hyperbolic group, see Lemma 2.2; or more directly because G is a group of diagonalizable 2×2 matrices).

First consider the case in which G is a free group with basis $X=\{x,y,\dots\}$. Given an integer $p>1$, let $G\to\mathbb{Z}/p\mathbb{Z}$ be the homomorphism which sends x to

1 and all other elements of X to 0. The kernel H of this homomorphism is free on $Y = \{x^p, y, x^{-1}yx, \dots, x^{1-p}yx^{p-1}, \dots\}$. Clearly, the automorphism of H which exchanges y and x^p and fixes all other elements of Y cannot be extended to an automorphism of G, because x^p is primitive in H but not in G. By Theorem 2.7, $\mathrm{Comm}(G)$ is not finitely generated.

It is convenient to translate this argument to topological language. The group G is the fundamental group of a rose R, with petals indexed by the elements of X. Consider the regular degree-p cover $\widetilde{R}$ of R, in which a petal (say x) has been unfolded p times to a "gynoecium" (central circle) $\tilde{x}$. Consider another petal y of R, and its lift $\tilde{y}$. The graph $\widetilde{R}$ is homotopy equivalent to a rose, so admits a homotopy equivalence φ that exchanges $\tilde{x}$ and $\tilde{y}$ while fixing (up to homotopy) the other petals. Then φ cannot be induced by a homotopy equivalence of R, because it fixes (up to homotopy) some lift of y while moves another.

Consider now the case in which $G = \pi_1(S)$ where S is a compact closed surface of negative Euler characteristic. By Lemma 2.1 we may assume that S is orientable. Given an integer $p > 1$, let $\widetilde{S} \to S$ a regular degree-p cover of S. Clearly $\widetilde{S}$ is of strictly more negative Euler characteristic.

Consider two handles x, x' of $\widetilde{S}$ covering the same handle of S, and a handle y that covers a different handle of S. Let T be a neighbourhood of x, y and a path connecting x to y that is homeomorphic to a punctured 2-handlebody. Let φ be the homeomorphism of $\widetilde{S}$ that exchanges x and y and is homotopic to the identity outside of T. Again, φ is not induced by a homeomorphism of S, since it moves x while it fixes its conjugate x' . Therefore, the automorphism induced by φ on $\pi_1(\widetilde{S})$ cannot be extended to an automorphism of $\pi_1(S)$. As above, Theorem 2.7 completes the proof. □

3. Free products of groups

We prove in this section that many free products have infinitely generated commensurator.

Lemma 3.1. *Let H be a finite-index subgroup of G; assume G is generated by the union of two subgroups A, B and has the unique root property; let $\varphi : H \to H$ be an automorphism. If $\varphi \neq id$, but $\varphi\downharpoonright_{H\cap A} = \mathrm{id}$, $\varphi\downharpoonright_{H\cap B} = \mathrm{id}$, then φ does not extend to an automorphism of G.*

Proof. Write $n = |G : H|$, and let $\psi : G \to G$ be an extension of φ. Take an arbitrary element $a \in A$. Then $a^{n!} \in H \cap A$, and so $\psi(a^{n!}) = a^{n!}$. Since G has the unique root property, we get $\psi(a) = a$, that is ψ is the identity on A. Analogously ψ is the identity on B, and hence $\psi = \mathrm{id}$, a contradiction. □

Theorem 3.2. *Suppose that two nontrivial groups A and B have the unique root property, and at least one of them has finite quotients of arbitrarily large prime order. Then* $\mathrm{Comm}(A * B)$ *is not finitely generated.*

Proof. Write $G = A * B$, and assume without loss of generality that A has arbitrarily large quotients. Consider a normal subgroup $H \triangleleft G$ of finite index $n > 1$ and containing B, e.g. the kernel of the map $A * B \to Q * 1$ for a finite quotient Q of A. By Kurosh's theorem, there exists a nontrivial splitting of the form $H = (H \cap A) * (H \cap B) * C$ with $C \neq 1$. Let b be a nontrivial element of $H \cap B$; there is some, because $H \cap B = B$ is nontrivial. Consider the automorphism φ of H, which is the identity on $H \cap A$ and on $H \cap B$ and is conjugation by b on C.

By Lemma 3.1, this φ does not extend to G. We conclude by Theorem 2.7. □

This gives another proof of Corollary A for free groups of rank $n \geq 2$: if $G = F_n$, take $A = \mathbb{Z}$ and $B = F_{n-1}$ and apply Theorem 3.2. Another proof of Corollary A for surface groups follows from Theorem 4.2 or 4.4.

Note that the abstract commensurator of a free group admits an elegant description through automata, see [6]. Lemma 2.5 essentially says that, given a finite collection of elements in the commensurator of F_m, there exists a finite alphabet (with n letters in the lemma's notation) such that these elements are represented by automata on that alphabet.

4. Groups splitting over $\mathbb{Z}$

Following on Theorem 3.2, we now apply Theorem 2.7 to free products with amalgamation and HNN extensions. In the proof we will use certain automorphisms of G, called *Dehn twists*.

Lemma 4.1. *Let k be an integer, and consider the Baumslag-Solitar group*

$$G = \langle a, t \mid tat^{-1} = a^k \rangle.$$

Then $\mathrm{Comm}(G)$ *is not finitely generated.*

Proof. If $k = 0$, then G is infinite cyclic and the statement obviously holds; so assume $k \neq 0$. Let p be a prime $> k$. Consider the endomorphism $\psi : G \to G$ sending t to t and a to a^p. We prove that ψ is injective, and that $\psi(G)$ has index p in G; the conclusion then follows from Proposition 2.8.

We have $G = \mathbb{Z}[1/k] \rtimes \langle t \rangle$, and ψ is given by $\psi(x, t^i) = (px, t^i)$; so ψ is an injective endomorphism. Its image is $p\mathbb{Z}[1/k] \rtimes \langle t \rangle$, which has index p because p and k are coprime. □

Theorem 4.2. *Let $G = A*_C$, where C is an infinite cyclic group. If G has the unique root property, then* $\mathrm{Comm}(G)$ *is not finitely generated.*

Proof. The group G has the presentation $\langle A, t \,|\, t^{-1}Ct = D \rangle$, where t is stable letter and $C = \langle c \rangle$, $D = \langle d \rangle$ are associated subgroups of A.

Consider $n \geq 3$ and let H_n be the kernel of the homomorphism $G \to \mathbb{Z}/n\mathbb{Z}$ sending A to 0 and t to 1. Then H_n is also an HNN extension, which has the following presentation:

$$H_n = \left\langle K, s \,\middle|\, s^{-1}(t^{n-1}Ct^{1-n})s = D \right\rangle, \text{ where}$$
$$K = A \underset{C = tDt^{-1}}{*} tAt^{-1} \underset{tCt^{-1} = t^2Dt^{-2}}{*} t^2At^{-2} * \dots \underset{t^{n-2}Ct^{2-n} = t^{n-1}Dt^{1-n}}{*} t^{n-1}At^{1-n}$$

and the stable letter s corresponds to t^n in G. Consider the automorphism φ of H_n which fixes the base K of the HNN extension and sends s to sd. Suppose that φ can be extended to an automorphism ψ of G. Then, since $tAt^{-1} \leqslant K$, for any $a \in A$, we have $tat^{-1} = \varphi(tat^{-1}) = \psi(tat^{-1}) = \psi(t)\psi(a)\psi(t^{-1}) = \psi(t)a\psi(t)^{-1}$, and so $t^{-1}\psi(t) \in C_G(A)$.

Now either the HNN extension is ascending ($C = A$), in which case G is a Baumslag-Solitar group, and we are done by Lemma 4.1; or $C_G(A) = Z(A)$, and we get $\psi(t) = ta$ for some $a \in Z(A)\backslash\{1\}$. We then have $t^n d = sd = \varphi(s) = \psi(t^n) = (ta)^n$; hence

$$(\dagger) \qquad \underbrace{t^{-1}(t^{-1}(\dots(t^{-1}(t^{-1}}_{n-1}(a)ta)ta)\dots)ta)tad^{-1} = 1.$$

Another cyclic form of this equation is

$$(\ddagger) \qquad tata\dots tat(ad^{-1})\underbrace{t^{-1}t^{-1}\dots t^{-1}t^{-1}}_{n-1}a = 1.$$

Using normal form in HNN extensions we deduce from (†) that $a \in C$, and from (‡) that $ad^{-1} \in D$. Thus, $a = c^p = d^q$ for some nonzero p, q. Since $a \in Z(A)$ and $Z(A)$ is closed under taking roots (since G has unique root property), we get $c, d \in Z(A)$. In particular, $\langle c, d\rangle$ is a torsion-free Abelian group satisfying $c^p = d^q$. Therefore this group is cyclic, that is $c = z^l$ and $d = z^r$ for some $z \in Z(A)$ and $l, r \in \mathbb{Z}$. Thus, we have

$$(\S) \qquad a = z^{pl} \qquad \text{and} \qquad t^{-1}z^l t = z^r.$$

We now analyze Equation (†) deeper. Using (§), we successively deduce

$$\begin{aligned}
&a = z^{pl},\\
&t^{-1}(a)ta = z^{pl(1+(r/l))},\\
&t^{-1}(t^{-1}(a)ta)ta = z^{pl(1+(r/l)+(r/l)^2)},\\
&\vdots\\
&\underbrace{t^{-1}(\dots(t^{-1}(t^{-1}}_{n-2}(a)ta)ta)\dots)ta = z^{pl(1+(r/l)+\dots+(r/l)^{n-2})},
\end{aligned}$$

Finally, we obtain from (†) that

$$1 = t^{-1}(t^{-1}(\dots(t^{-1}(t^{-1}(a)ta)ta)\dots)ta)tad^{-1} = z^{pl(1+(r/l)+\dots+(r/l)^{n-1})-r},$$

so

$$pl(1 + (r/l) + \dots + (r/l)^{n-1}) = r.$$

Equivalently,

$$p(l^{n-1} + rl^{n-2} + \dots + r^{n-1}) = rl^{n-1}.$$

Note that $\gcd(r, l) = 1$, otherwise, using the unique root property of G, we could extract a root from $tz^l t^{-1} = z^r$ and get a wrong equation. Hence $(l^{n-1} + rl^{n-2} + \dots + r^{n-1})$ has no nontrivial common divisor neither with r, nor with l. Therefore $(l^{n-1} + rl^{n-2} + \dots + r^{n-1}) = \pm 1$. Since $n \geq 3$, this is possible only if $l = 1, r = -1$ or $l = -1, r = 1$. In that last case, G has the presentation $G = \langle A, t \mid t^{-1}zt = z^{-1}\rangle$. Then its index 2 subgroup H_2 has the presentation

$$H_2 = \Big\langle \Big(A \underset{z=tz^{-1}t^{-1}}{*} tAt^{-1}\Big), s \mid s^{-1}zs = z\Big\rangle,$$

where s corresponds to t^2 in G. Thus, if we replace G by H_2 we will have $l = r = 1$. Thus, after possible replacement, φ cannot be extended to an automorphism of G and we conclude by Theorem 2.7. □

Lemma 4.3. *Consider $G = G_1 *_C G_2$, where C is infinite cyclic. If G_2 is Abelian, assume furthermore that $G_2 = K \oplus L$ with $C \leqslant K$ and $|L| > 2$.*

Then G has a nontrivial automorphism φ which fixes G_1.

Proof. It is enough to define a nontrivial automorphism $\psi : G_2 \to G_2$, such that $\psi\!\downharpoonright_C = \mathrm{id}$. Then such ψ can be obviously extended to the desired φ.

If C does not lie in $Z(G_2)$, we define ψ as conjugation by a generator of C. If C lies in $Z(G_2)$ and G_2 is not Abelian, we take an element $g \in G_2 \setminus Z(G_2)$ and define ψ as conjugation by g. Consider finally G_2 Abelian, with $G_2 = K \oplus L$. If $2L \neq 0$, define $\psi : G_2 \to G_2$ by $\psi(x, y) = (x, -y)$ for $x \in K, y \in L$; while if $2L = 0$ then L is an $\mathbb{F}_2$-vector space of dimension > 1, so admits a non-trivial automorphism ψ'. Set then $\psi(x, y) = (x, \psi'(y))$. □

Theorem 4.4. *Let G be $A *_C B$, where C is an infinite cyclic subgroup distinct from A and B. Suppose that*

(1) *G has the unique root property;*
(2) *A/C^A maps homomorphically onto $\mathbb{Z}$;*
(3) *if B is Abelian, then B maps homomorphically onto $\mathbb{Z}$.*

Then $\mathrm{Comm}(G)$ *is not finitely generated.*

Note that (2) is satisfied as soon as G maps onto $\mathbb{Z}\times\mathbb{Z}$, and (3) is satisfied as soon as B is finitely generated.

Proof. We first show that we may assume additionally that the following condition is satisfied:

$$|B:C| \text{ is infinite.} \tag{4}$$

Suppose that the index $|B:C|$ is finite, so B is virtually cyclic. Since G is torsion-free, B is infinite cyclic. Let $1, b, b^2, \dots, b^{n-1}$ be representatives of B modulo C. Note that $n \geq 2$, since $B \neq C$. Let $\varphi : A *_C B \to \mathbb{Z}/n\mathbb{Z}$ send A onto $\mathbb{Z}/n\mathbb{Z}$ and B to 0. The kernel G_1 of φ can be presented as the free product of groups $b^{-i}Ab^i$ for $i \in \{0, 1, \dots, n-1\}$, amalgamated over the common subgroup C. Therefore $G_1 = A *_C B_1$, where B_1 is the free product of A^{b^i} for $i \in \{1, \dots, n-1\}$, amalgamated over C. Then $B_1 = A^b *_C V = A^b *_{C^b} V$ for some group V. It follows $B_1/V^{B_1} \cong A/C^A$ and so B_1 satisfies Condition (3). Moreover, B_1 satisfies Condition (4), since B_1 contains A^b and $|A^b : C| = |A : C|$ is infinite. Since G_1 has finite index in G, we have $\mathrm{Comm}(G) \cong \mathrm{Comm}(G_1)$. Therefore, replacing G by G_1 if necessary, we may assume that Conditions (1–4) are satisfied.

We then show that we may assume additionally that the following condition, which in required in Lemma 4.3, is satisfied:

$$\text{if } B \text{ is Abelian, then } B = K \oplus L, \text{ with } C \leqslant K \text{ and } |L| > 2. \tag{5}$$

Suppose that B is Abelian. By Condition (3), there is an epimorphism $\psi : B \to \mathbb{Z}$. Thus, $B = \ker\psi \oplus \mathbb{Z}$. If $C \leqslant \ker\psi$, we are done. If $C \not\leqslant \ker\psi$, then $\psi(C)$ has finite index in $\psi(B)$. Since C is infinite cyclic, we have $\ker\psi \cap C = \{0\}$. Denote $B_1 = \langle \ker\psi, C\rangle$. Then $B_1 = \ker\psi \oplus C$ and the index $n = |B : B_1|$ is finite. Hence B_1 satisfies Conditions (3–4). In particular, $\ker\psi$ is infinite. Therefore, B_1 satisfies Condition (5).

If $n = 1$, then $B = B_1$, and so B satisfies Conditions (3–5). Suppose then $n \geq 2$ and let $T = \{b_1, \dots, b_n\}$ be a transversal of B_1 in B. Consider $H = \langle A, B_1\rangle^G$. Then H has index n in G; hence $\mathrm{Comm}(G) \cong \mathrm{Comm}(H)$. Moreover, T is a transversal of H in G. Consider the induced decomposition of H as the fundamental group of a graph of groups (see [12]): it has the shape of a star; there is a central vertex with vertex group B_1 and n outer vertices with vertex groups A^b for $b \in T$. All edge groups are C. We can rewrite this decomposition in the form $H = B_1 *_C A_1$, where $A_1 = A *_C V$ for some V. The group A_1/C^{A_1} can be mapped homomorphically onto $A_1/V^{A_1} = A/C^A$ and so Condition (2) is satisfied by A_1.

In summary, without loss of generality we assume that Conditions (1–5) are satisfied for the original G.

We now show that for any prime number $p > 1$, there exists a normal subgroup H of index p in G, and an automorphism of H that does not extend to an automorphism of G. Then Theorem 2.7 will complete this proof.

By (2), the quotient group A/C^A can be homomorphically mapped onto $\mathbb{Z}$ and further onto $\mathbb{Z}/p\mathbb{Z}$. Let $N \triangleleft A$ be the kernel of the composition of these epimorphisms, and set $H = \langle N, B\rangle^G$. Then $C \leqslant H \triangleleft G$ and $|G : H| = p$. Consider the induced decomposition of H as the fundamental group of a graph of groups: it has the shape of a star; there is a central vertex with the vertex group N and p outer vertices with the vertex groups B^a for a in a transversal of N in A.

In particular, $H = U *_{C^a} B^a$ for some $a \notin A$ and some subgroup U containing B and N. By Lemma 4.3, there is a non-trivial automorphism φ of H fixing U. We conclude by Lemma 3.1 that φ cannot be extended to an automorphism of G. □

To prove Corollary B, we recall a theorem by Paulin:

Theorem 4.5 ([10])**.** *Suppose G is a word-hyperbolic group with infinite* $\mathrm{Out}(G)$*. Then G splits over a virtually cyclic group.*

Proof of Corollary B. By Theorem 4.5, G splits over a virtually cyclic subgroup, that is $G = A *_C B$ or $G = A *_C$, where C is virtually cyclic. Since G is torsion-free, $C = 1$ or $C = \mathbb{Z}$. If $C = 1$, we apply Theorem 3.2. If G is an HNN extension, we apply Theorem 4.2.

If $C = \mathbb{Z}$ then, since G maps onto $\mathbb{Z}^2$, its quotient G/C^G maps onto $\mathbb{Z}$. Since $G/C^G = A/C^A * B/C^B$, one of the groups A/C^A or B/C^B maps onto $\mathbb{Z}$. If A or B is Abelian, it is cyclic, since G is a torsion-free hyperbolic group. We conclude by Theorem 4.4. □

5. Examples

We conclude in this section with a few examples showing that additional conditions are required on a hyperbolic group or on a free product with amalgamation to ensure that its commensurator is infinitely generated.

The following construction was generously indicated to us by Marc Lackenby. Consider a complicated-enough knot $K \subset S^2 \times S^1$; namely, the mapping torus of a complicated-enough braid $\tilde{K} \subset S^2 \times [0,1]$. Let μ be a small loop in $S^2 \times S^1 \setminus K$ around K.

Set $\Delta = \pi_1(S^2 \times S^1 \setminus K)$. Then, for n large enough, $\Gamma := \Delta/\langle \mu^n \rangle^\Delta$ is hyperbolic [2], and is a non-arithmetic lattice in $G := \mathrm{PSL}_2(\mathbb{C})$. By rigidity (see the Introduction), $\mathrm{Comm}(\Gamma) = \mathrm{Comm}_G(\Gamma)$; and by [7, Theorem IX.1.B], Γ has finite index in $\mathrm{Comm}(\Gamma)$, so in particular $\mathrm{Comm}(\Gamma)$ is finitely generated. We deduce:

Theorem 5.1 (=Theorem C(1))**.** *There exist word-hyperbolic groups with finitely generated commensurator.*

(Note of course that Γ is not torsion-free).

Recall that a group G is called *complete* if it has trivial center and no outer automorphisms. A group is called *perfect* if it equals its own commutator subgroup. A subgroup C of a group G is called *malnormal* if $C \cap g^{-1}Cg = 1$ for every $g \in G \setminus C$. We will use the following result of V.N. Obraztsov (see Corollary 3 in [8] and its proof).

Theorem 5.2 ([8])**.** *There exists a 2-generated simple complete torsion-free group G in which every proper subgroup is infinite cyclic.*

We note that such a group G has maximal cyclic subgroups; indeed otherwise it would contain an infinite ascending sequence of cyclic subgroups; its union cannot be cyclic, and so it must coincide with G. This is impossible since G is finitely generated.

Lemma 5.3. *Let G be a group as in Theorem 5.2. Then every maximal cyclic subgroup of G is malnormal. Moreover, G has the unique root property.*

Proof. Let $\langle z \rangle$ be a maximal cyclic subgroup in G and suppose that it is not malnormal, that is $\langle z \rangle \cap g^{-1}\langle z \rangle g \neq 1$ for some $g \in G \setminus \langle z \rangle$. Then $z^s = g^{-1}z^t g$ for some nonzero s, t. Moreover, the subgroup $\langle g, z \rangle$ is larger than $\langle z \rangle$, so it is noncyclic and therefore equals G.

If $g^{-1}zg \notin \langle z\rangle$, then $\langle g^{-1}zg, z\rangle = G$ and hence z^s lies in the center of G, a contradiction.

If $g^{-1}zg \in \langle z\rangle$, then $g^{-1}zg = z^k$ for some k. If $|k| \geq 2$, then $\langle z\rangle$ is not maximal, a contradiction. If $|k| = 1$, then g^2 lies in the center of $G = \langle g, z\rangle$, again a contradiction.

Now we prove that G has the unique root property. Suppose that for some $x, y \in G$ holds $x^n = y^n$, $n \neq 0$. If x, y generate a cyclic group, then clearly $x = y$. If they generate a noncyclic group, then $\langle x, y\rangle = G$. But then x^n lies in the center of G, so $x^n = 1$, and so $x = 1$. Similarly $y = 1$. □

Theorem 5.4 (=Theorem C(2))**.** *There exists a 3-generated group $G = G_1 \underset{u_1=u_2}{*} G_2$ such that*

1. *G is torsion-free;*
2. *$G/[G,G] = \mathbb{Z}$ and $u_i \notin [G,G]$;*
3. *G has the unique root property;*
4. $\mathrm{Comm}(G) = \mathrm{Aut}(G)$*;*
5. $\mathrm{Aut}(G)$ *is generated by inner automorphisms, a Dehn twist along $\langle u_i\rangle$ and possibly one extra automorphism which interchanges G_1 and G_2. In particular,* $\mathrm{Aut}(G)$ *is finitely generated.*

Proof. Let H_1, H_2 be two groups as in Theorem 5.2. In each H_i we choose an element h_i, generating a maximal cyclic subgroup. We set $G_i = H_i \times A_i$, where $A_i = \langle a_i\rangle$ is an infinite cyclic group, take $u_i = h_i a_i$ and define $G = G_1 \underset{u_1=u_2}{*} G_2$.

We denote by u the image of u_i in G. Note that the centralizer of the subgroup $\langle u\rangle$ in G has the following structure: $C_G(u) = \langle u\rangle \times Z$, where $Z = \langle A_1, A_2\rangle$. Since $A_i \cap \langle u_i\rangle = 1$, we have $Z = A_1 * A_2 \cong F_2$.

Remark. Using Lemma 5.3 one can prove the following important property: if for some $g \in G$ we have that $g^{-1}u^s g = u^t$ for some nonzero s, t, then $s = t$ and $g \in C_G(u)$.

We are now ready to prove the statements. (1) is weaker than (3).

(2) This statement follows from the fact that H_1, H_2 are perfect.

(3) Assume the converse: there are two different elements $x, y \in G$ such that $x^n = y^n$. We will analyze the action of x and y on the Bass-Serre tree T associated with the decomposition $G = G_1 \underset{u_1=u_2}{*} G_2$. Clearly, x, y are either both elliptic or both hyperbolic. For any edge e of T let $\alpha(e)$ and $\omega(e)$ denote the initial and the terminal vertices of e respectively.

Case 1. Suppose that x, y are both elliptic. If they stabilize the same vertex of T, then (after conjugation) we may assume that $x, y \in G_i$ for some $i = 1, 2$. Then, using Lemma 5.3, we conclude $x = y$.

Suppose that x and y do not stabilize the same vertices of T. We choose the shortest path $p = e_1e_2\dots e_m$ in T such that $x \in \mathrm{Stab}(\alpha(e_1))$ and $y \in \mathrm{Stab}(\omega(e_m))$. Then this path is stabilized by $x^n(=y^n)$, in particular, e_1 is stabilized by x^n. By conjugating and renaming the factors, we can assume that $\mathrm{Stab}(\alpha(e_1)) = G_1$, $\mathrm{Stab}(\omega(e_1)) = G_2$ and $\mathrm{Stab}(e_1) = G_1 \cap G_2 = \langle u\rangle$. Since $x \in G_1$, we have $x = za_1^k$ for some $z \in H_1$, $k \in \mathbb{Z}$. And since $x^n \in G_1 \cap G_2$, we have $x^n = z^n a_1^{kn} = u^{kn} = h_1^{kn}a_1^{kn}$. In particular, $z^n = h_1^{kn}$ and so $z = h_1^k$ by Lemma 5.3. This implies that $x = h_1^k a_1^k = u_1^k \in G_1 \cap G_2 = \mathrm{Stab}(e_1)$, a contradiction to the minimality of the path p.

Case 2. Suppose that x, y are both hyperbolic. Since $x^n = y^n$, the axes of x and y coincide and $x^{-1}y$ and $x^{-2}y^2$ stabilize this axis. By conjugating we may assume that $x^{-1}y$ and $x^{-2}y^2$ lie in $G_1 \cap G_2$. Thus $y = xu^k$ for some $k \in \mathbb{Z}$ and so $y^2 = x^2 \cdot x^{-1}u^k x u^k$. Hence $x^{-1}u^k x \in G_1 \cap G_2$. By the remark at the beginning of this proof, we conclude that $x \in C_G(u)$. Similarly, $y \in C_G(u)$. Since $C_G(u) = \langle u \rangle \times Z \cong \langle u \rangle \times F_2$ has the unique root property, we conclude from $x^n = y^n$ that $x = y$.

(4,5) First we describe finite index subgroups of G. Let B be a subgroup of finite index m in G, and let N be a normal subgroup of finite index in G such that $N \leqslant B$. Since H_i does not contain proper finite index subgroups, we have $G_i \cap N = (H_i \times \langle a_i \rangle) \cap N = H_i \times \langle a_i^{m_i} \rangle$ for some $m_i \in \mathbb{Z}$. Then N contains the normal closure of $\langle H_1, H_2 \rangle$ in G. The factor group of G by this normal closure is isomorphic to $\mathbb{Z}$. Therefore B is normal and coincides with the preimage of $m\mathbb{Z}$.

We claim that $B = (H_1 \times \langle a_1^m \rangle) \underset{u_1^m = u_2^m}{*} (H_2 \times \langle a_2^m \rangle)$. Simplifying notations we write $G_{i,m} = H_i \times \langle a_i^m \rangle$ and $G(m) = G_{1,m} \underset{u_1^m = u_2^m}{*} G_{2,m}$. Thus we want to prove that $B = G(m)$.

It is enough to prove that $G(m)$ is normal in G (then clearly $G/G(m) \cong \mathbb{Z}/m\mathbb{Z}$ and so $B = G(m)$). Note that $G(m) = \langle a_1^m, a_2^m, H_1, H_2 \rangle$ and $G = \langle a_1, a_2, H_1, H_2 \rangle$. Preparing to conjugate, we deduce from the equations $h_1 a_1 = h_2 a_2$ and $[h_i, a_i] = 1$ the following:

$$a_1 a_2^{-1} = h_1^{-1} h_2 \in H_1 H_2 \leqslant G(m),$$
$$a_1^{-1} a_2 = h_1 h_2^{-1} \in H_1 H_2 \leqslant G(m).$$

Then for $\varepsilon \in \{-1, 1\}$ we have

$$a_1^{\varepsilon} a_2^m a_1^{-\varepsilon} = (a_1^{\varepsilon} a_2^{-\varepsilon}) a_2^m (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} \in G(m),$$
$$a_1^{\varepsilon} H_2 a_1^{-\varepsilon} = (a_1^{\varepsilon} a_2^{-\varepsilon}) a_2^{\varepsilon} H_2 a_2^{-\varepsilon} (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} = (a_1^{\varepsilon} a_2^{-\varepsilon}) H_2 (a_1^{\varepsilon} a_2^{-\varepsilon})^{-1} \leqslant G(m).$$

By symmetry we get $a_2^{\varepsilon} a_1^m a_2^{-\varepsilon} \in G(m)$ and $a_2^{\varepsilon} H_1 a_2^{-\varepsilon} \leqslant G(m)$. This completes the proof that $G(m)$ is normal in G and so $B = G(m)$. Thus, for every natural m there is a unique subgroup of index m in G; it has the form

$$(\#) \qquad G(m) = G_{1,m} \underset{u_1^m = u_2^m}{*} G_{2,m}.$$

We now investigate which isomorphisms can appear in $\mathrm{Comm}(G)$. Let n, m be two natural numbers and let $\alpha : G(n) \to G(m)$ be an isomorphism. We claim that $G_{i,n}$ is nonsplittable over a cyclic subgroup. Indeed, suppose $G_{i,n} = K *_L M$, where L is a cyclic group. If one of the indices $|K : L|$ or $|M : L|$ is larger than 2, then $G_{i,n}$ and hence its direct factor H_i would contain a noncyclic free group, contradicting the properties of H_i. If $|K : L| = |M : L| = 2$, then $G_{i,n} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ or $G_{i,n} \cong \mathbb{Z} *_{2\mathbb{Z}=2\mathbb{Z}} \mathbb{Z}$, again absurd in regard of Theorem 5.2. An analogous reasoning shows that $G_{i,n}$ cannot be a nontrivial HNN extension over a cyclic group.

This implies that $\alpha(G_{i,n})$ is also nonsplittable over a cyclic subgroup and so is conjugate to $G_{1,m}$ or to $G_{2,m}$.

Case 1. Suppose that $\alpha(G_{1,n})$ is conjugate to $G_{1,m}$ and $\alpha(G_{2,n})$ is conjugate to $G_{2,m}$. Composing α with an appropriate conjugation, we can assume that $\alpha(G_{1,n}) \leqslant G_{1,m}$ and $\alpha(G_{2,n}) \leqslant g G_{2,m} g^{-1}$ for some $g \in G(m)$. We prove that $\alpha(G_{2,n}) \leqslant G_{2,m}$. We can assume that g, written in reduced form with respect to the amalgamated product (#), is either empty or starts with an element of $G_{2,m} \setminus \langle u^m \rangle$ and ends with an element of $G_{1,m} \setminus \langle u^m \rangle$.

Suppose that g is nonempty and write it in reduced form: $g = g_1 g_2 \dots g_{2k-1} g_{2k}$, where $g_i \in G_{1,m} \setminus \langle u^m \rangle$ if i is even and $g_i \in G_{2,m} \setminus \langle u^m \rangle$ if i is odd. The element $\alpha(u^n)$ lies in $\alpha(G_{1,n}) \cap \alpha(G_{2,n}) = G_{1,m} \cap g G_{2,m} g^{-1}$, hence it can be written as $\alpha(u^n) = g_1 g_2 \dots g_{2k-1} g_{2k} v g_{2k}^{-1} g_{2k-1}^{-1} \dots g_2^{-1} g_1^{-1}$ for some $v \in G_{2,m}$ and the reduced form of this product consists of only one factor which lies in $G_{1,m}$. Therefore $v \in \langle u^m \rangle$ and $g_i \in C_{G_{2,m}}(u^m) \setminus \langle u^m \rangle$ for odd i and $g_i \in C_{G_{1,m}}(u^m) \setminus \langle u^m \rangle$ for even i. This implies

(a) $g u^m g^{-1} = u^m$;

(b) $\alpha(G_{1,m}) \cap \alpha(G_{2,m}) = \langle u^m \rangle$;

(c) if $w \in \langle u^m \rangle$, then the reduced form of gwg^{-1} with respect to (#) is w;

(d) if $w \in G_{2,m} \setminus \langle u^m \rangle$, then the reduced form of gwg^{-1} is $g_1 g_2 \dots g_{2k-1} g_{2k} w g_{2k}^{-1} g_{2k-1}^{-1} \dots g_2^{-1} g_1^{-1}$; it starts and ends with elements from $G_{2,m} \setminus \langle u^m \rangle$ and contains at least one element from $G_{1,m} \setminus \langle u^m \rangle$.

Using this we prove that the group generated by $G_{1,m}$ and $gG_{2,m}g^{-1}$ does not contain elements of $G_{2,m} \setminus \langle u^m \rangle$, and that will contradict the surjectivity of α. Let z be an arbitrary element of $\langle \alpha(G_{1,n}), \alpha(G_{2,n}) \rangle$. We write z as $z = z_1 z_2 \dots z_l$, so that z_i lie alternately in $\alpha(G_{1,n})$ or in $\alpha(G_{2,n})$ and l is minimal. First suppose that $l > 1$. Then $z_i \notin \langle u^m \rangle$, otherwise one can unify two consecutive factors of $z_1 z_2 \dots z_l$ and decrease l. Therefore the following hold:

(i) If $z_i \in \alpha(G_{1,n})$, then $z_i \in G_{1,n} \setminus \langle u^m \rangle$.

(ii) If $z_i \in \alpha(G_{2,n})$, then $z_i \in g(G_{2,n} \setminus \langle u^m \rangle)g^{-1}$ by (a). By (c–d) the reduced form of z_i with respect to (#) starts and ends with elements from $G_{2,m} \setminus \langle u^m \rangle$ and contains at least one element from $G_{1,m} \setminus \langle u^m \rangle$.

Therefore the normal form of z is the product of normal forms of z_i's, and so $z \notin G_{2,m} \setminus \langle u^m \rangle$.

If $l = 1$, then either $z \in \langle u^m \rangle$, or as above $z \notin G_{2,m} \setminus \langle u^m \rangle$. In both cases $z \notin G_{2,m} \setminus \langle u^m \rangle$.

We have reached a contradiction. Thus g is empty and so $\alpha(G_{i,n}) \leqslant G_{i,m}$ for $i = 1, 2$.

Case 2. Suppose that $\alpha(G_{1,n})$ is conjugate to $G_{1,m}$ and $\alpha(G_{2,n})$ is also conjugate to $G_{1,m}$. Composing α with an appropriate conjugation, we can assume that, say, $\alpha(G_{1,n}) \leqslant G_{1,m}$ and $\alpha(G_{2,n}) \leqslant gG_{1,m}g^{-1}$ for some $g \in G(m)$. Then arguing as in Case 1 we obtain a contradiction independently of whether g is empty or not.

All other possible cases can be considered similarly. Thus (after an appropriate conjugation), we may assume that $\alpha(G_{1,n}) = G_{1,m}$ and $\alpha(G_{2,n}) = G_{2,m}$ or $\alpha(G_{1,n}) = G_{2,m}$ and $\alpha(G_{2,n}) = G_{1,m}$. In particular, $\alpha(u^n) = u^{\varepsilon m}$ for some $\varepsilon \in \{-1, 1\}$. We consider the first case (the second case is similar).

Since H_i has no infinite cyclic quotients, we obtain $\alpha(H_i) = H_i$. Since α carries the center of $G_{i,n}$ to the center of $G_{i,m}$, we have $\alpha(a_i^n) = a_i^{\sigma m}$ for some $\sigma \in \{-1, 1\}$. Since H_i is complete, $\alpha\downharpoonright_{H_i}$ is conjugation by an element $w_i \in H_i$. Therefore, $\alpha(u^n) = \alpha(h_i^n a_i^n) = w_i h_i^n w_i^{-1} a_i^{\sigma m}$. On the other hand $\alpha(u^n) = u^{\varepsilon m} = h_i^{\varepsilon m} a_i^{\varepsilon m}$. Thus, we have $w_i h_i^n w_i^{-1} = h_i^{\varepsilon m}$ and $\sigma = \varepsilon$. By Lemma 5.3, $w_i = h_i^{k_i}$ for some k_i and so $n = \varepsilon m$, which implies $n = m$ and $\sigma = \varepsilon = 1$ since $m, n \in \mathbb{N}$. Then $\alpha\downharpoonright_{G_{i,m}}$ is conjugation by w_i, which is the same as conjugation by $h_i^{k_i} a_i^{k_i} = u_i^{k_i}$. Thus, α is a product of two Dehn twists.

All inner automorphisms and Dehn twists, and the (possible) permutation of factors of $G(n)$ can be lifted to the corresponding automorphisms of G. Thus properties (3) and (4) are proven.

Finally we prove that G is 3-generated. Recall that h_i generates a maximal cyclic subgroup in H_i. First we choose an element $y_i \in H_i \setminus \langle h_i \rangle$, $i = 1, 2$, and then take a generator x_i of a maximal cyclic subgroup of H_i containing y_i. Clearly, $x_i \in H_i \setminus \langle h_i \rangle$ and also $h_i \in H_i \setminus \langle x_i \rangle$.

We claim that the subgroup $F = \langle x_1, x_2, u_1 \rangle$ coincides with G. In the proof we will use the equations $h_1 a_1 = u_1 = u_2 = h_2 a_2$. We have $[x_i, u_i] = [x_i, h_i a_i] = [x_i, h_i] \in H_i$. By Lemma 5.3, the subgroup $\langle x_i \rangle$ is malnormal in H_i and so $[x_i, h_i] \notin \langle x_i \rangle$. Then, by Theorem 5.2, $\langle x_i, [x_i, u_i] \rangle = H_i$. In particular, $H_i \leqslant F$. Then $A_i = \langle a_i \rangle = \langle h_i^{-1} u_i \rangle \leqslant F$ and hence $G = \langle H_1, H_2, A_1, A_2 \rangle = F$. □

Note that G from the proof of Theorem 5.4 cannot be generated by 2 elements. Indeed, if G were 2-generated, then its homomorphic image $H_1 \underset{h_1=h_2}{*} H_2$ would be also 2-generated. But this is impossible in view of [13, Corollary 1], which states that if B is an amalgamated product of type $\overset{n}{\underset{i=1}{*_C}} B_i$ where $C \neq 1$, $C \neq B_i$, and C is malnormal in B, then $\mathrm{rank}(B) \geq n+1$.

6. Acknowledgments

This work started in the "profinite groups" conference in Oberwolfach, in 2008. Conversations with Martin Bridson, Marc Lackenby, Volodymyr Nekrashevych, and Alexander Ol'shanskij are gratefully acknowledged. The second named author thanks the MPIM at Bonn for its support and excellent working conditions during the fall 2008, while this research was conducted.

References

[1] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
[2] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque No. 272 (2001), 208 pp.
[3] B. Farb and M. Handel, *Commensurations of* $\mathrm{Out}(F_n)$, Publ. Math. Inst. Hautes Études Sci. **105** (2007), 1–48.
[4] N. Ivanov, *Automorphisms of Complexes of Curves and of Teichmüller Spaces*, Inter. Math. Res. Not., No. **14** (1997), 651–666.
[5] C.J. Leininger and D. Margalit, *Abstract commensurators of braid groups*, J. Algebra **299** (2006), n. 2, 447–455.
[6] O. Macedońska, V.V. Nekrashevych and V.I. Sushchansky, *Commensurators of groups and reversible automata*, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki **12** (2000), 36–39.
[7] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1990.
[8] V.N. Obraztsov, *On infinite complete groups*, Communications in Algebra, **22** (1994), 5875–5887.
[9] A.Yu. Ol'shanskiĭ, *On the Bass-Lubotzky question about quotients of hyperbolic groups*, J. Algebra **226** (2000), 807–817.
[10] F. Paulin, *Outer automorphisms of hyperbolic groups and small actions on* $\mathbb{R}$*-trees*, Arboreal Group Theory (R. C. Alperin, ed.), MSRI Publ., vol. **19**, Springer-Verlag, 1991, pp. 331–343.
[11] C.E. Röver, *Abstract commensurators of groups acting on rooted trees*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000). Geom. Dedicata **94** (2002), 45–61.
[12] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.
[13] R.Weidmann. *On the rank of amalgamated products and product knot groups*. Math. Ann. **312** (1999), 761-771.
[14] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Math., Vol. **81**, Birkhäuser, 1984.

(L.B.) Mathematisches Institut, Bunsenstrasse 3–5, G.-A. Universität zu Göttingen, 37073 Göttingen, Germany
E-mail address: laurent.bartholdi@gmail.com

(O.B.) Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, Russia, and

Mathematisches Institut der H.-H. Universität Düsseldorf, Universitätsstrasse 1, 40225 Düsseldorf, Germany
E-mail address: Oleg_Bogopolski@yahoo.com