A recursive presentation for Mihailova's subgroup

O. Bogopolski
Institute of Mathematics of
Siberian Branch of Russian Academy of Sciences,
Novosibirsk, Russia
e-mail: groups@math.nsc.ru

E. Ventura
Dept. Mat. Apl. III,
Universitat Politècnica de Catalunya,
Manresa, Barcelona, Catalunya
e-mail: enric.ventura@upc.edu

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Abstract

We give an explicit recursive presentation for Mihailova's subgroup $M(H)$ of $F_n \times F_n$ corresponding to a finite, concise and Peiffer aspherical presentation $H = \langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$. This partially answers a question of R.I. Grigorchuk, [8, Problem 4.14]. As a corollary, we construct a finitely generated recursively presented orbit undecidable subgroup of $\mathrm{Aut}(F_3)$.

1 Introduction

For all the paper, let $n \geqslant 2$, let F_n be the free group with basis $\{x_1, \ldots, x_n\}$, and let $H = \langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$ be a finite presentation of a quotient H of F_n (although most of what follows will depend on the specific presentation, we shall make the usual abuse of notation which consists on denoting by H both the group and its given presentation).

K.A. Mihailova, in her influential paper [11], associated to the presentation H the *Mihailova subgroup* of $F_n \times F_n$, namely

$$M(H) = \{(w_1, w_2) \in F_n \times F_n \mid w_1 =_H w_2\} \leqslant F_n \times F_n,$$

i.e. the subgroup of pairs of words in F_n determining the same element in H. It is clear that (x_i, x_i) and $(1, R_j)$ belong to $M(H)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and it is not difficult to see that, in fact, these pairs generate $M(H)$. The important observation made in [11] says that the membership problem for $M(H)$ in $F_n \times F_n$ is solvable (i.e. there exists an algorithm to decide whether a given $(w_1, w_2) \in F_n \times F_n$ belongs to $M(H)$ or not) if and only if the word problem for H is solvable.

By a result of P.S. Novikov [13] and W.W. Boone [3] (see also [4]), there exist finitely presented groups with unsolvable word problem. Thus, there also exist finitely generated subgroups of $F_n \times F_n$ with unsolvable membership problem.

Clearly, $M(H)$ has solvable word problem for every H (because $F_n \times F_n$ also does). In particular, $M(H)$ is recursively presented. More interestingly, F.J. Grunewald proved, in [9, Theorem B], that if H is infinite then $M(H)$ cannot be finitely presented. In [1], G. Baumslag and J.E. Roseblade completely described the structure of finitely presented subgroups of $F_n \times F_n$, a result that was later reproved by H. Short [14] and M.R. Bridson and D.T. Wise [5], and that implies Grunewald's result.

In this context, a natural problem is to look for recursive presentations for Mihailova's group $M(H)$, in terms of the original presentation H. This was recently posted as Problem 4.14 in [8] by R.I. Grigorchuk: "What kind of presentations can be obtained for Mihailova's subgroups of $F_n \times F_n$ determined by finite automata?"

The main result in the present paper (Theorem 1.1 below) gives a partial answer to this problem: under certain technical conditions on the initial H we give an explicit recursive presentation for $M(H)$ with finitely many generators and a one-parametric family of relations.

Theorem 1.1 *Let F_n be the free group on $x_1, \ldots, x_n$, and let $H = \langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$ be a finite, concise and Peiffer aspherical presentation. Then Mihailova's group $M(H) \leqslant F_n \times F_n$ admits the following presentation*

$$\big\langle d_1, \ldots, d_n, t_1, \ldots, t_m \;\mid\; [t_j, d^{-1} t_i^{-1} r_i\, d],\; [t_i,\; \mathrm{root}(r_i)]\;\; (1 \leqslant i, j \leqslant m,\; d \in D_n) \big\rangle,$$

where D_n is the free group with basis $d_1, \ldots, d_n$, where r_i denotes the word in D_n obtained from R_i by replacing each x_k to d_k, and where $\mathrm{root}(r_i)$ *denotes the unique element $s_i \in D_n$ such that r_i is a positive power of s_i but s_i itself is not a proper power.*

In this presentation the elements d_i and t_j correspond, respectively, to the elements (x_i, x_i) and $(1, R_j)$ of $M(H)$.

As a corollary we deduce the existence of a finitely generated, orbit undecidable subgroup of $\mathrm{Aut}(F_3)$ (see [2] for details), which has the recursive presentation given in Theorem 1.1.

The structure of the paper is the following. In Section 2 we recall some definitions and discuss some properties of concise and Peiffer aspherical presentations that will be used later. In Section 3 we prove Theorem 1.1. And in Section 4 we recall the relationship between Mihailova's subgroup and orbit undecidability, recently discovered in [2], and deduce the announced corollary (Theorem 4.2).

2 Asphericity

As stated, let $\langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$ be a finite presentation. Formally, $R_1, \ldots, R_m$ is a list of words in the alphabet $\{x_1, \ldots, x_n\}^{\pm 1}$ which may contain the trivial element, possible repetitions, and even possible members conjugated to each other or to the inverse of each other.

A presentation $\langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$ is called *concise* if every relation R_i is non-trivial and reduced, and every two relations R_i, R_j, $i \neq j$, are not conjugate to each other, or to the inverse of each other. Given an arbitrary finite presentation, $\langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$, one can always reduce the relations and eliminate some of them, to obtain another presentation of the same group, which is concise. We call this a *concise refinement* of $\langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$.

Now, we recall the definition of Peiffer transformations. Consider some elements $U_1, \ldots, U_l \in F_n$, some relators $R_{i_1}, \ldots, R_{i_l} \in \{R_1, \ldots, R_n\}$, and some numbers $\varepsilon_1, \ldots, \varepsilon_l \in \{-1, 1\}$, such that the equation

$$(U_1 R_{i_1}^{\varepsilon_1} U_1^{-1}) \cdots (U_l R_{i_l}^{\varepsilon_l} U_l^{-1}) = 1$$

holds in F_n. In this situation, the sequence of elements $(U_1R_{i_1}^{\varepsilon_1}U_1^{-1},\dots,U_lR_{i_l}^{\varepsilon_l}U_l^{-1})$ of F_n is called an *identity among relations* of length l. For $l=0$ we have the *empty* identity among relations, ( ).

In such a sequence, let us replace two consecutive terms, say $U_pR_{i_p}^{\varepsilon_p}U_p^{-1}$ and $U_{p+1}R_{i_{p+1}}^{\varepsilon_{p+1}}U_{p+1}^{-1}$ for some $1\leqslant p\leqslant l-1$, by the new ones $U_{p+1}R_{i_{p+1}}^{\varepsilon_{p+1}}U_{p+1}^{-1}$ and $(U_{p+1}R_{i_{p+1}}^{-\varepsilon_{p+1}}U_{p+1}^{-1}U_p)R_{i_p}^{\varepsilon_p}(U_p^{-1}U_{p+1}R_{i_{p+1}}^{\varepsilon_{p+1}}U_{p+1}^{-1})$. Since the product of the two old terms do coincide with that of the two new ones, the new sequence is again an identity among relations. This transformation is called a *Peiffer transformation of the first kind* or, shortly, an *exchange*.

Suppose now that in the sequence $(U_1R_{i_1}^{\varepsilon_1}U_1^{-1},\dots,U_lR_{i_l}^{\varepsilon_l}U_l^{-1})$ there are two consecutive terms, say $U_pR_{i_p}^{\varepsilon_p}U_p^{-1}$ and $U_{p+1}R_{i_{p+1}}^{\varepsilon_{p+1}}U_{p+1}^{-1}$ for some $1\leqslant p\leqslant l-1$, whose product equals 1. Then, we can obtain a new identity among relations by just deleting these two terms. This transformation and the inverse one are called *Peiffer transformations of the second kind* or shortly, *deletion* and *insertion*, respectively.

Definition 2.1 We say that a presentation is *Peiffer aspherical* if every identity among relations can be carried to the empty one by a sequence of Peiffer transformations.

In particular, a presentation admitting identities among relations of odd length is automatically not Peiffer aspherical.

A large class of Peiffer aspherical presentations can be obtained by using Theorems 3.1 and 4.2, and Lemma 5.1 from [6]. They state, respectively, that Peiffer asphericity is preserved under certain HNN extensions, under free products, and under Tietze transformations.

In the next section we shall argue using Peiffer asphericity. However, for completeness, we mention that in the literature there are (at least) three concepts of asphericity for presentations, which do not agree in general: *Peiffer asphericity* (called *combinatorial asphericity* in [6], see Proposition 1.5 there); *diagrammatical asphericity* defined in [6] like Peiffer asphericity but without allowing insertions (and also considered in Chapter III.10 of [10]); and *topological asphericity*.

Let $H=\langle x_1,\dots,x_n\,|\,R_1,\dots,R_m\rangle$ be a presentation and $\mathcal{K}(H)$ be the two-dimensional CW-complex with a single 0-cell, n 1-cells corresponding to the generators $x_1,\dots,x_n$, and m 2-cells each one being attached to the 1-skeleton along the path determined by the spelling of the corresponding relation. The presentation H is said to be *topologically aspherical* if $\pi_2(\mathcal{K}(H))=0$. As was indicated in Proposition 1.1 of [6], this is equivalent to the triviality of the second homology group of the universal cover of $\mathcal{K}(H)$.

The relations between these three concepts are as follows (for more details, see the introduction and Proposition 1.3 of [6]):

(i) topological asphericity implies Peiffer asphericity,

(ii) diagrammatical asphericity implies Peiffer asphericity,

(iii) for presentations where every relation is reduced, topological asphericity is equivalent to Peiffer asphericity plus conciseness and "no relator being a proper power".

3 Proof of Theorem 1.1

Back to Mihailova's construction for $H=\langle x_1,\dots,x_n\,|\,R_1,\dots,R_m\rangle$, we recall that $M(H)\leqslant F_n\times F_n$ is generated by (x_i,x_i) and $(1,R_j)$, $i=1,\dots,n$, $j=1,\dots,m$. So, letting F_{n+m} be the free group with basis $\{d_1,\dots,d_n,t_1,\dots,t_m\}$, we have an epimorphism $\pi\colon F_{n+m}\to M(H)$ defined by $d_i\mapsto(x_i,x_i)$ and $t_j\mapsto(1,R_j)$, $i=1,\dots,n$, $j=1,\dots,m$. Now, for proving Theorem 1.1 we have to show that $\ker\pi$ is precisely the normal closure of the relations shown in the pretended presentation for $M(H)$. Note that the images of elements $d_1,\dots,d_n$ generate the diagonal subgroup of $F_n\times F_n$, denoted $Diag(F_n\times F_n)$, which is isomorphic to F_n; hence, π restricts to an isomorphism from $D_n=\langle d_1,\dots,d_n\rangle\leqslant F_{n+m}$ onto $Diag(F_n\times F_n)\leqslant M(H)\leqslant F_n\times F_n$.

We will keep the following notational convention in the proof: capital letters will always mean words on $x_1,\dots,x_n$; with this in mind, if u is a word on $d_1,\dots,d_n$, then its capitalization U will denote the word obtained from u by replacing each occurrence of d_i to x_i. Thus, U is just the projection of $\pi(u)$ to the first (or the second) coordinate.

Proof of Theorem 1.1. Recall that in the statement, r_j is the word in D_n obtained from R_j by replacing each x_i to d_i, $j=1,\dots,m$.

Let $\mathcal{N}$ be the normal closure (in the free group F_{n+m}) of the recursive family of commutators

$$\{[t_j,d^{-1}t_i^{-1}r_i\,d],\ \ [t_i,\ \mathrm{root}(r_i)]\mid i,j=1,\dots,m,\quad d\in D_n\}.$$

Our goal is to show that $\mathcal{N}=\ker\pi$. The inclusion $\mathcal{N}\leqslant\ker\pi$ is straightforward from the following computations:

$$\pi([t_j,d^{-1}t_i^{-1}r_i\,d])=[(1,R_j),(u,u)^{-1}(R_i,1)(u,u)]=[(1,R_j),(u^{-1}R_iu,1)]=(1,1),$$

$$\pi([t_i,\ \mathrm{root}(r_i)])=[(1,R_i),(\mathrm{root}(R_i),\mathrm{root}(R_i))]=(1,1).$$

In order to prove $\ker\pi\leqslant\mathcal{N}$, we shall use the following strategy: to each word $w\in\ker\pi$ we will associate an identity among relations for the presentation $\langle x_1,\dots,x_n\mid R_1,\dots,R_m\rangle$ of H, in such a way that if $w\neq 1$ then the associated identity is non-empty; then we will show that, after applying an arbitrary Peiffer transformation, the resulting identity among relations is again the one associated to some other word $w'\in\ker\pi$ satisfying, additionally, that $w^{-1}w'\in\mathcal{N}$.

Having seen this, let $w\in\ker\pi$ and consider the associated identity among relations. Since, by hypothesis, the presentation $\langle x_1,\dots,x_n\mid R_1,\dots,R_m\rangle$ is Peiffer aspherical, there exists a sequence of Peiffer transformations reducing such identity to the empty one. Now, repeatedly using the result mentioned in the previous paragraph, we obtain a list of words (ending with the trivial one because the last identity is empty), w, w', w'', $\dots,1$, and such that the difference between every two consecutive ones belongs to $\mathcal{N}$. This shows that $w\in\mathcal{N}$ concluding the proof.

So, we are reduced to construct such an association. Let $w\in\ker\pi\leqslant F_{n+m}$ and write it in the form $w=u_1t_{i_1}^{\varepsilon_1}u_2\cdots u_lt_{i_l}^{\varepsilon_l}u_{l+1}$, where $l\geqslant 0$ and $u_1,\dots,u_{l+1}$ are words in $d_1,\dots,d_n$. Then, projecting $\pi(w)$ to each coordinate, we have

$$U_1U_2\cdots U_{l+1}=1\quad\text{ and }\quad U_1R_{i_1}^{\varepsilon_1}U_2\cdots U_lR_{i_l}^{\varepsilon_l}U_{l+1}=1. \tag{1}$$

Denote the accumulative products by $\mathbb{U}_i=U_1U_2\cdots U_i$, $i=1,\dots,l+1$ (note that $\mathbb{U}_{l+1}=1$). By (1), we have

$$\mathbb{U}_1R_{i_1}^{\varepsilon_l}\mathbb{U}_1^{-1}\cdot\mathbb{U}_2R_{i_2}^{\varepsilon_2}\mathbb{U}_2^{-1}\cdot\ldots\cdot\mathbb{U}_lR_{i_l}^{\varepsilon_l}\mathbb{U}_l^{-1}=1$$

in the free group F_n. In other words,

$$(\mathbb{U}_1R_{i_1}^{\varepsilon_l}\mathbb{U}_1^{-1},\mathbb{U}_2R_{i_2}^{\varepsilon_2}\mathbb{U}_2^{-1},\dots,\mathbb{U}_lR_{i_l}^{\varepsilon_l}\mathbb{U}_l^{-1}) \tag{2}$$

is an identity among relations for the presentation $\langle x_1,\dots,x_n\,|\,R_1,\dots,R_m\rangle$ of H. This is the *identity associated to* $w\in\ker\pi$. Note that if this identity is empty, that is $l=0$, then $w=u_1\in\langle d_1,\dots,d_n\rangle\cap\ker\pi$ and so $w=1$.

Let us analyze the situation when we apply an arbitrary Peiffer transformation to this identity.

Case 1: Consider the exchange which, for some $1 \leqslant p \leqslant l-1$, replaces the consecutive terms

$$\mathbb{U}_p R_{i_p}^{\varepsilon_p} \mathbb{U}_p^{-1} \quad \text{and} \quad \mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1},$$

in (2), by the terms

$$\mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1} \quad \text{and} \quad (\mathbb{U}_{p+1} R_{i_{p+1}}^{-\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1} \mathbb{U}_p) R_{i_p}^{\varepsilon_p} (\mathbb{U}_p^{-1} \mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1}), \tag{3}$$

respectively. We claim that the identity among relations obtained in this way is precisely the one corresponding to the word

$$w' = v_1 t_{i_1}^{\varepsilon_1} \cdots v_{p-1} t_{i_{p-1}}^{\varepsilon_{p-1}} v_p t_{i_{p+1}}^{\varepsilon_{p+1}} v_{p+1} t_{i_p}^{\varepsilon_p} v_{p+2} t_{i_{p+2}}^{\varepsilon_{p+2}} \cdots v_l t_{i_l}^{\varepsilon_l} v_{l+1},$$

where

$$\begin{array}{lll} v_1 = u_1, & v_p = u_p u_{p+1}, & v_{p+3} = u_{p+3}, \\ \vdots & v_{p+1} = r_{i_{p+1}}^{-\varepsilon_{p+1}} u_{p+1}^{-1}, & \vdots \\ v_{p-1} = u_{p-1}, & v_{p+2} = u_{p+1} r_{i_{p+1}}^{\varepsilon_{p+1}} u_{p+2}, & v_{l+1} = u_{l+1}. \end{array}$$

And we also claim that $w^{-1}w' \in \mathcal{N}$. This second assertion is easy to verify since we can obtain back w from w' by permuting the two consecutive subwords $u_{p+1} t_{i_{p+1}}^{\varepsilon_{p+1}} r_{i_{p+1}}^{-\varepsilon_{p+1}} u_{p+1}^{-1}$ and $t_{i_p}^{\varepsilon_p}$. But the commutator of these two words is an element of $\mathcal{N}$: for $\varepsilon_{p+1} = -1$ this is immediate; and for $\varepsilon_{p+1} = 1$ it follows from the facts that, modulo $\mathcal{N}$, t_{i_p} (and so $t_{i_p}^{\varepsilon_p}$) commutes with $u_{p+1}(t_{i_{p+1}}^{-1} r_{i_{p+1}})^{\pm 1} u_{p+1}^{-1}$, but also $t_{i_{p+1}}$ commutes with $t_{i_{p+1}}^{-1} r_{i_{p+1}}$ (and so, $t_{i_{p+1}}^{-1}$ with $r_{i_{p+1}}$). Therefore, w' equals w modulo $\mathcal{N}$.

To see the first part of the claim, let us capitalize the v_i's:

$$\begin{array}{lll} V_1 = U_1, & V_p = U_p U_{p+1}, & V_{p+3} = U_{p+3}, \\ \vdots & V_{p+1} = R_{i_{p+1}}^{-\varepsilon_{p+1}} U_{p+1}^{-1}, & \vdots \\ V_{p-1} = U_{p-1}, & V_{p+2} = U_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} U_{p+2}, & V_{l+1} = U_{l+1}. \end{array}$$

And let us compute the $\mathbb{V}_i = V_1 V_2 \cdots V_i$'s:

$$\begin{array}{lll} \mathbb{V}_1 = \mathbb{U}_1, & \mathbb{V}_p = \mathbb{U}_{p+1}, & \mathbb{V}_{p+3} = \mathbb{U}_{p+3}, \\ \vdots & \mathbb{V}_{p+1} = \mathbb{U}_{p+1} R_{i_{p+1}}^{-\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1} \mathbb{U}_p, & \vdots \\ \mathbb{V}_{p-1} = \mathbb{U}_{p-1}, & \mathbb{V}_{p+2} = \mathbb{U}_{p+2}, & \mathbb{V}_{l+1} = \mathbb{U}_{l+1}. \end{array}$$

Finally, the identity among relations associated to w' is

$$\begin{array}{rcl} (\mathbb{V}_1 R_{i_1}^{\varepsilon_1} \mathbb{V}_1^{-1} & = & \mathbb{U}_1 R_{i_1}^{\varepsilon_1} \mathbb{U}_1^{-1}, \\ & \vdots & \\ \mathbb{V}_{p-1} R_{i_{p-1}}^{\varepsilon_{p-1}} \mathbb{V}_{p-1}^{-1} & = & \mathbb{U}_{p-1} R_{i_{p-1}}^{\varepsilon_{p-1}} \mathbb{U}_{p-1}^{-1}, \\ \mathbb{V}_p R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{V}_p^{-1} & = & \mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1}, \\ \mathbb{V}_{p+1} R_{i_p}^{\varepsilon_p} \mathbb{V}_{p+1}^{-1} & = & \mathbb{U}_{p+1} R_{i_{p+1}}^{-\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1} \mathbb{U}_p R_{i_p}^{\varepsilon_p} \mathbb{U}_p^{-1} \mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1}, \\ \mathbb{V}_{p+2} R_{i_{p+2}}^{\varepsilon_{p+2}} \mathbb{V}_{p+2}^{-1} & = & \mathbb{U}_{p+2} R_{i_{p+2}}^{\varepsilon_{p+2}} \mathbb{U}_{p+2}^{-1}, \\ & \vdots & \\ \mathbb{V}_l R_{i_l}^{\varepsilon_l} \mathbb{V}_l^{-1} & = & \mathbb{U}_l R_{i_l}^{\varepsilon_l} \mathbb{U}_l^{-1}), \end{array}$$

which does coincide with the identity among relations obtained from (2) after applying the Peiffer transformation (3).

Case 2: Consider the deletion which, for some $1 \leqslant p \leqslant l-1$, deletes the consecutive terms

$$\mathbb{U}_p R_{i_p}^{\varepsilon_p} \mathbb{U}_p^{-1} \quad \text{and} \quad \mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1}, \tag{4}$$

in (2), assuming that its product equals 1. We claim that the identity among relations obtained in this way is precisely the one corresponding to the word

$$w' = v_1 t_{i_1}^{\varepsilon_1} \cdots v_{p-1} t_{i_{p-1}}^{\varepsilon_{p-1}} v_p t_{i_{p+2}}^{\varepsilon_{p+2}} v_{p+1} \cdots v_{l-2} t_{i_l}^{\varepsilon_l} v_{l-1},$$

where

$$\begin{array}{ccc} v_1 = u_1, & & v_{p+1} = u_{p+3}, \\ \vdots & v_p = u_p u_{p+1} u_{p+2}, & \vdots \\ v_{p-1} = u_{p-1}, & & v_{l-1} = u_{l+1}. \end{array}$$

And we also claim that $w^{-1}w' \in \mathcal{N}$. This second assertion follows from the hypothesis that $(\mathbb{U}_p R_{i_p}^{\varepsilon_p} \mathbb{U}_p^{-1}) \cdot (\mathbb{U}_{p+1} R_{i_{p+1}}^{\varepsilon_{p+1}} \mathbb{U}_{p+1}^{-1}) = 1$. In fact, conciseness implies that $i_p = i_{p+1}$, $\varepsilon_p = -\varepsilon_{p+1}$ and so $\mathbb{U}_p^{-1}\mathbb{U}_{p+1} = U_{p+1}$ commutes with $R_{i_{p+1}}$; hence, u_{p+1} commutes with $r_{i_{p+1}}$ and so $u_{p+1} \in \langle \mathrm{root}(r_{i_{p+1}}) \rangle$. Now w' can be obtained from w by replacing the subword $t_{i_p}^{\varepsilon_p} u_{p+1} t_{i_{p+1}}^{\varepsilon_{p+1}}$ to u_{p+1}. But $(t_{i_p}^{\varepsilon_p} u_{p+1} t_{i_{p+1}}^{\varepsilon_{p+1}})^{-1} u_{p+1} \in \mathcal{N}$ since $t_{i_{p+1}}$ commutes with $\mathrm{root}(r_{i_{p+1}})$ modulo $\mathcal{N}$.

To see the first part of the claim, let us capitalize the v_i's:

$$\begin{array}{ccc} V_1 = U_1, & & V_{p+1} = U_{p+3}, \\ \vdots & V_p = U_p U_{p+1} U_{p+2}, & \vdots \\ V_{p-1} = U_{p-1}, & & V_{l-1} = U_{l+1}. \end{array}$$

And let us compute the $\mathbb{V}_i = V_1 V_2 \cdots V_i$'s:

$$\begin{array}{ccc} \mathbb{V}_1 = \mathbb{U}_1, & & \mathbb{V}_{p+1} = \mathbb{U}_{p+3}, \\ \vdots & \mathbb{V}_p = \mathbb{U}_{p+2}, & \vdots \\ \mathbb{V}_{p-1} = \mathbb{U}_{p-1}, & & \mathbb{V}_{l-1} = \mathbb{U}_{l+1}. \end{array}$$

Finally, the identity among relations associated to w' is

$$\begin{array}{rcl} (\mathbb{V}_1 R_{i_1}^{\varepsilon_1} \mathbb{V}_1^{-1} & = & \mathbb{U}_1 R_{i_1}^{\varepsilon_1} \mathbb{U}_1^{-1}, \\ & \vdots & \\ \mathbb{V}_{p-1} R_{i_{p-1}}^{\varepsilon_{p-1}} \mathbb{V}_{p-1}^{-1} & = & \mathbb{U}_{p-1} R_{i_{p-1}}^{\varepsilon_{p-1}} \mathbb{U}_{p-1}^{-1}, \\ \mathbb{V}_p R_{i_{p+2}}^{\varepsilon_{p+2}} \mathbb{V}_p^{-1} & = & \mathbb{U}_{p+2} R_{i_{p+2}}^{\varepsilon_{p+2}} \mathbb{U}_{p+2}^{-1}, \\ \mathbb{V}_{p+1} R_{i_{p+3}}^{\varepsilon_{p+3}} \mathbb{V}_{p+1}^{-1} & = & \mathbb{U}_{p+3} R_{i_{p+3}}^{\varepsilon_{p+3}} \mathbb{U}_{p+3}^{-1}, \\ & \vdots & \\ \mathbb{V}_{l-2} R_{i_l}^{\varepsilon_l} \mathbb{V}_{l-2}^{-1} & = & \mathbb{U}_l R_{i_l}^{\varepsilon_l} \mathbb{U}_l^{-1}), \end{array}$$

which coincides with the identity among relations obtained from (2) after applying the Peiffer transformation (4).

Case 3: Consider an insertion, and argue in a similar way as in Case 2.

This concludes the proof. □

4 A recursively presented orbit undecidable subgroup of $\mathrm{Aut}(F_3)$

In [2], O. Bogopolski, A. Martino and E. Ventura studied the conjugacy problem for extensions of groups. In that context, the notion of orbit decidability is crucial and we recall it here.

Let F be a group, and $A \leqslant \mathrm{Aut}(F)$. We say that A is *orbit decidable* if and only if there exists an algorithm such that, given $u, v \in F$, decides whether v is conjugate to $\alpha(u)$ for some $\alpha \in A$.

The main result in [2] states that, given a short exact sequence of groups

$$1 \to F \to G \to P \to 1$$

with some conditions on F and P, the group G has solvable conjugacy problem if and only if the action subgroup

$$A_G = \{\gamma_g \colon F \to F,\, x \mapsto g^{-1}xg \mid g \in G\} \leqslant \mathrm{Aut}(F)$$

is orbit decidable (see [2, Theorem 3.1] for details).

In particular, this applies to the case where F and P are finitely generated free groups, giving a characterization of the solvability of the conjugacy problem within the family of [f.g. free]-by-[f.g. free] groups. This family of groups is interesting because C.F. Miller, back in the 1970's, already showed the existence of [f.g. free]-by-[f.g. free] groups with unsolvable conjugacy problem (see [12]). Via [2, Theorem 3.1], this can be restated by saying that $\mathrm{Aut}(F_n)$ contains finitely generated orbit undecidable subgroups (for some n).

Question 6 in the last section of [2] asks whether finitely presented subgroups $A \leqslant \mathrm{Aut}(F_n)$ are orbit decidable or not. The answer is known to be positive in rank 2 (every finitely generated subgroup of $\mathrm{Aut}(F_2)$ is orbit decidable, see [2, Proposition 6.21]), but open for bigger rank. The comment made in [2] after this question says that if H is a finitely generated group with unsolvable word problem, then Mihailova's group $M(H)$ is isomorphic to an orbit undecidable subgroup of $\mathrm{Aut}(F_3)$. And, as mentioned in the introduction, this subgroup is then finitely generated, and recursively presented, but it cannot be finitely presented.

In the rest of the paper, we will recall how $M(H)$ can be embedded into $\mathrm{Aut}(F_3)$, in such a way that the image becomes an orbit undecidable subgroup of $\mathrm{Aut}(F_3)$. Then we will choose an appropriate H and prove Theorem 4.2 by applying Theorem 1.1 to $A = M(H)$.

Of course, Theorem 4.2 does not answer the above mentioned Question 6, but shows its tightness in the sense that orbit undecidability is already showing up in the class of one-parametric recursively presented subgroups of $\mathrm{Aut}(F_3)$.

First, let $F_3 = \langle q, a, b \mid \quad \rangle$ be the free group on $\{q, a, b\}$, and let us embed $F_2 \times F_2$ into $\mathrm{Aut}(F_3)$ in the following natural way. For every $u, v \in \langle a, b\rangle$, consider the automorphism

$$\begin{array}{rcl} {}_u\theta_v \colon F_3 & \to & F_3 \\ q & \mapsto & uqv \\ a & \mapsto & a \\ b & \mapsto & b. \end{array}$$

Clearly, ${}_{u_1}\theta_1 \cdot {}_{u_2}\theta_1 = {}_{u_1u_2}\theta_1$ and ${}_1\theta_{v_1} \cdot {}_1\theta_{v_2} = {}_1\theta_{v_2v_1}$, which means that $\{\,{}_u\theta_1 \mid u \in \langle a, b\rangle\} \simeq F_2$ and $\{\,{}_1\theta_v \mid v \in \langle a, b\rangle\} \simeq F_2^{\mathrm{op}} \simeq F_2$. It is also clear that ${}_u\theta_1 \cdot {}_1\theta_v = {}_u\theta_v = {}_1\theta_v \cdot {}_u\theta_1$. So, we have an embedding $\theta \colon F_2 \times F_2 \simeq F_2^{\mathrm{op}} \times F_2^{\mathrm{op}} \hookrightarrow \mathrm{Aut}(F_3)$ given by $(u, v) \mapsto {}_{u^{-1}}\theta_v$, whose image is

$$F_2 \times F_2 \simeq B = \langle {}_{a^{-1}}\theta_1,\, {}_{b^{-1}}\theta_1,\, {}_1\theta_a,\, {}_1\theta_b\rangle = \{\,{}_u\theta_v \mid u,\, v \in \langle a, b\rangle\} \leqslant \mathrm{Aut}(F_3).$$

As shown in [2, Section 7.2], the element $qaqbq$ satisfies the technical condition required in [2, Proposition 7.3]. Hence, we have

Lemma 4.1 (7.3 in [2]) *For the above defined subgroup* $B \leqslant \mathrm{Aut}(F_3)$ *and for every subgroup* $A \leqslant B$*, undecidability of the membership problem for* A *in* B *implies orbit undecidability for* A *in* $\mathrm{Aut}(F_3)$.

We are ready to deduce the main result of this section.

Theorem 4.2 *There exists a finitely generated (and not finitely presented) orbit undecidable subgroup* $A \leqslant \mathrm{Aut}(F_3)$ *admitting a one-parametric recursive presentation as in Theorem 1.1.*

Proof. In [7], D.J. Collins and C.F. Miler III proved that there exists a finite, concise and Peiffer aspherical presentation $\langle x_1, \ldots, x_n \,|\, R_1, \ldots, R_m \rangle$ of a group H with unsolvable word problem. The corresponding Mihailova's group $M(H)$ is a subgroup of $F_n \times F_n$ and the membership problem for $M(H)$ in $F_n \times F_n$ is unsolvable.

Now, denoting $A = M(H)$ and using a finite index embedding of $F_n \times F_n$ in $B \cong F_2 \times F_2$, we have that $A \leqslant B$ and the membership problem for A in B is unsolvable. By Lemma 4.1, A is an orbit undecidable subgroup of $\mathrm{Aut}(F_3)$.

Moreover, as it was discussed in the introduction, A is finitely generated, and is not finitely presented. But Theorem 1.1 provides an explicit one-parametric recursive presentation for A. This concludes the proof. □

We end by reproducing [2, Question 6] again:

Question 4.3 *Does there exist a finitely presented orbit undecidable subgroup of* $\mathrm{Aut}(F_n)$*, for* $n \geqslant 3$ *?*

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