Quartic Thue Equations

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Abstract

We will give upper bounds upon the number of integral solutions to binary quartic Thue equations. We will also study the geometric properties of a specific family of binary quartic Thue equations to establish sharper upper bounds.

Key words: Thue Equation; Linear Forms in Logarithms; The Thue-Siegel Principle

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1. Introduction

In this paper, we will consider irreducible binary quartic forms with integer coefficients; i.e. polynomials of the shape

$$F(x,y) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4.$$

In [1], the first author showed that when the so-called catalecticant invariant

$$J_F = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4 - 72a_0a_2a_4 + 27a_0a_3^2$$

vanishes and F splits in $\mathbb{R}$, the equation

$$|F(x,y)| = 1 \tag{1}$$

has at most 12 solutions in integers x, y. In this paper we will give upper bounds for the number of integral solutions to (1) with large discriminant and no restriction on J. We will use some ideas of Stewart [14] to prove

Theorem 1.1. *Let $F(x,y)$ be an irreducible binary form with integral coefficients and degree 4. The Diophantine equation (1) has at most 61 solutions in integers x and y (with (x,y) and $(-x,-y)$ regarded as the same), provided that the discriminant of F is greater than D_0, where D_0 is an effectively computable constant.*

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We also combine some analytic methods from [14] with some geometric methods from [12] to show that

Theorem 1.2. *Let $F(x, y)$ be an irreducible binary form with integral coefficients and degree 4 that splits in $\mathbb{R}$. Then the Diophantine equation (1) has at most 37 solutions in integers x and y (with (x, y) and $(-x, -y)$ regarded as the same), provided that the discriminant of F is greater than D_0, where D_0 is an effectively computable constant.*

We remark here that D_0 can be computed effectively. To use our method (linear forms in logarithms) to prove Theorem 1.2, we need to take $D_0 > 10^{500}$. However, to prove Theorem 1.1, using the Thue-Siegel principle, we don't really need to take D_0 very large. Here we choose to work with the same D_0 to be consistent. Propositions 2.3 and 2.4 together with Theorem 6.9 give an algorithm to compute D_0.

Note that if (x, y) is a solution to (1) then $(-x, -y)$ is also a solution to (1). So here we will only count the solutions with $y \geq 0$.

The equation

$$F(x, y) = x^4 - 4x^3y - x^2y^2 + 4xy^3 + y^4 = 1$$

has exactly 8 solutions $(x, y) = (0, 1), (1, 0), (1, 1), (-1, 1), (4, 1), (-1, 4), (8, 7), (-7, 8)$ (see [11] for a proof). The authors are not aware of any binary quartic forms $F(x, y)$ for which the equation $F(x, y) = 1$ has more than 8 solutions.

Let

$$F(x, y) = a_0(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)(x - \alpha_4 y).$$

We call forms F_1 and F_2 equivalent if they are equivalent under $GL_2(\mathbb{Z})$-action; i.e. if there exist integers a_1 , a_2 , a_3 and a_4 such that

$$F_1(a_1x + a_2y, a_3x + a_4y) = F_2(x, y)$$

for all x, y, where $a_1a_4 - a_2a_3 = \pm 1$. We denote by N_F the number of solutions in integers x and y of the Diophantine equation (1). If F_1 and F_2 are equivalent then $N_{F_1} = N_{F_2}$ and $D_{F_1} = D_{F_2}$.

Suppose there is a solution (x_0, y_0) to the equation (1). Since

$$\gcd(x_0, y_0) = 1,$$

there exist integers x_1, $y_1 \in \mathbb{Z}$ with

$$x_0y_1 - x_1y_0 = 1.$$

Then

$$F^*(1,0) = 1,$$

where,

$$F^*(x,y) = F(x_0x + x_1y, y_0x + y_1y).$$

Therefore, F^* is a monic form equivalent to F. From now on we will assume F is monic.

In this paper we give an upper bound for the number of integral solutions to $F(x,y) = \pm 1$. For the equation

$$F(x,y) = h$$

of degree 4, one may use an argument of Bombieri and Schmidt [2] to prove that if N is a given bound in the special case $h = 1$, then $N4^{\nu}$ is a corresponding bound in the general case, where ν is the number of distinct prime factors of h.

2. Heights

For any algebraic number α, we define the (naive) height of α, denoted by $H(\alpha)$, by

$$H(\alpha) = H(f(x)) = \max(|a_n|, |a_{n-1}|, \ldots, |a_0|)$$

where $f(x) = a_nx^n + \ldots + a_1x + a_0$ is the minimal polynomial of α. Suppose that over $\mathbb{C}$,

$$f(x) = a_n(x - \alpha_1)\ldots(x - \alpha_n).$$

We put

$$M(\alpha) = |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|).$$

$M(\alpha)$ is known as the Mahler measure of α. We have the following result of Landau:

Lemma 2.1. *Let α be an algebraic number of degree n. then*

$$M(\alpha) \leq (n+1)^{1/2}H(\alpha).$$

For any polynomial G in $\mathbb{C}[z_1,\dots,z_n]$ that is not identically zero the Mahler measure $M(G)$ is defined by

$$M(G)=\exp\int_0^1 dt_1\dots\int_0^1 dt_n\log\left|G(e^{2\pi it_1},\dots,e^{2\pi it_n})\right|.$$

Thus if $n=1$ and $G(z)=a_n(z-\alpha_1)\dots(z-\alpha_n)$ with $a_n\neq 0$, by Jensen's theorem,

$$M(G)=|a_n|\prod_{i=1}^n\max(1,|a_i|).$$

In [8], Mahler showed, for polynomial G of degree n and discriminant D_G, that

$$M(G)\geq\left(\frac{D_G}{n^n}\right)^{\frac{1}{2n-2}}. \tag{2}$$

Following Matveev [9, 10], we will define the absolute logarithmic height of an algebraic number. Let $\mathbb{Q}(\alpha_1)^\sigma$ be the embeddings of the real number field $\mathbb{Q}(\alpha_1)$ in $\mathbb{R}$, $1\leq\sigma\leq n$, where $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ are roots of $F(x,1)=0$. We respectively have n Archimedean valuations of $\mathbb{Q}(\alpha_1)$:

$$|\rho|_\sigma=\left|\rho^{(\sigma)}\right|,\quad 1\leq\sigma\leq n.$$

We enumerate simple ideals of $\mathbb{Q}(\alpha)$ by indices $\sigma>n$ and define non-Archimedean valuations of $\mathbb{Q}(\alpha)$ by the formulas

$$|\rho|_\sigma=(\mathrm{Norm}\ \mathfrak{p})^{-k},$$

where

$$k=\mathrm{ord}_{\mathfrak{p}}(\alpha),\ \mathfrak{p}=\mathfrak{p}_\sigma,\ \sigma>n,$$

for any $\rho\in\mathbb{Q}^*(\alpha)$. Then we have the *product formula* :

$$\prod_1^\infty|\rho|_\sigma=1,\ \rho\in\mathbb{Q}(\alpha).$$

Note that $|\rho|_\sigma\neq 1$ for only finitely many ρ. We should also remark that if $\sigma_2=\bar{\sigma}_1$, i.e.,

$$\sigma_2(x)=\bar{\sigma}_1(x)\qquad\text{for}\qquad x\in\mathbb{Q}(\alpha),$$

then the valuations $|\,.\,|_{\sigma_1}$ and $|\,.\,|_{\sigma_2}$ are equal. We define the *absolute logarithmic height* of ρ as

$$h(\rho)=\frac{1}{2n}\sum_{\sigma=1}^\infty\left|\log|\rho|_\sigma\right|.$$

Lemma 2.2. *Suppose α is an algebraic number of degree n over $\mathbb{Q}$. Then*

$$h(\alpha) = \frac{1}{n}\log M(\alpha).$$

Proof. It is well-known that

$$\prod_{\sigma} \max(1, |\alpha|_{\sigma}) = M(\alpha).$$

Since

$$h(\rho) = \frac{1}{2n}\sum_{\sigma=1}^{\infty} \left|\log|\rho|_{\sigma}\right|,$$

by the product formula,

$$h(\alpha) = \frac{2}{2n}\log\prod_{\sigma}\max(1, |\alpha|_{\sigma}).$$

Therefore,

$$h(\alpha) = \frac{1}{n}\log M(\alpha).$$

□

Let α and β be two algebraic numbers. Then the following inequalities hold (see [3]):

$$h(\alpha+\beta) \leq \log 2 + h(\alpha) + h(\beta) \tag{3}$$

and

$$h(\alpha\beta) \leq h(\alpha) + h(\beta). \tag{4}$$

Let us call strongly equivalent the polynomials $f(x)$ and $f^*(x) \in \mathbb{Z}$ if $f^*(x) = f(x+a)$ for some $a \in \mathbb{Z}$. Two algebraic integers α and α' are called (strongly) equivalent if their minimal polynomials are (strongly) equivalent.

Proposition 2.3. *(Győry [5]) Suppose that $f(x)$ is a monic polynomial in $\mathbb{Z}[x]$ with degree $n \geq 2$ and non-zero discriminant D. There is a polynomial $f^*(x) \in \mathbb{Z}$ strongly equivalent to $f(x)$ so that*

$$H\left(f^*(x)\right) < \exp\{n^{4n^{12}}|D|^{6n^8}\} < \exp\exp\{4\left(\log|D|\right)^{13}\}.$$

This allows us to assume $H\left(F(x,1)\right) < \exp\{4^{4^{13}}|D|^{6(4^8)}\}$, for our quartic form $F(x,y)$. In fact, from now on, we will work with a monic irreducible quartic binary form $F(x,y)$ so that $H\left(F(x,1)\right)$ satisfies the above inequality.

Proposition 2.4. *(Győry [6]) Suppose that $f(x)$ is a monic polynomial in $\mathbb{Z}[x]$ with degree $n \geq 2$ and non-zero discriminant D. Then for every constant $\chi > 9(n-1)(n-2)/2$ there exists a polynomial $f^*(x) \in \mathbb{Z}$ strongly equivalent to $f(x)$ which satisfies*

$$H\left(f^*(x)\right) < \exp(cD^{\chi}),$$

where $c = c(n, \chi)$ is a positive computable constant.

A much more precise estimate is given for $H(f)$ in terms of $D(f)$ by Evertse [4]. It is, however, partially ineffective.

Proposition 2.5. *(Evertse [4]) Let $F(x, y)$ be a binary form with degree $n \geq 2$ and non-zero discriminant D. Assume that $H\left(F(x, 1)\right) \leq H\left(G(x, 1)\right)$ for every $G(x, y)$ equivalent to $F(x, y)$. Then*

$$H\left(F(x, 1)\right) \leq c\left|D\right|^{21/(r-1)},$$

where c is an ineffective constant depending on n.

Lemma 2.6. *(Mahler [8]) If a and b are distinct zeros of polynomial $P(x)$ with degree n, then we have*

$$|a - b| \geq \sqrt{3}(n+1)^{-n} M(P)^{-n+1},$$

where $M(P)$ is the Mahler measure of P.

Since $M(P) \leq (n+1)^{1/2} H(P)$, we have

$$|a - b| \geq \sqrt{3}(n+1)^{-(2n+1)/2} H(P)^{-n+1}.$$

3. The Thue-Siegel Principle

Let α be an algebraic number of degree n and f be its minimal polynomial over the integers. Let t and τ be positive numbers such that $t < \sqrt{2/n}$ and $\sqrt{2 - nt^2} < \tau < t$, and put $\lambda = \frac{2}{t-\tau}$ and

$$A_1 = \frac{t^2}{2 - nt^2}\left(\log M(\alpha) + \frac{n}{2}\right).$$

Suppose that $\lambda < n$. A rational number $\frac{x}{y}$ is said to be a very good approximation to α if

$$|\alpha - x/y| < \left(4\, e^{A_1} \max(|x|, |y|)\right)^{-\lambda}.$$

The following result of Bombieri and Schmidt [2] is based on a classical work of Thue and Siegel.

Proposition 3.1. *(Thue-Siegel principle) If α is of degree $n \geq 3$ and x/y and x'/y' are two very good approximations to α then*

$$\log\left(4e^{A_1}\right) + \log\left(\max(|x'|, |y'|)\right) \leq \gamma^{-1}\left(\log\left(4e^{A_1}\right) + \log\left(\max(|x|, |y|)\right)\right),$$

where $\gamma = \frac{nt^2+\tau^2-2}{n-1}$.

We also need the following refinement of an inequality of Lewis and Mahler [7]:

Lemma 3.2. *Let F be a binary form of degree $n \geq 3$ with integer coefficients and nonzero discriminant D. For every pair of integers (x, y) with $y \neq 0$*

$$\min_{\alpha}\left|\alpha - \frac{x}{y}\right| \leq \frac{2^{n-1}n^{n-1/2}\left(M(F)\right)^{n-2}|F(x,y)|}{|D(F)|^{1/2}|y|^n},$$

where the minimum is taken over the zeros α of $F(z, 1)$.

Proof. This is Lemma 3 of [14]. □

4. Large Solutions

We will now estimate the number of solutions (x, y) of (1) with $y > M(F)^2$. Suppose that (x, y) is an integral solution to (1). Then we have

$$(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)(x - \alpha_4 y) = \pm 1.$$

Therefore, for some $1 \leq i \leq 4$,

$$|x - \alpha_i y| < 1.$$

Definition. We say the pair of solutions (x, y) is *r*elated to α_i if

$$|x - \alpha_i y| = \min_{1\leq j\leq 4}|x - \alpha_j y|.$$

Suppose (x_1, y_1), (x_2, y_2), ... are the solutions to (1) which are related to α_i with $y_j > M(F)^2$, for $j = 1, 2, \ldots$, ordered so that $y_1 \leq y_2 \leq \ldots$. By Lemma 3.2,

$$\left|\alpha_i - \frac{x_j}{y_j}\right| \leq \frac{2^{10}M(F)^2}{|D(F)|^{1/2}y_j^4} \tag{5}$$

for $j = 1, 2, \ldots$. Therefore,

$$\left|\frac{x_{j+1}}{y_{j+1}} - \frac{x_j}{y_j}\right| \leq \frac{2^{11}M(F)^2}{|D(F)|^{1/2}y_j^4}$$

Since $|x_{j+1}y_j - x_jy_{j+1}| \geq 1$, assuming $D > 2^{22}$, we have

$$\frac{y_j^3}{M(F)^2} \leq y_{j+1}. \tag{6}$$

To each solution (x_j, y_j), we associate a real number $\delta_j > 1$ by

$$y_j = M(F)^{1+\delta_j}. \tag{7}$$

From (6), we have

$$3\delta_j \leq \delta_{j+1}.$$

Therefore,

$$3^{j-1} \leq \delta_j. \tag{8}$$

Moreover, if the pairs of solutions (x_k, y_k) and (x_{k+l}, y_{k+l}) are both related to α_i then

$$3^l\delta_k \leq \delta_{k+l}. \tag{9}$$

Let us now apply the Thue-Siegel principle (Proposition 3.1) with

$$t = \sqrt{\frac{2}{4.01}}$$

and

$$\tau = 1.2\sqrt{2 - 4t^2} = 0.12\,t.$$

Then

$$\lambda = \frac{2}{t - \tau} = \frac{2}{0.88t} < 3.22,$$
$$A_1 = 100\,(\log(M(F)) + 2)$$

and

$$\gamma^{-1} < 1368, \tag{10}$$

where, $\gamma = \frac{4t^2+\tau^2-2}{3}$. Since we have assumed $\left|\alpha_i - \frac{x_j}{y_j}\right| < 1$,

$$|x_j| < |y_j|(|\alpha_i| + 1) \leq 2M(F)y_j,$$

whereby

$$H(x_j, y_j) < 2M(F)y_j.$$

By (2) and since $D > 10^{500}$, we have

$$8e^{A_1} = 8e^{200}M(F)^{100} < M(F)^{102}, \tag{11}$$

so by (7),

$$(4e^{A_1}H(x_j, y_j))^{\lambda} < M(F)^{(103+\delta_j)\lambda}. \tag{12}$$

From (5),

$$\left|\alpha_i - \frac{x_j}{y_j}\right| < M(F)^{-4\delta_j}.$$

Hence, $\frac{x_j}{y_j}$ is a very good approximation to α_i whenever

$$4\delta_j \geq (103 + \delta_j)\lambda.$$

Since $\lambda \leq 3.22$, if $\delta_j > 414$ then $\frac{x_j}{y_j}$ is a very good approximation to α_i. So by (8), whenever

$$k > 1 + \frac{\log 415}{\log 3},$$

$\frac{x_k}{y_k}$ is a very good approximation to α_i. This means there are at most 6 large solutions (x_1,y_1),..., (x_6, y_6) to (1) which are related to α_i for which $\frac{x_1}{y_1}$,..., $\frac{x_6}{y_6}$ are not good approximations to α_i. Suppose that there are l pairs of solutions (x_7, y_7), ..., (x_{6+l}, y_{6+l}) $(l > 1)$ which are both related to α_i, and for which $\frac{x_j}{y_j}$ are very good approximations to α_i. Then by the Thue-Siegel principle (Lemma (3.1)) and (10),

$$\log\left(4e^{A_1}\right) + \log y_{7+l} \leq 1368\left(\log\left(4e^{A_1}\right) + \log(2M(F)y_8)\right),$$

and so, by (11),

$$\log y_{7+l} \leq 1368\,(103\log M(F) + \log(y_8)) - 102\log M(F) + \log(2).$$

Since $\delta_8 > 414$, by (7) and (9),

$$3^{l-1}\delta_8 \leq \delta_{7+l} < 1368\,\delta_8 + 139435 < 336\,\delta_8.$$

Thus,

$$l \leq \frac{\log 336}{\log 3} + 1 \leq 6.30.$$

This means there are at most 12 large solution related to each root of $F(x,1)$.

5. Small Solutions

Here we will count the number of solutions to (1) with $1 \leq y \leq M(F)^2$. We will follow Stewart's [14] results for Thue inequalities with arbitrary degree and sharpen them for quartic Thue equations. Suppose that Y_0 is a fixed positive number. For each root α_i of $F(x,1)$, let $(x^{(i)}, y^{(i)})$ be the solution to (1) related to α_i with the largest value of y among those with $1 \leq y \leq Y_0$. Let $\mathfrak{X}$ be the set of solutions of (1) with $1 \leq y \leq Y_0$ minus the elements $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$, $(x^{(3)}, y^{(3)})$, $(x^{(4)}, y^{(4)})$. From inequality (60) of [14], we have

$$\left(\left(\frac{2}{7}\right)^4 M(F)\right)^{|\mathfrak{X}|} \leq Y_0^4, \tag{13}$$

where $|\mathfrak{X}|$ denotes the cardinality of $\mathfrak{X}$. By (2), when $D > 10^{500}$, we have

$$\left(\frac{2}{7}\right)^4 M(F) \geq M(F)^{64/65}.$$

By (13),

$$|\mathfrak{X}| < 4\frac{65 \log Y_0}{64 \log M(F)}. \tag{14}$$

So when $Y_0 = M(F)^2$, we have $|\mathfrak{X}| \leq 8$. Therefore the number of small solutions does not exceed 12.

We have seen that there are at most 48 large solutions and 12 small ones to (1), when the discriminant is large. Since we assumed the quartic form $F(x,y)$ is monic, $(1,0)$ is also a solution to (1). Thus, the proof of Theorem 1.1 is complete.

In the next section, we will consider quartic forms $F(x,y)$ for which all roots of $F(x,1)$ are real. There we will call a solution (x,y) a large solution if $y > M(F)^6$.

Lemma 5.1. *There are at most* 14 *solutions to (1) with* $1 \leq y \leq M(F)^6$.

Proof. Choose $\theta > 0$ such that

$$\frac{65}{16}\left(\frac{8}{3} + \theta\right) < 11.$$

From (13), we conclude that (1) has at most 10 solutions with $1 \leq y < M(f)^{\frac{8}{3}+\theta}$. Further, by (6), equation (1) has at most 4 solutions with $M(f)^{\frac{8}{3}+\theta} \leq y < M(f)^6$. So altogether (1) has at most 14 solutions with $1 \leq y < M(f)^6$. $\square$

6. Forms With Real Roots

In this section, we will assume α_i, the roots of $F(x, 1)$, are real.

Define

$$\phi_m(x, y) = \log\left|\frac{D^{\frac{1}{12}}(x - y\alpha_m)}{|f'(\alpha_m)|^{\frac{1}{3}}}\right| \tag{15}$$

and

$$\phi(x, y) = (\phi_1(x, y), \phi_2(x, y), \phi_3(x, y), \phi_4(x, y)).$$

Let

$$\|\phi(x, y)\|$$

be the L_2 norm of the vector $\phi(x, y)$.

Lemma 6.1. *Suppose that (x, y) is a solution to the equation $F(x, y) = 1$ for the binary form F in Theorem 1.2. If*

$$|x - \alpha_i y| = \min_{1\leq j\leq 4} |x - \alpha_j y|$$

Then

$$\|\phi(x, y)\| \leq 6 \log\frac{1}{|x - \alpha_i y|} + 4\log\left(\frac{D^{\frac{1}{12}}(5)^4 M(F)^3}{\sqrt{3}}\right).$$

Proof. Let us assume that

$$\left|x - \alpha_{s_j} y\right| < 1, \qquad \text{for } 1 \leq j \leq p$$

and

$$|x - \alpha_{b_k} y| \geq 1, \qquad \text{for } 1 \leq k \leq 4 - p,$$

where $1 \leq p, s_j, b_k \leq 4$. We have

$$\prod_k |x - \alpha_{b_k} y| = \frac{1}{\prod_j \left|x - \alpha_{s_j} y\right|}.$$

Therefore, for any $1 \leq k \leq 4 - p$, we have

$$\log|x - \alpha_{b_k} y| \leq p \log\frac{1}{|x - \alpha_i y|}.$$

Since

$$|x - \alpha_i y| = \min_{1\leq j\leq 4} |x - \alpha_j y|,$$

we also have

$$\left|\log\left|x-\alpha_{s_j}y\right|\right| \leq \left|\log\left|x-\alpha_i y\right|\right|.$$

From here, we conclude that

$$\begin{aligned}\|\phi(x,y)\| &\leq \sum_{m=1}^{4}\log\left|\frac{D^{\frac{1}{12}}}{|f'(\alpha_m)|^{\frac{1}{3}}}\right| + (4-p)p\,|\phi_i(x,y)| + p\,|\phi_i(x,y)| \\ &= \sum_{m=1}^{4}\log\left|\frac{D^{\frac{1}{12}}}{|f'(\alpha_m)|^{\frac{1}{3}}}\right| + (5p-p^2)\,|\phi_i(x,y)|.\end{aligned}$$

The function $f(p) = 5p - p^2$ is at most 6 for $p \in \{1,2,3,4\}$. Our proof is complete by recalling the fact that if a and b are distinct zeros of $f(x) = F(x,1)$, then by Lemma 2.6, we have

$$|a-b| \geq \frac{\sqrt{3}}{5^4}M(f)^{-3}. \tag{16}$$

□

6.1. Exponential Gap Principle

Here, our goal is to show

Theorem 6.2. *Suppose that (x_1,y_1), (x_2,y_2) and (x_3,y_3) are three pairs of non-trivial solutions to (1) with*

$$|x_j - \alpha_4 y_j| < 1$$

and $|y_j| > M(F)^6$, for $j \in \{1,2,3\}$. If $r_1 \leq r_2 \leq r_3$ then

$$r_3 > \exp\left(\frac{r_1}{6}\right) 2\sqrt{3}\log^4\frac{1+\sqrt{5}}{2},$$

where $r_j = \|\phi(x_j,y_j)\|$.

We note that for three pairs of solutions in Theorem 6.2, the three points $\phi_1 = \phi(x_1,y_1)$, $\phi_2 = \phi(x_2,y_2)$ and $\phi_3 = \phi(x_3,y_3)$ form a triangle Δ. To establish Theorem 6.2, we will find a lower bound and an upper bound for the area of Δ. Then comparing these bounds, Theorem 6.2 will be proved. The length of each side of Δ is less than $2r_3$. Lemma 6.3 gives an upper bound for the height of Δ. Suppose that $(x,y) \neq (1,0)$ is a solution to (1) and let $t = \frac{x}{y}$. We have

$$\phi(x,y) = \phi(t) = \sum_{i=1}^{4} \log \frac{|t-\alpha_i|}{|f'(\alpha_i)|^{\frac{1}{3}}} \mathbf{b_i},$$

where,

$$\mathbf{b_1} = \frac{1}{4}(3,-1,-1,-1), \qquad \mathbf{b_2} = \frac{1}{4}(-1,3,-1,-1),$$
$$\mathbf{b_3} = \frac{1}{4}(-1,-1,3,-1), \qquad \mathbf{b_4} = \frac{1}{4}(-1,-1,-1,3),$$

Without loss of generality, we will suppose that for the solution (x,y) we have

$$|x - \alpha_4 y| < 1.$$

We may write

$$\phi(x,y) = \phi(t) = \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|f'(\alpha_i)|^{\frac{1}{3}}} \mathbf{c_i} + E_4 \mathbf{b_4}, \tag{17}$$

where, for $1 \leq i \leq 3$,

$$\mathbf{c_i} = \mathbf{b_i} + \frac{1}{3}\mathbf{b_4}, \quad E_4 = \log \frac{|t-\alpha_4|}{|f'(\alpha_4)|^{\frac{1}{3}}} - \frac{1}{3}\sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|f'(\alpha_i)|^{\frac{1}{3}}}$$

One can easily observe that

$$\mathbf{c_i} \perp \mathbf{b_4}, \text{ for } 1 \leq i \leq 4.$$

Lemma 6.3. *Let*

$$\mathbf{L_4} = \sum_{i=1}^{3} \log \frac{|\alpha_4 - \alpha_i|}{|f'(\alpha_i)|^{\frac{1}{3}}} \mathbf{c_i} + z\mathbf{b_4}, \quad z \in \mathbb{R}.$$

Suppose that $(x,y) \neq (1,0)$ is a solution to (1) with

$$|x - \alpha_4 y| = \min_{1 \leq j \leq 4} |x - \alpha_j y|$$

and $y \geq M(F)^6$. Then the distance between $\phi(x,y)$ and the line $\mathbf{L_4}$ is less than

$$\exp\left(\frac{-r}{6}\right),$$

where $r = \|\phi(x,y)\|$.

Proof. The distance between $\phi(x, y)$ and $\mathbf{L_4}$ is equal to

$$\left\| \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \mathbf{c_i} \right\|,$$

where $t = \frac{x}{y}$. If $|t-\alpha_i| > |\alpha_4-\alpha_i|$, then

$$\left| \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \right| = \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \leq \log \left(\frac{|t-\alpha_4|}{|\alpha_4-\alpha_i|} + 1 \right) < \frac{|t-\alpha_4|}{|\alpha_i-\alpha_4|}.$$

If $|t-\alpha_i| < |\alpha_4-\alpha_i|$, then

$$\left| \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \right| = \log \frac{|\alpha_4-\alpha_i|}{|t-\alpha_i|} \leq \log \left(\frac{|t-\alpha_4|}{|t-\alpha_i|} + 1 \right) < \frac{|t-\alpha_4|}{|\alpha_i-t|}.$$

Note that when $i \neq 3$, either

$$|t-\alpha_i| > |\alpha_4-\alpha_i|$$

or

$$|t-\alpha_i| > |\alpha_3-\alpha_i|.$$

Therefore, for $i \neq 3$,

$$\left| \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \right| < \frac{|t-\alpha_4|}{m},$$

where $m = \min_{i \neq j}\{|\alpha_j-\alpha_i|\}$. Moreover, since we assumed t is closer to α_4,

$$|t-\alpha_3| \geq \frac{|\alpha_4-\alpha_3|}{2}.$$

Consequently,

$$\left| \log \frac{|t-\alpha_3|}{|\alpha_4-\alpha_3|} \right| < \frac{2|t-\alpha_4|}{m}.$$

Therefore

$$\left\| \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \mathbf{c_i} \right\| < 4\sqrt{\frac{2}{3}} \frac{|u|}{m}, \tag{18}$$

where $u = t - \alpha_4$. On the other hand, by Lemma 6.1

$$r - 4\log\left(\frac{D^{\frac{1}{12}} 5^4 M(F)^3}{\sqrt{3}}\right) \leq 6\log\frac{1}{|x-\alpha_4 y|},$$

which implies

$$\log|yu| < \frac{-r}{6} + \frac{16}{25}\log\left(\frac{D^{\frac{1}{12}}5^4M(F)^3}{\sqrt{3}}\right).$$

Therefore,

$$|u| < \exp\left(\frac{-r}{6}\right)\frac{\exp\left(\frac{16}{25}\log\left(\frac{D^{\frac{1}{12}}5^4M(F)^3}{\sqrt{3}}\right)\right)}{|y|}$$

Comparing this with(18), since $|y| > M(F)^6$ and (by (2)) we have

$$D^{1/12} < 4^{1/3}M(F)^{1/12},$$

our proof is complete (note that by (2.6), $m \geq \frac{\sqrt{3}}{5^4M(f)^3}$). □

Lemma 6.3 shows that the height of Δ is at most

$$2\ \exp\left(\frac{-r_1}{6}\right).$$

Therefore, the area of Δ is less than

$$2r_3\exp\left(\frac{-r_1}{6}\right). \tag{19}$$

To estimate the area of Δ from below, we appeal to Pohst's lower bound for units. Since

$$F(x,y) = (x-\alpha_1y)(x-\alpha_2y)(x-\alpha_3y)(x-\alpha_4y) = \pm 1,$$

we conclude that $x-\alpha_iy$ is a unit in $\mathbb{Q}(\alpha_i)$ when (x,y) is a solution to (1). Suppose that (x_1,y_1) and (x_2,y_2) are two pairs of non-trivial solutions to (1). Then

$$\phi(x_1,y_1)-\phi(x_2,y_2) = \left(\log\frac{x_1-\alpha_1y}{x_2-\alpha_1y_2},\ldots,\log\frac{x_1-\alpha_4y_1}{x_2-\alpha_4y_2}\right) = \vec{e}.$$

Since $\frac{x_1-\alpha_iy}{x_2-\alpha_iy_2}$ is a unit in $\mathbb{Q}(\alpha_i)$, we have

$$\|\vec{e}\| \geq 4\log^2\frac{1+\sqrt{5}}{2}$$

(see exercise 2 on page 367 of [13]). Now we can estimate each side of Δ from below to conclude that the area of the triangle Δ is greater than

$$16\,\frac{\sqrt{3}}{4}\log^4\frac{1+\sqrt{5}}{2}.$$

Comparing this with (19) we conclude that

$$2\,r_3\exp\left(\frac{-r_1}{6}\right) > 16\,\frac{\sqrt{3}}{4}\log^4\frac{1+\sqrt{5}}{2}.$$

Theorem 6.2 is immediate from here.

6.2. Geometry Of The Curve $\phi(t)$

In order to study the curve $\phi(t)$, we will consider some well-known geometric properties of the unit group U of $\mathbb{Q}(\alpha)$, where α is a root of $F(x,1)=0$.

Theorem 6.4 (Dirichlet's Unit Theorem)**.** *Let K be an algebraic number field of degree n. Let r be the number of real conjugate fields of K and $2s$ the number of complex conjugate fields of K. Then the ring of integers O_K contains $r+s-1$ fundamental units $\epsilon_1,\ldots,\epsilon_{r+s-1}$ such that each unit of O_K can be expressed uniquely in the form $u\epsilon_1^{n_1}\ldots\epsilon_{r+s-1}^{n_{r+s-1}}$, where u is a root of unity in O_K and $n_1,\ldots,n_{r+s-1}$ are integers.*

For a real algebraic number field $\mathbb{Q}(\alpha)$ of degree 4, in Dirichlet's Unit Theorem we have $r=4$ and $s=0$. By Dirichlet's unit theorem, we have a sequence of mappings

$$\tau : U \longmapsto V \subset \mathbb{R}^4 \tag{20}$$

and

$$\log : V \longmapsto \Lambda, \tag{21}$$

where V is the image of the map τ, Λ is a 3-dimensional lattice, τ is the obvious restriction of the embedding of $\mathbb{Q}(\alpha)$ in $\mathbb{R}^4$, and the mapping log is defined as follows:
For $(x_1,x_2,x_3,x_4)\in V$,

$$\log(x_1,x_2,x_3,x_4) = (\log|x_1|,\log|x_2|,\log|x_3|,\log|x_4|).$$

If (x,y) is a pair of solutions to (1) then

$$(x-\alpha_j y)$$

is a unit in $\mathbb{Q}(\alpha_i)$. Suppose that

$$\lambda_2, \lambda_3, \lambda_4$$

are fundamental units of $\mathbb{Q}(\alpha_i)$ and are chosen so that

$$\log\left(\tau(\lambda_2)\right), \log\left(\tau(\lambda_3)\right), \log\left(\tau(\lambda_4)\right)$$

form a *reduced* basis for the lattice Λ. Let us assume that

$$\left\|\log\left(\tau(\lambda_2)\right)\right\| \leq \left\|\log\left(\tau(\lambda_3)\right)\right\| \leq \left\|\log\left(\tau(\lambda_4)\right)\right\|.$$

$$\phi(x,y) = \phi(1,0) + \sum_{k=2}^{4} m_k \log\left(\tau(\lambda_k)\right) \; m_k \in \mathbb{Z} \tag{22}$$

Lemma 6.5. *For every fixed integer m, there are at most 6 solutions (x, y) to (1) for which in (22), $m_4 = m$.*

Proof. Let S be the 3-dimensional affine space of all points $\phi(1,0)+\sum_{i=2}^{4} \mu_i \log\left(\tau(\lambda_i)\right)$ ($\mu_i \in \mathbb{R}$). Let $\mu_4 = m$. Then the points

$$\phi(1,0) + \sum_{i=2}^{3} \mu_i \log\left(\tau(\lambda_i)\right) + m \log\left(\tau(\lambda_4)\right)$$

form a linear subvariety S_1 of S. Let

$$\vec{N} = (N_1, N_2, N_3, N_4) \in S$$

be the normal vector of S_1. Then the number of times that the curve $\phi(t)$ intersects S_1 equals the number of solutions in t to

$$\vec{N}.\phi(t) = 0, \tag{23}$$

where $\vec{N}.\phi(t)$ is the inner product of two vectors $\vec{N}$ and $\phi(t)$. We have

$$\frac{d}{dt}\left(\vec{N}.\phi(t)\right) = \frac{P(t)}{F(t)},$$

where

$$F(t) = (t-\alpha_1)(t-\alpha_2)(t-\alpha_3)(t-\alpha_4)$$

and $P(t)$ is a polynomial of degree 3. Therefore, since

$$\lim_{t \to \alpha_i^+} \log |t - \alpha_i| = -\infty$$

and

$$\lim_{t \to \alpha_i^-} \log |t - \alpha_i| = -\infty,$$

the derivative has at most 3 zeros and consequently, the equation (23) can not have more than 6 solutions. □

Definition of the set $\mathfrak{A}$. Assume that equation (1) has more than 6 solutions. Then we can list 6 solutions (x_i, y_i) $(1 \leq i \leq 6)$, so that $r_i = \|\phi(x_i, y_i)\|$ are the smallest among all $\|\phi(x, y)\|$, where (x, y) varies over all non-trivial pairs of solutions. We call the set of all these 6 solutions $\mathfrak{A}$.

Corollary 6.6. *Let $(x, y) \notin \mathfrak{A}$ be a solution to (1). Then*

$$\|\log(\tau(\lambda_2))\| \leq \|\log(\tau(\lambda_3))\| \leq \|\log(\tau(\lambda_4))\| \leq 2\,\|\phi(x, y)\|\,.$$

Proof. Since we have assumed that $\|\log(\tau(\lambda_2))\| \leq \|\log(\tau(\lambda_3))\| \leq \|\log(\tau(\lambda_4))\|$, it is enough to show that $\|\log(\tau(\lambda_4))\| \leq \|\phi(x, y)\|$. By Lemma 6.5, there is at least one solution $(x_0, y_0) \in \mathfrak{A}$ so that

$$\phi(x, y) - \phi(x_0, y_0) = \sum_{i=2}^{4} k_i \log(\tau(\lambda_i))\,,$$

with $k_4 \neq 0$. Since $\{\log(\tau(\lambda_i))\}$ is a reduced basis for the lattice Λ in (21), we conclude that

$$\begin{aligned} \|\log(\tau(\lambda_4))\| &< \|\phi(x, y) - \phi(x_0, y_0)\| \\ &\leq 2\,\|\phi(x, y)\|\,. \end{aligned}$$

□

Lemma 6.7. *Suppose $(x, y) \notin \mathfrak{A}$. Then for $r(x, y) = \|\phi(x, y)\|$, we have*

$$r(x, y) \geq \frac{1}{2} \log\left(\frac{|D|^{1/12}}{2}\right).$$

Proof. Let $(x', y') \in \mathfrak{A}$ be a pair of solutions to equation (1) and α_i and α_j be two distinct roots of quartic polynomial $F(x, 1)$. We have

$$\begin{aligned}
\left|e^{\phi_i(x',y')-\phi_i(x,y)} - e^{\phi_j(x',y')-\phi_j(x,y)}\right| &= \left|\frac{x'-y'\alpha_i}{x-y\alpha_i} - \frac{x'-y'\alpha_j}{x-y\alpha_j}\right| \\
&= \frac{|\alpha_i-\alpha_j|\,|xy'-yx'|}{|x-y\alpha_i||x-y\alpha_j|} \\
&\geq \frac{|\alpha_i-\alpha_j|}{|x-y\alpha_i||x-y\alpha_j|}.
\end{aligned}$$

The last inequality follows from the fact that $|xy' - yx'|$ is a non-zero integer. Since $|\phi_i| < \|\phi\| = r$ and $r(x', y') < r(x, y)$, we may conclude

$$\left(2e^{2r(x,y)}\right)^6 \geq \prod_{1\leq i<j\leq 4} \left|\frac{x'-y'\alpha_i}{x-y\alpha_i} - \frac{x'-y'\alpha_j}{x-y\alpha_j}\right| \geq \sqrt{D}.$$

□

Let us define $T_{i,j}(t) := \log\left|\frac{(t-\alpha_i)(\alpha_4-\alpha_j)}{(t-\alpha_j)(\alpha_4-\alpha_i)}\right|$, so that for a pair of solutions $(x, y) \neq (1, 0)$,

$$\begin{aligned}
T_{i,j}(x,y) = T_{i,j}(t) &= \log\left|\frac{\alpha_4-\alpha_i}{\alpha_4-\alpha_j}\right| + \log\left|\frac{t-\alpha_j}{t-\alpha_i}\right| \\
&= \log\left|\frac{\alpha_4-\alpha_i}{\alpha_4-\alpha_j}\right| + \log\left|\frac{x-\alpha_j y}{x-\alpha_i y}\right| \\
&= \log|\lambda_{i,j}| + \sum_{k=2}^{4} m_i \log\frac{|\lambda_k|}{|\lambda'_k|},
\end{aligned} \tag{24}$$

where $t = \frac{x}{y}$,

$$\lambda_{i,j} = \log\left|\frac{\alpha_4-\alpha_i}{\alpha_4-\alpha_j}\right|$$

and λ_k and λ'_k are fundamental units in $\mathbb{Q}(\alpha_j)$ and $\mathbb{Q}(\alpha_i)$, respectively. Note that the $m_k \in \mathbb{Z}$ in (22) and (24) are the same integers. We will end this section by giving an upper bound for $|T|$ and will estimate $|T|$ from below in the next section.

Lemma 6.8. *Let (x, y) be a pair of solutions to (1) with $|y| > M(F)^6$. Then there exists a pair (i, j) for which*

$$|T_{i,j}(x, y)| < \exp\left(\frac{-r}{6}\right),$$

where $r = \|\phi(t)\|$.

Proof. Let us define

$$\beta_i = \begin{cases} \alpha_i & \text{if } i \leq 3 \\ \beta_{i-3} & \text{if } i \geq 4. \end{cases}$$

Note that

$$\begin{aligned}
& \sum_{k=1}^{2}\sum_{i=1}^{3} \log^2\left|\frac{(t-\beta_i)(\alpha_4-\beta_{i+k})}{(\alpha_4-\beta_i)(t-\beta_{i+k})}\right| \\
= \quad & 4\sum_{i=1}^{3}\log^2\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| - 4\sum_{i\neq j}\log\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right|\log\left|\frac{(t-\alpha_j)}{(\alpha_4-\alpha_j)}\right| \\
= \quad & 4\sum_{i=1}^{3}\log^2\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| - 2\sum_{i=1}^{3}\log\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right|\sum_{j\neq i}\log\left|\frac{(t-\alpha_j)}{(\alpha_4-\alpha_j)}\right| \\
= \quad & 4\sum_{i=1}^{3}\log^2\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| - 2\sum_{i=1}^{3}\log\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right|\log\left|\frac{(\alpha_4-\alpha_i)}{y^4 f'(\alpha_4)(t-\alpha_4)(t-\alpha_i)}\right| \\
= \quad & 6\sum_{i=1}^{3}\log^2\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| - 2\log\left|\frac{1}{y^n f'(\alpha_4)(t-\alpha_n)}\right|\sum_{i=1}^{3}\log\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| \\
= \quad & 6\sum_{i=1}^{3}\log^2\left|\frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)}\right| - 2\log^2\left|\frac{1}{y^4 f'(\alpha_4)(t-\alpha_4)}\right|
\end{aligned}$$

On the other hand, from the proof of Lemma 6.3 the distance between $\phi(x, y)$ and the line

$$\mathbf{L_4} = \sum_{i=1}^{3}\log\frac{|\alpha_4-\alpha_i|}{|f'(\alpha_i)|^{\frac{1}{3}}}\mathbf{c_i} + z\mathbf{b_4}, \quad z \in \mathbb{R}$$

is equal to $\left\|\sum_{i=1}^{3}\log\frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|}\mathbf{c_i}\right\|$ and by the definition of $\mathbf{c_i}$ in section 6.1, we have

$$
\begin{aligned}
& \left\| \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \mathbf{c_i} \right\|^2 \\
= & \left\| \sum_{i=1}^{3} \log \left(\frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} - \frac{1}{3} \left| \log \frac{1}{y^4 f'(\alpha_4)(t-\alpha_4)} \right| \right) \mathbf{e_i} \right\|^2 \\
= & \sum_{i=1}^{3} \log^2 \left(\frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} - \frac{1}{3} \left| \log \frac{1}{y^4 f'(\alpha_4)(t-\alpha_4)} \right| \right) \\
= & \sum_{i=1}^{3} \log^2 \left| \frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)} \right| - \frac{1}{3} \log \left| \frac{1}{y^4 f'(\alpha_4)(t-\alpha_4)} \right| \sum_{i=1}^{3} \log \left| \frac{(t-\alpha_i)}{(\alpha_4-\alpha_i)} \right|
\end{aligned}
$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^3$. So there must be a pair (i, j), for which

$$
\begin{aligned}
& \log^2 \left| \frac{(t-\alpha_i)(\alpha_4-\alpha_j)}{(t-\alpha_j)(\alpha_4-\alpha_i)} \right| \\
< & \frac{1}{6} \sum_{k=1}^{2} \sum_{i=1}^{3} \log^2 \left| \frac{(t-\beta_i)(\alpha_4-\beta_{i+k})}{(\alpha_4-\beta_i)(t-\beta_{i+k})} \right| \\
= & \left\| \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|} \mathbf{c_i} \right\|^2 .
\end{aligned}
$$

Therefore, by Lemma 6.3

$$
|T_{i,j}(x,y)| = \left| \log \left| \frac{(t-\alpha_i)(\alpha_4-\alpha_j)}{(t-\alpha_j)(\alpha_4-\alpha_i)} \right| \right| < \exp\left(\frac{-r}{6} \right).
$$

□

6.3. Linear Forms In Logarithms

Theorem 6.9 (Matveev)**.** *Suppose that $\mathbb{K}$ is a real algebraic number field of degree d. We are given numbers $\alpha_1, \ldots \alpha_n \in \mathbb{K}^*$ with absolute logarithm heights $h(\alpha_j)$. Let $\log \alpha_1$, … , $\log \alpha_n$ be arbitrary fixed non-zero values of the logarithms. Suppose that*

$$
A_j \geq \max\{dh(\alpha_j), |\log \alpha_j|\}, \quad 1 \leq j \leq n.
$$

Now consider the linear form

$$L = b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n,$$

with $b_1, \ldots, b_n \in \mathbb{Z}$ *and with the parameter* $B = max\{1, \max\{b_j A_j / A_n : \ 1 \leq j \leq n\}\}$. *Put*

$$\Omega = A_1 \ldots A_n,$$

$$C(n) = \frac{16}{n!} e^n (2n+2)(n+2)(4n+4)^{n+1} (\frac{1}{2} en),$$

$$C_0 = \log(e^{4.4n+7} n^{5.5} d^2 \log(en)),$$

$$W_0 = \log(1.5eBd \log(ed)).$$

If $b_n \neq 0$*, then*

$$\log |L| > -C(n) C_0 W_0 d^2 \Omega.$$

Proof. See [10] for the proof. $\square$

Let index σ be the isomorphism from $\mathbb{Q}(\alpha_i)$ to $\mathbb{Q}(\alpha_j)$ such that $\sigma(\alpha_i) = \alpha_j$. We may assume that $\sigma(\lambda_i) = \lambda'_i$ for $i = 2, 3, 4$. Let (x_1, y_1) , (x_2, y_2) , (x_3, y_3),(x_4, y_4),(x_5, y_5) be five distinct large solutions to (1) with $(x_k, y_k) \notin \mathfrak{A}$,

$$y_k > M(F)^6$$

and

$$|x_k - \alpha_4 y_k| = \min_{1 \leq i \leq 4} |x_k - \alpha_i y_k| \quad k \in \{1, 2, 3, 4, 5\}$$

and $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5$ where $r_k = \|\phi(x_k, y_k)\|$. We will apply Matveev's lower bound to

$$T_{i,j}(x_5, y_5) = \log |\lambda_{i,j}| + \sum_{k=2}^{4} m_k \log \frac{|\lambda_k|}{|\lambda'_k|},$$

where (i, j) is chosen so that Lemma 6.8 is satisfied and $m_k \in \mathbb{Z}$. In the above representation, λ_k are multiplicatively dependent if and only if $\lambda_{i,j}$ is a unit. If $\lambda_{i,j}$ is a unit then we can write $T_{i,j}(x, y)$ as a linear form in 3 logarithms. Since theorem 6.9 gives a better lower bound for linear forms in 3 logarithms, we will assume that $\lambda_{i,j}$, λ_2, λ_3 and λ_4 are multiplicatively independent and $T_{i,j}(x, y)$ is a linear form in 4 logarithms.

Suppose that λ is a unit in the number field and λ' is its algebraic conjugate. We have

$$h(\lambda') = h(\lambda) = \frac{1}{8}\left|\log\left(\tau(\lambda)\right)\right|_1,$$

where h is the logarithmic height and $|\ |_1$ is the L_1 norm on $\mathbb{R}^4$ and the mappings τ and log are defined in (20) and (21) . So we have

$$h(\lambda) = \frac{1}{8}\left|\log\left(\tau(\lambda)\right)\right|_1 \leq \frac{\sqrt{4}}{8}\left\|\log\left(\tau(\lambda)\right)\right\|,$$

where $\|\|$ is the L_2 norm on $\mathbb{R}^4$. Since α_4, α_i and α_j have degree 4 over $\mathbb{Q}$, the number field $\mathbb{Q}(\alpha_4, \alpha_i, \alpha_j)$ has degree $d \leq 24$ over $\mathbb{Q}$. So when λ is a unit

$$\max\{dh(\frac{\lambda}{\lambda'}), \left|\log(\left|\frac{\lambda}{\lambda'}\right|)\right|\} \leq \max\{24h(\frac{\lambda}{\lambda'}), |\log(\left|\frac{\lambda}{\lambda'}\right|)|\} \leq 12\left\|\log\left(\tau(\lambda)\right)\right\|. \tag{25}$$

Therefore, to apply Theorem 6.9 to $T_{i,j}(x, y)$, by Corollary 6.6, we may take

$$A_i = 24r_1, \ \text{ for } \ 2 \leq i \leq 4.$$

By Lemma 2.2, Proposition 2.3 (see the comment after this proposition), (3) and (4), we may take

$$\frac{A_1}{24} = 2\log 2 + 4^{4^{13}+1}D^{393216}$$

(note that α_1, α_i , α_j are algebraic conjugates and the degree of α_1 is 4). To estimate B, we note that since λ_i $(2 \leq i \leq 4)$ form a reduced basis for the lattice Λ, we have

$$\begin{aligned} m_i\left\|\log\tau(\lambda_i)\right\| &\leq \|\phi(x_5, y_5)\| + \|\phi(1, 0)\| \\ &\leq r_5 + 2\log D^{1/12} + 2\log\frac{5^4 M(F)^3}{\sqrt{3}} \\ &\leq r_5 + 2\log D^{1/12} + 2\log\frac{5^{11/2}H(F)^3}{\sqrt{3}}, \end{aligned}$$

where the inequalities are from Lemmas 2.1 and (16). Therefore, by Proposition 2.3,

$$B = \max\{1, \max\{b_j A_j/A_1 : \ 1 \leq j \leq n\}\} < r_5.$$

Theorem 6.9 implies that for a constant number K,

$$\log T_{i,j}(x_5, y_5) > -K\, D^{393216} r_1^3 \log r_5.$$

Comparing this with Lemma 6.8, we have

$$\left(\frac{-r_5}{6}\right) > -K\, D^{393216} r_1^3 \log r_5,$$

or

$$\frac{r_5}{\log r_5} < 6K\, D^{393216} r_1^3.$$

Thus we may compute the constant number K_1, so that

$$r_5 < K_1\, D^{393216} r_1^3, \tag{26}$$

This is because r_5 is large enough by Lemma 6.7. Using Lemma 6.2 twice, we obtain

$$r_5 > \exp\left(\frac{2\sqrt{3}}{6} \exp(r_1/6) \log^4 \frac{1+\sqrt{5}}{2}\right) 2\sqrt{3} \log^4 \frac{1+\sqrt{5}}{2}.$$

Comparing with (26, we get a contradiction. For by Lemma 6.7,

$$r_1 \geq \frac{1}{2} \log\left(\frac{|D|^{\frac{1}{12}}}{2}\right).$$

Thus, there are at most 16 solutions $(x, y) \notin \mathfrak{A}$ with $y > M(F)^6$. By Lemma 5.1 and since $|\mathfrak{A}| = 6$, counting the solution $(1, 0)$, Theorem 1.2 is proven.

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References

[1] S. Akhtari, The method of Thue-Siegel for binary quartic forms, to appear in Acta. Arith. (2009).

[2] E. Bombieri, W. M. Schmidt, On Thue's equation, Invent. Math. 88 (1987), 69-81.

[3] Y. Bugeaud, Bornes effectives pour les solutions des équations en S-units et des équations de Thue-Mahler, J. Number Theory 71(1998), 227-244.

[4] J.H. Evertse. Estimates for reduced binary forms, J. Reine Angew. Math. 434 (1993), 159-190.

[5] K. Győry, Sur les polynomial a coefficients entiers et the discriminant donne II, Pub. Math. Debrecen 21 (1974), 125-144.

[6] K. Győry, Sur les polynomial a coefficients entiers et the discriminant donne III, Pub. Math. Debrecen 23 (1976), 141-165.

[7] D. Lewis, K. Mahler, Representation of integers by binary forms, Acta Arith. 6 (1961), 333-363.

[8] K. Mahler, An inequality for the discriminant of a polynomial, Michigan Math. J. 11 (1964), 257-262.

[9] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, Izv. Ross. Akad. Nauk Ser. Mat. 62 (1998), 81-136, translation in Izv. Math. 62 (1998), 723-772.

[10] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), 125-180, translation in Izv. Math. 64 (2000), 1217-1269.

[11] M. Mignotte, A. Pethö and R. Roth, Complete solutions of a family of quartic Thue and index form equations, Math. Comp. 65 (1996), 341-354.

[12] R. Okazaki. Geometry of a cubic Thue equation, Publ. Math. Debrecen. 61 (2002),267-314.

[13] M. Pohst and H. Zassenhaus. Algorithmic Algebraic Number Theory, Cambridge University Press (1989).

[14] C. L. Stewart, On the number of solutions of polynomial congruences and Thue equations, Journal of American Math. Soc., 4 (1991), 793-838.