CUSPIDAL CLASS NUMBER OF A TOWER OF MODULAR CURVES $X_1(Np^n)$

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Abstract. We consider a cuspidal class number, which is the order of a subgroup of full cuspidal divisor class group of $X_1(Np^n)$ with $p \nmid N$ and $n \geq 1$. By studying the second generalized Bernoulli numbers, we obtain results similar to ones ([1], [9]) about the relative class numbers of cyclotomic $\mathbb{Z}_p$-extension of an abelian number field.

1. Introduction

Let G be a finite abelian group with a surjective homomorphism $r : G \to (\mathbb{Z}/N\mathbb{Z})^\times$ for an integer $N > 0$. Let χ be a character on G. A generalized k-th Bernoulli number $B_{k,\chi,G}$ can be defined for χ such that

$$B_{k,\chi,G} = N^{k-1} \sum_{g \in G} \chi(g) B_k \left(\frac{r(g)}{N} \right),$$

where $B_k(x)$ is the Bernoulli polynomial defined by the formula

$$B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{n-r}.$$

For $G = (\mathbb{Z}/N\mathbb{Z})^\times$, $B_{k,\chi,G}$ is the usual generalized Bernoulli numbers $B_{k,\chi}$. In many different contexts, those generalized Bernoulli numbers have been related to an index of the Stickelberger ideal of order k in the group ring $\mathbb{Z}[G]$, which is generated by a Stickelberger element

$$\theta = N^{k-1} \sum_{g \in G} B_k \left(\frac{r(g)}{N} \right) g \in \mathbb{Q}[G]$$

or by its variation.

When $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $k = 1$, the relative class number h_N^- of $\mathbb{Q}(\zeta_N)$ can be written as a product of $B_{1,\chi}$ for odd Dirichlet characters χ. More precisely, one has

$$h_N^- = Q_N w_N \prod_{\chi:odd} -\frac{1}{2} B_{1,\chi}, \tag{1.1}$$

where Q_N is the unit index and w_N is the number of roots of unity. The relative class number h_N^- is turned out to be an index of minus part of Stickelberger ideal of order 1. To the cyclotomic fields $\mathbb{Q}(\zeta_{Np^n})$, two non-negative integers μ and λ

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are associated in order to express p-adic valuation $v_p(h^-_{Np^n})$ of the relative class numbers $h^-_{Np^n}$. In fact, one has

$$v_p(h^-_{Np^n}) = \mu p^n + \lambda n + \nu \text{ for all } n \gg 1 \text{ and some } \nu \in \mathbb{Z}. \tag{1.2}$$

It has been conjectured that $\mu = 0$ and it was proved by Ferrero and Washington ([1]).

Let ℓ be an odd prime number different from p. The ℓ-adic valuation $v_\ell(h^-_{Np^n})$ of the relative class numbers was also determined by Washington ([9]). He has shown that there exists a constant δ_ℓ such that

$$v_\ell(h^-_{Np^n}) = \delta_\ell \text{ for all } n \gg 1 \tag{1.3}$$

by verifying

Theorem 1.1 ([9]). *Let N be fixed and $p \nmid N$. For almost all odd Dirichlet characters χ of conductor Np^n, we have*

$$v_\ell(B_{1,\chi}) = 0.$$

From the relative class number formula and the argument of Kummer, one can conclude the formula (1.3).

In this paper, we consider the case that $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $k = 2$. This case is studied by Kubert-Lang ([4]) for an odd prime power $N = p^n$ and by J. Yu ([12]) for general $N > 4$.

For an integer $N > 4$, a cusp on the modular curve $X_1(N)$ is said to be *of first type* if it is projected down to the cusp 0 on the modular curve $X_0(q)$ for all prime divisor q of N. We consider the group $\mathfrak{F}_1^0(N)$ of functions on $X_1(N)$ whose divisors are supported on the cusps of the first type and the group $\mathfrak{D}_1^0(N)$ of divisors with degree 0 on $X_1(N)$ that are supported on the cusps of the first type. We set

$$\mathcal{C}_1^0(N) = \mathfrak{D}_1^0(N)/\mathrm{div}\mathfrak{F}_1^0(N).$$

Note that $\mathcal{C}_1^0(N)$ is a subgroup of the full cuspidal divisor class group on the modular curve $X_1(N)$. Let $h_1^0(N)$ be the order of $\mathcal{C}_1^0(N)$. Then one has an analogue of the formula (1.1) as follows:

Theorem 1.2 ([12]). *For a prime number p, we set*

$$L_p(N) = \phi\left(\frac{N}{p^{v_p(N)}}\right) p^{v_p(N)-1} - 2v_p(N) + \epsilon_p(N)$$

where

$$\epsilon_p(N) = \begin{cases} 2 - \phi\left(\frac{N}{p^{v_p(N)}}\right) & \text{if } N \text{ is not a prime power} \\ 2 & \text{if } N = p^n > 4 \text{ and } p \geq 3 \\ 3 & \text{if } N = 2^n > 4\ . \end{cases}$$

Then we have

$$h_1^0(N) = \prod_{p|N} p^{L_p(N)} \prod_{\substack{\chi:even \\ \chi\neq 1}} \left[\frac{1}{4} B_{2,\chi_1} \prod_{p|N} (1 - p^2\chi_1(p))\right], \tag{1.4}$$

where χ_1 is the primitive Dirichlet character which induces the character χ.

Remark 1.1. As explained in [11], a minor error has to be corrected in the statement of [12, Theorem 5].

The purpose of present paper is to determine p-adic and ℓ-adic valuations of $h_1^0(Np^n)$, $p \nmid N$ for all sufficiently large n. We obtain similar results to the formulae (1.2) and (1.3) as follows:

Theorem 1.3. *Let $\xi = \xi_p(N)$ be given in (2.11). We set*

$$\kappa = \begin{cases} \phi(N) & \text{if } p \text{ is odd} \\ p\phi(N) & \text{if } p = 2 \end{cases},$$

and

$$\tau = \begin{cases} (p-1)\phi(N/\ell^{v_\ell(N)})(\ell^{v_\ell(N)-1} - 1) & \text{if } \ell \mid N \\ 0 & \text{if } \ell \nmid N. \end{cases}.$$

There exists two integers c_p and c_ℓ such that

(1) $v_p(h_1^0(Np^n)) = \kappa p^{n-1} + (\lambda + \xi - 3)n + c_p$ *for all* $n \gg 1$.
(2) $v_\ell(h_1^0(Np^n)) = \tau p^{n-1} + c_\ell$ *for all* $n \gg 1$.

Remark 1.2. When $N = p^n$ is a prime power and p is a regular prime, the primary decomposition of $\mathcal{C}_1^0(N)$ is determined in [11]. Note that in this case, $\xi = \lambda = 0$.

The p-adic valuation of $h_1^0(Np^n)$ or $B_{2,\chi}$ can be computed in a very similar way as done in [2], [10] and in the next section we follow the argument. On the other hand, in order to get ℓ-adic valuation, we prove an analogue of a theorem of Washington for the generalized Bernoulli number of higher order. In other words, we show:

Theorem 1.4. *Let N be a fixed integer with $p \nmid N$. Let $\{\chi\}$ be a set of Dirichlet characters of conductor $p^n N$ with $n \geq 1$, and $\chi(-1) \neq (-1)^k$. Then we have*

$$v_\ell\left(\frac{B_{k,\chi}}{k}\right) = 0 \text{ for almost all } \chi.$$

In [8], using the following formula (see Proposition 3.1)

$$\frac{B_{k,\chi}}{k} = G(\chi) \int_{-i\infty}^{i\infty} \frac{\sum_{r=1}^{Np^n-1} \chi^{-1}(r) e^{2\pi i z}}{1 - e^{2\pi i N p^n z}} z^{k-1} dz. \tag{1.5}$$

a proof of Theorem 1.1 is obtained from a homological formulation such as abelian modular symbols on punctured cylinders and a certain homological equi-distribution property. In the present paper, we count on elementary calculations rather than devise a homological description.

Remark 1.3. Let G be a Cartan group $C(N)$ and $k = 2$. If $N = p^n$ is a prime power, the full cuspidal class number $h(N)$ of the modular curve $X(N)$ is represented by a product of terms involving $B_{2,\chi,C(N)}$, the Cartan-Bernoulli number. In fact, computing the index of a Stickelberger ideal of order 2, Kubert and Lang ([4]) also obtained the class number formula: For $p \neq 2, 3$,

$$h(p^n) = \frac{6p^{3n}}{|C(p^n)|} \prod_{\chi:even} \frac{1}{4} B_{2,\chi,C(p^n)}.$$

Thanks to the results [5], one has the formula

$$B_{k,\chi,C(N)} = N^{1-k} \frac{G(\chi, r)}{G(\chi_{\mathbb{Z}})} B_{k,\chi_{\mathbb{Z}}},$$

where $G(\chi, r) = \sum_{g \in C(N)} \chi(g) e^{2\pi i r(g)/N}$, $\chi_{\mathbb{Z}}$ is the restriction of χ to $(\mathbb{Z}/N\mathbb{Z})^\times$, and $G(\chi_{\mathbb{Z}})$ is the Gauss sum of $\chi_{\mathbb{Z}}$ with respect to the additive character $r \mapsto e^{2\pi i r/N}$. Since $|G(\chi, r)|^2$ is a p-power([3, Lemma 4.2]), Theorem 1.4 enables us to obtain a similar result as above. In other words, $v_\ell(h(p^n))$ is bounded for all $n \gg 1$ and $\ell > 3$.

For a p-adic integer α, we let $(\alpha)_n$ be the n-th partial sum of p-adic expansion of α. We say two sequences $\{a_n\}$, $\{b_n\}$ of p-adic numbers are *equivalent* if $v_p(a_n/b_n)$ are eventually constant as $n \to \infty$ and denote them by $a_n \overset{p}{\sim} b_n$. In Section 2, we fix two embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. For an integer $L > 0$, we set $\zeta_L = e^{2\pi i/L}$ and for a Dirichlet character ψ with the conductor f, $G(\psi)$ is the Gauss sum $G(\psi) = \sum_{r=1}^{f-1} \psi(r)\zeta_f^r$.

2. The p-part of the cuspidal class number

Let $q_0 = 4$ if $p = 2$ and $q_0 = p$ if $p \geq 3$. Let χ be a Dirichlet character of conductor Nq_0p^n, and θ and π be the first and the second factors of χ respectively in the sense of [2]. In other words, θ and π are the restrictions of χ to $(\mathbb{Z}/Nq_0\mathbb{Z})^\times$ and $\frac{1+p\mathbb{Z}}{1+p^n\mathbb{Z}}$ respectively. From the cuspidal class number formula (1.4), we have $L_p(Nq_0p^n) = \phi(N)q_0p^{n-1} - 2n + \epsilon_p(Np^n)$ and since $\prod_\chi (1 - \chi_1(p)p^2) \overset{p}{\sim} 1$ we obtain

$$h_1^0(Nq_0p^n) \overset{p}{\sim} p^{\phi(N)q_0p^{n-1}-2n} \prod_{\substack{\chi:even \\ \pi \neq 1}} \frac{1}{4} B_{2,\chi_1} \prod_{q|N} (1 - \chi_1(q)q^2). \tag{2.1}$$

From the equation $L(s, \chi) = \prod_{q|N}(1 - \chi_1(q)q^{-s})L(s, \chi_1)$, we obtain

$$B_{n,\chi} = B_{n,\chi_1} \prod_{q|N} (1 - \chi_1(q)q^{n-1}).$$

The formula (2.1) is written as

$$h_1^0(Nq_0p^n) \overset{p}{\sim} p^{\phi(N)q_0p^{n-1}-2n} \prod_{\substack{\chi:even \\ \pi \neq 1}} \frac{1}{4} B_{2,\chi} \prod_{q|N} \frac{1 - \chi_1(q)q^2}{1 - \chi_1(q)q}. \tag{2.2}$$

We consider the first product of generalized Bernoulli numbers in (2.2). The following discussion is similar to [2], [10]. Let k be the number field obtained by adjoining θ to $\mathbb{Q}$ and $\mathfrak{o}$ be the ring of integers. It is a well-known fact ([10]) that if $\theta \neq 1$, there exists a power series $P_\theta(T) \in \mathfrak{o}[[T-1]]$ such that

$$2P_\theta(\zeta_\chi(1+q_0)^{1-n}) = -(1 - \chi\omega^{-n}(p)p^{n-1}) \frac{B_{n,\chi\omega^{-n}}}{n}$$

for $\chi(-1) = 1$, $\zeta_\chi = \chi(1+q_0)^{-1}$ and $n \geq 1$. In particular, for a Dirichlet character χ with $\pi \neq 1$, we have

$$P_{\theta\omega^2}(\zeta_\pi(1+q_0)^{-1}) = -\frac{B_{2,\chi}}{4}.$$

And that if $\theta = 1$, then $P_1(T)$ can be written as

$$P_1(T) = G(T) \left(1 - \frac{1+q}{T}\right)^{-1}, \tag{2.3}$$

where $G(T) \in \mathbb{Z}_p[[T-1]]^\times$. Using this, the first product in the formula (2.2) can be written as

$$\prod_{\substack{\chi:even\\ \pi\neq 1}} \frac{1}{4}B_{2,\chi} \overset{p}{\sim} \prod_{\substack{\theta:even\\ \pi\neq 1}} P_\theta(\zeta_\pi(1+q_0)^{-1}). \tag{2.4}$$

Note that ζ_π runs over all p^n-th roots of unity. Dividing the above product into two parts, namely a product over $\theta = 1$ and a product over $\theta \neq 1$, the formula (2.4) is equivalent to

$$\prod_{\substack{\zeta^{p^n}=1\\ \zeta\neq 1}} P_1(\zeta(1+q_0)^{-1}) \prod_{\zeta^{p^n}=1} \prod_{\substack{\theta:even\\ \theta\neq 1}} P_\theta(\zeta(1+q_0)^{-1}). \tag{2.5}$$

Now we set

$$P(T) = \prod_{\substack{\theta:even\\ \theta\neq 1}} P_\theta(T).$$

Note that we have $P(T) \in \mathbb{Z}_p[[T-1]]$ and there exists two non-negative integers μ and λ such that the power series $P(T)$ has the factorization

$$P(t) = p^\mu Q(t) \text{ with } Q(T) \equiv (T-1)^\lambda U(T) \pmod p$$

where $Q(T), U(T) \in \mathbb{Z}_p[[T-1]]$ with $U(0) \in \mathbb{Z}_p^\times$. The celebrated theorem due to Ferrero and Washington([1]) shows that $\mu = 0$. Furthermore, when p is an odd prime number and $N = 1$, we have $\lambda = 0$ if and only if p is regular, that is, $p \nmid h^-(\mathbb{Q}(\zeta_p))$.

From (2.3), we have

$$\prod_{\substack{\zeta^{p^n}=1\\ \zeta\neq 1}} |P_1(\zeta(1+q_0)^{-1})|_p = \prod_{\zeta} |1-\zeta^{-1}(1+q_0)^2|_p^{-1} = |p^n|_p^{-1}. \tag{2.6}$$

For the second product in (2.5), one can easily deduce that

$$\prod_{\zeta^{p^n}=1} |Q(\zeta(1+q_0))^{-1}|_p \overset{p}{\sim} p^{\lambda n}. \tag{2.7}$$

In total, from (2.6) and (2.7) we have

$$\prod_{\substack{\chi:even\\ \pi\neq 1}} \frac{1}{4}B_{2,\chi} \overset{p}{\sim} p^{(\lambda-1)n} \tag{2.8}$$

Next we consider the second product in the formula (2.2). For a prime $q \mid N$, let $N^{(q)}$ be the integer obtained by removing q factors of N and F_q be the order of q in $(\mathbb{Z}/N^{(q)}q_0p^n\mathbb{Z})^\times/\{\pm 1\}$ and $E_q = \frac{\phi(N^{(q)}q_0p^n)}{2F_q}$. As observed in [11], one obtains

$$\prod_{\substack{\chi:even\\ \chi\neq 1}} \frac{1-\chi_1(q)q^2}{1-\chi_1(q)q} = \frac{(1+q^{F_q})^{E_q}}{1+q}. \tag{2.9}$$

Since $q^{F_q} \equiv \pm 1 \pmod{q_0}$, we have $\omega(q)^{F_q} = \pm 1$. Furthermore if $\omega(q)^{F_q} = 1$, then obviously we have

$$v_p(1+q^{F_q}) = v_p(1+\langle q\rangle^{F_q}) = v_p(2).$$

On the other hand, if $\omega(q)^{F_q} = -1$, then we have

$$v_p(1+q^{F_q}) = v_p(1-\langle q\rangle^{F_q}) = v_p(F_q \log_p\langle q\rangle) + v_p(2).$$

Let f_q be the order of q in $(\mathbb{Z}/N^{(q)}q_0\mathbb{Z})^\times/\{\pm 1\}$ and $e_q = \frac{\phi(N^{(q)}q_0)}{2f_q}$. Since we have $F_q = p^{n-v_p(\log_p\langle q\rangle)}f_q$ and $E_q = p^{v_p(\log_p\langle q\rangle)}e_q$, we obtain $v_p(1+q^{F_q}) = n + v_p(f_q) + v_p(2)$ and from (2.9) we have

$$\prod_{q|N}\prod_{\substack{\chi:even\\ \chi\neq 1}} \frac{1-\chi_1(q)q^2}{1-\chi_1(q)q} \overset{p}{\sim} \prod_{\substack{q|N\\ q^{f_q}\equiv -1(p)}} p^{E_q n}. \tag{2.10}$$

We set

$$\xi_p(N) = \sum_{\substack{q|N\\ q^{f_q}\equiv -1(p)}} E_q = \sum_{\substack{q|N\\ q^{f_q}\equiv -1(p)}} p^{v_p(\log_p\langle q\rangle)}e_q. \tag{2.11}$$

Putting together (2.2), (2.8), and (2.10), for an odd prime p we obtain

$$h_1^0(Nq_0p^n) \overset{p}{\sim} p^{\phi(N)q_0p^{n-1}+(\lambda_p(N)+\xi_p(N)-3)n}.$$

3. The non-p-part of cuspidal class number

Let ℓ be an odd prime number different from p. We also consider a similar formula as (2.2) as follows.

$$h_1^0(Np^n) \overset{\ell}{\sim} \prod_{q|N} q^{L_q(Np^n)} \prod_{\substack{\chi:even\\ \pi\neq 1}} \frac{1}{4}B_{2,\chi} \prod_{q|N} \frac{1-\chi_1(q)q^2}{1-\chi_1(q)q}. \tag{3.1}$$

Since $1-\chi_1(q)q^2$ and $1-\chi_1(q)q$ for $q|N$ are equivalent to 1, we have

$$h_1^0(Np^n) \overset{\ell}{\sim} \ell^{L'_\ell(Np^n)} \prod_{\substack{\chi:even\\ \pi\neq 1}} \frac{1}{4}B_{2,\chi}, \tag{3.2}$$

where

$$L'_\ell(Np^n) = \begin{cases} \phi(N^{(\ell)})(\ell^{v_p(N)-1}-1)(p-1)p^{n-1} & \text{if } \ell \mid N \\ 0 & \text{if } \ell \nmid N. \end{cases}$$

Hence it remains to show Theorem 1.4. The main idea is to modify the method of Washington ([9]) to apply to the generalized Bernoulli numbers of higher order by following the discussion in [8], where a homological argument has been developed to obtain a conceptual interpretation of Washington's proof.

For a periodic function λ of period N, we define a rational function $R_\lambda(t)$ so that

$$R_\lambda(t) = \frac{\sum_{r=1}^{N-1}\lambda(r)t^r}{1-t^N}.$$

For $q = e^{2\pi i z}, z \in \mathbb{C}$ and a polynomial $P(z)$, we have a meromorphic function $R_\lambda(q)P(z)$ on $\mathbb{C}$ with poles $z = \frac{r}{N}$, $r \in \mathbb{Z}$ where the residue is given by

$$\operatorname{Res}\left(\frac{r}{N}; R_\lambda(q)P(z)\right) = \frac{\widehat{\lambda}(r)}{N}P\left(\frac{r}{N}\right),$$

where $\operatorname{Res}\big(z_0; R_\lambda(q)P(z)\big)$ is the residue of $R_\lambda(q)P(z)$ at $z = z_0$. Here $\widehat{\lambda}$ is a periodic function which is the Fourier transform of λ defined by

$$\widehat{\lambda}(r) = \sum_{s=1}^{N} \lambda(s)e^{\frac{2\pi i rs}{N}}.$$

From now on, we assume that $\lambda(N) = 0$ i.e. $\sum_{s=1}^{N} \widehat{\lambda}(s) = 0$. With this assumption we observe that $R_\lambda(e^{2\pi iz})P(z)$ is exponentially decreasing as $\Im(z) \to \pm\infty$. Therefore the contour integral $\int_{x-i\infty}^{x+i\infty} R_\lambda(e^{2\pi iz})P(z)dz$ is well-defined for $x \in \mathbb{R} - \frac{1}{N}\mathbb{Z}$. Furthermore if $\frac{r-1}{N} < x < \frac{r}{N}$ and $\frac{r}{N} < y < \frac{r+1}{N}$, then we have

$$\begin{aligned}(3.3) \qquad &\int_{x-i\infty}^{x+i\infty} R_\lambda(e^{2\pi iz})P(z)dz - \int_{y-i\infty}^{y+i\infty} R_\lambda(e^{2\pi iz})P(z)dz \\ &= \operatorname{Res}\big(\frac{r}{N}; R_\lambda(q)P(z)\big) = \frac{\widehat{\lambda}(r)}{N} P\left(\frac{r}{N}\right).\end{aligned}$$

When $P(z) = z^k$, we obtain the following special value formula of Dirichlet L-functions.

Proposition 3.1. *For $k \geq 0$, we have*

$$L(-k, \lambda) = N^k \int_{-i\infty}^{i\infty} R_{\widehat{\lambda}}(e^{2\pi iz}) z^k dz. \tag{3.4}$$

Proof. We have the expression

$$L(s, \lambda) = N^{-s} \sum_r \lambda(r) \zeta\left(s, \frac{r}{N}\right) \tag{3.5}$$

which enables us to get the functional equation (See [6]) of $L(s, \lambda)$ as follows:

$$(e^{2\pi is} - 1)\Gamma(s)L(s, \lambda) = \left(\frac{2\pi i}{N}\right)^s \left(L(1-s, \widehat{\lambda}) - (-1)^{s-1} L(1-s, \widehat{\lambda} \circ -1)\right),$$

Since $\lim_{s \to -k} (e^{2\pi is} - 1)\Gamma(s) = \dfrac{(-1)^k 2\pi i}{k!}$ for $k \geq 0$, we have

$$L(-k, \lambda) = \frac{N^k}{(2\pi i)^{k+1}} ((-1)^k L(k+1, \lambda \circ -1) + L(k+1, \lambda)). \tag{3.6}$$

Since we have $L(k+1, \lambda) = \frac{1}{k!} \int_0^\infty R_\lambda(e^{-y}) y^k dy$, we obtain the proposition. □

Let ψ be a Dirichlet character. It is well-known that for an integer a relatively prime to the conductor of ψ and for integers $k \geq 0$, one has

$$(a^{k+1}\psi(a) - 1)L(-k, \psi) \in \mathbb{Z}[\psi].$$

Furthermore if we consider Dirichlet characters ψ of conductor Np^n, $n \geq 0$ then we have

$$v_\ell(a^{k+1}\psi(a) - 1) = 0 \text{ for all } n \gg 1.$$

Hence we are able to conclude that $L(-k, \psi)$ is ℓ-integral for all characters ψ of sufficiently large conductors, say all conductors Np^n with $n \geq m_1$.

In order to treat the case of $N = 1$, we let λ_0 be a periodic function with period g, a prime number different from p and ℓ, which is defined by

$$\begin{aligned}\lambda_0(r) = &\quad 1 \quad \text{if } g \nmid r \\ &= 1 - g \text{ if } g \mid r.\end{aligned}$$

Observe that $\widehat{\lambda_0}(g) = 0$ and $\widehat{\lambda_0}(r) = -g$ if $r < g$. We have the formula

Proposition 3.2. *Let χ and λ be Dirichlet characters of conductor p^n and N respectively. When $N > 1$, we have*

$$L(-k,\chi\lambda) = N^k \sum_{r=1}^{p^n-1} \chi(r) \int_{\frac{r}{p^n}-i\infty}^{\frac{r}{p^n}+i\infty} R_{\widehat{\lambda}}(q) \left(z - \frac{r}{p^n}\right)^k dz \tag{3.7}$$

When $N = 1$, we have

$$L(-k,\chi) = \frac{1}{1-g^{k+1}\chi(g)} L(-k,\chi\lambda_0) \tag{3.8}$$

$$= \frac{g^k}{1-g^{k+1}\chi(g)} \sum_{r=1}^{p^n-1} \chi(r) \int_{\frac{r}{p^n}-i\infty}^{\frac{r}{p^n}+i\infty} R_{\widehat{\lambda_0}}(q) \left(z - \frac{r}{p^n}\right)^k dz \tag{3.9}$$

Proof. Let $N > 1$. We start with the formula (3.4)

$$L(-k,\chi\lambda) = N^k \int_{-i\infty}^{i\infty} R_{\widehat{\chi\lambda}}(q) z^k dz.$$

Note that $\widehat{\chi\lambda} = G(\chi\lambda)\chi^{-1}\lambda^{-1}$. Since $R_{\chi^{-1}\lambda^{-1}}(q)$ can be written as

$$R_{\chi^{-1}\lambda^{-1}}(q) = \frac{1}{G(\chi)} \sum_{r=0}^{p^n-1} \chi(r) R_{\lambda^{-1}}(q\zeta_n^r)$$

and $G(\chi\lambda) = G(\chi)G(\lambda)$, we obtain the formula (3.7). When $N = 1$, we have (3.8) by the definition of λ_0 and by the following formula

$$\widehat{\chi\lambda_0} = G(\chi)\chi^{-1}\widehat{\lambda_0}.$$

The formula (3.9) can be obtained in a same way as before and we conclude the proposition. □

From now on we let χ be a Dirichlet character of p-power conductor and

$$\lambda = \begin{cases} \text{Dirichlet character of conductor } N & \text{if } N > 1 \\ \lambda_0 & \text{if } N = 1, \end{cases}$$

and L be the period of λ. In other words,

$$L = N \text{ if } N > 1 \text{ and } L = g \text{ if } N = 1.$$

Let $W = \mu_{p-1}$ if p is odd and $W = \mu_4$ if $p = 2$. Observe that we have the decomposition

$$\mathbb{Z}_p^\times = W \times (1 + 2p\mathbb{Z}_p).$$

Let $k_0 = \mathbb{Q}(\lambda, \chi|_W)$ be the finite extension of $\mathbb{Q}$ adjoining the values λ and $\chi(W)$. Set $k_n = k_0(\mu_{p^n})$, $k_\infty = k_0(\mu_{p^\infty})$ and let $\mathfrak{L}$ be a prime in k_∞ over ℓ. The extension k_∞/k_0 is unramified at $\mathfrak{L}\cap k_0$. Let H be the decomposition group of $\mathfrak{L}$ and $k = k_\infty^H$. Then $\mathfrak{L}$ is inert in k_∞/k and for all sufficiently large n, say $n \geq m_2$, we have $k_{n+1} \neq k_n$ and $\mathfrak{L}$ is inert in k_∞/k_n. For $\sigma \in \mathrm{Gal}(k_\infty/k_n)$, we have

$$\sigma L(-k,\chi\lambda) = L(-k,\chi^\sigma\lambda).$$

Recall that $L(-k,\chi\lambda)$ is ℓ-integral for all Dirichlet characters χ of which conductor is p^n with $n \geq m_1$. Now we set

$$m_0 = \max\{m_1, m_2\}.$$

For each $\eta \in W$ and an integer $m > 0$, we set

$$R_{\lambda,\eta,m}(q) = \sum_{s=1}^{p^m-1} \zeta_m^{s\eta^{-1}} R_\lambda(q\zeta_m^s) = \sum_{\substack{r\equiv -\eta (p^m)\\ 1\le r<p^m N}} \frac{\lambda(r)q^r}{1-q^{p^m L}}.$$

Observe that if we define a periodic function $\lambda_{\eta,m}$ of a period Np^m such that

$$\lambda_{\eta,m}(r) = \begin{cases} \lambda(r) & \text{if } r \equiv -\eta \pmod{p^m} \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$R_{\lambda,\eta,m}(q) = R_{\lambda_{\eta,m}}(q).$$

Proposition 3.3. *Let the conductor of χ is p^n with $n > m_0$. If we have*

$$L(-k,\chi\lambda) \equiv 0 \pmod{\mathfrak{L}},$$

then for all $m \ge m_0$ with $n > 2m$ and $\alpha \in 1 + p^m\mathbb{Z}_p$ we have

$$L^k \sum_{\eta\in W} \chi(\eta) \int_{\frac{(\alpha\eta)_n}{p^n}-i\infty}^{\frac{(\alpha\eta)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta,m}(q) \left(z - \frac{(\alpha\eta)_n}{p^n}\right)^k \equiv 0 \pmod{\mathfrak{L}}. \tag{3.10}$$

Proof. Applying the trace $\mathrm{Tr} = \mathrm{Tr}_{k_n/k_{n-m}}$ to the congruence $L(-k,\chi\lambda) \equiv 0 \pmod{\mathfrak{L}}$ after multiplying $\chi(\alpha^{-1})$ for an $\alpha \in 1 + p\mathbb{Z}_p$, we have

$$\mathrm{Tr}(\chi(\alpha^{-1})L(-k,\chi\lambda)) \equiv 0 \pmod{\mathfrak{L}}. \tag{3.11}$$

Observe that we have

$$\mathrm{Tr}(\chi(x)) = \begin{cases} [k_n : k_{n-m}]\chi(x) & \text{if } \chi(x) \in k_{n-m} \\ 0 & \text{otherwise.} \end{cases}$$

We start with the formulae (3.7) and (3.9). Since $\chi(x) \in k_{n-m}$ if and only if $x \in 1 + p^{n-m}\mathbb{Z}_p$, the formula (3.11) implies that

$$L^k \sum_{\eta\in W} \chi(\eta) \sum_{r\in\alpha\frac{1+p^{n-m}\mathbb{Z}_p}{1+p^n\mathbb{Z}_p}} \chi(r) \int_{\frac{(r\eta)_n}{p^n}-i\infty}^{\frac{(r\eta)_n}{p^n}+i\infty} R_\lambda(q)\left(z - \frac{(r\eta)_n}{p^n}\right)^k dz \equiv 0 \pmod{\mathfrak{L}}.$$

Setting $r = \alpha(1 + p^{n-m}s)$ with $0 \le s < p^m$, we have $\chi(1 + p^{n-m}s) = \zeta_m^s$ and the last congruence becomes

$$L^k \sum_{\eta} \chi(\eta) \sum_{s=0}^{p^m-1} \zeta_m^s \int_{\frac{(\alpha\eta(1+p^{n-m}s))_n}{p^n}-i\infty}^{\frac{(\alpha\eta(1+p^{n-m}s))_n}{p^n}+i\infty} R_\lambda(q)$$

$$\times \left(z - \frac{(\alpha\eta(1+p^{n-m}s))_n}{p^n}\right)^k dz \equiv 0 \pmod{\mathfrak{L}}.$$

Set $t_{n,m} = (\alpha\eta + p^{n-m}s)_n - (\alpha\eta)_n$. Changing the domain of integration and setting $s \mapsto (\alpha\eta)^{-1}s$, we have

$$L^k \sum_{\eta} \chi(\eta) \sum_{s=0}^{p^m-1} \zeta_m^{s(\alpha\eta)^{-1}} \int_{\frac{(\alpha\eta)_n}{p^n}-i\infty}^{\frac{(\alpha\eta)_n}{p^n}+i\infty} R_\lambda(q\zeta_n^{t_{n,m}}) \left(z - \frac{(\alpha\eta)_n}{p^n}\right)^k dz \equiv 0 \,(\bmod \mathfrak{L}).$$

Observe that $t_{n,m} \equiv p^{n-m}s \,(\bmod p^n)$. In total, after choosing $\alpha \in 1 + p^m\mathbb{Z}_p$, we conclude the proposition. □

Proof of Theorem 1.4. Now we show the following statement: Let χ be a Dirichlet character of conductor p^n and $\chi\lambda(-1) \neq (-1)^k$. Then we have

$$L(-k, \chi\lambda) \not\equiv 0 \,(\bmod \mathfrak{L}) \text{ for almost all } \chi.$$

In order to reach a contradiction in the end, we assume the contrary that there exist infinitely many χ such that

$$L(-k, \chi\lambda) \equiv 0 \,(\bmod \mathfrak{L}).$$

By Proposition 3.3, the formula (3.10) holds for each m with $m \geq m_0$ and infinitely many $n \geq 2m$. Observe that we have $\frac{(-\alpha\eta)_n}{p^n} = 1 - \frac{(\alpha\eta)_n}{p^n}$, and therefore for each $\alpha \in 1 + p^m\mathbb{Z}_p$ we have

$$\begin{aligned}\int_{\frac{(-\alpha\eta)_n}{p^n}-i\infty}^{\frac{(-\alpha\eta)_n}{p^n}+i\infty} & R_{\widehat{\lambda},\eta,m}(q) \left(z - \frac{(-\alpha\eta)_n}{p^n}\right)^k dz \\ &= (-1)^k \int_{\frac{(\alpha\eta)_n}{p^n}-i\infty}^{\frac{(\alpha\eta)_n}{p^n}+i\infty} R_{\widehat{\lambda},-\eta,m}(q^{-1}) \left(z - \frac{(\alpha\eta)_n}{p^n}\right)^k dz.\end{aligned}$$

Also observe that

$$R_{\widehat{\lambda},-\eta,m}(q^{-1}) = -\lambda(-1) R_{\widehat{\lambda},\eta,m}(q).$$

With the same notation in Proposition 3.3, the formula (3.10) becomes

$$2L^k \sum_{\eta \in W} \chi(\eta) \int_{\frac{(\alpha\eta)_n}{p^n}-i\infty}^{\frac{(\alpha\eta)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta,m}(q) \left(z - \frac{(\alpha\eta)_n}{p^n}\right)^k \equiv 0 \,(\bmod \mathfrak{L}).$$

Let M be the constant defined in (4.3). We choose $m > 0$ so that $p^m L > M$ and a character χ of which conductor is large enough to choose α and β as given in Proposition 4.2. Then we have

$$I(\alpha) := 2L^k \sum_{\eta \in W} \chi(\eta) \int_{\frac{(\alpha\eta)_n}{p^n}-i\infty}^{\frac{(\alpha\eta)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta,m}(q) z^k dz \equiv 0 \,(\bmod \mathfrak{L}), \tag{3.12}$$

and same formula for β. Now we consider the difference

$$I(\alpha) - I(\beta) \equiv 0 \,(\bmod \mathfrak{L}).$$

Let $W = \{\eta_1, \cdots, \eta_R\}$. By Lemma 4.2, for each $j = 2, \cdots, R$, $\frac{(\alpha\eta_j)_n}{p^n}$ and $\frac{(\beta\eta_j)_n}{p^n}$ are in a same interval $\left(\frac{s_j}{Lp^m}, \frac{s_j+1}{Lp^m}\right)$ for some $1 \leq s_j < Lp^m$, and we have

$$\int_{\frac{(\alpha\eta_j)_n}{p^n}-i\infty}^{\frac{(\alpha\eta_j)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta_j,m}(q)z^k dz = \int_{\frac{(\beta\eta_j)_n}{p^n}-i\infty}^{\frac{(\beta\eta_j)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta_j,m}(q)z^k dz.$$

On the other hand, we have $\frac{(\alpha\eta_1)_n}{p^n} < \frac{1}{Lp^m} < \frac{(\beta\eta_1)_n}{p^n}$ and

$$\int_{\frac{(\alpha\eta_1)_n}{p^n}-i\infty}^{\frac{(\alpha\eta_1)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta_1,m}(q)z^k dz - \int_{\frac{(\beta\eta_1)_n}{p^n}-i\infty}^{\frac{(\beta\eta_1)_n}{p^n}+i\infty} R_{\widehat{\lambda},\eta_1,m}(q)z^k dz$$
$$= \operatorname{Res}\left(\frac{1}{Lp^m}; R_{\widehat{\lambda},\eta_1,m}(q)z^k\right).$$

Since we have

$$\operatorname{Res}\left(\frac{1}{Lp^m}; R_{\widehat{\lambda},\eta,m}(q)z^k\right) = \frac{\widehat{\widehat{\lambda}}(\overline{p^m})\zeta_L^{-\overline{p^m}(-\eta)_m}\zeta_{Lp^m}^{(-\eta)_m}}{(Lp^m)^{k+1}}$$

and $\widehat{\widehat{\lambda}}(\overline{p^m}) = L\lambda(-\overline{p^m})$, we obtain the following absurd congruence

$$I(\alpha) - I(\beta) = \frac{\lambda(-\overline{p^m})\zeta_L^{-\overline{p^m}(-\eta_1)_m}\zeta_{Lp^m}^{(-\eta_1)_m}}{p^{m(k+1)}} \equiv 0 \ (\bmod\ \mathfrak{L}).$$

From this contradiction, we deduce the theorem. ☐

4. Equi-distribution of p-adic integers

We quote a proposition due to Ferrero and Washington. In [7], the reader can find a proof using compactness of the set $[0,1]^r$. We give another proof using Fourier expansion of a suitable elementary function.

Proposition 4.1 ([9], Proposition 1)**.** *Let $\gamma_1, \cdots, \gamma_r$ be $\mathbb{Q}$-linearly independent p-adic integers, $\delta > 0$, $m > 0$ an integer, $d > 0$ an integer prime to p, and $(y_1, \cdots, y_r) \in (0,1)^r$. For all sufficiently large n, there exists $\alpha \equiv 1 \ (\bmod\ p^m)$ so that*

$$\left|\frac{(\alpha\gamma_j)_n}{p^n} - y_j\right| \leq \delta \text{ and } (\alpha\gamma_j)_n \equiv 0 \ (\bmod\ d) \text{ for all } j \leq r.$$

Proof. Set $z_i = \frac{y_i}{d}$ and $\epsilon = \frac{1}{d}\min(\delta, 1-y_i, y_i)$. Let $\mathbf{x} = (x_i) \in [0,1)^r$. Define

$$f(\mathbf{x}) = \begin{cases} \prod_{i=1}^r \sin(2\pi\epsilon^{-1}(x_i - z_i)) & \text{if } |x_i - z_i| \leq \epsilon \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

We have the Fourier expansion

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^r} c_{\mathbf{n}} e^{2\pi i \mathbf{x}\cdot\mathbf{n}}.$$

Since we have the evaluation of the integration

$$\int_{\alpha-\epsilon}^{\alpha+\epsilon} \sin\left(2\pi\epsilon^{-1}(x-\alpha)\right) e^{2\pi n i x} dx = \frac{\epsilon(e^{2n\pi i(\alpha+\epsilon)} - e^{2n\pi i(\alpha-\epsilon)})}{2\pi(n^2\epsilon^2 - 1)},$$

we obtain that

$$c_{\mathbf{n}} = O\left(\frac{1}{|\mathbf{n}|^{2r}}\right).$$

For $\beta \in \mathbb{Z}_p^\times$, set $\mathbf{x}_n(\beta) = \left(\frac{(\beta\gamma_1)_n}{p^n}, \cdots, \frac{(\beta\gamma_r)_n}{p^n}\right)$ and $Z_n = \frac{d^{-1}+p^m\mathbb{Z}_p}{1+p^n\mathbb{Z}_p}$. In order to verify the proposition, first we show that

$$\frac{1}{p^{n-m}} \sum_{\beta \in Z_n} |f(\mathbf{x}_n(\beta))|^2 > 0 \text{ for } n \gg 1. \tag{4.1}$$

For $\mathbf{n} = (n_i) \in \mathbb{Z}^r$, we set $\sigma_{\mathbf{n}} = \sum_{i=1}^r n_i\gamma_i$. By the Fourier expansion of $f(\mathbf{x})$, we have

$$\frac{1}{p^{n-m}} \sum_{\beta \in Z_n} |f(\mathbf{x}_n(\beta))|^2 = \sum_{\mathbf{n}} |c_{\mathbf{n}}|^2 + \lim_{M\to\infty} \delta_{n,M},$$

where

$$\delta_{n,M} = \sum_{\substack{|\mathbf{n}|,|\mathbf{m}|<M,\\ \mathbf{n}\neq\mathbf{m}\\ \sigma_{\mathbf{n}}\equiv\sigma_{\mathbf{m}}(p^{n-m})}} e^{2\pi i \frac{d^{-1}(\sigma_{\mathbf{n}}-\sigma_{\mathbf{m}})}{p^n}} c_{\mathbf{n}}\bar{c}_{\mathbf{m}}.$$

Observe that $\lim_{n\to\infty} \delta_{n,M} = 0$ since $\{\gamma_i\}$ is linearly independent over $\mathbb{Q}$. Since the sum $\sum_{\mathbf{n}} c_{\mathbf{n}}$ converges absolutely, we have $\lim_{n\to\infty}\lim_{M\to\infty} \delta_{n,M} = 0$ and verify (4.1). In sum, there exists $\beta \in d^{-1} + p^m\mathbb{Z}_p$ such that

$$\left|\frac{(\beta\gamma_j)_n}{p^n} - z_j\right| \le \epsilon \text{ for each } j. \tag{4.2}$$

Now we set $\alpha = d\beta$. From the inequality (4.2), we have $0 < d(\beta\gamma_j) < p^n$ and, hence we conclude that

$$(\alpha\gamma_j)_n = d(\beta\gamma_j)_n \equiv 0 \,(\bmod\, d).$$

Clearly we also have

$$\left|\frac{(\alpha\gamma_j)_n}{p^n} - y_j\right| \le \delta \text{ for each } j.$$

This finish the proposition. □

Let $\{\eta_1, \cdots, \eta_R\} = W \subset \mathbb{Z}_p^\times$ and $U = \{\eta_1, \cdots, \eta_r\}$ be a maximal independent subset of W over $\mathbb{Q}$. Obviously $r = \phi(p-1)$ for Euler phi function ϕ and $\eta_{r+1}, \cdots, \eta_R$ are $\mathbb{Z}$-linear combinations of $\eta_1, \cdots, \eta_r$, say $\eta_j = \sum_{i=1}^r a_{ji}\eta_i$, $a_{ij} = a_{ij}(U) \in \mathbb{Z}$ for $j = r+1, \cdots, R$. Then we set

$$M = \max\left\{|a_{ji}(U)| \,\middle|\, \text{maximal } \mathbb{Q}\text{-linearly independent } U \subseteq W\right\}. \tag{4.3}$$

Now we state a lemma to control contours of the integrations in Section 3.

Lemma 4.2. *Let $N > M$ and $m > 0$ be an integer. For all sufficiently large integer n, there exists p-adic integers $\alpha, \beta \equiv 1 \,(\bmod\, p^m)$ so that*

$$\frac{(\alpha\eta_1)_n}{p^n} < \frac{1}{N} < \frac{(\beta\eta_1)_n}{p^n}$$

and

$$\frac{s_j}{N} < \frac{(\alpha\eta_j)_n}{p^n}, \frac{(\beta\eta_j)_n}{p^n} < \frac{s_j+1}{N}$$

for $j = 2\cdots R$ and $1 \le s_j < N$. Furthermore we also have

$$(\alpha\eta_j)_n, (\beta\eta_j)_n \equiv 0 \,(\bmod\, p) \textit{ for all } j = 1, \cdots, R.$$

Proof. We form an $r \times (R-r)$ matrix $A = (a_{ji})$. By changing the sign of η_j, we may assume that the first non-zero entry of the row $(a_{j1}, a_{j2}, \cdots, a_{jr})$ is positive for all $j = r+1, \cdots, R$. We form a linear map

$$P(x_1, \cdots, x_r) = (x_1, \cdots, x_R) = (x_1, \cdots, x_r)\,(I|A)$$

for $x_1, \cdots, x_r \in \mathbb{R}$, and for $r \times r$ identity matrix I and an $r \times R$ block matrix $(I|A)$ formed by I and A. Now we want to show that there exists $x_1, \cdots, x_r$ and $y_1, \cdots, y_r$ such that $P(x_1, \cdots, x_r), P(y_1, \cdots, y_r) \in (0,1)^R$, and $0 < x_1 < \frac{1}{N} < y_1 < 1$ and x_i and y_i are in the same interval $\left(\frac{s_i}{N}, \frac{s_i+1}{N}\right)$ for $i = 2, \cdots, R$ and for some $1 \le s_i < N$. In fact, since $N > a_{j1} > 0$ for each $j = r+1, \cdots, R$, we can choose small enough real numbers $1 \gg z_2 \gg z_3 \cdots \gg z_r > 0$ so that

$$P\left(\frac{1}{N}, z_2, \cdots, z_r\right) \in (0,1)^R.$$

We choose x_1, y_1 close enough to $\frac{1}{N}$, and choose the points x_j, y_j close enough to z_j for each $j = 2, \cdots, R$ so that x_j, y_j, and z_j are included in a same interval, say $\left(\frac{s_i}{N}, \frac{s_i+1}{N}\right)$.

By Proposition 4.1, we are able to choose two p-adic integers α and β so that $\alpha, \beta \equiv 1 \,(\bmod\, p^m)$ and $\frac{(\alpha\eta_j)_n}{p^n}$, $\frac{(\beta\eta_j)_n}{p^n}$ are close enough to x_j, y_j for $j = 1, \cdots, r$ respectively. By the choice of x_j and y_j, we obtain that

$$P\left(\frac{(\alpha\eta_1)_n}{p^n}, \cdots, \frac{(\alpha\eta_r)_n}{p^n}\right) \in (0,1)^R.$$

Furthermore, we have $\frac{(\alpha\eta_1)_n}{p^n} < \frac{1}{N}$ and

$$0 < \sum_{i=1}^{r} a_{ji}(\alpha\eta_j)_n < p^n \text{ for } j = r+1, \cdots, R.$$

Since $\alpha\eta_j \equiv \sum_{i=1}^{r} a_{ji}(\alpha\eta_j)_n \,(\bmod\, p^n)$, we conclude that

$$(\alpha\eta_j)_n = \sum_{i=1}^{r} a_{ji}(\alpha\eta_j)_n \text{ and } (\alpha\eta_j)_n \equiv 0 \,(\bmod\, p).$$

We do the same for y_j and conclude the proposition. □

References

[1] Ferrero, B. and Washington, L.: The Iwasawa invariant μ_p vanishes for abelian number fields, Ann. of Math. (2) 109 (1979), no. 2, 377–395.

[2] Iwasawa, K.: Lectures on p-adic L-functions, Ann. of Math. Stud. 74. Princeton Univ. Press 1972.

[3] Kubert, D. and Lang, S.: Distributions on toroidal groups, Math. Z. 148, 33–51 (1976).

[4] Kubert, D. and Lang, S.: The index of Stickelberger ideals of oerder 2 and cuspidal class numbers, Math. Ann. 237, 213–232(1978).

[5] Kubert, D. and Lang, S.: Cartan-Bernoulli numbers as values of L-Series, Math. Ann. 240. 21–26 (1979).

[6] Lang, S.: Introduction to modular forms, Grundlehren der Mathematischen Wissenschaften, 222. Springer-Verlag, Berlin

[7] Oesterlé, J.: Travaux de Ferrero et Washington sur le Nombre de Classes d'Idéaux des Corps Cyclotomiques, Séminaire Bourbaki, 31e année, 1978/79, n. 535.

[8] Sun, H.-S.: Homological interpretation of a Theorem of Washington, J. Number Theory 127 (2007), no. 1, 47–63.

[9] Washington, L.: The non-p–part of the class number in a cyclotomic $\mathbb{Z}_p$–extension, Invent. Math. 49 (1978), 87–97

[10] Washington, L.: Introduction to cyclotomic fields, Second edition. Graduate Texts in Mathematics, 83. Springer-Verlag, New York, 1997
[11] Yang, Y.: Modular units and cuspidal divisor class groups of $X_1(N)$, arXiv:0803.3732
[12] Yu, J.: A cuspidal class number formula for the modular curves $X_1(N)$, Math. Ann. 252, 197–216 (1980)

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