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ON THE DERIVED ALGEBRA OF A CENTRALISER

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ABSTRACT. Let $\mathfrak{g}$ be a classical Lie algebra, $e \in \mathfrak{g}$ a nilpotent element and $\mathfrak{g}_e \subset \mathfrak{g}$ the centraliser of e. We prove that $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ if and only if e is rigid. It is also shown that if $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$, then the nilpotent radical of $\mathfrak{g}_e$ coincides with $[\mathfrak{g}(1)_e, \mathfrak{g}_e]$, where $\mathfrak{g}(1)_e \subset \mathfrak{g}_e$ is an eigenspace of a characteristic of e with the eigenvalue 1.

INTRODUCTION

Let $\mathfrak{g}$ be a semisimple (or reductive) Lie algebra over an algebraically closed field $\mathbb{F}$ ($\mathrm{char}\,\mathbb{F} = 0$) and $x \in \mathfrak{g}$. The main objects of our study here are the centraliser $\mathfrak{g}_x$ and its derived algebra $[\mathfrak{g}_x, \mathfrak{g}_x]$. There are two natural questions: for what elements x we have $x \in [\mathfrak{g}_x, \mathfrak{g}_x]$ and a stronger one, when $\mathfrak{g}_x = [\mathfrak{g}_x, \mathfrak{g}_x]$. Evidently only a nilpotent element x can satisfies any of these conditions.

Let $e \in \mathfrak{g}$ be a nilpotent element. By the Jacobson-Morozov theorem it can be included into an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\mathfrak{g}$. Set $\mathfrak{g}(\lambda) := \{\xi \in \mathfrak{g} \mid \mathrm{ad}\,(h){\cdot}\xi = \lambda\xi\}$ and $\mathfrak{g}(\lambda)_e = \mathfrak{g}(\lambda) \cap \mathfrak{g}_e$. Nilpotent elements e such that $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$ were studied in [4], where they are called *compact*, and in [7], where they are called *reachable*. Extending results of [7], we show that if e is reachable and $\mathfrak{g}$ is a classical Lie algebra, then $\mathfrak{g}(\lambda{+}1)_e = [\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e]$ for all λ, see Theorem 10.

The irreducible components of the algebraic varieties $\mathfrak{g}^{(m)} = \{\xi \in \mathfrak{g} \mid \dim \mathfrak{g}_\xi = m\}$ are called the *sheets* of $\mathfrak{g}$. Their description was obtained by Borho and Kraft in [1], [2] in terms of the so-called parabolic induction. One of the basic results is that each sheet contains a unique nilpotent orbit. If a nilpotent orbit coincides with a sheet, it is said to be *rigid*. A nilpotent element is said to be *rigid*, if its orbit is rigid. If $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$, then e is rigid by an almost trivial reason (see Proposition 11). We prove that in the classical Lie algebras the converse is true. This answers a question put to me by A. Premet at the Ascona conference in August (2009). In the exceptional Lie algebras there are rigid elements such that $\mathfrak{g}_e \neq [\mathfrak{g}_e, \mathfrak{g}_e]$, see Remark 3. Interest in $[\mathfrak{g}_e, \mathfrak{g}_e]$ is motivated by a connection with finite W-algebras $\mathbf{U}(\mathfrak{g}, e)$ and their 1-dimensional representations. Recall that $\mathbf{U}(\mathfrak{g}, e)$ is a deformations of the universal enveloping algebras $\mathbf{U}(\mathfrak{g}_e)$ and the commutators $[\xi, \eta]$ of $\xi, \eta \in \mathfrak{g}_e$ naturally appear in the commutator relation of $\mathbf{U}(\mathfrak{g}, e)$, see e.g. [8, Section 3.4].

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As was explained to me by A. Premet, the equality $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ implies that $\mathbf{U}(\mathfrak{g}, e)$ has at most one non-trivial 1-dimensional representation.

1. Basis of a centraliser

In this section we fix a basis of a centraliser, which is used throughout the paper, and state a few easy useful facts. Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let e be a nilpotent element of $\hat{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$. Let k be the number of Jordan blocks of e and $W \subseteq \mathbb{V}$ a (k-dimensional) complement of $\operatorname{Im} e$ in $\mathbb{V}$. Let d_i denote the size of the ith Jordan block of e. We always assume that the Jordan blocks are ordered such that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$ so that e is represented by the partition $(d_1, \ldots, d_k)$ of $\dim \mathbb{V}$. Choose a basis $w_1, w_2, \ldots, w_k$ in W such that the vectors $e^j{\cdot}w_i$ with $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant d_i{-}1$ form a basis of $\mathbb{V}$, and put $\mathbb{V}[i] := \operatorname{span}\{e^j{\cdot}w_i \mid j \geqslant 0\}$. Note that $e^{d_i}{\cdot}w_i = 0$ for all i.

If $\xi \in \hat{\mathfrak{g}}_e$, then $\xi(e^j{\cdot}w_i) = e^j{\cdot}\xi(w_i)$, hence ξ is completely determined by its values on W. The only restriction on $\xi(w_i)$ is that $e^{d_i}{\cdot}\xi(w_i) = \xi(e^{d_i}{\cdot}w_i) = 0$. Since vectors $e^s{\cdot}w_i$ form a basis of $\mathbb{V}$, the centraliser $\hat{\mathfrak{g}}_e$ has a basis $\{\xi_i^{j,s}\}$ such that

$$\begin{cases} \xi_i^{j,s}(w_i) = e^s{\cdot}w_j, \\ \xi_i^{j,s}(w_t) = 0 \ \text{ for } t \neq i, \end{cases} \qquad 1 \leqslant i, j \leqslant k, \ \text{ and } \ \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j{-}1 \, .$$

It is convenient to assume that $\xi_i^{j,s} = 0$ whenever s does not satisfy the above restrictions. An example of $\xi_i^{j,1}$ with $i > j$ and $d_j = d_i{+}1$ is shown in Figure 1.

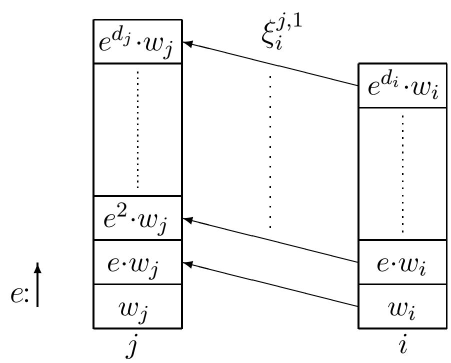

FIGURE 1.

The composition rule shows that the basis elements $\xi_i^{j,s}$ satisfy the following commutator relation:

$$[\xi_i^{j,s}, \xi_p^{q,t}] = \delta_{q,i}\xi_p^{j,t+s} - \delta_{j,p}\xi_i^{q,s+t}, \tag{1}$$

where $\delta_{i,j} = 1$ if $i = j$ and is zero otherwise.

An $\mathfrak{sl}_2$-triple $\{e, h, f\}$ can be chosen in such a way that $h{\cdot}w_i = (1 - d_i)w_i$. Then

$$[h, \xi_i^{j,s}] = (d_i - d_j) + 2s. \tag{2}$$

Using this equality, it is not difficult to describe h-eigenspaces $\mathfrak{g}(\lambda)$ in terms of $\xi_i^{j,s}$. For example, $\mathfrak{g}(1)_e$ is generated by $\xi_i^{j,0}$ with $d_j = d_i - 1$ and $\xi_i^{j,1}$ with $d_j = d_i + 1$.

Let $(\ ,\ )_{\mathbb{V}}$ be a non-degenerate symmetric or skew-symmetric bilinear form on $\mathbb{V}$, i.e., $(v, w)_{\mathbb{V}} = \varepsilon(w, v)_{\mathbb{V}}$, where $v, w \in \mathbb{V}$ and $\varepsilon = +1$ or -1. Let $\sigma : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ be a linear mapping such that $(x{\cdot}v, w)_{\mathbb{V}} = -(v, \sigma(x){\cdot}w)_{\mathbb{V}}$ for all $v, w \in \mathbb{V}$ and $x \in \hat{\mathfrak{g}}$. Then σ in an involutive automorphism of $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_0 \oplus \mathfrak{m}$ be the symmetric decomposition of $\hat{\mathfrak{g}}$ corresponding to the σ-eigenvalues ± 1. The elements $x \in \mathfrak{m}$ have the property that $(x{\cdot}v, w)_{\mathbb{V}} = (v, x{\cdot}w)_{\mathbb{V}}$ for all $v, w \in \mathbb{V}$.

Set $\mathfrak{g} := \hat{\mathfrak{g}}_0$ and let e be a nilpotent element of $\mathfrak{g}$. Since $\sigma(e) = e$, the centraliser $\hat{\mathfrak{g}}_e$ of e in $\hat{\mathfrak{g}}$ is σ-stable and $(\hat{\mathfrak{g}}_e)_0 = \hat{\mathfrak{g}}_e^{\sigma} = \mathfrak{g}_e$. This yields the $\mathfrak{g}_e$-invariant symmetric decomposition $\hat{\mathfrak{g}}_e = \mathfrak{g}_e \oplus \mathfrak{m}_e$.

Lemma 1. *In the above setting, suppose that $e \in \hat{\mathfrak{g}}_0$ is a nilpotent element. Then the cyclic vectors $\{w_i\}$ and thereby the spaces $\{\mathbb{V}[i]\}$ can be chosen such that there is an involution $i \mapsto i'$ on the set $\{1, \ldots, k\}$ satisfying the following conditions:*

- $d_i = d_{i'}$;
- $(\mathbb{V}[i], \mathbb{V}[j])_{\mathbb{V}} = 0$ *if* $i \neq j'$;
- $i = i'$ *if and only if* $(-1)^{d_i}\varepsilon = -1$.

Proof. This is a standard property of the nilpotent orbits in $\mathfrak{sp}(\mathbb{V})$ and $\mathfrak{so}(\mathbb{V})$, see, for example, [3, Sect. 5.1] or [5, Sect. 1]. □

1.1. **Basis in the orthogonal and symplectic cases.** Let $\{w_i\}$ be a set of cyclic vectors chosen according to Lemma 1. Consider the restriction of the $\mathfrak{g}$-invariant form $(\ ,\ )_{\mathbb{V}}$ to $\mathbb{V}[i] + \mathbb{V}[i']$. Since $(w, e^s{\cdot}v)_{\mathbb{V}} = (-1)^s(e^s{\cdot}w, v)_{\mathbb{V}}$, a vector $e^{d_i-1}{\cdot}w_i$ is orthogonal to all vectors $e^s{\cdot}w_{i'}$ with $s > 0$. Therefore $(w_{i'}, e^{d_i-1}{\cdot}w_i)_{\mathbb{V}} = (-1)^{d_i-1}(e^{d_i-1}{\cdot}w_{i'}, w_i)_{\mathbb{V}} \neq 0$. There is a (unique up to a scalar) vector $v \in \mathbb{V}[i]$ such that $(v, e^s{\cdot}w_{i'})_{\mathbb{V}} = 0$ for all $s < d_i - 1$. It is not contained in $\operatorname{Im} e$, otherwise it would be orthogonal to $e^{d_i-1}{\cdot}w_{i'}$ too and hence to $\mathbb{V}[i']$. Therefore there is no harm in replacing w_i by v. Let us always choose the cyclic vectors w_i in such a way that $(w_i, e^s{\cdot}w_{i'})_{\mathbb{V}} = 0$ for $s < d_i - 1$ and normalise them according to:

$$(w_i, e^{d_i-1}{\cdot}w_{i'})_{\mathbb{V}} = \pm 1 \ \text{ and } \ (w_i, e^{d_i-1}{\cdot}w_{i'})_{\mathbb{V}} > 0 \ \text{ if } \ i \leqslant i'. \tag{3}$$

Then $\mathfrak{g}_e$ is generated (as a vector space) by the vectors $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j'}^{i',d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i, j and s in the following way

$$(e^{d_j-s}{\cdot}w_j, e^{s-1}{\cdot}w_{j'})_{\mathbb{V}} = -\varepsilon(i,j,s)(w_i, e^{d_i-1}{\cdot}w_{i'})_{\mathbb{V}}.$$

Elements $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j'}^{i',d_i-s}$ form a basis of $\mathfrak{m}_e$. In the following we always normalise w_i as above and enumerate the Jordan blocks such that $i' \in \{i, i+1, i-1\}$ keeping inequalities $d_i \geqslant d_j$ for $i < j$. In this basis $\{e^s{\cdot}w_i\}$ the matrix of the restriction of the Killing form to $\mathbb{V}[i] + \mathbb{V}[i']$ is anti-diagonal with entries ± 1.

2. Reachable nilpotent elements

Let $\mathfrak{g}$ be a reductive Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. We include it into an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ and let $\mathfrak{g}(\lambda)$ stand for the $\mathrm{ad}\,(h)$-eigenspace with eigenvalue λ. Set $\mathfrak{g}(\lambda)_x := \mathfrak{g}_x \cap \mathfrak{g}(\lambda)$. We fix a non-degenerate invariant bilinear form κ on $\mathfrak{g}$.

In [7] an element $x \in \mathfrak{g}$ is called reachable if $x \in [\mathfrak{g}_x, \mathfrak{g}_x]$. Clearly each reachable element is nilpotent. Since $\mathfrak{g}(0)_e$ is a reductive subalgebra, the representation of $\mathfrak{g}(0)_e$ on $\mathfrak{g}(1)_e$ is completely reducible. Therefore $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$ if and only if $e \in [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$. Following [7], we continue to study the derived algebra of a centraliser $\mathfrak{g}_e$ for reachable e.

Lemma 2. *Let $\hat{f}(\xi, \eta) = \kappa(f, [\xi, \eta])$ be a skew-symmetric form on $\mathfrak{g}(1)$. Then $\hat{f}$ is non-degenerate on $\mathfrak{g}(1)_e$.*

Proof. If $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$ are two irreducible representations of any subalgebra $\mathfrak{sl}_2 \subset \mathfrak{g}$ and $\dim \mathfrak{a}_1 \neq \dim \mathfrak{a}_2$, then necessarily $\kappa(\mathfrak{a}_1, \mathfrak{a}_2) = 0$. Applying this to the $\mathfrak{sl}_2$-triple $\{e, h, f\}$ we get that κ defines a non-degenerate pairing between $\mathfrak{g}(-1)_f$ and $\mathfrak{g}(1)_e$. It remains to notice that $\kappa(f, [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]) = \kappa([f, \mathfrak{g}(1)_e], \mathfrak{g}(1)_e) = \kappa(\mathfrak{g}(-1)_f, \mathfrak{g}(1)_e)$. □

Remark 1. In the following we need only the fact that $\hat{f}$ is non-zero on $\mathfrak{g}(1)_e$. But a proof of the weaker statement is not any easier.

Suppose that $e \in \hat{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$ is given by a partition $((d+1)^m, d^n)$ with both m and n being non-zero. Then $\hat{\mathfrak{g}}(0)_e = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ by [5, Sect. 3]. One can easily compute that

$$\hat{\mathfrak{g}}(1)_e \cong \mathbb{F}^m \otimes (\mathbb{F}^n)^* \oplus (\mathbb{F}^m)^* \otimes \mathbb{F}^n \quad \text{and} \quad \hat{\mathfrak{g}}(2)_e \cong \mathfrak{gl}_m \oplus \mathfrak{gl}_n$$

as $\hat{\mathfrak{g}}(0)_e$-modules. Let $e = e_{d+1} + e_d$ be a decomposition of e according to the size of Jordan blocks, i.e., e_d is given by the rectangular partition (d^n) and e_{d+1} by $((d+1)^m)$. Here $e_d{\cdot}w_i = 0$, if w_i generates a Jordan block of size $d+1$, and $e_{d+1}{\cdot}w_j = 0$, if w_j generates a Jordan block of size d. As a representation of $\mathfrak{g}(0)_e$ the subspace $\hat{\mathfrak{g}}(2)_e$ decomposes as $\mathfrak{sl}_m \oplus \mathfrak{sl}_l \oplus \mathbb{F}e_d \oplus \mathbb{F}e_{d+1}$, where $[\hat{\mathfrak{g}}(0)_e, e_d] = [\hat{\mathfrak{g}}(0)_e, e_{d+1}] = 0$.

Lemma 3. *Keep the above assumptions and notation. Then $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e] = \mathfrak{sl}_m \oplus \mathfrak{sl}_n \oplus \mathbb{F}(me_1 - ne_2)$ as a $\hat{\mathfrak{g}}(0)_e$-module.*

Proof. First, we show that the "$\mathfrak{sl}$-parts" of $\hat{\mathfrak{g}}(2)_e$ are contained in $[\hat{\mathfrak{g}}_e, \hat{\mathfrak{g}}_e]$. Suppose that $m > 1$, otherwise $\mathfrak{sl}_m$ is zero. According to (2), $\xi_{m+1}^{2,1}, \xi_1^{m+1,0} \in \hat{\mathfrak{g}}(1)_e$. Hence $[\xi_{m+1}^{2,1}, \xi_1^{m+1,0}] =$

$\xi_1^{2,1} \in [\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$. By the "zero-trace" reason, $\xi_1^{2,1} \in \mathfrak{sl}_m$ for the irreducible $\hat{\mathfrak{g}}(0)_e$-subrepresentation $\mathfrak{sl}_m \subset \hat{\mathfrak{g}}(2)_e$. Since $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ is $\hat{\mathfrak{g}}(0)_e$-invariant, the whole subspace $\mathfrak{sl}_m$ is contained in it. In order to prove the inclusion for the "$\mathfrak{sl}_n$-part" (in case $n > 1$), we take $[\xi_1^{m+1,0}, \xi_{m+2}^{1,1}] = \xi_{m+2}^{m+1,1}$.

By Lemma 2, the skew-symmetric form $\hat{f}$ is non-degenerate on $\hat{\mathfrak{g}}(1)_e$. The subspace $\mathfrak{sl}_m \oplus \mathfrak{sl}_n \subset \hat{\mathfrak{g}}(2)_e$, being a non-trivial $\hat{\mathfrak{g}}(0)_e$-module, is orthogonal to f (with respect to κ), hence $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ contains at least one non-zero vector of the form $ae_1 + be_2$. Recall that $\hat{\mathfrak{g}}(1)_e = \mathbb{F}^m \otimes (\mathbb{F}^n)^* \oplus (\mathbb{F}^m)^* \otimes \mathbb{F}^n$ as a representation of $\hat{\mathfrak{g}}(0)_e$. Since $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e] \subset \Lambda^2(\hat{\mathfrak{g}}(1)_e)$ and $\dim \Lambda^2(\hat{\mathfrak{g}}(1)_e)^{\hat{\mathfrak{g}}(0)_e} \leqslant 1$, the subspace of $\hat{\mathfrak{g}}(0)_e$-invariant vectors in $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ is at most one dimensional. In other words, the subspace $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ contains at most one element commuting with $\hat{\mathfrak{g}}(0)_e$. Taking $y = [\xi_{m+1}^{1,1}, \xi_1^{m+1,0}]$ we get an element $y = \xi_1^{1,1} - \xi_{m+1}^{m+1,1}$ in $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ and $(mn)y - (me_d - ne_{d+1}) \in \mathfrak{sl}_m \oplus \mathfrak{sl}_n$. □

Corollary 4. *If $d = 1$, then $e_d = 0$ and $\hat{\mathfrak{g}}(2)_e = [\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$. Otherwise $[\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e]$ is of codimension 1 in $\hat{\mathfrak{g}}(2)_e$.*

In the following we are going to deal with the orthogonal and symplectic Lie algebras and freely use results and assumptions of subsection 1.1.

Lemma 5. *Suppose $\mathfrak{g}$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$ and $e \in \mathfrak{g}$ is given by a rectangular partition d^k.*

- *If $(-1)^d \varepsilon = 1$, then $\mathfrak{g}(0)_e = \mathfrak{sp}_k$ and $\mathfrak{g}(2m) = S^2\mathbb{F}^k$ for even $m < d$, $\mathfrak{g}(2m) = \Lambda^2\mathbb{F}^k$ for odd $m < d$.*
- *If $(-1)^d \varepsilon = -1$, then $\mathfrak{g}(0)_e = \mathfrak{so}_k$ and $\mathfrak{g}(2m) = \Lambda^2\mathbb{F}^k$ for even $m < d$, $\mathfrak{g}(2m) = S^2\mathbb{F}^k$ for odd $m < d$.*

Proof. The assertions concerning $\mathfrak{g}(0)_e$ follow, for example, from [5, Sect. 3, Prop. 2]. Recall that $W \subset \mathbb{V}$ is a k-dimensional complement of $\operatorname{Im} e$. and each $\xi \in \mathfrak{g}_e$ is completely determined by its values on W. Therefore $\mathfrak{g}(2m)_e$ can be identified with the set of $\xi \in \operatorname{Hom}(W, e^m{\cdot}W)$ such that $(\xi(w), e^{d-m-1}w')_{\mathbb{V}} = -(w, e^{d-m-1}{\cdot}\xi(w'))_{\mathbb{V}}$. Identifying W and $e^m{\cdot}W$ by means of e^m, we see that $(\xi(w), w') = -(w, \xi(w')$ for a non-degenerate symmetric or skew-symmetric bilinear form $(w, w') := (w, e^{d-1}{\cdot}w')_{\mathbb{V}}$ on W, and that is the only condition on ξ. In case the form is symmetric, we get $\mathfrak{g}(2m)_e \cong \Lambda^2\mathbb{F}^k$, and if it is skew-symmetric, then $\mathfrak{g}(2m)_e \cong S^2\mathbb{F}^k$. □

Lemma 6. *Let $\mathfrak{g}$ be either $\mathfrak{so}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$ and $e \in \mathfrak{g}$ a nilpotent element defined by a partition $((d+1)^m, d^n)$. Let e_d, e_{d+1} be as in Lemma 3. Then $me_d - ne_{d+1} \in [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$.*

Proof. By Lemma 5, the subspace $\mathfrak{g}(2)_e$ decomposes into a direct sum of irreducible $\mathfrak{g}(0)_e$-representations as follows $\mathfrak{g}(2)_e = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathbb{F}e_d \oplus \mathbb{F}e_{d+1}$, where e_d, e_{d+1} are the same central vectors as in the $\mathfrak{gl}(\mathbb{V})$ case and $\mathfrak{a}_1 = \mathfrak{g} \cap \mathfrak{sl}_n$, $\mathfrak{a}_2 = \mathfrak{g} \cap \mathfrak{sl}_m$. The exact description of subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ depends on $\mathfrak{g}$ and the parity of d, see Lemma 5. In any case they both contain no non-zero $\mathfrak{g}(0)_e$-invariant vectors. Using Lemma 3, we get an inclusion $[\mathfrak{g}(1)_e, \mathfrak{g}(1)_e] \subset \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathbb{F}(me_d - ne_{d+1})$. The first two subspaces, $\mathfrak{a}_1, \mathfrak{a}_2$, are orthogonal to f, but the whole commutator is not, due to Lemma 2. Hence $me_d - ne_{d+1} \in [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$. $\square$

It is natural to suggest that the following conditions are equivalent:

(i) e is reachable and $\mathfrak{g}(0)_e$ is semisimple;

(ii) $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$.

Clearly, condition (i) is necessary for (ii). Below we prove that in the classical Lie algebras it is also sufficient.

Given $x \in \hat{\mathfrak{g}}$ and a non-negative integer q let $x^q \in \hat{\mathfrak{g}}$ be the qth power of x (as a matrix). Note that if $\mathfrak{g}$ is either $\mathfrak{so}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$, the number q is odd, and $x \in \mathfrak{g}$, then also $x^q \in \mathfrak{g}$. This remains true for a product $x_1 \dots x_q$ of q elements $x_i \in \mathfrak{g}$.

Theorem 7. *Suppose that $\mathfrak{g}$ is a simple classical Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Then conditions* (i) *and* (ii) *are equivalent.*

Proof. Actually, we need to show only that (i) implies (ii).

Suppose first that $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$. Then $\mathfrak{g}(0)_e$ is semisimple exactly in two cases: $e = 0$ and e is regular. In the first of them there is nothing to prove. In the second case, where $\mathfrak{g}_e$ is commutative, e is not reachable.

Suppose $\mathfrak{g}$ is either $\mathfrak{so}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$ and $e \in \mathfrak{g}$ satisfies (i). According to the description of reachable nilpotent elements [7, Theorem 2.1.(4)], e has Jordan blocks of sizes $(d, d-1, \dots, 1)$ with positive multiplicities $(r_d, \dots, r_1)$. Let $e = e_d + e_{d-1} + \dots + e_1$ be a decomposition of e according to the sizes of Jordan blocks. We assume that $e_i{\cdot}\mathbb{V}[t] = 0$ if $\dim \mathbb{V}[t] \neq i$. Then $e_1 = 0$ and $e_i e_j = 0$ for $i \neq j$. Since $\mathfrak{g}(0)_e$ is semisimple, it is contained in $[\mathfrak{g}_e, \mathfrak{g}_e]$. Using Lemma 5, it is not difficult to see that the nilpotent radical of $\mathfrak{g}_e$ contains $\mathfrak{g}(0)_e$-subrepresentations of the form $\mathbb{F}^{r_i} \otimes \mathbb{F}^{r_j}$, $\Lambda^2\mathbb{F}^r$, and $S^2\mathbb{F}^r$ ($r = r_i, r_j$) of algebras $\mathfrak{sp}_{r_i}$, $\mathfrak{so}_{r_j}$. Each non-trivial representation appears also in $[\mathfrak{g}(0)_e, \mathfrak{g}_e]$. Trivial representations, or elements commuting with $\mathfrak{g}(0)_e$, are either $\mathfrak{so}_r$-invariant vectors in $S^2\mathbb{F}^r$ (correspondingly, $\mathfrak{sp}_r$-invariant vectors in $\Lambda^2\mathbb{F}^r$) or $\mathbb{F}\otimes\mathbb{F}$. Vectors of the first type are e_i^{2s+1}, vectors of the second type come from pairs $r_i = r_j = 1$ with $i \neq j$ as $\xi_p^{q,s} \pm \xi_q^{p,s'}$, where pth Jordan block is the unique block of size i and qth Jordan block is the unique block of size j.

Recall that $e_1 = 0$. Lemma 6 implies that $e_2 \in [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$. Applying the same lemma to the partition $(3^{r_3}, 2^{r_2})$, we obtain that $r_2 e_3 - r_3 e_2$ is contained in $[\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$. Hence $e_3 \in [\mathfrak{g}_e, \mathfrak{g}_e]$. Continuing through all sizes $3, 4, \dots, d$ of Jordan blocks we prove that all e_i are elements of the derived algebra $[\mathfrak{g}_e, \mathfrak{g}_e]$.

Consider the matrix product $e^{2s}e_i \in \mathfrak{g}$. Since $e = e_d + e_{d-1} + \ldots + e_1$ and $e_ie_j = 0$ for $i \neq j$, we obtain that $e^{2s}e_i = e_i^{2s+1}$. Because e is a central element in $\mathfrak{g}_e$, all its powers commute with $\mathfrak{g}_e$. Thus $e^{2s}[\xi, \eta] = [e^{2s}\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}_e$. Moreover $e^{2s}\xi \in \mathfrak{g}_e$. Since $e_i \in [\mathfrak{g}_e, \mathfrak{g}_e]$, we have

$$e_i^{2s+1} = e^{2s}e_i \in e^{2s}[\mathfrak{g}_e, \mathfrak{g}_e] = [e^{2s}\mathfrak{g}_e, \mathfrak{g}_e] \subset [\mathfrak{g}_e, \mathfrak{g}_e]$$

for all s.

It remains to deal with pairs (i, j), where $i \neq j$ and $r_i = r_j = 1$. Let $t \mapsto t'$ be the same involution on the set of Jordan blocks as in Lemma 1. Suppose that the Jordan block of size i has number p and the Jordan block of size j has number q. Then $p' = p$ and $q' = q$. In particular, i and j have the same parity. Assume that $i > j$. Then trivial $\mathfrak{g}(0)_e$-representations associated with the pair (i, j) are generated by the vectors

$$x(s) := \xi_p^{q,s} + (-1)^{s+1}\xi_q^{p,i-j+s} \quad \text{with} \quad 0 \leqslant s \leqslant j - 1.$$

Note that $x(s + 1) = [e_i, x(s)]$. Thus we only need to show that $x(0)$ is contained in the derived subalgebra.

The Jordan block number $p + 1$ has size $i - 1$. Without any doubt, $i - 1$ has different from i parity. Hence $(p + 1)' = p + 2$ and $p + 2 < q$. Take two elements $y, z \in \mathfrak{g}_e$:

$$y = \xi_p^{p+1,0} - \xi_{p+2}^{p,1}, \quad z = \xi_{p+1}^{q,0} - \xi_q^{p+2,i-1-j}$$

and compute their commutator according to (1):

$$[z, y] = \xi_{p+1}^{q,0}\xi_p^{p+1,0} - z\xi_{p+2}^{p,1} - y\xi_{p+1}^{q,0} - \xi_{p+2}^{p,1}\xi_q^{p+2,i-1-j} = \xi_p^{q,0} - \xi_q^{p,i-j} = x(0).$$

This completes the proof. □

In [7] a question was raised whether the properties $e \in [\mathfrak{g}_e, \mathfrak{g}_e]$ and $\mathfrak{g}(\lambda + 1) = [\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e]$ are equivalent. The positive answer was given for $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$ and $\lambda \geqslant 1$, see [7, Theorem 4.5]. Here we prove the equivalence for $\mathfrak{sp}(\mathbb{V})$ and $\mathfrak{so}(\mathbb{V})$. Of course, if $\mathfrak{g}(2)_e = [\mathfrak{g}(1)_e, \mathfrak{g}(1)_e]$, then e is reachable.

Lemma 8. *For any reductive Lie algebra $\mathfrak{g}$ and any nilpotent element $e \in \mathfrak{g}$ we have $\mathfrak{g}(1)_e = [\mathfrak{g}(0)_e, \mathfrak{g}(1)_e]$.*

Proof. Let $\mathfrak{t} \subset \mathfrak{g}(0)_e$ be a maximal torus. Then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a reductive subalgebra and $e \in \mathfrak{h}$. Moreover $\mathfrak{h}_e = \mathfrak{h} \cap \mathfrak{g}_e$ contains no semisimple elements. In other words, e is a *distinguished* nilpotent element in $\mathfrak{h}$. Therefore $e \in \mathfrak{h}$ is even cf. [3, Theorem 8.2.3] and $\mathfrak{h}(1)_e = 0$. It follows that $\mathfrak{g}(1)_e$ contains no non-zero $\mathfrak{g}(0)_e$-invariant vectors and $\mathfrak{g}(1)_e = [\mathfrak{g}(0)_e, \mathfrak{g}(1)_e]$. □

Lemma 9. *Suppose $\mathfrak{g}$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$ and $e \in \mathfrak{g}$ is reachable. Then $[\mathfrak{g}(1)_e, \mathfrak{g}(1)_e] = \mathfrak{g}(2)_e$.*

Proof. Since e is reachable, it has Jordan blocks of sizes $(d, d-1, \ldots, 1)$ (with positive multiplicities). Recall that $\mathfrak{g}$ is a symmetric subalgebra of $\hat{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$, see Section 1 for more details. In other words, $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{m}$ is a $\mathbb{Z}_2$-grading and $\hat{\mathfrak{g}}_e = \mathfrak{g}_e \oplus \mathfrak{m}_e$. Suppose $\xi_i^{j,s}, \xi_a^{b,t}$ are non-commuting elements of $\hat{\mathfrak{g}}(1)_e$. Commutator relation (1) implies that either $j = a$ or $i = b$. Without loss of generality we may (and will) assume that $i = b$. Let d_i be the size of the ith Jordan block and d_j the size of the jth Jordan block. By (2), d_i and d_j have different parity and the same holds for Jordan blocks with numbers i and a. Take an element $x \in \mathfrak{g}(1)_e$ such that $x = \xi_i^{j,s} + \varepsilon(i,j,s)\xi_{j'}^{i',s'}$. Because of the parity conditions $j' \neq i$ and $i' \neq a$. Therefore $[x, \xi_a^{i,t}] = [\xi_i^{j,s}, \xi_a^{i,t}]$. It follows that

$$[\mathfrak{g}(1)_e, \hat{\mathfrak{g}}(1)_e] = [\hat{\mathfrak{g}}(1)_e, \hat{\mathfrak{g}}(1)_e] = \hat{\mathfrak{g}}(2)_2 = \mathfrak{g}(2)_e \oplus \mathfrak{m}(2)_e.$$

To conclude the proof note that $\hat{\mathfrak{g}}(1)_e = \mathfrak{g}(1)_e \oplus \mathfrak{m}(1)_e$ and $[\mathfrak{m}(1)_e, \mathfrak{g}(1)_e] \subset \mathfrak{m}(2)_e$. Hence $[\mathfrak{g}(1)_e, \mathfrak{g}(1)_e] = \mathfrak{g}(2)_e$. □

Lemma 10. *Suppose that $\mathfrak{g}$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$ and $e \in \mathfrak{g}$ is reachable. Then $[\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e] = \mathfrak{g}(\lambda+1)_e$ for all $\lambda \geqslant 0$.*

Proof. If λ is odd, the proof of Lemma 9 goes practically without changes. Let $\xi_i^{j,s}, \xi_a^{b,t}$ be non-commuting elements of $\hat{\mathfrak{g}}(1)_e$ and $\mathfrak{g}(\lambda)_e$, without specifying which vector lies in what subspace. Assuming λ is odd, one can say that the sizes of the ith and jth Jordan blocks are of different parity and the sizes of the ath and bth Jordan blocks are also of different parity. Since $\xi_i^{j,s}$ and $\xi_a^{b,t}$ do not commute, using (1) we get that either $j = a$ or $i = b$ and one still may assume that $i = b$. Proceeding as in the proof of Lemma 9, we get

$$[\mathfrak{g}(1)_e, \hat{\mathfrak{g}}(\lambda)_e] + [\hat{\mathfrak{g}}(1)_e, \mathfrak{g}(\lambda)_e] = \hat{\mathfrak{g}}(\lambda)_e.$$

Now notice that $\hat{\mathfrak{g}}(\lambda)_e = \mathfrak{g}(\lambda)_e \oplus \mathfrak{m}(\lambda)_e$ for all λ and $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$.

Suppose now that λ is even and $\hat{x} = \xi_i^{j,s} \in \hat{\mathfrak{g}}(1)_e$, $y = \xi_a^{b,t} \in \hat{\mathfrak{g}}(\lambda)_e$ are non-commuting elements. According to (2), sizes of Jordan blocks with numbers a and b are of the same parity. In particular, if $i \in \{a, b\}$, then $j, j' \notin \{a, b, a', b'\}$ and if $j \in \{a, b\}$, then $i, i' \notin \{a, b, a', b'\}$. Take again $x = \hat{x} + \varepsilon(i,j,s)\xi_{j'}^{i',s'}$. Then $[\hat{x}, y] = [x, y]$ apart from two cases $\{i, i'\} = \{a, b\}$ and $\{j, j'\} = \{a, b\}$, where $[\hat{x}, y]$ is either $\xi_{i'}^{j,s+t}$ or $-\xi_i^{j',s+t}$. Let $\mathfrak{a} \subset \hat{\mathfrak{g}}_e$ be a subspace generated by all $\xi_i^{j,t}$ such that $|d_i - d_j| = 1$. We have shown that $\mathfrak{g}(\lambda+1)_e \subset [\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e] + \mathfrak{a} + \mathfrak{m}_e$. Note that $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{g}) \oplus (\mathfrak{a} \cap \mathfrak{m})$. Hence it remains to prove that $\mathfrak{a} \cap \mathfrak{g}(\lambda+1)_e \subset [\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e]$.

Suppose $|d_i - d_j| = 1$ and let $x(s) = \xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j'}^{i',s'}$ be an element of $\mathfrak{g}_e$. Set $D := \min(d_i, d_j)$. Then $x(D) \in \mathfrak{g}(1)_e$ and, by (2), $[h, x(s)] = (1 + 2(D-s))x(s)$. Our goal is to show that each $x(s)$ with $2(D-s) = \lambda$ lies in $[\mathfrak{g}(1)_e, \mathfrak{g}(\lambda)_e]$. This will complete the proof.

Since the sizes of the ith and jth Jordan blocks are of different parity, either $i' = i$ or $j' = j$. Without loss of generality, we may (and will) assume that $i = i'$. Then necessary $j \neq j'$. Assume first that $t = D - s$ is odd. Then $\xi_i^{i,t} \in \mathfrak{g}_e$. Using (1), it is straightforward to compute that $[x(D), \xi_i^{i,t}] = x(s)$. Here $\xi_i^{i,t} \in \mathfrak{g}(\lambda)_e$ as required. Assume now that t is even. Then $y = \xi_j^{j,t} - \xi_{j'}^{j',t} \in \mathfrak{g}(\lambda)_e$ and again $x(s) = [y, x(D)]$. □

Remark 2. It is also possible to verify the equality $\mathfrak{g}(\lambda+1)_e = [\mathfrak{g}(\lambda)_e, \mathfrak{g}(1)_e]$ directly, by writing down bases of $\mathfrak{g}(\lambda)_e$, $\mathfrak{g}(1)_e$ and computing the commutators.

3. Rigid nilpotent elements

The irreducible components of the quasi-affine varieties $\mathfrak{g}^{(m)} = \{\xi \in \mathfrak{g} \mid \dim \mathfrak{g}_\xi = m\}$ are called the sheets of $\mathfrak{g}$. Their description was obtained by Borho and Kraft [1], [2]. One of the main results is that each sheet contains exactly one nilpotent orbit. Nilpotent orbits, which coincide with sheets, are said to be rigid, and all their elements are said to be rigid as well. In the classical Lie algebras the classification of rigid nilpotent elements was obtained by Kempken [6, Subsection 3.3]. (Note that rigid orbits are called *original* in [6].) Namely, an element e of $\mathfrak{so}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$ is rigid if and only if it is given by a partition $(d^{r_d}, (d-1)^{r_{d-1}}, \ldots, 1^{r_1})$ with all multiplicities r_i being positive and

- if $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ and d_i is odd, then $r_i \neq 2$;
- if $\mathfrak{g} = \mathfrak{sp}(\mathbb{V})$ and d_i is even, then $r_i \neq 2$.

In view of [5, Sect. 3], the last two conditions mean that $\mathfrak{g}(0)_e$ has no factors isomorphic to $\mathfrak{so}_2$ and therefore is semisimple.

Proposition 11. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $x \in \mathfrak{g}$ such that $\mathfrak{g}_x = [\mathfrak{g}_x, \mathfrak{g}_x]$. Then x is a rigid nilpotent element.*

Proof. Assume that e is not rigid. Then any sheet S, containing e, contains also non-nilpotent elements and they form a non-empty open subset in S. In particular, there is a curve $x : \mathbb{F} \to S$ such that $\dim \mathfrak{g}_{x(t)} = \dim \mathfrak{g}_e$ for all $t \in \mathbb{F}$, $\lim_{t \to 0} x(t) = e$, and $x(t)$ is not nilpotent for all $t \neq 0$. Thereby $[\mathfrak{g}_{x(t)}, \mathfrak{g}_{x(t)}] \neq \mathfrak{g}_{x(t)}$ for all non-zero values of t. In the limit, dimension of the commutant cannot increase. Hence $\dim[\mathfrak{g}_e, \mathfrak{g}_e] < \dim \mathfrak{g}_e$. This contradiction completes the proof. □

Theorem 12. *Let $\mathfrak{g}$ be a simpe classical Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Then $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ if and only if e is rigid.*

Proof. Due to Proposition 11, we need to prove only one implication. Assume that e is rigid. In type A this means that $e = 0$ and there is nothing to prove.

Suppose that $\mathfrak{g}$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$. Comparing descriptions of Kempken [6], reproduced above, and of Panyushev [7, Theorem 2.1.(4)], we conclude that e is reachable. It follows from the Kempken's results, that $\mathfrak{g}(0)_e$ is semisimple. Therefore we have: e is reachable and $\mathfrak{g}(0)_e$ is semisimple. This is exactly condition (i), and the equality $\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$ holds by Theorem 7. □

Remark 3. In exceptional Lie algebras there are rigid nilpotent elements such that $e \notin [\mathfrak{g}_e, \mathfrak{g}_e]$. The simplest example is provided by a short root vector e in the simple Lie algebra of type G_2. Since $\dim(\mathfrak{g}_e) = 6$, this element is rigid. Here $\mathfrak{g}(1)_e = 0$ and therefore e is not reachable.

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