FORMAL MULTIPLICATIONS, BIALGEBRAS OF DISTRIBUTIONS AND NON-ASSOCIATIVE LIE THEORY

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Abstract. We describe the general non-associative version of Lie theory that relates unital formal multiplications (formal loops), Sabinin algebras and non-associative bialgebras.

Starting with a formal multiplication we construct a non-associative bialgebra, namely, the bialgebra of distributions with the convolution product. Considering the primitive elements in this bialgebra gives a functor from formal loops to Sabinin algebras. We compare this functor to that of Mikheev and Sabinin and show that although the brackets given by both constructions coincide, the multioperator does not. We also show how identities in loops produce identities in bialgebras. While associativity in loops translates into associativity in algebras, other loop identities (such as the Moufang identity) produce new algebra identities. Finally, we define a class of unital formal multiplications for which Ado's theorem holds and give examples of formal loops outside this class.

A by-product of the constructions of this paper is a new identity on Bernoulli numbers. We give two proofs: one coming from the formula for the non-associative logarithm, and the other (due to D. Zagier) using generating functions.

1. Introduction

The Lie theory describes the relationship among three types of algebraic structures: Lie groups, Lie algebras and Hopf algebras. In brief, we have the following triangle: for a finite-dimensional Lie group G, the Hopf algebra of distributions on G supported at the unit is nothing else but the universal enveloping algebra of the Lie algebra of G.

Strictly speaking, Lie algebras correspond directly not to Lie groups, but rather to *formal* Lie groups (for instance, via the Campbell-Baker-Hausdorff formula). The local methods of Lie theory are not sufficient to establish that finite-dimensional formal groups give rise to Lie groups: this follows from the existence of a faithful representation for every finite-dimensional Lie algebra. This phenomenon is even more apparent in the Lie theory of non-associative multiplications, where many natural examples of multiplications on manifolds are of local nature and do not have evident extensions to global operations.

In the present paper we describe the non-associative version of the correspondence between formal groups, Lie algebras and Hopf algebras.

The main step towards generalizing the Lie theory to this context was done by Sabinin and Mikheev [SM87, MS90] who defined algebraic structures tangent to arbitrary local analytic loops (multiplications). These structures, now known as Sabinin algebras, can be integrated under some convergence conditions to local loops: essentially, they are the analog of Lie algebras in the non-associative setting. Shestakov and Umirbaev later showed [SU02] that the set of primitive elements in any bialgebra has the structure of a Sabinin algebra, and it was proved by the second author of the present paper [PI07], that each Sabinin algebra arises in this way. The main purpose of the present paper is to show how the Lie theory for non-associative formal multiplications can be constructed by first passing from a formal multiplication to the corresponding bialgebra of distributions, and then to the Sabinin algebra of the primitive elements of the latter. We compare this construction to the direct geometric argument of Sabinin and Mikheev and show that these two constructions do not give precisely the same result: they produce Sabinin algebras with coinciding brackets but different multioperators.

There are two aspects of the non-associative Lie theory that are absent from the usual Lie theory. Firstly, in the non-associative context it is rather usual to consider a class of multiplications satisfying certain identity. We show how these identities translate into identities in the corresponding bialgebras of distributions. The second novelty is that while Lie groups are always locally isomorphic to linear groups, this property (or,

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rather, its appropriate generalization) no longer holds for general loops. We discuss this phenomenon and give examples of formal loops that do not satisfy this property.

The paper has the following structure. In the next section we show that the category of unital formal multiplications is equivalent to that of irreducible cocommutative and coassociative bialgebras. In Section 3 we consider how identities in formal loops correspond to identities in bialgebras. In Section 4 we show that the primitive operations of Shestakov and Umirbaev give an equivalence between the category of irreducible cocommutative and coassociative bialgebras and the category of Sabinin algebras. Section 5 contains a comparison of two functors from formal loops to Sabinin algebras: the Sabinin algebra of the primitive elements in the algebra of distributions on a formal loop and the the Sabinin algebra as defined by Sabinin and Mikheev. In Section 6 we discuss linear formal loops (those for which Ado's theorem holds). Finally, in the appendix we give the formulae for the non-associative exponential and logarithm and describe an identity on Bernoulli numbers.

2. Formal multiplications and bialgebras of distributions

In what follows all coalgebras are always assumed to be cocommutative. We refer to [Abe80] for the basics on coalgebras.

2.1. **Formal maps.** Let V be a vector space over a field k of characteristic zero. We shall write $k[V]_i$ for the ith symmetric power of V and $k[V]$ for the symmetric algebra of V. Recall that the space $k[V]$ is also a coalgebra: the coproduct

$$\Delta : k[V] \to k[V] \otimes k[V]$$

is defined by the condition that all elements of V are primitive, and the counit $\epsilon : k[V] \to k$ sends an element of $k[V]$ to its degree 0 component. We denote by π_V the projection of $k[V]$ onto its primitive part $k[V]_1 = V$.

Elements of the dual space $k[V]^*$ will be referred to as *formal functions on* V, and those of $k[V]$ as *formal distributions on* V. A *formal map* θ *from* V *to* W is a linear map

$$\theta : k[V] \to W$$

with $\theta(1) = 0$.

Proposition 2.1. *Any formal map* $\theta\colon k[V] \to W$ *induces a unique coalgebra morphism* $\theta' : k[V] \to k[W]$ *with* $\pi_W \theta' = \theta$.

Proof. Define the coalgebra morphism θ' by

$$\theta'(\mu) = \sum_{n=0}^{\infty} \frac{1}{n!} \theta(\mu_{(1)}) \cdots \theta(\mu_{(n)}) = \epsilon(\mu)1 + \theta(\mu) + \cdots$$

and observe that by [Abe80, Corollary 2.4.17 (i)] any coalgebra morphism θ' from $k[V]$ to $k[W]$ is determined by its projection $\pi_W \theta'$. □

The algebra $k[V_1 \times \cdots \times V_n]$ is canonically isomorphic to $k[V_1] \otimes \cdots \otimes k[V_n]$. In order to work with formal maps from products of vector spaces the following notation will be of help.

The map $\pi_{V_i} : k[V_i] \to V_i$ will be denoted by $\boldsymbol{x}_i$ and the null map $k[V_i] \to V_i$ for any i will be denoted simply by $\mathbf{0}$ (the absence of the index i should not lead to confusion). The induced coalgebra morphism $\boldsymbol{x}_i'$ is the identity map on $k[V_i]$, and $\mathbf{0}'(\mu) = \epsilon(\mu)1$. Given a formal map

$$G\colon k[V_1 \times \cdots \times V_n] \to W$$

and formal maps $\theta_i\colon k[U_i] \to V_i$ for $1 \le i \le n$ we write $G(\theta_1, \ldots, \theta_n)$ for the map $G \circ \theta_1' \otimes \cdots \otimes \theta_n'$.

With this notation the $\boldsymbol{x}_i$ can be treated as variables. In particular, G can be also written as $G(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. If

$$G(\ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \ldots) = G(\ldots, \boldsymbol{x}_{i-1}, \mathbf{0}, \boldsymbol{x}_{i+1}, \ldots)$$

we say that G does not depend on $\boldsymbol{x}_i$ and omit this variable altogether; the domain of definition of G will always be clear from the context.

If $V_1 = \cdots = V_n = V$ the notation $G(\boldsymbol{x}, \dots, \boldsymbol{x})$ stands for the composition of G with the map $k[V] \to k[V_1 \times \cdots \times V_n]$ induced by the diagonal $V \to V_1 \times \cdots \times V_n$:

$$\mu \mapsto \sum G(\mu_{(1)}, \dots, \mu_{(n)}).$$

Similarly one defines $G(\boldsymbol{x}_{i_1}, \dots, \boldsymbol{x}_{i_n})$ when there are various groups of repeating indices among the i_k.

2.2. **Formal multiplications.** Formal multiplications are a special form of formal maps.

Definition 2.2. *A formal multiplication on V is a formal map*

$$F : k[V \times V] \to V.$$

A formal multiplication on V is said to be unital (or a formal loop) if

$$F|_{k[V]\otimes 1} = \pi_V = F|_{1\otimes k[V]}.$$

Since

$$F \in \mathrm{Hom}(k[V] \otimes k[V], V) \cong \prod_{i,j=0}^{\infty} \mathrm{Hom}(k[V]_i \otimes k[V]_j, V)$$

with our notation we can write any unital formal multiplication F as an infinite formal sum

$$F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} + \sum_{i,j\geq 1} F_{i,j}(\boldsymbol{x}, \boldsymbol{y})$$

with $F_{i,j} \in \mathrm{Hom}(k[V]_i \otimes k[V]_j, V)$, or equivalently

$$F(\mu_1 \otimes \mu_2) = \pi_V(\mu_1)\epsilon(\mu_2) + \epsilon(\mu_1)\pi_V(\mu_2) + \sum_{i,j\geq 1} F_{i,j}(\mu_1 \otimes \mu_2).$$

We say that $F_{i,j}(\boldsymbol{x}, \boldsymbol{y})$ is of degree i in $\boldsymbol{x}$ and j in $\boldsymbol{y}$. Sometimes we shall write $\boldsymbol{xy}$ for a unital formal multiplication $F(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 2.3. *Let $F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{xy}$ be a unital formal multiplication. There exist formal multiplications $\boldsymbol{x}\backslash\boldsymbol{y}$ and $\boldsymbol{x}/\boldsymbol{y}$ such that*

(1) $\boldsymbol{x}\backslash(\boldsymbol{xy}) = \boldsymbol{y} = \boldsymbol{x}(\boldsymbol{x}\backslash\boldsymbol{y})$ *and*
(2) $(\boldsymbol{yx})/\boldsymbol{x} = \boldsymbol{y} = (\boldsymbol{y}/\boldsymbol{x})\boldsymbol{x}$.

Proof. Write $F(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} + \sum_{i,j\geq 1} F_{i,j}(\boldsymbol{x}, \boldsymbol{y})$. Given a formal multiplication $D(\boldsymbol{x}, \boldsymbol{y})$ we have that

$$F(\boldsymbol{x}, D(\boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{y} \text{ if and only if } D(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{x} - \sum_{i,j\geq 1} F_{i,j}(\boldsymbol{x}, D(\boldsymbol{x}, \boldsymbol{y})).$$

The latter recurrence determines a unique solution $D(\boldsymbol{x}, \boldsymbol{y})$ that in addition satisfies $D(\boldsymbol{0}, \boldsymbol{y}) = F(\boldsymbol{0}, D(\boldsymbol{0}, \boldsymbol{y})) = \boldsymbol{y}$. By the same argument, there exists then a unique solution $H(\boldsymbol{x}, \boldsymbol{y})$ to the equation $D(\boldsymbol{x}, H(\boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{y}$. By construction

$$H(\boldsymbol{x}, \boldsymbol{y}) = F(\boldsymbol{x}, D(\boldsymbol{x}, H(\boldsymbol{x}, \boldsymbol{y}))) = F(\boldsymbol{x}, \boldsymbol{y})$$

so $\boldsymbol{x}\backslash\boldsymbol{y} = D(\boldsymbol{x}, \boldsymbol{y})$ satisfies the required conditions. In a similar way one proves the existence of $\boldsymbol{x}/\boldsymbol{y}$. □

By Proposition 2.1, any unital formal multiplication $F(\boldsymbol{x}, \boldsymbol{y})$ induces a product

$$F' : k[V] \otimes k[V] \to k[V].$$

Whenever we consider $k[V]$ as an algebra with multiplication induced by F we shall denote it by $k[F]$; similarly, we shall write $k[F]^*$ for $k[V]^*$. The unit $\delta\colon k \to k[F]$ is defined by $\alpha \mapsto \alpha 1$.

Proposition 2.4. $(k[F], \Delta, \epsilon, F', \delta)$ *is an irreducible unital bialgebra.*

Proof. By construction

$$F'(\mu, 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \pi_V(\mu_{(1)}) \cdots \pi_V(\mu_{(n)}) = \mu = F'(1, \mu)$$

for any $\mu \in k[F]$. Since F' is a coalgebra morphism, the proposition follows. □

In what follows we shall assume that all bialgebras in question are unital.

2.3. **The equivalence of categories.** The category of unital formal multiplications is, in fact, equivalent to that of irreducible bialgebras.

Definition 2.5. *Let F and H be unital formal multiplications on vector spaces V and W respectively. A formal map θ from V to W is called a homomorphism from F to H (notation: $\theta\colon F \to H$) if*

$$H(\theta(\boldsymbol{x}), \theta(\boldsymbol{y})) = \theta(F(\boldsymbol{x}, \boldsymbol{y}))$$

or, equivalently, $H(\theta'(\mu_1) \otimes \theta'(\mu_2)) = \theta(F'(\mu_1 \otimes \mu_2))$ for any $\mu_1, \mu_2 \in k[V]$.

Homomorphisms are the morphisms in the category of unital formal multiplications. It follows directly from the definitions that a homomorphism of formal multiplications $\theta\colon F \to H$ induces a homomorphism of bialgebras

$$\theta'\colon k[F] \to k[H].$$

Proposition 2.6. *The category of unital formal multiplications and the category of irreducible unital bialgebras are equivalent.*

Proof. First, let us show that any irreducible bialgebra is isomorphic to $k[F]$ for some unital formal multiplication F. The coalgebra structure of such bialgebras is very well known: in [SU02] it is proved (see Theorem 3.2) that every such bialgebra is isomorphic as a coalgebra to $k[V]$ for some V.

Given a product $m : k[V] \otimes k[V] \to k[V]$, its primitive (that is, linear) part

$$k[V] \otimes k[V] \xrightarrow{m} k[V] \xrightarrow{\pi_V} V$$

is a formal multiplication F_m. The fact that $m(x, 1) = m(1, x) = x$ means that F_m is unital. It follows from the construction that $k[F_m]$ coincides with the bialgebra $k[V]$ equipped with the product m.

Now, if $\psi : k[F] \to k[H]$ is a homomorphism of bialgebras, its primitive part $\theta = \pi_V \psi$ is a homomorphism $F \to H$ and, clearly, $\psi = \theta'$. □

2.4. **Analytic loops and formal loops.** Let V be a finite-dimensional vector space with a basis $\{e_1, \dots, e_n\}$ and let $\{x_1, \dots, x_n\}$ be the dual basis of V^*. The symmetric algebra $k[V]$ is spanned by monomials in the e_i. The dual space $k[V]^*$ is also an algebra with the *convolution* product: for $f, g \in k[V]^*$ it is defined as

$$(f * g)(m) = \sum_{m_1 m_2 = m} f(m_1) g(m_2)$$

where m, m_1, m_2 are basis monomials of $k[V]$. This algebra can be identified with the algebra $k[[x_1, \dots, x_n]]$ of formal power series in the x_i, where the products of the x_i are thought of as convolutions.

Elements of $k[V]$ can be understood as elements in $(k[V]^*)^* = k[[x_1, \dots, x_n]]^*$. For any monomial $x_1^{a'_1} \cdots x_n^{a'_n}$ we have that

$$e_1^{a_1} \cdots e_n^{a_n} (x_1^{a'_1} \cdots x_n^{a'_n}) = x_1^{a'_1} \cdots x_n^{a'_n} (e_1^{a_1} \cdots e_n^{a_n})$$

which, by definition of the convolution product equals to $a_1! \cdots a_n!$ if the ordered set of exponents $(a_1, \dots, a_n)$ coincides with $(a'_1, \dots, a'_n)$ and 0 otherwise. Therefore

$$e_1^{a_1} \cdots e_n^{a_n} = \partial_1^{a_1} \cdots \partial_n^{a_n}|_{(0,\dots,0)}$$

and $k[V]$ is the coalgebra of all distributions (linear functionals on analytic functions) on V with support at zero. In particular, the constant polynomial 1 corresponds to the evaluation at 0 (also known as the Dirac delta).

Now, let G be an analytic local loop defined on a neighborhood of the origin (which plays the role of the unit) in V. Having chosen a basis in V, we may write the product $F(x, y)$ on G as an n-tuple

$$(F_1(x, y), \dots, F_n(x, y))$$

of power series in $2n$ variables, that is, an element of $V \otimes k[V \times V]^*$, satisfying $F(x, 0) = x$ and $F(0, y) = y$. Under the natural isomorphism

$$V \otimes k[V \times V]^* \cong \mathrm{Hom}(k[V \times V], V)$$

this condition is equivalent to $F|_{k[V] \otimes 1} = \pi_V = F|_{1 \otimes k[V]}$. Therefore, analytic local loops give rise to formal multiplications.

The product F on the analytic local loop G induces a product on $k[V]$ which sends $\mu_1 \otimes \mu_2$ to $\mu_1 \cdot \mu_2 \colon f \mapsto \mu_1 \otimes \mu_2(f \circ F)$. This product is a coalgebra map, and it gives $k[V]$ the structure of an irreducible unital bialgebra. Since

$$\mu_1 \otimes \mu_2(x_i \circ F) = \mu_1 \otimes \mu_2(F_i),$$

the primitive part of $\mu_1 \cdot \mu_2$ is $F(\mu_1 \otimes \mu_2)$ and, as a consequence,

$$\mu_1 \cdot \mu_2 = F'(\mu_1 \otimes \mu_2).$$

Our definition of the bialgebra of distributions corresponding to a formal loop is motivated by this observation.

More generally, any analytic map $\theta \colon V \to W$ defined on a neighbourhood of 0 and such that $\theta(0) = 0$ induces a coalgebra morphism θ' on distributions given by $\theta'(\mu)(f) = \mu(f \circ \theta)$ for any analytic function f and any distribution μ. Note that θ gives rise to a formal map from V to W; if the distributions on V are identified with $k[V]$ and formal power series with $k[V]^*$, this formal map induces the same map as θ'.

3. Identities

3.1. **Identities in formal loops and in bialgebras.** A *formal group* $F \colon k[V] \otimes k[V] \to V$ is a formal loop that in addition satisfies the identity $F(F(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{z}) = F(\boldsymbol{x}, F(\boldsymbol{y}, \boldsymbol{z}))$. The consequence of this identity is that $F' \circ (F' \otimes \mathrm{Id}) = F' \circ (\mathrm{Id} \otimes F')$ which implies that $k[F]$ is associative. In our approach groups do not play any special role and the bialgebras of distributions considered here will be non-associative in general. However, the principle that identities on loops produce identities on distributions works in general and provides interesting examples of identities in non-associative bialgebras.

Consider the set of formal maps

$$\boldsymbol{x}\boldsymbol{y} = F(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x}\backslash\boldsymbol{y}, \quad \boldsymbol{x}/\boldsymbol{y}, \quad \boldsymbol{x}\backslash\mathbf{e}, \quad \text{and} \quad \mathbf{e}/\boldsymbol{x} \tag{1}$$

with $\mathbf{e} = \mathbf{0}$ and $F(\boldsymbol{x}, \boldsymbol{y})$ is a unital formal multiplication on a vector space V (since $\boldsymbol{x}\backslash\mathbf{e}$ and $\mathbf{e}/\boldsymbol{x}$ do not depend on the variable $\boldsymbol{y}$ we may consider them as defined on $k[V]$). We may compose these formal maps to obtain new formal maps, such as $F(\boldsymbol{x}, \boldsymbol{x}\backslash\mathbf{e})$ (which is equal to $\mathbf{e}$), $\mathbf{e}/(\boldsymbol{x}\backslash\mathbf{e})$ (equals to $\boldsymbol{x}$), $F(\boldsymbol{x}, F(\boldsymbol{y}, F(\boldsymbol{x}, \boldsymbol{z})))$, $F(F(F(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}), \boldsymbol{z})$, and so on.

Let $x_1, \ldots, x_n$ be a set of generators for the free loop on n letters, and let $V_1, \ldots, V_n$ be n copies of V. Since $F(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{x} = F(\mathbf{0}, \boldsymbol{x})$, by Proposition 2.3 to each word $w(x_1, \ldots, x_n)$ we can assign a formal map

$$w : k[V_1] \otimes \ldots \otimes k[V_n] \to V \tag{2}$$

by substituting $\boldsymbol{x}_i$ for each occurrence of x_i in $w(x_1, \ldots, x_n)$, and understanding the products and divisions as in (1) above. For instance, to the word $x(y(xz))$ we assign the map $\boldsymbol{x}(\boldsymbol{y}(\boldsymbol{x}\boldsymbol{z})) = F(\boldsymbol{x}, F(\boldsymbol{y}, F(\boldsymbol{x}, \boldsymbol{z})))$.

Definition 3.1. *Given two words $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ in the free loop on n letters, we say that F satisfies the identity*

$$u(x_1, \ldots, x_n) \sim v(x_1, \ldots, x_n)$$

if the maps $u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ and $v(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ coincide.

Any map w as in (2) induces $w' \colon k[V_1] \otimes \cdots \otimes k[V_n] \to k[V]$. The map w can be recovered from w' by taking the primitive part. In particular, a untial formal multiplication F satisfies the identity $u(x_1, \ldots, x_n) \sim v(x_1, \ldots, x_n)$ if and only if $u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)' = v(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)'$.

Operations $\boldsymbol{x}\backslash\boldsymbol{y}$ and $\boldsymbol{x}/\boldsymbol{y}$ induce the corresponding operations on distributions; these operations were first considered in [PI07]. We shall simply write $\mu\backslash\nu$ and μ/ν to denote these operations. Since any formal unital multiplication $\boldsymbol{x}\boldsymbol{y} = F(\boldsymbol{x}, \boldsymbol{y})$ satisfies $\boldsymbol{x}\backslash(\boldsymbol{x}\boldsymbol{y}) = \boldsymbol{y} = \boldsymbol{x}(\boldsymbol{x}\backslash\boldsymbol{y})$ and $(\boldsymbol{x}\boldsymbol{y})/\boldsymbol{y} = \boldsymbol{x} = (\boldsymbol{x}/\boldsymbol{y})\boldsymbol{y}$ we have that for any $\mu, \nu \in k[F]$

$$\sum \mu_{(1)}\backslash(\mu_{(2)}\nu) = \epsilon(\mu)\nu = \sum \mu_{(1)}(\mu_{(2)}\backslash\nu)$$
$$\sum (\mu\nu_{(1)})/\nu_{(2)} = \epsilon(\nu)\mu = \sum (\mu/\nu_{(1)})\nu_{(2)}.$$

A particular choice of u and v is

$$u(x, y, z) = x(y(xz))$$

and

$$v(x, y, z) = ((xy)x)z.$$

The identity $u \sim v$ is called the *Moufang identity*. The corresponding u' and v' are $u'(\mu,\nu,\eta) = \sum \mu_{(1)}(\nu(\mu_{(2)}\eta))$ and $v'(\mu,\nu,\eta) = \sum((\mu_{(1)}\nu)\mu_{(2)})\eta$. This shows that F is a formal Moufang loop if and only if

$$\sum \mu_{(1)}(\nu(\mu_{(2)}\eta)) = \sum((\mu_{(1)}\nu)\mu_{(2)})\eta \tag{3}$$

in $k[F]$.

In [PI07] w' was called the *linearization* of w. It was proved that certain bialgebras constructed from Malcev algebras satisfy the identity (3) above. That was surprising since the construction of those bialgebras [PIS04] had no relation with Moufang loops. Distributions provide a natural connection between identities in loops and identities in bialgebras. In case that we consider a local analytic loop G, any word $u(x_1,\dots,x_n)$ induces by evaluation a map $u\colon G\times\cdots\times G \to G$ and a coalgebra map u' on distributions which agrees with the map u' defined above. Therefore, G satisfies the identity $u \sim v$ in the usual sense if and only if the formal loop corresponding to G satisfies the identity $u \sim v$.

3.2. **Right alternativity.** Another example of an identity on formal multiplications is

$$x(yy) = (xy)y$$

called *right alternativity*. The corresponding bialgebra identity reads

$$\sum \mu(\nu_{(1)}\nu_{(2)}) = \sum(\mu\nu_{(1)})\nu_{(2)}. \tag{4}$$

It was proved by Sabinin and Mikheev [SM85] that the right alternativity in a formal loop implies the identity

$$x(y^k y^l) = (xy^k)y^l \tag{5}$$

for all $k, l \geq 0$ (see also [Sab99]).

The importance of the right alternativity for Lie theory was understood first by Sabinin and Mikheev [SM87, MS90]. They realized that this algebraic property for a local loop is satisfied if and only if the loop comes from a flat connection as a so-called *geodesic loop*, and showed that, in fact, the multiplication in any local loop can be modified so as to become right alternative.

Given a local loop $(G,\cdot)$ with the unit e, one can define the *canonical* flat connection ∇ on the tangent bundle to G in a neighbourhood of e as follows. For $a, b \in G$ two points in a small neighbourhood of e, the parallel transport of the tangent space T_bG to T_aG is induced by a self-map of G that sends x to $a\cdot(b\backslash x)$. There is a new product on G given by

$$a \times b = \exp_a(a \log b), \tag{6}$$

where $\exp_a$ is the exponential map of the connection ∇ at the point a and $\log_a$ is its inverse (we write simply exp and log for $\exp_e$ and $\log_e$ respectively) and $a \log b$ stands for the parallel transport of the vector $\log b \in \mathrm{T}_eG$ to T_aG. The local loop $(G,\times)$ is then right alternative. Note that the canonical connection for the loop $(G,\times)$ is the same as that of $(G,\cdot)$. The original local loop $(G,\cdot)$ can be reconstructed from $(G,\times)$ and the operation $\Phi(a,b)$ defined by

$$a \times \Phi(a,b) = a \cdot b. \tag{7}$$

The right alternative modification can also be defined for formal loops. Given a formal loop F on a vector space V, the *formal canonical connection* of F is the restriction of F to the subspace

$$k[V]\otimes V \subset k[V]\otimes k[V].$$

Let us say that two formal loops on the same vector space are *similar* if their formal canonical connections coincide.

Lemma 3.2. *Each formal loop is similar to a unique right alternative formal loop.*

Proof. A unital formal multiplication can be written as

$$F(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y} + q_1(\boldsymbol{x},\boldsymbol{y}) + q_2(\boldsymbol{x},\boldsymbol{y}) + \dots$$

where $q_j(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{\infty} F_{i,j}(\boldsymbol{x},\boldsymbol{y})$. Specifying the canonical connection for F is the same thing as specifying $q_1(\boldsymbol{x},\boldsymbol{y})$.

Given $q_1(\boldsymbol{x},\boldsymbol{y})$, the right alternative formal loop similar to F can be reconstructed inductively. Assume that the $q_i(\boldsymbol{x},\boldsymbol{y})$ with $i<n$ are known, and consider the equation $F(F(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y}) = F(\boldsymbol{x},F(\boldsymbol{y},\boldsymbol{y}))$, modulo the terms of degree $> n$ in $\boldsymbol{y}$. A simple calculation shows that, apart from the (compositions of the) q_i with $i<n$, this equation contains the term $q_n(\boldsymbol{x},\boldsymbol{y})$ with coefficient 2 on the left-hand side and 2^n on the right-hand side. Therefore, for $n>1$ we see that q_n can be expressed via the q_i with $i<n$. □

Let $\Phi : k[V]\otimes k[V]\to V$ be a formal multiplication such that $\Phi|_{1\otimes k[V]} = \pi_V$ and $\Phi|_{k[V]_{\geq 1}\otimes(1\oplus V)} = 0$. In other words,

$$\Phi(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{y} + \sum_{i\geq 1, j\geq 2}\Phi_{i,j}(\boldsymbol{x},\boldsymbol{y}),$$

with $\Phi_{i,j}(\boldsymbol{x},\boldsymbol{y})$ of degree i in $\boldsymbol{x}$ and of degree j in $\boldsymbol{y}$. Call such a multiplication a *similarity.*

Lemma 3.3. *Two formal loops F_1 and F_2 are similar if and only if there is a similarity Φ such that $F_1(\boldsymbol{x},\boldsymbol{y}) = F_2(\boldsymbol{x},\Phi(\boldsymbol{x},\boldsymbol{y}))$.*

The proof is straightforward. For the "if" part compare the corresponding homogeneous terms $(F_1)_{i,j}$ and $(F_2)_{i,j}$; for the "only if" part define Φ inductively by the degree. □

These notions have their versions for bialgebras. If Φ is a similarity, we obtain a coalgebra morphism $\Phi' : k[V]\otimes k[V]\to k[V]$, which satisfies

$$\Phi'(\mu_1,1)=\epsilon(\mu_1)1,\ \ \Phi'(1,\mu_2)=\mu_2\ \text{ and }\ \Phi'(\mu_1,\alpha)=\epsilon(\mu_1)\alpha \tag{8}$$

for all $\mu_1,\mu_2\in k[V]$ and $\alpha\in V$. Conversely, the primitive part of a coalgebra morphism Φ' satisfying these conditions is a similarity.

Let F_1 and F_2 be two similar formal loops with $F_1(\boldsymbol{x},\Phi(\boldsymbol{x},\boldsymbol{y})) = F_2(\boldsymbol{x},\boldsymbol{y})$. Denote by $\times$ and $\cdot$ and the products in $k[F_1]$ and in $k[F_2]$ respectively. Then we have

$$\sum \mu_{(1)}\times\Phi'(\mu_{(2)},\nu) = \mu\cdot\nu. \tag{9}$$

If $k[V]$ has two different bialgebra products $\times$ and $\cdot$ such that there exists a map Φ' satisfying (8) and (9), we say that the products $\times$ and $\cdot$ are *similar* and that Φ' is a *(bialgebra) similarity* between them.

Lemma 3.4. *If $\times$ and $\cdot$ are two similar products on $k[V]$, then for any $\mu\in k[V]$ and $\alpha\in V$*

$$\mu\times\alpha = \mu\cdot\alpha.$$

The proof is an immediate consequence of (8) and (9).

4. Bialgebras of distributions and Sabinin algebras

4.1. **Shestakov–Umirbaev's functor** *УШ***.** Let $\mathcal{S}$ be a set. Denote by $k\{\mathcal{S}\}$ the unital free non-associative algebra generated by the elements of $\mathcal{S}$. The algebra $k\{\mathcal{S}\}$ can be given a structure of an irreducible bialgebra: the comultiplication is defined by the condition that all the elements of $\mathcal{S}$ are primitive; the counit $\epsilon\colon k\{\mathcal{S}\}\to k$ is the homomorphism that sends 1 to 1 and all the elements of $\mathcal{S}$ to 0.

Recall that instead of the antipodes, non-associative bialgebras have operations of left and right division $\backslash$ and $/$. In $k\{\mathcal{S}\}$ they are as follows. Starting with $1\backslash v = v$, $a\backslash v = -av$ for any generator $a\in\mathcal{S}$ and $v\in k\{\mathcal{S}\}$ we use induction on the degree $|u|$ of of u to define a bilinear operation $u\backslash v$ so that

$$\sum u_{(1)}\backslash(u_{(2)}v) = \epsilon(u)v.$$

We also have $\sum u_{(1)}(u_{(2)}\backslash v) = \epsilon(u)v$. Indeed, by induction on $|u|$, we get

$$uv = \sum u_{(1)}\left(u_{(2)}\backslash u_{(3)}v\right) = \sum u_{(1)}(u_{(2)}\backslash v) + \sum_{|u_{(1)}|<|u|}\epsilon(u_{(1)})u_{(2)}v$$
$$= -\epsilon(u)v + uv + \sum u_{(1)}(u_{(2)}\backslash v).$$

Hence $\sum u_{(1)}(u_{(2)}\backslash v) = \epsilon(u)v$. Similarly we define a bilinear operation u/v that satisfies

$$\sum (uv_{(1)})v_{(2)} = \epsilon(v)u = \sum (u/v_{(1)})v_{(2)}.$$

Apart from the generators and their linear combinations, the algebra $k\{\mathcal{S}\}$ has many other primitive elements. All these elements were described by Shestakov and Umirbaev in [SU02].

Let $u = ((x_1x_2)\cdots)x_m$ and $v = ((y_1y_2)\cdots)y_n$ with x_i and y_j primitive. The *primitive operations* $p(u, v, z)$ are defined by

$$p(x_1,\dots,x_m; y_1,\dots,y_n; z) = p(u, v, z) = \sum (u_{(1)}v_{(1)})\backslash(u_{(2)}, v_{(2)}, z)$$

where $(x, y, z) = (xy)z - x(yz)$ denotes the associator and z is primitive. In [SU02] it is shown that the $p(u, v, z)$ are primitive, and, moreover, that each primitive element of $k\{\mathcal{S}\}$ can be obtained from the generators by applying repeatedly the commutators and the operations $p(u, v, z)$, and taking linear combinations.

Since $p(x_1,\dots,x_m; y_1,\dots,y_n; z)$ are just polynomial expressions in $x_1,\dots,x_m$, $y_1,\dots,y_n$, z, they make sense in any algebra; given a non-associative algebra A we shall consider them as new $(m + n + 1)$–ary operations obtained from the product on A. When evaluating in an arbitrary algebra A, the compact notation $p(u, v, z)$ for the operation $p(x_1,\dots,x_m; y_1,\dots,y_n; z)$ may be misleading since it suggests that we should first evaluate $u = ((x_1x_2)\cdots)x_m$ and $v = ((y_1y_2)\cdots)y_n$ and then apply a ternary operation $p(u, v, z)$. In order to avoid confusion, we shall write $p(\underline{u}, \underline{v}, z)$ when working in a non-associative, not necessarily free, algebra A. The relation

$$(u, v, z) = \sum u_{(1)}v_{(1)} \cdot p(\underline{u}_{(2)}, \underline{v}_{(2)}, z)$$

also makes sense in any algebra A even if it is not a bialgebra. This is a consequence of the corresponding identity in $k\{\mathcal{S}\}$. Observe that when $u, v, u_{(1)}$ and $v_{(1)}$ are not used as the arguments of p, they become products in A, so we do not need to underline them.

Shestakov and Umirbaev related their work with the results of Mikheev and Sabinin on local loops [SM87, MS90]. Namely, in [SU02] they defined for any non-associative algebra A the operations

$$\begin{aligned}
\langle 1; y, z\rangle &= \langle y, z\rangle = -[y, z] = -yz + zy \\
\langle x_1,\dots,x_m; y, z\rangle &= \langle \underline{u}; y, z\rangle = -p(\underline{u}, y, z) + p(\underline{u}, z, y) \\
\Phi^{SU}(x_1,\dots,x_m; y_1,\dots,y_n) &= \\
&\frac{1}{m!}\frac{1}{n!}\sum_{\tau\in S_m,\sigma\in S_n} p(x_{\tau(1)},\dots,x_{\tau(m)}; y_{\sigma(1)},\dots,y_{\sigma(n-1)}; y_{\sigma(n)})
\end{aligned}$$

with $u = ((x_1x_2)\cdots)x_m$, S_m the symmetric group on m letters and $m \geq 1, n \geq 2$. With these operations A turns out to be a Sabinin algebra [SU02] so we have a functor from non-associative algebras to Sabinin algebras

$$A \mapsto \text{УШ}(A)$$

that generalizes the usual functor from associative algebras to Lie algebras given by assigning to an associative algebra its commutator algebra. The primitive elements of any bialgebra W form a Sabinin subalgebra of $\text{УШ}(W)$.

One is then naturally led to ask whether every Sabinin algebra is isomorphic to a Sabinin algebra of the primitive elements in some irreducible bialgebra. An affirmative answer (with a modified version of the operations $p(; ;)$ and, hence, of the functor УШ) was given in [PI07]. Given a Sabinin algebra $(V, \langle ; , \rangle, \Phi')$ the corresponding bialgebra is denoted by $U(V)$ and has the following universal property: any homomorphism of Sabinin algebras from V to a unital algebra A extends to a unique homomorphism of unital algebras $U(V) \to A$. The algebra $U(V)$ was called in [PI07] *the universal enveloping algebra* of V.

There is a Poincaré–Birkhoff–Witt Theorem for the universal enveloping algebras of Sabinin algebras: as a coalgebra, $U(V)$ is isomorphic to $k[V]$. Moreover, the algebra $U(V)$ is filtered and the corresponding associated graded algebra is commutative and associative: it is isomorphic to the symmetric algebra $S(V)$. If we start with an irreducible bialgebra W, $\mathrm{Prim}(W)$ is a Sabinin subalgebra of $\text{УШ}(W)$ and if $\{e_1, e_2, \dots, e_\alpha, \dots\}$ is a basis of $\mathrm{Prim}(W)$ then

$$\{((e_{i_1}e_{i_2})\cdots)e_{i_k} \mid 0 \leq i_1 \leq \cdots \leq i_k, k \geq 0\}$$

is a basis of W (*Poincaré–Birkhoff–Witt basis*). The universal property of the enveloping algebras gives an isomorphism

$$U(\mathrm{Prim}(W)) \cong W$$

of bialgebras, which identifies the respective Poincaré–Birkhoff–Witt bases. In this way irreducible bialgebras can be classified in terms of the Sabinin algebra of their primitive elements. In the sequel we shall often write irreducible bialgebras as pairs $(k[V],\cdot)$ where $\cdot$ is a product on the coalgebra $k[V]$. Sometimes, for clarity, we shall also indicate the product explicitly while working with primitive operations and the bialgebra divisions.

One useful consequence of the Poincaré–Birkhoff–Witt Theorem is the following

Lemma 4.1. *Any irreducible bialgebra is additively spanned by elements of the form $x^n = ((xx)x\dots)x$ with $n \geq 0$ and x primitive.*

4.2. **Similarity of bialgebras and the primitive operations.** In a Sabinin algebra the identities for the brackets do not involve the multioperator, and vice versa. Here we shall see how to modify a product in a bialgebra so that the bracket operations defined via the Shestakov-Umirbaev operations do not change and so that Φ^{SU} takes any prescribed form.

Proposition 4.2. *Let $k[V]$ be a bialgebra with respect to two similar products $\cdot$ and $\times$. Then for any $\mu \in k[V]$ and $\alpha,\beta \in V$*

$$\langle \underline{\mu}; \alpha, \beta\rangle^{\cdot} = \langle \underline{\mu}; \alpha, \beta\rangle^{\times}.$$

Proof. Let Φ' be the similarity between $\times$ and $\cdot$. By the definition of $p^{\cdot}(\underline{\mu},\underline{\nu},\alpha)$, (9) and Lemma 3.4, these operations can be written in terms of $\times$ and Φ' as

$$\begin{aligned} p^{\cdot}(\underline{\mu},\underline{\nu},\alpha) &= \sum (\mu_{(1)}\cdot\nu_{(1)})\backslash^{\times}(\mu_{(2)},\nu_{(2)},\alpha)^{\cdot} \\ &= \epsilon(\mu)\epsilon(\nu)\alpha - \sum (\mu_{(1)}\times\Phi'(\mu_{(2)},\nu_{(1)}))\backslash^{\times}(\mu_{(3)}\times\Phi'(\mu_{(4)},\nu_{(2)}\times\alpha)) \end{aligned}$$

Hence, by (8) we have

$$\begin{aligned} p^{\cdot}(\underline{\mu},\alpha,\beta) &= -\sum (\mu_{(1)}\times\Phi'(\mu_{(2)},\alpha_{(1)}))\backslash^{\times}(\mu_{(3)}\times\Phi'(\mu_{(4)},\alpha_{(2)}\times\beta)) \\ &= -\sum (\mu_{(1)}\times\alpha)\backslash^{\times}(\mu_{(2)}\times\beta) \\ &\quad -\sum \mu_{(1)}\backslash^{\times}(\mu_{(2)}\times\Phi'(\mu_{(3)},\alpha\times\beta)) \\ &= -\sum (\mu_{(1)}\times\alpha)\backslash^{\times}(\mu_{(2)}\times\beta) - \Phi'(\mu,\alpha\times\beta). \end{aligned}$$

It follows that

$$\begin{aligned} -p^{\cdot}(\underline{\mu},\alpha,\beta) + p^{\cdot}(\underline{\mu},\beta,\alpha) &= \sum (\mu_{(1)}\times\alpha)\backslash^{\times}(\mu_{(2)}\times\beta) - \\ &\quad -\sum (\mu_{(1)}\times\beta)\backslash^{\times}(\mu_{(2)}\times\alpha) + \Phi'(\mu,[\alpha,\beta]) \\ &= \sum (\mu_{(1)}\times\alpha)\backslash^{\times}(\mu_{(2)}\times\beta) - \\ &\quad -\sum (\mu_{(1)}\times\beta)\backslash^{\times}(\mu_{(2)}\times\alpha) + \epsilon(\mu)[\alpha,\beta], \end{aligned}$$

an expression that does not depend on the particular Φ'. □

Proposition 4.3. *Let $k[V]$ be a bialgebra with respect to the product $\cdot$. Given any set of multilinear operations*

$$\Phi = \{\Phi_{i,j} : k[V]_i \otimes k[V]_j \to V\}$$

for $i \geq 1$ and $j \geq 2$ there exists a product $\times$ on $k[V]$ similar to $\cdot$, such that the operations Φ^{SU} in $(k[V],\times)$ coincide with the operations Φ.

Proof. Extend the definition of the $\Phi_{i,j}$ to the cases $i = 0$ and $j = 1$ by setting $\Phi_{0,j}$ and $\Phi_{i,1}$ to be identically zero. Take

$$\Psi(x,1) = \epsilon(x)1$$

for any $x \in k[V]$ and define the bialgebra similarity

$$\Psi\colon k[V]\otimes k[V] \to k[V]$$

inductively by

$$\Psi(x,b^{m+1}) = \sum x_{(1)}\backslash^{\cdot}\Big((x_{(2)}\cdot\Psi(x_{(3)},b^m_{(1)}))\cdot\big(\epsilon(x_{(4)})\epsilon(b^m_{(2)})b - \Phi(\underline{x}_{(4)};\underline{b}^m_{(2)},b)\big)\Big)$$

for any $x \in k[V]$ and $b \in V$. Here b^{m+1} stands for $((b \cdot b) \dots) \cdot b$. According to Lemma 4.1 this determines Ψ completely. It is easy to check that $\Psi(x,b) = \epsilon(x)b$, and an induction on m shows that $\Psi(1,b^m) = b^m$, $\Delta(\Psi(x,b^m)) = \Psi \otimes \Psi(\Delta(x,b^m))$ and $\epsilon(\Psi(x,b^m)) = \epsilon(x)\epsilon(b^m)$.

Define a new product $\times$ on $k[V]$ by setting

$$x \times y = \sum x_{(1)} \cdot \Psi(x_{(2)}, y).$$

In $(k[V], \times)$, on one hand,

$$(x, b^m, b) = (x_{(1)} \times b^m_{(1)}) \times \Phi^{SU}(\underline{x}_{(2)}; \underline{b}^m_{(2)}, b),$$

and, on the other hand,

$$\begin{aligned}(x, b^m, b) &= \sum (x_{(1)} \cdot \Psi(x_{(2)}, b^m)) \cdot b - \sum x_{(1)} \cdot \Psi(x_{(2)}, b^{m+1}) \\ &= \sum (x_{(1)} \cdot \Psi(x_{(2)}, b^m_{(1)})) \cdot \Phi(\underline{x}_{(3)}; \underline{b}^m_{(2)}, b) \\ &= \sum (x_{(1)} \times b^m_{(1)}) \times \Phi(\underline{x}_{(2)}; \underline{b}^m_{(2)}, b).\end{aligned}$$

Using these two ways of computing $\sum (x_{(1)} \times b^m_{(1)}) \backslash^{\times} (x_{(2)}, b^m_{(2)}, b)$ we get $\Phi^{SU} = \Phi$ as desired. □

4.3. **The equivalence of categories.** It is known from [PI07] that the category of irreducible bialgebras is equivalent to that of Sabinin algebras. The proof given in [PI07], however, uses primitive operations different from the original operations $p(x_1, \dots, x_m; y_1, \dots, y_n; z)$ considered by Shestakov and Umirbaev. Here we shall show that the functor that assigns to an irreducible bialgebra its subspace of primitive elements with the operations defined in the Section 4.1 also gives an equivalence of categories.

Lemma 4.4. *Let W be an irreducible bialgebra, $(\mathrm{Prim}(W), \langle\,;\,,\,\rangle', \Phi')$ the Sabinin subalgebra of its primitive elements and $\mathcal{V} = \{e_1, \dots, e_\alpha, \dots\}$ - a basis of the vector space $\mathrm{Prim}(W)$. Let $k\{\mathcal{V}\}$ be the unital free non-associative algebra on $\mathcal{V}$ and I the ideal of $k\{\mathcal{V}\}$ generated by*

$$\langle u; a, b\rangle - \langle \underline{u}; a, b\rangle' \quad \text{and} \quad \Phi^{SU}(u; v) - \Phi'(\underline{u}, \underline{v})$$

for any $a, b \in \mathrm{Prim}(W)$ and u, v right-normed[1] *monomials in the e_i. Then $W \cong k\{\mathcal{V}\}/I$.*

Proof. Denote by $\bar{W}$ be the algebra $k\{\mathcal{V}\}/I$ and by $\pi\colon k\{\mathcal{V}\} \to \bar{W}$ the quotient map. Since $\mathcal{V}$ is a basis of $\mathrm{Prim}(W)$, there is an epimorphism $k\{\mathcal{V}\} \to W$ which vanishes on I, and, hence, factors through an epimorphism $\varphi\colon \bar{W} \to W$. In order to show that φ is an isomorphism, we exhibit a vector space basis of $\bar{W}$ which is sent by φ to the Poincaré–Birkhoff–Witt basis of W.

By definition, in $\bar{W}$ we have

$$ua \cdot b - ub \cdot a = -\sum u_{(1)} \langle \underline{u}_{(2)}; a, b\rangle' \tag{10}$$

for any right-normed monomial u in $\pi(\mathcal{V})$ and $a, b \in \pi(\mathcal{V})$. It follows that any two right-normed monomials in $\pi(\mathcal{V})$ that differ only by a permutation of their variables, are equal in $\bar{W}$ modulo monomials of smaller degree. Using this fact, together with the definition of Φ^{SU} we see that, modulo the right-normed monomials of lower order

$$uv \cdot a - u \cdot va \equiv \sum (u_{(1)} v_{(2)}) \Phi'(\underline{u}_{(2)}; \underline{v}_{(2)}) \equiv 0 \tag{11}$$

for any pair of right-normed monomials u and v in $\pi(\mathcal{V})$ and $a \in \pi(\mathcal{V})$.

Using the induction on the degree of the monomials we now can deduce that $\bar{W}$ admits a Poincaré–Birkhoff–Witt type set of linear generators $((\bar{e}_{i_1} \bar{e}_{i_2}) \cdots) \bar{e}_{i_k}$ where $0 \le i_1 \le \cdots \le i_k, k \ge 0$ and $\bar{e}_\alpha = \pi(e_\alpha)$. Since φ sends this set to a Poincaré–Birkhoff–Witt basis of W then it must be a basis of $\bar{W}$ and $\bar{W} \cong W$. □

Theorem 4.5. *The functor from the category of irreducible bialgebras to that of Sabinin algebras, which assigns to a bialgebra W the Sabinin subalgebra $\mathrm{Prim}(W)$ of УШ(W) is an equivalence of categories.*

[1] that is, of the form $((e_{i_1} e_{i_2}) \dots)$, or, in other words, with all opening brackets to the left of the first argument.

Proof. We will show that the functor $W \mapsto \mathrm{Prim}(W)$ is (1) faithful, (2) full and (3) essentially surjective.

(1) Recall that any irreducible bialgebra, as a coalgebra, is isomorphic to $k[V]$ where V is the space of the primitive elements. Therefore, by Corollary 2.4.17 in [Abe80] any homomorphism $W \to W'$ of bialgebras, with W irreducible, is determined by its restriction to $\mathrm{Prim}(W)$. This implies that the functor $W \to \mathrm{Prim}(W)$ is faithful.

(2) This is a consequence of Lemma 4.4.

(3) It was shown in [PI07] that given a Sabinin algebra $(V, \langle ; , \rangle, \Phi')$ there exists an irreducible cocommutative unital bialgebra $(k[V], \cdot)$ such that the operations $\langle ; , \rangle$ are recovered as

$$(x \cdot a) \cdot b - (x \cdot b) \cdot a = -\sum x_{(1)} \cdot \langle \underline{x}_{(2)}; a, b\rangle.$$

Now, by Proposition 4.3 the product $\cdot$ can always be modified in such a way that the operations $\langle ; , \rangle$ remain the same and that the multioperator on V takes any desired form. □

5. Sabinin algebras and formal multiplications

In this section we show directly, following the method of Sabinin and Mikheev, that the category of Sabinin algebras and that of unital formal multiplications are equivalent. As a result, we shall have two constructions of a Sabinin algebra associated with a formal multiplication: via the primitive elements in the bialgebra of distributions, described in the preceding two sections, and the direct construction of the present section. These two constructions, however, do not coincide. We shall prove that the operations $\langle ; , \rangle$ are the same in both cases and exhibit a formal multiplication for which the two multioperators are different.

5.1. **The geometry of the operations in a Sabinin algebra.** For a Lie group G the left multiplication by elements of G gives a flat connection (the *canonical connection*) on the tangent bundle of G. All covariant derivatives of the torsion tensor of the canonical connection vanish and the torsion tensor itself coincides on the tangent space to the unit, up to sign, with the bracket of the Lie algebra of G.

A generalization of this approach led Sabinin and Mikheev to the first successful general treatment of the non-associative Lie theory. They observed that an infinitesimal loop satisfying the right alternative identity is, essentially, the same thing as a germ of a flat affine connection. It is known that an (analytic) flat affine connection can be reconstructed locally from its torsion tensor and its covariant derivatives; therefore, these tensors provide analogues of the Lie brackets for right alternative infinitesimal loops. The identities for these operations are the universal identities satisfied by the covariant derivatives of the torsion tensor of a flat affine connection; their explicit form is well-known.

Any infinitesimal loop determines a unique right alternative infinitesimal loop; and the difference between the two is measured by the operation $\Phi(a, b)$ defined by the equation (7). If this operation is analytic, it is reconstructed from its Taylor series in the normal coordinates (local coordinates on the loop coming from the tangent space via the exponential map). The homogeneous terms of this Taylor series form a set of multilinear operations (multioperator) which complements the derivatives of the torsion tensor as a part of the structure of a Sabinin algebra.

These constructions can be translated into the formal setting with minimal effort, as we shall now see.

5.2. **The torsion of a formal flat connection and the Mikheev-Sabinin brackets.** A *formal vector field* is a linear map $A : k[V] \to V$. The product of a formal vector field A with a formal function f is given by

$$fA \colon \mu \mapsto \sum f\big(\mu_{(1)}\big) A\big(\mu_{(2)}\big).$$

This action provides the formal vector fields with the structure of a free $k[V]^*$–module. In fact, any set $\{A_i\}_i$ of formal vector fields such that $\{A_i(1)\}$ is a basis of V gives a $k[V]^*$–basis of $\mathrm{Hom}(k[V], V)$.

A formal vector field A gives a derivation D_A of the algebra $k[V]^*$ of formal functions into itself:

$$D_A(f) = A(f) \colon \mu \mapsto \sum f\big(\mu_{(1)} A(\mu_{(2)})\big)$$

where the product on $k[V]$ is that of the symmetric algebra. We have $(fA)(g) = f \cdot A(g)$. Formal vector fields form a Lie algebra with the Lie bracket $[A, B]$ given by

$$[A, B] \colon \mu \mapsto \sum B\big(\mu_{(1)} A(\mu_{(2)})\big) - A\big(\mu_{(1)} B(\mu_{(2)})\big).$$

Clearly $[D_A, D_B] = D_{[A,B]}$. We also have that

$$[A, fB] = A(f)B + f[A, B].$$

A *formal flat affine connection* is a linear map $k[V] \otimes V \to V$ whose restriction to $1 \otimes V$ is the identity. For a given formal connection, $\mu \in k[V]$ and $v \in V$, we write $\mu * v$ for the image of $\mu \otimes v$. The vector field $v^*\colon \mu \mapsto \mu * v$ is said to be *adapted* to the *tangent vector* v. There always exists a unique "inverse" map $k[V] \otimes V \to V$ sending $\mu \otimes u$ to an element that we denote by $\mu \backslash^* u$ and such that $\sum \mu_{(1)} \backslash^* (\mu_{(2)} * v) = \epsilon(\mu)v = \sum \mu_{(1)} * (\mu_{(2)} \backslash^* v)$.

The covariant differentiation with respect to the formal vector field A is defined as

$$\nabla_A(B)\colon \mu \mapsto \sum B\big(\mu_{(1)} A(\mu_{(2)})\big) - \big(\mu_{(1)} A(\mu_{(2)})\big) * \big(\mu_{(3)} \backslash^* B(\mu_{(4)})\big).$$

Proposition 5.1. *Let A, B be formal vector fields, f a formal function and $v, w \in V$. Then*

(1) $\nabla_{fA}(B) = f\nabla_A(B)$,
(2) $\nabla_A(fB) = A(f)B + f\nabla_A(B)$,
(3) $\nabla_{v^*}(w^*) = 0$.

Proof. We shall only prove (3). By definition

$$\begin{aligned}\nabla_{v^*}(w^*)(\mu) &= \sum w^*(\mu_{(1)} v^*(\mu_{(2)})) - (\mu_{(1)} v^*(\mu_{(2)})) * (\mu_{(3)} \backslash^* w^*(\mu_{(4)}))\\ &= \sum (\mu_{(1)}(\mu_{(2)} * v)) * w - (\mu_{(1)}(\mu_{(2)} * v)) * (\mu_{(3)} \backslash^* (\mu_{(4)} * w))\\ &= \sum (\mu_{(1)}(\mu_{(2)} * v)) * w - (\mu_{(1)}(\mu_{(2)} * v)) * (\epsilon(\mu_{(3)})w) = 0.\end{aligned}$$

□

The torsion of two formal vector fields A and B is defined in the usual way

$$T(A, B) = \nabla_A(B) - \nabla_B(A) - [A, B].$$

In the case of adapted vector fields x^*, y^* with $x, y \in V$ we get

$$T(x^*, y^*) = -[x^*, y^*].$$

Now, assume that a unital formal multiplication F is given on V and denote by $\mu_1 \cdot \mu_2$ the corresponding product on distributions. As mentioned in Section 3.2, it gives rise to a formal connection simply by restricting F to $k[V] \otimes V$. The action of the adapted vector fields on functions is easily derived from the product $\mu_1 \cdot \mu_2$ on $k[V]$.

Lemma 5.2. *Let $\gamma\colon k[V] \to k[V]$ a linear map that satisfies*

$$\Delta(\gamma(\mu)) = \sum \gamma(\mu_{(1)}) \otimes \mu_{(2)} + \mu_{(1)} \otimes \gamma(\mu_{(2)}).$$

Then

$$\gamma(\mu) = \sum \mu_{(1)} \pi_V\big(\gamma(\mu_{(2)})\big).$$

Proof. Let S denote the antipode of the symmetric algebra $k[V]$ considered as a Hopf algebra. Since

$$\begin{aligned}\Delta\Big(\sum S(\mu_{(1)})\gamma(\mu_{(2)})\Big) &= \sum S(\mu_{(1)})\gamma(\mu_{(2)}) \otimes S(\mu_{(3)})\mu_{(4)}\\ &\quad + S(\mu_{(3)})\mu_{(4)} \otimes S(\mu_{(1)})\gamma(\mu_{(2)})\\ &= \sum S(\mu_{(1)})\gamma(\mu_{(2)}) \otimes 1 + 1 \otimes \sum S(\mu_{(1)})\gamma(\mu_{(2)})\end{aligned}$$

we have that $\sum S(\mu_{(1)})\gamma(\mu_{(2)})$ is primitive. Considering the degrees of the terms in this expression, we see that $\sum S(\mu_{(1)})\gamma(\mu_{(2)}) = \pi_V(\gamma(\mu))$. Thus

$$\gamma(\mu) = \sum \mu_{(1)} S(\mu_{(2)})\gamma(\mu_{(3)}) = \sum \mu_{(1)} \pi_V(\gamma(\mu_{(2)})).$$

□

Lemma 5.3. *For any $x \in V$ and $f \in k[V]^*$ we have*

$$x^*(f)(\mu) = f(\mu \cdot x).$$

Proof. Notice that $\gamma\colon \mu \mapsto \mu \cdot x$ satisfies $\Delta(\gamma(\mu)) = \sum \gamma(\mu_{(1)}) \otimes \mu_{(2)} + \mu_{(1)} \otimes \gamma(\mu_{(2)})$. This implies that

$$x^*(f)(\mu) = \sum f(\mu_{(1)} x^*(\mu_{(2)})) = \sum f(\mu_{(1)} \pi_V(\mu_{(2)} \cdot x)) = f(\mu \cdot x).$$

□

If T is the torsion tensor of this connection, then setting

$$\langle x_1, \dots, x_n; y, z\rangle_F = \nabla_{x_1^*} \cdots \nabla_{x_n^*} T(y^*, z^*)(1)$$

we obtain an $n+2$-linear operation on V for all $n \geq 0$. In case that G is an analytic local loop, the corresponding affine flat connection is determined by its adapted vector fields v^*, $v \in \mathrm{T}_e G$. For any analytic function f on G and any distribution μ with support at the identity e, the construction of Mikheev and Sabinin provides $\mu(v^*(f)) = \mu(g)$ with $g\colon a \mapsto v(f \circ L_a)$, so $\mu(v^*(f)) = (\mu \cdot v)(f)$ in the bialgebra of distributions of G with support at the identity. Under the identification of analytic functions with elements of $k[V]^*$, Lemma 5.3 shows that the definition of adapted vector fields that we present agrees with this one. Therefore, the formal connection, torsion and bracket operations that we define agree with the corresponding constructions by Mikheev and Sabinin. In [SM87] they proved

Proposition 5.4. *Assigning the set of operations $\langle x_1, \dots, x_n, y, z\rangle_F$ to a formal multiplication F gives a functor from the category of formal loops to that of Sabinin algebras with trivial multioperator.*

The torsion tensor also admits a simple interpretation in terms of the product $\mu_1 \cdot \mu_2$.

Lemma 5.5. *For any $x, y \in V$ and $\mu \in k[V]$ it holds*

$$T(x^*, y^*)(\mu) = \pi_V\left((\mu \cdot y) \cdot x - (\mu \cdot x) \cdot y\right).$$

Proof. We have that

$$\begin{aligned} T(x^*, y^*)(\mu) = -[x^*, y^*](\mu) &= \sum x^*(\mu_{(1)} y^*(\mu_{(2)})) - y^*(\mu_{(1)} x^*(\mu_{(2)})) \\ &= \sum x^*(\mu_{(1)} \pi_V(\mu_{(2)} \cdot y)) - y^*(\mu_{(1)} \pi_V(\mu_{(2)} \cdot x)) \\ &= \sum x^*(\mu \cdot y) - y^*(\mu \cdot x) = \pi_V((\mu \cdot y) \cdot x - (\mu \cdot x) \cdot y) \end{aligned}$$

□

Recall that a set of multilinear brackets $\langle x_1, \dots, x_n; y, z\rangle$ on V can be defined via the Shestakov-Umirbaev operations.

Theorem 5.6. *The operations $\langle x_1, \dots, x_n; y, z\rangle$ of Shestakov and Umirbaev identically coincide with the operations $\langle x_1, \dots, x_n; y, z\rangle_F$ of Mikheev and Sabinin.*

Proof. Let $\{v_i\}_i$ be a basis of V and define formal functions $\{f_i\}_i$ such that

$$\langle \underline{\mu}; y, z\rangle = \sum_i f_i(\mu) v_i.$$

The map $\gamma\colon \mu \mapsto (\mu \cdot z) \cdot y - (\mu \cdot y) \cdot z$ satisfies the condition in Lemma 5.2 so

$$\begin{aligned} \sum \mu_{(1)} T(y^*, z^*)(\mu_{(2)}) = (\mu \cdot z) \cdot y - (\mu \cdot y) \cdot z &= \sum \mu_{(1)} \cdot \langle \underline{\mu}_{(2)}; y, z\rangle \\ &= \sum \sum_i f_i(\mu_{(2)}) \mu_{(1)} \cdot v_i = \sum \sum_i f_i(\mu_{(1)}) \mu_{(2)} \pi_V(\mu_{(3)} \cdot v_i) \\ &= \sum \sum_i \mu_{(1)}(f_i(\mu_{(2)}) v_i^*(\mu_{(3)})) \\ &= \sum \mu_{(1)} \left(\sum_i f_i v_i^*(\mu_{(2)}) \right). \end{aligned}$$

This proves that

$$T(y^*, z^*) = \sum_i f_i v_i^*.$$

The covariant differentiation of the torsion T is then given by

$$\nabla_{x_1^*}\cdots\nabla_{x_n^*}T(y^*,z^*)=\sum_i x_1^*\cdots x_n^*(f_i)v_i^*$$

and the operations of Mikheev and Sabinin are recovered as

$$\begin{aligned}\langle x_1,\dots,x_n;y,z\rangle_F &= \nabla_{x_1^*}\cdots\nabla_{x_n^*}T(y^*,z^*)(1)\\ &= \sum_i f_i(((x_1\cdot x_2)\cdots)\cdot x_n)v_i\\ &= \langle x_1,\dots,x_n;y,z\rangle\end{aligned}$$

as desired. □

5.3. **Multioperators.** For a local analytic loop $(G,\cdot)$, the multioperator is a series of operations on the tangent space $V=\mathrm{T}_eG$ at the identity of G given by

$$\Phi'(x,y)=\log\Phi(\exp x,\exp y)$$

where Φ is as in (7). The homogeneous components of Φ' are linear maps

$$\Phi'_{i,j}:k[V]_i\otimes k[V]_j\to V.$$

Each $\Phi'_{i,j}$ can be thought of either as a multilinear map $V^{\otimes i+j}\to V$ which is totally symmetric in two groups of variables, namely, the first i and the last j variables, or as a polynomial map in two variables and bidegree (i,j). In the language of Section 3.2 Φ' is a similarity.

The construction works for arbitrary formal loops if, instead of the exponential map $\mathrm{T}_eG\to G$ one uses the exponential series as defined in the Appendix. In particular, let us consider it for the formal loop of *non-associative polynomials.*

Let $\mathcal{S}$ be a set and $k\{\mathcal{S}\}$ - the unital free non-associative algebra generated by the elements of $\mathcal{S}$. Denote by $\mathcal{R}$ the ideal in $k\{\mathcal{S}\}$ generated by $\mathcal{S}$. There is a unital formal multiplication on $\mathcal{R}$ sending $x\otimes y$ to $x+y+xy$, where the product xy is taken in $\mathcal{R}$.

For $\alpha,\beta\in\mathcal{R}$ write $a=\exp\alpha$, $b=\exp\beta$ and $\Phi'=\sum\Phi'_{i,j}(\alpha;\beta)$. (Here we treat $\Phi'_{i,j}$ as a function of two variables α,β which is of degree i in α and j in β.) Then (7) has the form

$$\exp_a(a\Phi')=\sum_{k=0}^{\infty}\frac{1}{k!}((a\Phi')\cdots)\Phi'=ab.$$

This formula may be seen as a recursive definition of Φ'. For instance, expanding a as a series in α we see that

$$\begin{aligned}\Phi'_{1,3}&=-\frac{1}{12}[\beta,(\alpha,\beta,\beta)]-\frac{1}{6}p_{1,2}(\alpha;\beta^2;\beta)\\ \Phi'_{2,3}&=-\frac{1}{12}(\alpha,\beta,(\alpha,\beta,\beta))+\frac{1}{12}(\alpha,(\alpha,\beta,\beta),\beta)\\ &\qquad-\frac{1}{24}[v,p_{2,1}(\alpha^2;\beta;\beta)]-\frac{1}{12}p_{2,2}(\alpha^2;\beta^2;\beta).\end{aligned}$$

These expressions are essentially different from the multioperator of Shestakov and Umirbaev

$$\Phi^{SU}_{i,j}=p_{i,j-1}(\alpha^i,\beta^{j-1},\beta)/i!j!.$$

In general, we do not have such a closed formula for the Sabinin-Mikheev multioperator.

6. Linear formal loops

Any finite–dimensional unital algebra A over the real numbers defines a local loop in a neighborhood on the identity 1. By translation $x\mapsto x-1$ we obtain a local loop in a neighborhood of 0. The product xy of this local loop is related with the product $x*y$ of A by

$$xy=x+y+x*y.$$

This formula, in fact, defines a unital formal multiplication on A considered as a vector space. We shall denote this formal loop by G. Note that the existence of the identity in A is not relevant here, so A can be taken to be non-unital.

As a vector space, A can be identified with $\mathrm{Prim}(k[G])$ and, hence, there are two ways to give the structure of a Sabinin algebra to A: using the Shestakov-Umirbaev operations in $k[G]$ and in the algebra $(A,*)$.

Theorem 6.1. *$\mathrm{Prim}(k[G])$ and $\text{УШ}(A,*)$ coincide as Sabinin algebras.*

Proof. Let $(A^{\#},*) = k1 \oplus A$ the algebra obtained by adding a formal unit element 1 to A.

The map $\pi_{A^{\#}}: k[G] \to A^{\#}$ $\mu \mapsto \epsilon(\mu)1 + \pi_A(\mu)$ which assigns to a distribution its component of degree ≤ 1 is, in fact, a homomorphism of algebras. Indeed, $G(\mu_1,\mu_2) = \epsilon(\mu_2)\pi_A(\mu_1) + \epsilon(\mu_1)\pi_A(\mu_2) + \pi_A(\mu_1) * \pi_A(\mu_2)$ so $\pi_{A^{\#}}(G'(\mu_1,\mu_2)) = \epsilon(\mu_1)\epsilon(\mu_2)1 + G(\mu_1,\mu_2) = \epsilon(\mu_1)\epsilon(\mu_2)1 + \epsilon(\mu_2)\pi_A(\mu_1) + \epsilon(\mu_1)\pi_A(\mu_2) + \pi_A(\mu_1) * \pi_A(\mu_2) = \pi_{A^{\#}}(\mu_1) * \pi_{A^{\#}}(\mu_2)$. Since the Shestakov-Umirbaev operations are functorial with respect to algebra homomorphisms then they coincide on $\mathrm{Prim}(k[G]) = A$.

□

Corollary 6.2. *With the previous notation $k[G] \cong U(\text{УШ}(A))$ as bialgebras.*

Definition 6.3. *A formal loop F is called linear if there exists a finite-dimensional vector space A with a bilinear product $x * y$ and a homomorphism*

$$\psi\colon F \to G$$

with $G(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}+\boldsymbol{y}+\boldsymbol{x}*\boldsymbol{y}$, where $\boldsymbol{x}*\boldsymbol{y}$ stands for the formal map $k[A]\otimes k[A] \to A$, $\mu_1\otimes\mu_2 \mapsto \pi_A(\mu_1)*\pi_A(\mu_2)$, such that the induced $\Psi\colon k[G]^* \to k[F]^*$, $g \mapsto g \circ \psi'$ is an epimorphism.

Lemma 6.4. *Let F, G be formal loops and $\psi\colon F \to G$ a homomorphism of formal loops. Then $\Psi\colon k[G]^* \to k[F]^*$, $g \mapsto g(\psi')$ is surjective if and only if $\psi'\colon k[F] \to k[G]$ is injective.*

Proposition 6.5. *Let F be a formal loop. Then F is linear if and only if there exists a finite–codimensional ideal I of the algebra $k[F]$ with $I \cap \mathrm{Prim}(k[F]) = 0$.*

Proof. Suppose that F is linear. Then there exist $G(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}+\boldsymbol{y}+\boldsymbol{x}*\boldsymbol{y}$, with $x*y$ bilinear, and $\psi\colon F \to G$ homomorphism such that the induced homomorphism $\psi'\colon k[F] \to k[G]$ is injective. In the proof of Theorem 6.1 we saw that the identity map on A extends to surjective homomorphism $\pi_{A^{\#}}: k[G] \to (A^{\#},*)$. The kernel I of the composition $\pi_{A^{\#}}\psi'$ is a finite–codimensional ideal of $k[F]$ with $I \cap \mathrm{Prim}(k[F]) = 0$.

Conversely, assume that I is a finite–codimensional ideal of $k[F]$ such that $I \cap \mathrm{Prim}(k[F]) = 0$. Consider $A = k[F]/I$, with the product denoted by $*$, and $G(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}+\boldsymbol{y}+\boldsymbol{x}*\boldsymbol{y}$ the corresponding unital formal multiplication. Using the universal property of $k[F]$ we see that the monomorphism $\mathrm{Prim}(k[F]) \to A \cong \mathrm{Prim}(k[G])$ of Sabinin algebras induces a homomorphism of bialgebras $\psi'\colon k[F] \to k[G]$ with injective restriction to $\mathrm{Prim}(k[F])$. By Theorem 2.4.11 in [Abe80], this map ψ' must be injective too. The proposition now follows from Lemma 6.4. □

In [PIS04] it was proved that any Moufang formal loop is linear, a result that extends Ado's theorem to Malcev algebras. However, there exist formal multiplications that are not linear. Important examples come from Bruck loops. A *Bruck loop* is a loop that satisfies the Bol identity

$$a(b(ac)) = (a(ba))c$$

(which implies that $L_a^{-1} = L_{a^{-1}}$ for some a^{-1}) and the *automorphic inverse property*

$$(ab)^{-1} = a^{-1}b^{-1}$$

for all a, b, c. The bialgebra of distributions $k[F]$ of a formal Bruck loop satisfies

$$\sum \mu_{(1)}(\nu(\mu_{(2)}\eta)) = \sum (\mu_{(1)}(\nu\mu_{(2)}))\eta$$

and there exists a map S such that

$$\mu\backslash\nu = S(\mu)\nu$$

and

$$S(\mu\nu) = S(\mu)S(\nu)$$

for all $\mu, \nu, \eta \in k[F]$. In this case all the operations of the Sabinin algebra $\mathrm{Prim}(k[F])$ are easily described in terms of a Lie triple system, and conversely, any Lie triple system provides a formal Bruck loop law. However, Lie triple systems that are not nilpotent do not provide linear multiplications [PI08].

6.1. **Some examples.** Let us consider the bilinear product on k^3

$$x * y = (x_1y_1 + x_2y_3 + x_3y_2, x_1y_2 + x_2y_1, x_1y_3 + x_3y_1)$$

and the formal multiplications

$$\begin{aligned} G(x, y) &= x + y + x * y \\ F((x_2, x_3), (y_2, y_3)) &= \frac{1}{1 + x_2y_3 + x_3y_2}(x_2 + y_2, x_3 + y_3) \end{aligned}$$

The map

$$\phi = \Big(\frac{x_2}{1 + x_1}, \frac{x_3}{1 + x_1}\Big)$$

defines a homomorphism $\phi\colon G \to F$ of formal loops. It induces a surjective homomorphism $\phi'\colon k[G] \to k[F]$ determined by $\phi'(\partial_1) = 0$, $\phi'(\partial_2) = \partial_2$ and $\phi'(\partial_3) = \partial_3$, where ∂_i is the basis vector of k^3 corresponding to the coordinate x_i. These formulae come from considering $(V, (\ ,\ ))$ a two–dimensional vector space with a bilinear form of maximal Witt index, $A = \mathbb{R}e \oplus V$ the Jordan algebra with the product

$$(\alpha e + a)(\beta e + b) = (\alpha\beta + (a, b))e + \alpha b + \beta a$$

and the formal loop determined by A. The subspace $\mathbb{R}e$ may be thought of as a normal subloop and F as the quotient of G by $\mathbb{R}e$. Although the formal loop G is linear, we shall see that F is not . To simplify the notation involved in our computations, we shall identify $k[G]$ with $U(\text{УШ}(A))$.

Lemma 6.6. *Let A be a Jordan algebra, then in $U(\text{УШ}(A))$ we have*

$$p(a, \underline{xc}, b) = -\sum p(c, \underline{x}_{(1)}, p(a, \underline{x}_{(2)}, b)) + \epsilon(x)(a, c, b)$$

and

$$(a, xc, b) = (a, x, b)c + x(a, c, b)$$

for any primitive a, b, c and any $x \in U(\text{УШ}(A))$.

Proof. The map $x \mapsto (a, x, b)$ is a derivation of any Jordan algebra so in A

$$(a, xc, b) = \begin{cases} \sum x_{(1)}p(a, \underline{x}_{(2)}c, b) + \sum (x_{(1)}c)p(a, \underline{x}_{(2)}, b) \\ (a, x, b)c + x(a, c, b) = \sum c\big(x_{(1)}p(a, \underline{x}_{(2)}, b)\big) + x(a, c, b) \end{cases}$$

thus

$$\begin{aligned} \sum x_{(1)}p(a, \underline{x}_{(2)}c, b) &= \sum -(x_{(1)}, c, p(a, \underline{x}_{(2)}, b)) + x(a, c, b) \\ &= \sum -x_{(1)}p(\underline{x}_{(2)}, c, p(a, \underline{x}_{(3)}, b)) + \sum x_{(1)}\epsilon(x_{(2)})(a, c, b) \end{aligned}$$

Dividing by $x_{(1)}$ we get the first equality. The second equality follows from the first one by reversing our argument in $U(\text{УШ}(A))$ (notice that $U(\text{УШ}(A))$ is commutative). □

Theorem 6.7. *The formal loop*

$$F((x_2, x_3), (y_2, y_3)) = \frac{1}{1 + x_2y_3 + x_3y_2}(x_2 + y_2, x_3 + y_3)$$

is not linear.

Proof. Any finite–codimensional ideal of $k[F]$ that meets trivially the primitive elements provides a finite–codimensional ideal of $k[G]$ that contains ∂_1 and with trivial intersection with $k\partial_2 + k\partial_3$. With the identification $k[G] \cong U(\text{УШ}(A))$ we obtain a finite–codimensional ideal I of $U = U(\text{УШ}(A))$ with $e \in I$ and $V \cap I = 0$. Let us show that this is not possible. We will fix $a, b \in V$ with $(a, a) = 0 = (b, b)$ and $(a, b) = 2$.

Since A is a Jordan algebra, A is commutative and power–associative, so the formal loop determined by A also is. The universal enveloping algebra U is commutative and the powers x^n are well–defined for any $x \in \mathrm{Prim}(U)$. The dimension of U/I is finite so we can find a linear combination $a^N + \alpha_1 a^{N-1} + \cdots + \alpha_{N-1}a \in I$.

By the previous Lemma, we conclude that $a^N \in I$. We also assume that N is minimal with respect to this property.

In A the powers a^m vanish if $m \geq 2$. In such a case the relation $(a^m, b, b) = \sum a^m_{(1)} p(\underline{a}^m_{(2)}, b, b)$ implies that

$$p(\underline{a}^m, b, b) = -map(\underline{a}^{m-1}, b, b) = \cdots = (-1)^{m-1} 4m! a^{m-2}$$

and we obtain $p(a, b, b) = 2b, p(\underline{a}^2, b, b) = -8e, p(\underline{a}^3, b, b) = 24a$ and $p(\underline{a}^m, b, b) = 0$ if $m \geq 4$. Let us use these formulae to prove that $a^N \in I$ implies $a \in I$, which is not possible because $V \cap I = 0$. In case that $N = 2$, in $U(A)$ we have that modulo I

$$\begin{aligned} 0 &\equiv (a^2, b, b)a = p(\underline{a}^2, b, b)a + 2ap(a, b, b) \cdot a = -8ea + 4ab \cdot a \\ &\equiv 4a^2 b - 4(a, a, b) \equiv -4(a, a, b) = 8a. \end{aligned}$$

In case that $N \geq 3$ then

$$\begin{aligned} 0 &\equiv (a^N, b, b)a = Na^{N-1} p(a, b, b) \cdot a + \binom{N}{2} a^{N-2} p(\underline{a}^2, b, b) \cdot a \\ &\quad + \binom{N}{3} a^{N-3} p(\underline{a}^3, b, b) \cdot a \\ &= 2Na^{N-1} b \cdot a - 4N(N-1)a^{N-2} e \cdot a + 4N(N-1)(N-2)a^{N-1} \\ &\equiv -2N(a, a^{N-1}, b) + 4N(N-1)(N-2)a^{N-1} \\ &= 4N(N-1)^2 a^{N-1} \end{aligned}$$

so $a^{N-1} \in I$, a contradiction with the minimality of N. □

Operations $\langle\, ;\, ,\, \rangle$ on Jordan algebras are determined by a Lie triple system. The same relation holds for Bol algebras with trivial binary product. This indicates that a formal loop determined by a Jordan algebra is similar to a formal Bruck loop [PI07].

Proposition 6.8. *If A is a Jordan algebra, in УШ(A) we have*

$$\langle \underline{xc}; a, b\rangle = \sum \langle \underline{x}_{(1)}; c, \langle \underline{x}_{(2)}; a, b\rangle\rangle \quad \textit{and} \quad \langle c; a, b\rangle = -(a, c, b)$$

if $|x| \geq 1$.

Proof. Since $U($УШ$(A))$ is commutative, by Lemma 6.6

$$\begin{aligned} (xc, a, b) - (xc, b, a) &= (xc)a \cdot b - (xc)(ab) - (xc)b \cdot a + (xc)(ba) \\ &= -(b, xc, a) = -(b, x, a)c - x(b, c, a) \end{aligned}$$

and

$$\begin{aligned} \sum (xc)_{(1)} \cdot \langle \underline{xc}_{(2)}; b, a\rangle &= (xc, a, b) - (xc, b, a) = -(b, x, a)c - x(b, c, a) \\ &= \sum x_{(1)} \langle \underline{x}_{(2)}; b, a\rangle \cdot c - x(b, c, a) \end{aligned}$$

so

$$\begin{aligned} \sum x_{(1)} \cdot \langle \underline{x}_{(2)} c; b, a\rangle &= \sum -x_{(1)} c \cdot \langle \underline{x}_{(2)}; b, a\rangle \\ &\quad + \sum x_{(1)} \langle \underline{x}_{(2)}; b, a\rangle \cdot c - x(b, c, a) \\ &= \sum -(c, x_{(1)}, \langle \underline{x}_{(2)}; b, a\rangle) - x(b, c, a) \\ &= \sum x_{(1)} \langle \underline{x}_{(2)}; c, \langle \underline{x}_{(3)}; b, a\rangle\rangle - x(b, c, a) \end{aligned}$$

as desired. □

The exponential. Let $\widehat{\mathcal{R}}$ be the algebra of non-associative power series in some set of variables with coefficients in k and with no constant term. Given $X \in \widehat{\mathcal{R}}$ we define $\exp X \in 1 + \widehat{\mathcal{R}}$ as

$$\exp X = 1 + X + \frac{X^2}{2!} + \frac{X^2 X}{3!} + \frac{((X^2)X)X}{4!} + \dots$$

It is readily seen that $\exp X$ is the value at $t = 1$ of the solution of the differential equation

$$\frac{da}{dt} = aX$$

with the initial condition $a(0) = 1$.

One may think of the algebra $\widehat{\mathcal{R}}$ is the tangent space at 1 to the multiplicative loop $1 + \widehat{\mathcal{R}}$. Right multiplication by $b \in 1 + \widehat{\mathcal{R}}$ defines a parallel transport of $\widehat{\mathcal{R}}$ to $b + \widehat{\mathcal{R}}$. More generally, the canonical connection on $1 + \widehat{\mathcal{R}}$ is defined by transporting $b + X \in b + \widehat{\mathcal{R}}$ to $c + c(b\backslash X) \in c + \widehat{\mathcal{R}}$ for all $b, c \in 1 + \widehat{\mathcal{R}}$.

Curves of the form $\exp Xt$ are the geodesics of the canonical connection that pass through 1. It is equally easy to write down the geodesics that pass through $b \in 1 + \widehat{\mathcal{R}}$. For $X \in \widehat{\mathcal{R}}$ define $\exp_b X$ as

$$\exp_b X = b + X + \frac{X(b\backslash X)}{2!} + \frac{(X(b\backslash X))(b\backslash X)}{3!} + \frac{((X(b\backslash X))(b\backslash X))(b\backslash X)}{4!} + \dots.$$

Then $\exp_b Xt$ satisfies the equation

$$\frac{da}{dt} = a(b\backslash X)$$

with the initial condition $a(0) = b$.

It is easily verified that, just as in the associative case, $X \in \widehat{\mathcal{R}}$ is primitive if and only if $\exp X \in 1 + \widehat{\mathcal{R}}$ is group-like, that is,

$$\Delta \exp X = \exp X \otimes \exp X.$$

This property, however, does not define the exponential series uniquely; see, for instance, [GH03].

The logarithm. The power series $\log(1 + x)$ is defined by $\exp(\log(1 + x)) = 1 + x$. The coefficients of $\log(1 + x)$ can be found as follows.

Assume that $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(x)$, the algebra of non-associative power series in one variable x. (One can forget altogether about the variable and think of the non-associative monomials in x as of rooted binary plane trees.)

Write $X = \sum_\tau X_\tau \tau$ where the sum runs over all non-associative monomials τ. Then $\exp X$ can be written as

$$\exp X = \sum_{\tau = (\dots(\tau_1\tau_2)\dots)\tau_k} \frac{X_{\tau_1} X_{\tau_2} \dots X_{\tau_k}}{k!} \cdot \tau.$$

Writing $\exp X = \sum_\tau a_\tau \tau$ we have

$$\begin{aligned} a_{(..(\tau_1\tau_2)\dots)\tau_k} &= X_{(..(\tau_1\tau_2)\dots)\tau_k} + \frac{1}{2!} X_{(..(\tau_1\tau_2)\dots)\tau_{k-1}} X_{\tau_k} \\ &\quad + \frac{1}{3!} X_{(..(\tau_1\tau_2)\dots)\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} + \dots. \end{aligned}$$

Also,

$$\begin{aligned} a_{(..(\tau_1\tau_2)\dots)\tau_{k-1}} X_{\tau_k} &= X_{(..(\tau_1\tau_2)\dots)\tau_{k-1}} X_{\tau_k} + \frac{1}{2!} X_{(..(\tau_1\tau_2)\dots)\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} + \dots, \\ a_{(..(\tau_1\tau_2)\dots)\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} &= X_{(..(\tau_1\tau_2)\dots)\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} \\ &\quad + \frac{1}{2!} X_{(..(\tau_1\tau_2)\dots)\tau_{k-3}} X_{\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} + \dots \end{aligned}$$

and so on.

Recall that the Bernoulli numbers B_k satisfy the identity

$$\sum_{k=0}^{n-1} \frac{B_k}{k!(n-k)!} = 0.$$

It follows that

$$X_{(..(\tau_1\tau_2)\dots)\tau_k} = a_{(..(\tau_1\tau_2)\dots)\tau_k} + \frac{B_1}{1!} a_{(..(\tau_1\tau_2)\dots)\tau_{k-1}} X_{\tau_k} + \frac{B_2}{2!} a_{(..(\tau_1\tau_2)\dots)\tau_{k-2}} X_{\tau_{k-1}} X_{\tau_k} + \dots + \frac{B_{k-1}}{(k-1)!} a_{\tau_1} X_{\tau_2} \dots X_{\tau_k}.$$

Now, set $a_x = 1$ and $a_\tau = 0$ for $\tau \neq x$. Then the X_τ are the coefficients of the power series $\log(1+x)$. Setting $\tau = (..((x\tau_1)\tau_2)\dots)\tau_k$ we see that

$$X_\tau = \frac{B_k}{k!} X_{\tau_1} \dots X_{\tau_k}.$$

Given a binary rooted plane tree τ define B_τ and $\tau!$ inductively as follows.

For $\tau = x$ set $B_\tau = \tau! = 1$. If $\tau \neq x$, there is only one way of writing τ as a product $(\dots((x\tau_1)\tau_2)\dots)\tau_k$. Set

$$B_\tau = B_k \cdot B_{\tau_1} \dots B_{\tau_k}$$

and

$$\tau! = k!\tau_1! \dots \tau_k!.$$

With this notation we have

$$\log(1+x) = \sum_\tau \frac{B_\tau}{\tau!} \cdot \tau.$$

Identities related to sums over trees. This expression for the coefficients of the non-associative logarithm implies certain identities on Bernoulli numbers. Imposing the associativity condition on $\widehat{\mathcal{R}}$, we turn our exponential into the usual exponential series; therefore, our logarithm becomes the usual logarithm. All monomials τ with $\deg \tau = n$ are sent to the monomial x^n. We obtain

$$\sum_{\deg \tau = n} \frac{B_\tau}{\tau!} = \frac{(-1)^{n+1}}{n}. \tag{12}$$

A direct proof of (12), together with a generalization of it, was communicated to us by D. Zagier.

Choose arbitrary weights $\beta_1, \beta_2, \dots$ and for $n \geq 1$ define λ_n as $\sum \beta_\tau$, where the sum runs over plane rooted trees τ of degree n and β_τ is defined as $\beta_{i_1} \cdots \beta_{i_k}$ if the vertices of τ have $i_1, \dots, i_k$ outgoing branches. Since each such tree consists of a root which is joined to the roots of some (ordered) collection of plane rooted trees, say $\tau_1, \dots, \tau_r$ of degrees $n_1, \dots, n_r \geq 1$ with $\sum_i n_i = n-1$, we have

$$\lambda_1 = 1\,, \qquad \lambda_n \;=\; \sum_{r \geq 1} \beta_r \sum_{\substack{n_1,\dots,n_r \geq 1 \\ n_1+\dots+n_r = n-1}} \lambda_{n_1} \cdots \lambda_{n_r} \qquad \text{if } n > 1\,.$$

Hence the generating function $L = L(x) = \sum_{n=1}^{\infty} \lambda_n x^n$ satisfies the functional equation

$$L \;=\; x\left(1 + \sum_{r=1}^{\infty} \beta_r\, L^r\right),$$

or

$$\frac{L}{\mathcal{B}(L)} \;=\; x\,,$$

where $\mathcal{B}(t) = 1 + \sum_{r=1}^{\infty} \beta_r\, t^r$. For instance, if all $\beta_r = 1$ then $\mathcal{B}(t) = \dfrac{1}{1-t}$ and therefore $L(1-L) = x$ or $L = \frac{1}{2}(1-\sqrt{1-4x})$, the standard generating function for the number $\binom{2n}{n}/(n+1)$ of plane rooted trees of degree n (= number of length n bracketings = nth Catalan number). If $\beta_r = B_r/r!$ then we have instead $\mathcal{B}(t) = \dfrac{t}{e^t - 1}$ and hence $x = e^L - 1$ or $L = \log(1+x)$, giving $\lambda_n = (-1)^{n-1}/n$, that is, the formula (12).

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