

RAMANUJAN CONGRUENCES FOR SIEGEL MODULAR FORMS

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ABSTRACT. We determine conditions for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. We extend these results to Siegel modular forms of degree 2 and as an application, we establish Ramanujan-type congruences for explicit examples of Siegel modular forms.

1. INTRODUCTION AND STATEMENT OF RESULTS

Congruences in the coefficients of automorphic forms have been the subject of much study. A famous early example involves the partition function $p(n)$ which counts the number of ways of writing n as a sum of non-increasing positive integers. Ramanujan established

$$(1.1) \quad \begin{aligned} p(5n + 4) &\equiv 0 \pmod{5} \\ p(7n + 5) &\equiv 0 \pmod{7} \\ p(11n + 6) &\equiv 0 \pmod{11}, \end{aligned}$$

which are now simply called *Ramanujan congruences*. More generally, an elliptic modular form with Fourier coefficients $a(n)$ is said to have a *Ramanujan-type congruence at $b \pmod{p}$* if $a(pn + b) \equiv 0 \pmod{p}$, where p is a prime. Ahlgren and Boylan [1] build on work by Kiming and Olsson [9] to prove that (1.1) are the only such congruences for the partition function. Nevertheless, congruences of non-Ramanujan-type also exist, as Ono [13] demonstrates. (See also Chapter 5 of Ono [14] for an account of congruences for the partition function.) The existence and non-existence of Ramanujan-type congruences for elliptic modular forms have recently been studied by Cooper, Wage, and Wang [4] and Sinick [20]. See also [5], which generalizes [1] to provide a method to find all Ramanujan-type congruences in certain weakly holomorphic modular forms.

In this paper, we investigate Ramanujan-type congruences for Siegel modular forms of degree 2. Throughout, $Z := \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ is a variable in the Siegel upper half space of degree 2, $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi iz}$, $q' := e^{2\pi i\tau'}$, and $\mathbb{D} := (2\pi i)^{-2} \left(4 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau'} - \frac{\partial^2}{\partial z^2} \right)$ is the generalized theta operator, which acts on Fourier expansions of Siegel modular

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forms as follows:

$$\mathbb{D} \left(\sum_{\substack{T=^tT \geq 0 \\ T \text{ even}}} a(T) e^{\pi i \operatorname{tr}(TZ)} \right) = \sum_{\substack{T=^tT \geq 0 \\ T \text{ even}}} \det(T) a(T) e^{\pi i \operatorname{tr}(TZ)},$$

where tr denotes the trace, and where the sum is over all symmetric, semi-positive definite, integral, and even 2×2 matrices. Additionally, we always let $p \geq 5$ be a prime and (for simplicity) we always assume that the weight k is an even integer.

Definition 1.1. *A Siegel modular form $F = \sum a(T) e^{\pi i \operatorname{tr}(TZ)}$ with p -integral rational coefficients has a Ramanujan-type congruence at $b \pmod{p}$ if $a(T) \equiv 0 \pmod{p}$ for all T with $\det T \equiv b \pmod{p}$.*

Note that such congruences at $0 \pmod{p}$ have already been studied in [3] and our main result in this paper complements [3] by giving the case $b \not\equiv 0 \pmod{p}$.

Theorem 1.2. *Let $F(Z) = \sum_{\substack{n,r,m \in \mathbb{Z} \\ n,m,4nm-r^2 \geq 0}} A(n,r,m) q^n \zeta^r q^m$ be a Siegel modular form of degree 2 and even weight k with p -integral rational coefficients and let $b \not\equiv 0 \pmod{p}$. Then F has a Ramanujan-type congruence at $b \pmod{p}$ if and only if*

$$(1.2) \quad \mathbb{D}^{\frac{p+1}{2}}(F) \equiv - \left(\frac{b}{p} \right) \mathbb{D}(F) \pmod{p},$$

where $\left(\frac{\cdot}{p} \right)$ is the Legendre symbol. Moreover, if $p > k$, $p \neq 2k - 1$, and there exists an $A(n,r,m)$ with $p \nmid \gcd(n,m)$ such that $A(n,r,m) \not\equiv 0 \pmod{p}$, then F does not have a Ramanujan-type congruence at $b \pmod{p}$.

Remarks:

- (1) *If F in Theorem 1.2 has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$, then it also has such congruences at $b' \pmod{p}$ whenever $\left(\frac{b}{p} \right) = \left(\frac{b'}{p} \right)$, i.e., there are $\frac{p-1}{2}$ or $p-1$ such congruences.*
- (2) *The condition $p \neq 2k-1$ in the second part of Theorem 1.2 is necessary since there are Siegel modular forms F of weight $\frac{p+1}{2}$ such that $F \not\equiv 0 \pmod{p}$ and $\mathbb{D}(F) \equiv 0 \pmod{p}$. For example, let F be the Siegel Eisenstein series of weight 4 normalized by $a\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1$ and take $p = 7$. Such Siegel modular forms satisfy (1.2) for any b and hence have Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$. The condition that there exists an $A(n,r,m) \not\equiv 0 \pmod{p}$ where $p \nmid \gcd(n,m)$ is also necessary since there exist Siegel modular forms F of weight $p-1$ such that $F \equiv 1 \pmod{p}$ (see Theorem 4.5 of [12]). Such forms have Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$.*

In Section 2, we investigate congruences of Jacobi forms and, in particular, we establish criteria for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. In Section 3, we use such congruences for Jacobi forms to prove

Theorem 1.2. Using our results, it is now a finite computation to find Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$ for any Siegel modular form. We give several explicit examples. Finally, we present a construction of Siegel modular forms that have Ramanujan-type congruences at $b \pmod{p}$ for arbitrary primes $p \geq 5$.

2. CONGRUENCES AND FILTRATIONS OF JACOBI FORMS

Let $J_{k,m}$ be the vector space of Jacobi forms of even weight k and index m (for details on Jacobi forms, see Eichler and Zagier [6]). The heat operator $L_m := (2\pi i)^{-2} \left(8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right)$ is a natural tool in the theory of Jacobi forms and plays an important role in this Section. In particular, if $\phi = \sum c(n, r) q^n \zeta^r$, then

$$(2.1) \quad L_m \phi := L_m(\phi) = \sum (4nm - r^2) c(n, r) q^n \zeta^r.$$

Set

$$\tilde{J}_{k,m} := \{ \phi \pmod{p} : \phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}_{(p)}[[q, \zeta]] \},$$

where $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$ denotes the local ring of p -integral rational numbers. If $\phi \in \tilde{J}_{k,m}$, then we denote its filtration modulo p by

$$\Omega(\phi) := \inf \left\{ k : \phi \pmod{p} \in \tilde{J}_{k,m} \right\}.$$

Recall the following facts on Jacobi forms modulo p :

Proposition 2.1 (Sofer [21]). *Let $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$ and $\psi(\tau, z) \in J_{k',m'} \cap \mathbb{Z}[[q, \zeta]]$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{p-1}$ and $m = m'$.*

Proposition 2.2 ([18]). *If $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$, then $L_m \phi \pmod{p} \in \tilde{J}_{k+p+1,m}$. Moreover, we have*

$$\Omega(L_m \phi) \leq \Omega(\phi) + p + 1,$$

with equality if and only if $p \nmid (2\Omega(\phi) - 1)m$.

We will now explore Ramanujan-type congruences for Jacobi forms.

Definition 2.3. *For $\phi(\tau, z) = \sum c(n, r) q^n \zeta^r \in \tilde{J}_{k,m}$, we say that ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if $c(n, r) \equiv 0 \pmod{p}$ whenever $4nm - r^2 \equiv b \pmod{p}$.*

Equation (2.1) implies that a Jacobi form ϕ has a Ramanujan-type congruence at $0 \pmod{p}$ if and only if $L_m^{p-1} \phi \equiv \phi \pmod{p}$. More generally, ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if and only if

$$L_m^{p-1} \left(q^{-\frac{b}{4m}} \phi \right) \equiv q^{-\frac{b}{4m}} \phi \pmod{p}.$$

Ramanujan-type congruences at $0 \pmod{p}$ for Jacobi forms have been considered in [17, 18]. The following proposition determines when Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ for Jacobi forms exist.

Proposition 2.4. *Let $\phi \in \tilde{J}_{k,m}$ and $b \not\equiv 0 \pmod{p}$. Then ϕ has a Ramanujan-type congruence at $b \pmod{p}$ if and only if $L_m^{\frac{p+1}{2}} \phi \equiv -\left(\frac{b}{p}\right) L_m \phi \pmod{p}$.*

Proof: If $\phi \in \mathbb{Z}_{(p)}[[q, \zeta]]$ and $f \in \mathbb{Z}_{(p)}[[q]]$, then $L_m(f\phi) = L_m(f)\phi + fL_m(\phi)$. This implies

$$\begin{aligned} L_m^{p-1} \left(q^{-\frac{b}{4m}} \phi \right) &= \sum_{i=0}^{p-1} \binom{p-1}{i} L_m^{p-1-i} \left(q^{-\frac{b}{4m}} \right) L_m^i \phi \\ &= \sum_{i=0}^{p-1} \binom{p-1}{i} (-b)^{p-1-i} q^{-\frac{b}{4m}} L_m^i \phi \\ &\equiv q^{-\frac{b}{4m}} \sum_{i=0}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}. \end{aligned}$$

In particular, ϕ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if and only if

$$(2.2) \quad 0 \equiv \sum_{i=1}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}.$$

Let $M_k^{(1)}$ denote the space of elliptic modular forms of weight k . Recall that every even weight $\phi \in J_{k,m}$ with p -integral coefficients can be written as

$$\phi = \sum_{j=0}^m f_j(\phi_{-2,1})^j (\phi_{0,1})^{m-j},$$

where $\phi_{-2,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]]$ and $\phi_{0,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]]$ are weak Jacobi forms of index 1 and weights -2 and 0 , respectively, and where each $f_j \in M_{k+2j}^{(1)}$ has p -integral rational coefficients and is uniquely determined (see §8 and §9 of [6] for details and also for the corresponding result for Jacobi forms of odd weight). Furthermore, by Proposition 2.2, for every i there exists $\psi_i \in J_{k+i(p+1),m}$ such that $L_m^i \phi \equiv \psi_i \pmod{p}$. Hence there exist $F_{i,j} \in M_{k+i(p+1)+2j}^{(1)}$ with p -integral rational coefficients such that

$$L_m^i \phi \equiv \psi_i \equiv \sum_{j=0}^m F_{i,j}(\phi_{-2,1})^j (\phi_{0,1})^{m-j} \pmod{p}$$

and hence (2.2) is equivalent to

$$0 \equiv \sum_{j=0}^m \left(\sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \right) (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \pmod{p}.$$

Since $(\phi_{-2,1})^j (\phi_{0,1})^{m-j}$ are linearly independent over $M_*^{(1)}$, we deduce that (2.2) is equivalent to $\sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \equiv 0 \pmod{p}$ for every j . Elliptic modular forms

modulo p have a natural direct sum decomposition (see Section 3 of [22] or Theorem 2 of [19]) graded by their weights modulo $p - 1$. Thus (2.2) is equivalent to

$$0 \equiv b^{p-1-i} F_{i,j} + b^{(p-1)/2-i} F_{i+(p-1)/2,j} \pmod{p}$$

and hence also

$$F_{i+(p-1)/2,j} \equiv -\left(\frac{b}{p}\right) F_{i,j} \pmod{p}$$

for all $0 \leq j \leq m$ and $1 \leq i \leq \frac{p-1}{2}$. This implies, for all $1 \leq i \leq \frac{p-1}{2}$,

$$\begin{aligned} L_m^{i+\frac{p-1}{2}} \phi &\equiv \sum_{j=0}^m F_{i+\frac{p-1}{2},j} (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \\ &\equiv \sum_{j=0}^m -\left(\frac{b}{p}\right) F_{i,j} (\phi_{-2,1})^j (\phi_{0,1})^{m-j} \\ &\equiv -\left(\frac{b}{p}\right) L_m^i \phi \pmod{p}. \end{aligned}$$

We conclude that

$$L_m^{\frac{p+1}{2}} \phi \equiv -\left(\frac{b}{p}\right) L_m \phi \pmod{p},$$

which completes the proof. \square

By (2.1), $L_m^p \phi \equiv L_m \phi \pmod{p}$. We call $L_m \phi, L_m^2 \phi, \dots, L_m^{p-1} \phi$ the *heat cycle* of ϕ and we say that ϕ is in its own heat cycle whenever $L_m^{p-1} \phi \equiv \phi \pmod{p}$. Assume $L_m \phi \not\equiv 0 \pmod{p}$ and $p \nmid m$. By Proposition 2.2, applying L_m to ϕ increases the filtration of ϕ by $p + 1$ except when $\Omega(\phi) \equiv \frac{p+1}{2} \pmod{p}$. If $\Omega(L_m^i \phi) \equiv \frac{p+1}{2} \pmod{p}$, then call $L_m^i \phi$ a *high point* and $L_m^{i+1} \phi$ a *low point* of the heat cycle. By Propositions 2.1 and 2.2,

$$(2.3) \quad \Omega(L_m^{i+1} \phi) = \Omega(L_m^i \phi) + p + 1 - s(p - 1)$$

where $s \geq 1$ if and only if $L_m^i \phi$ is a high point and $s = 0$ otherwise. The structure of the heat cycle of a Jacobi form is similar to the structure of the theta cycle of a modular form (see §7 of [8]). We will now prove a few basic properties:

Lemma 2.5. *Let $\phi \in \tilde{J}_{k,m}$ with $p \nmid m$ a prime such that $L_m \phi \not\equiv 0 \pmod{p}$.*

- (1) *If $j \geq 1$, then $\Omega(L_m^j \phi) \not\equiv \frac{p+3}{2} \pmod{p}$.*
- (2) *The heat cycle of ϕ has a single low point if and only if there is some $j \geq 1$ with $\Omega(L_m^j \phi) \equiv \frac{p+5}{2} \pmod{p}$. Furthermore, $L_m^j \phi$ is the low point.*
- (3) *If $j \geq 1$, then $\Omega(L_m^{j+1} \phi) \neq \Omega(L_m^j \phi) + 2$.*
- (4) *The heat cycle of ϕ either has one or two high points.*

Proof: (1) If $\Omega(L_m^j \phi) \equiv \frac{p+3}{2} \pmod{p}$, then by (2.3) for $1 \leq n \leq p-1$ we have

$$\Omega(L_m^{j+n} \phi) = \Omega(L_m^j \phi) + n(p+1).$$

In particular, $L_m^{j+p-1} \phi \not\equiv L_m^j \phi \pmod{p}$, which is impossible.

(2) If $\Omega(L_m^j \phi) \equiv \frac{p+5}{2} \pmod{p}$, then by (2.3), for $1 \leq n \leq p-2$ we have

$$\Omega(L_m^{j+n} \phi) = \Omega(L_m^j \phi) + n(p+1)$$

and

$$\Omega(L_m^j \phi) = \Omega(L_m^{j+p-1} \phi) = \Omega(L_m^j \phi) + (p-1)(p+1) - s(p-1)$$

where s must be $p+1$ and there can be no other low point. On the other hand, if there is a single low point, then the filtration must increase $p-2$ consecutive times. The only way this is possible is if the low point has filtration $\frac{p+5}{2} \pmod{p}$.

(3) By Proposition 2.2, $\Omega(L_m^{j+1} \phi) = \Omega(L_m^j \phi) + 2$ can only happen when $\Omega(L_m^j \phi) \equiv \frac{p+1}{2} \pmod{p}$. Suppose $\Omega(L_m^{j+1} \phi) = \Omega(L_m^j \phi) + 2 \equiv \frac{p+5}{2} \pmod{p}$. By part (2), this implies that the filtration increases $p-2$ more times before falling. Hence $L_m^{j+p-1} \phi \not\equiv L_m^j \phi \pmod{p}$, which is impossible.

(4) Suppose there are $t \geq 2$ high points $L_m^{i_j} \phi$ where $1 \leq i_1 < \dots < i_t \leq p-1$. By (2.3) and part (3) above, there are $s_j \geq 2$ such that

$$(2.4) \quad \Omega(L_m^{i_j+1} \phi) = \Omega(L_m^{i_j} \phi) + p+1 - s_j(p-1).$$

Hence

$$\Omega(L_m \phi) = \Omega(L_m^p \phi) = \Omega(L_m \phi) + (p-1)(p+1) - \sum_{j=1}^t s_j(p-1),$$

and so $\sum s_j = p+1$. By (2.4), $\Omega(L_m^{i_j+1} \phi) \equiv \frac{p+1}{2} + 1 + s_j \pmod{p}$ and so there will be $p-1-s_j$ increases before the next fall. That is, for $1 \leq j \leq t$, $i_{j+1} - i_j = p - s_j$ where we take $i_{t+1} = i_1 + p - 1$ for convenience. Thus

$$p-1 = i_{t+1} - i_1 = \sum_{j=1}^t (i_{j+1} - i_j) = \sum_{j=1}^t (p - s_j) = tp - (p+1),$$

i.e., $t = 2$. We conclude that the heat cycle of ϕ has at most two (i.e., one or two) high points. \square

The following Corollary of Proposition 2.4 is a key ingredient in the proof of Proposition 2.7 below.

Corollary 2.6. *If $\phi \in \tilde{J}_{k,m}$ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ and $L_m\phi \not\equiv 0 \pmod{p}$, then the heat cycle of ϕ has two low points which both have filtration congruent to 2 (mod p).*

Proof: Since $L_m^{\frac{p+1}{2}}\phi \equiv -\left(\frac{b}{p}\right)L_m\phi \pmod{p}$, we have $\Omega\left(L_m^{\frac{p+1}{2}}\phi\right) = \Omega(L_m\phi) = \Omega(L_m^p\phi)$. Hence there is a fall in the first half of the heat cycle and in the second half of the heat cycle. Furthermore, after a low point, the filtration increases $\frac{p-3}{2}$ times and then falls once. Thus, the filtration of the low points is 2 (mod p). \square

Our final result in this section gives the non-existence of Ramanujan-type congruences of Jacobi forms.

Proposition 2.7. *Let $\phi \in \tilde{J}_{k,m}$ where $k \geq 4$, $L_m(\phi) \not\equiv 0 \pmod{p}$ and let $b \not\equiv 0 \pmod{p}$. If $p > k$ and $p \nmid m$, then ϕ does not have a Ramanujan-type congruence at $b \pmod{p}$.*

Proof: Assume that ϕ has a Ramanujan-type congruence at $b \pmod{p}$. First suppose $k = \frac{p+1}{2}$. Then $\Omega(\phi) = \frac{p+1}{2}$ and so we must have $s \geq 1$ in (2.3). Since we need $\Omega(L_m\phi) \geq 0$, we must have $s = 1$ and hence $\Omega(L_m\phi) = \frac{p+5}{2}$. But by Lemma 2.5 (2), this implies there is only one low point, contrary to Corollary 2.6.

Now suppose $k \neq \frac{p+1}{2}$. Then $\Omega(L_m\phi) = k + p + 1$. There must be a low point of the heat cycle with filtration either $k + p + 1$ or k . By Corollary 2.6, either $k + 1 \equiv 2 \pmod{p}$ or $k \equiv 2 \pmod{p}$. Both of these alternatives are impossible since $p > k \geq 4$. \square

3. PROOF OF THEOREM 1.2 AND EXAMPLES

We employ the Fourier-Jacobi expansion of a Siegel modular form (as in [3]) to prove Theorem 1.2. Let $M_k^{(2)}$ denote the vector space of Siegel modular forms of degree 2 and even weight k (for details on Siegel modular forms, see for example Freitag [7] or Klingen [10]).

Proof of Theorem 1.2. Let $F \in M_k^{(2)}$ be as in Theorem 1.2 with Fourier-Jacobi expansion $F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'}$, i.e., $\phi_m \in J_{k,m}$. Let $b \not\equiv 0 \pmod{p}$.

Then F has a Ramanujan-type congruence at $b \pmod{p}$ if and only if for all m , ϕ_m has a Ramanujan-type congruence at b . By Proposition 2.4, it is equivalent that for all m

$$L_m^{\frac{p+1}{2}}\phi_m \equiv -\left(\frac{b}{p}\right)L_m\phi_m \pmod{p},$$

which is equivalent to (1.2), since

$$\mathbb{D}(F) = \sum_{m=0}^{\infty} L_m(\phi_m(\tau, z)) e^{2\pi i m \tau'}.$$

Now we turn to the second part of Theorem 1.2. Here we assume that $p > k$, $p \neq 2k-1$, and that there exists an $A(n, r, m)$ with $p \nmid \gcd(n, m)$ such that $A(n, r, m) \not\equiv 0 \pmod{p}$. Suppose that F has a Ramanujan-type congruence at $b \pmod{p}$. Then all Fourier-Jacobi coefficients ϕ_m have such a congruence at b . We would like to apply Proposition 2.7. First, $k \geq 4$, since F is non-constant and $M_k^{(2)} \subset \mathbb{C}$ if $k < 4$. Moreover, if $\phi_m \not\equiv 0 \pmod{p}$ with $p \nmid m$, then $\Omega(\phi_m) = k$ by Proposition 2.1 (since $p > k$ and F is non-constant modulo p) and $\Omega(L_m \phi_m) = k + p + 1$ by Proposition 2.2. In particular, $L_m \phi_m \not\equiv 0 \pmod{p}$ and Proposition 2.7 implies that such a ϕ_m does not have a Ramanujan-type congruence at $b \pmod{p}$. Hence, if $p \nmid m$, then $\phi_m \equiv 0 \pmod{p}$, i.e., $A(n, r, m) \equiv 0 \pmod{p}$. By assumption, there exists an $A(n, r, m) \not\equiv 0 \pmod{p}$ with $p \nmid \gcd(n, m)$, which is only possible if $p \mid m$ and hence $p \nmid n$. However, $F(\tau, z, \tau') = F(\tau', z, \tau)$ and $p \nmid n$ together yield the contradiction $A(n, r, m) = A(m, r, n) \equiv 0 \pmod{p}$. We conclude that F does not have a Ramanujan-type congruence at $b \pmod{p}$. \square

We will use Theorem 1.2 to discuss Ramanujan-type congruences for explicit examples of Siegel modular forms after reviewing a few facts on Siegel modular forms modulo p . Set

$$\widetilde{M}_k^{(2)} := \left\{ F \pmod{p} : F(Z) = \sum a(T) e^{\pi i \operatorname{tr}(TZ)} \in M_k^{(2)} \text{ where } a(T) \in \mathbb{Z}_{(p)} \right\}.$$

Recall the following two theorems on Siegel modular forms modulo p :

Theorem 3.1 (Nagaoka [12]). *There exists an $E \in M_{p-1}^{(2)}$ with p -integral rational coefficients such that $E \equiv 1 \pmod{p}$. Furthermore, if $F_1 \in M_{k_1}^{(2)}$ and $F_2 \in M_{k_2}^{(2)}$ have p -integral rational coefficients where $0 \not\equiv F_1 \equiv F_2 \pmod{p}$, then $k_1 \equiv k_2 \pmod{p-1}$.*

Theorem 3.2 (Böcherer and Nagaoka [2]). *If $F \in \widetilde{M}_k^{(2)}$, then $\mathbb{D}(F) \in \widetilde{M}_{k+p+1}^{(2)}$.*

Theorems 3.1 and 3.2 imply that that

$$(3.1) \quad G := \mathbb{D}^{\frac{p+1}{2}}(F) + \left(\frac{b}{p}\right) \mathbb{D}(F) \in \widetilde{M}_{k+\frac{(p+1)^2}{2}}^{(2)}.$$

Theorem 1.2 states that $F \in \widetilde{M}_k^{(2)}$ has a Ramanujan-type congruence at $b \not\equiv 0 \pmod{p}$ if and only if $G \equiv 0 \pmod{p}$ in (3.1). One can apply the following analog of Sturm's theorem for Siegel modular forms of degree 2 to verify that $G \equiv 0 \pmod{p}$ in (3.1) for concrete examples of Siegel modular forms.

Theorem 3.3 (Poor and Yuen [15]). *Let $F = \sum a(T) e^{\pi i \operatorname{tr}(TZ)} \in M_k^{(2)}$ be such that for all T with dyadic trace $w(T) \leq \frac{k}{3}$ one has that $a(T) \in \mathbb{Z}_{(p)}$ and $a(T) \equiv 0 \pmod{p}$. Then $F \equiv 0 \pmod{p}$.*

Remark: If $T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0$ is Minkowski reduced (i.e., $2|b| \leq a \leq c$), then $w(T) = a + c - |b|$. For more details on the dyadic trace $w(T)$, see Poor and Yuen [16].

The following table gives all Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ for Siegel cusp forms of weight 20 or less when $p \geq 5$. Let E_4, E_6, χ_{10} , and χ_{12} denote the usual generators of $M_k^{(2)}$ of weights 4, 6, 10, and 12, respectively, where the Eisenstein series E_4 and E_6 are normalized by $a\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1$ and where the cusp forms χ_{10} and χ_{12} are normalized by $a\left(\begin{pmatrix} 2 & \\ & 1 \end{pmatrix}\right) = 1$. Cris Poor and David Yuen kindly provided Fourier coefficients up to dyadic trace $w(T) = 74$ of the basis vectors for $M_k^{(2)}$ with $k \leq 20$. We used Magma to check that $G \equiv 0 \pmod{p}$ in (3.1) for each of the forms in (3.2) below. It is not difficult to verify that (up to scalar multiplication) no further Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ exist for Siegel cusp forms of weights 20 or less.

(3.2)

| | $b \not\equiv 0 \pmod{p}$ |
|--|--|
| χ_{12} | $b \equiv 1, 4 \pmod{5}$ and $b \equiv 2, 6, 7, 8, 10 \pmod{11}$ |
| $E_4\chi_{12}$ | $b \equiv 1, 4 \pmod{5}$ |
| $E_4\chi_{12} - E_6\chi_{10}$ | $b \equiv 3, 5, 6 \pmod{7}$ |
| $E_6\chi_{12}$ | $b \equiv 1, 4 \pmod{5}$ |
| $E_4^2\chi_{10} + 7E_6\chi_{12}$ | $b \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}$ |
| $E_4^2\chi_{12}$ | $b \equiv 1, 4 \pmod{5}$ |
| $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ | $b \equiv 2, 3, 8, 10, 12, 13, 14, 15, 18 \pmod{19}$ |

Remarks:

- (1) For $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ modulo 19 we have $G \in \widetilde{M}_{220}^{(2)}$ in (3.1) and we really do need Fourier coefficients up to dyadic trace $w(T) = \frac{220}{3}$, i.e., up to 74 in Theorem 3.3 to prove that $G \equiv 0 \pmod{19}$.
- (2) For Siegel modular forms in the Maass Spezialschar one could decide the existence and non-existence of their Ramanujan-type congruences also using Propositions 2.4 and 2.7 in combination with Maass' lift [11] (see also §6 of [6]). However, Theorem 1.2 is an essential tool in establishing such results for Siegel modular forms that are not in the Maass Spezialschar, such as $E_4^2\chi_{12}$ and $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ for example.

The following construction generates infinitely many Siegel modular forms with Ramanujan-type congruences. Note that this construction also works for elliptic

modular forms and for Jacobi forms by replacing \mathbb{D} with $\Theta := \frac{1}{2\pi i} \frac{d}{dz}$ and L_m , respectively. For any $F \in \widetilde{M}_k^{(2)}$ and any prime $p \geq 5$, set

$$\begin{aligned} F_0 &:= F - \mathbb{D}^{p-1} F \in \widetilde{M}_{k+p^2-1}^{(2)} \\ F_{+1} &:= \frac{1}{2} \left(\mathbb{D}^{p-1} F + \mathbb{D}^{\frac{p-1}{2}} F \right) \in \widetilde{M}_{k+p^2-1}^{(2)} \\ F_{-1} &:= \frac{1}{2} \left(\mathbb{D}^{p-1} F - \mathbb{D}^{\frac{p-1}{2}} F \right) \in \widetilde{M}_{k+p^2-1}^{(2)}. \end{aligned}$$

Clearly $F = F_0 + F_{+1} + F_{-1}$ and if $F = \sum a(T) e^{\pi i \operatorname{tr}(TZ)}$, then for $s = 0, \pm 1$, one finds that

$$(3.3) \quad F_s = \sum_{\left(\frac{\det(T)}{p}\right)=s} a(T) e^{\pi i \operatorname{tr}(TZ)}.$$

Hence F_s has Ramanujan-type congruences at all b with $\left(\frac{b}{p}\right) \neq s$. For example, if $F := \chi_{10}^2$, then a computation (in combination with Theorem 3.3) reveals that

$$\begin{aligned} F_0 &\equiv 3E_4^5 \chi_{12}^2 + 2E_4^4 E_6 \chi_{10} \chi_{12} && \pmod{5} \\ F_{+1} &\equiv E_4^6 \chi_{10}^2 + 4E_4^3 \chi_{10}^2 \chi_{12} + 4E_4^5 \chi_{12}^2 + 2E_4^4 E_6 \chi_{10} \chi_{12} + 3E_4^3 E_6^2 \chi_{10}^2 && \pmod{5} \\ F_{-1} &\equiv E_4^3 \chi_{10}^2 \chi_{12} + 3E_4^5 \chi_{12}^2 + E_4^4 E_6 \chi_{10} \chi_{12} + 2E_4^3 E_6^2 \chi_{10}^2 && \pmod{5}. \end{aligned}$$

Since $E_4 \equiv 1 \pmod{5}$, we actually have $F_0 \in \widetilde{M}_{28}^{(2)}$ and $F_{\pm 1} \in \widetilde{M}_{32}^{(2)}$.

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