Rank two filtered (φ, N)-modules with Galois descent data and coefficients

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Abstract

Let K be any finite extension of $\mathbb{Q}_p$, L any finite Galois extension of K, and E any finite large enough coefficient field containing L. We classify two-dimensional L-semistable E-representations of G_K, by listing the isomorphism classes of rank two weakly admissible filtered $(\varphi, N, L/K, E)$-modules.

1 Introduction

Let K be any finite extension of $\mathbb{Q}_p$ and $\rho : G_K \to GL_n(\bar{\mathbb{Q}}_p)$ any continuous n-dimensional representation of $G_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$. Let L be any finite Galois extension of K. The representation ρ is called L-semistable if it becomes semistable when restricted to G_L. The field of definition E of ρ is a finite extension of $\mathbb{Q}_p$ which may be extended to contain L. Let $k \geq 1$ be any integer. By a variant of fundamental work of Colmez and Fontaine ([CF00]), the category of L-semistable E-representations of G_K with Hodge-Tate weights in the range $\{0, 1, ..., k-1\}$ is equivalent to the category of weakly admissible filtered $(\varphi, N, L/K, E)$-modules D (Def. 1.1), such that $\mathrm{Fil}^0(L \otimes_{L_0} D) = L \otimes_{L_0} D$ and $\mathrm{Fil}^k(L \otimes_{L_0} D) = 0$. We classify two-dimensional L-semistable E-representations of G_K, by listing the isomorphism classes of rank two weakly admissible filtered $(\varphi, N, L/K, E)$-modules.

When $K \neq \mathbb{Q}_p$ interesting new phenomena occur, for example there exist disjoint infinite families of irreducible two-dimensional crystalline representations of G_K, sharing the same characteristic polynomial and filtration (Cor. 7.4). Such families have been constructed in [DO08] and their semisimplified modulo p reductions have been computed in [DO09].

Potentially semistable representations arise naturally in geometry. Deciding which isomorphism classes of filtered modules occur from certain geometric objects, e.g. Hilbert modular forms is an interesting open problem and we hope that this paper will contribute in this direction. Special cases of the problem have been treated by Fontaine and Mazur [FM95] when both E and K equal $\mathbb{Q}_p$ and $p \geq 5$, Breuil and Mézard [BM02] who initiated the subject with arbitrary coefficients, Savitt [SAV05] in cases where the representation becomes crystalline over tamely ramified extensions of $\mathbb{Q}_p$, and most recently by Ghate and Mézard [GM09] who treated almost all cases where $K = \mathbb{Q}_p$, assuming that E is large enough and $p \neq 2$. In this paper we assume that the coefficient field E is large enough, and make no further assumptions. The paper is organized as follows: in the rest of this introductory section we recall standard facts from p-adic Hodge theory and there is nothing original. In Section 2 we set up our main notations and prove a canonical form lemma for Frobenius and the monodromy operator (§2.1). We then proceed to determine the Galois descent data (§2.2). In Section 3 we construct the Galois-stable filtrations and in Section 4 we compute Hodge and Newton invariants. In Section 5 we provide the complete list of rank two weakly admissible filtered $(\varphi, N, L/K, E)$-modules, determine which are irreducible, non-split reducible or split-reducible, and describe their precise submodule structure. In Section 6 we list the isomorphism classes of rank two filters modules (§6.4), and in Section 7 we apply the results of previous sections to explore new phenomena occurring in the $K \neq \mathbb{Q}_p$ case, focusing on crystalline representations.

Contents

1.1 Fontaine's rings

Let $\mathbb{C}_p$ be the completion of $\bar{\mathbb{Q}}_p$ for the p-adic topology. The field $\mathbb{C}_p$ is algebraically closed and complete. Let $\widetilde{E} = \varprojlim_{x\mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, ..., x^{(n)}, ...)$ such that $(x^{(n+1)})^p = x^{(n)}$ for all $n \geq 0\}$ and let $\widetilde{E}^+$ be the set of $x = (x^{(0)}, x^{(1)}, ..., x^{(n)}, ...) \in \widetilde{E}$ with $v_E(x) := v_p(x^{(0)}) \geq 0$. Then $\widetilde{E}$ with addition and multiplication defined by

$$(x+y)^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m} \text{ and } (xy)^{(n)} = x^{(n)}y^{(n)}$$

for all $n \geq 0$ is an algebraically closed field of characteristic p and v_E is a valuation on $\widetilde{E}$ for which $\widetilde{E}$ is complete with valuation ring $\widetilde{E}^+$. Let $\widetilde{\mathbb{A}}^+$ be the ring of Witt vectors with $\widetilde{E}^+$-coefficients and let $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[\frac{1}{p}] = \{\sum_{k\gg-\infty} p^k[x_k],\ x_k \in \widetilde{E}^+\}$, where $[x] \in \widetilde{\mathbb{A}}^+$ is the Teichmüller lift of $x \in \widetilde{E}^+$. The ring $\widetilde{\mathbb{B}}^+$ is endowed with a ring epimorphism $\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$ given by $\theta(\sum_{k\gg-\infty} p^k[x_k]) = \sum_{k\gg-\infty} p^k x_k^{(0)}$. By functorial properties of Witt vectors the absolute Frobenius $\varphi : \widetilde{E}^+ \to \widetilde{E}^+$ lifts to a ring epimorphism $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$given by $\varphi(\sum_{k\gg-\infty} p^k[x_k]) = \sum_{k\gg-\infty} p^k[x_k^p]$. Let $\varepsilon = (\varepsilon^{(i)})_{i\geq 0} \in \widetilde{E}$ where $\varepsilon^{(0)} = 1$ and $\varepsilon^{(i)}$ is a primitive p^i-th root of 1 such that $\left(\varepsilon^{(i+1)}\right)^p = \varepsilon^{(i)}$ for all i. If $\pi = [\varepsilon] - 1$ and $\pi_1 = [\varepsilon^{\frac{1}{p}}] - 1$, we write $\omega = \frac{\pi}{\pi_1}$. The kernel of the epimorphism $\theta : \widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$ is the principal ideal generated by ω. The ring $\mathbb{B}_{dR}^+$ is defined to be the separated $\ker\theta$-adic completion of $\widetilde{\mathbb{B}}^+$, i.e. $\mathbb{B}_{dR}^+ = \varprojlim_n \widetilde{\mathbb{B}}^+/(\ker\theta)^n$. The series $\log([\varepsilon]) = -\sum_{n=1}^{\infty} \frac{(1-[\varepsilon])^n}{n}$ converges to some element $t \in \mathbb{B}_{dR}^+$ with the property that $gt = \chi(g)t$ for all $g \in G_{\mathbb{Q}_p}$, where $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character. We define $\mathbb{B}_{dR} = \mathbb{B}_{dR}^+[\frac{1}{t}]$. The ring $\mathbb{B}_{dR}$ is a field equipped with a decreasing, exhaustive and separated filtration given by $\mathrm{Fil}^j\mathbb{B}_{dR} = t^j\mathbb{B}_{dR}^+$ for all integers j. It contains a subring $\mathbb{B}_{cris}$ endowed with the induced Galois action and a Frobenius endomorphism φ which extends $\varphi : \widetilde{\mathbb{B}}^+ \to \widetilde{\mathbb{B}}^+$, such that $\varphi(t) = pt$. It has the property that $\mathbb{B}_{cris}^{G_K} = K_0$ for any finite extension K of $\mathbb{Q}_p$, where K_0 is the maximum unramified extension of $\mathbb{Q}_p$ inside K. Between $\mathbb{B}_{cris}$ and $\mathbb{B}_{dR}$ sits (non canonically) a ring $\mathbb{B}_{st} = \mathbb{B}_{cris}[X]$, where X is a polynomial variable over $\mathbb{B}_{cris}$. The ring $\mathbb{B}_{st}$ is equipped with a Frobenius which extends the Frobenius on $\mathbb{B}_{cris}$ and is such that $\varphi(X) = pX$. There is also a $\bar{\mathbb{Q}}_p$-linear monodromy operator $N = -\frac{d}{dX}$ which satisfies the equation $N\varphi = p\varphi N$. Let $\tilde{p} \in \tilde{E}^+$ be any element with $\tilde{p}^{(0)} = p$ and let

$$\log[\tilde{p}] = \log_p(p) - \sum_{n=1}^{\infty} \frac{(1-[\tilde{p}]/p)^{n-1}}{n}.$$

There exist Galois equivariant, $\mathbb{B}_{cris}$-linear embeddings of $\mathbb{B}_{st}$ in $\mathbb{B}_{dR}$ which map X to $\log[\tilde{p}]$. They require a choice of $\log_p(p)$ and we always assume that $\log_p(p) = 0$. The ring $\mathbb{B}_{st}$ is equipped with a Galois action which extends the Galois action on $\mathbb{B}_{cris}$. It has the properties that $\mathbb{B}_{st}^{G_K} = K_0$ for any finite extension K of $\mathbb{Q}_p$ and the map $K \otimes_{K_0} \mathbb{B}_{st}^{G_K} \to \mathbb{B}_{dR}$ is injective.

1.2 Potentially semistable representations

Let K be a finite extension of $\mathbb{Q}_p$ and V a $\mathbb{Q}_p$-linear representation of G_K. The fact that $\mathbb{B}_{dR}^{G_K} = K$ is part of a technical condition called regularity which implies that the K-vector space $D_{dR}(V) = (\mathbb{B}_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$ has dimension at most $\dim_{\mathbb{Q}_p}(V)$. The representation V is called de Rham if equality holds. All representations coming from geometry are de Rham. The K-space $D_{dR}(V)$ is equipped with a natural decreasing, exhaustive and separated filtration given by $\mathrm{Fil}^j D_{dR}(V) = (t^j \mathbb{B}_{dR}^+ \otimes_{\mathbb{Q}_p} V)^{G_K}$ for any integer j. An integer j is called a Hodge-Tate weight of a de Rham representation V if $\mathrm{Fil}^{-j} D_{dR}(V) \neq \mathrm{Fil}^{-j+1} D_{dR}(V)$, and is counted with multiplicity $\dim_K \left(\mathrm{Fil}^{-j} D_{dR}(V)/\mathrm{Fil}^{-j+1} D_{dR}(V)\right)$. There are $d = \dim_{\mathbb{Q}_p}(V)$ Hodge-Tate weights for V, counting multiplicities. A chosen inclusion of $\mathbb{B}_{st}$ in $\mathbb{B}_{dR}$ defines (non canonically) a filtration on $K \otimes_{K_0} D_{st}(V) = K \otimes_{K_0} (\mathbb{B}_{st} \otimes_{\mathbb{Q}_p} V)^{G_K}$ which is preserved by the Galois action. By the construction of the ring $\mathbb{B}_{st}$ the inequality $\dim_{K_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p}(V)$ always holds, and V is called semistable when equality holds. It is called potentially semistable if it becomes semistable when restricted to G_L, for some finite extension L of K. Crystalline representations are semistable and semistable representations are de Rham, with the converse inclusions being false. Potentially semistable representations are de Rham. The converse is a difficult theorem of Berger ([BE04b]), known as the p-adic monodromy theorem.

Let L be a finite Galois extension of K and E any finite extension of L. We write $D_{st}^L(V)$ instead of $D_{st}(V\,|_{G_L})$. Assume that V is equipped with an E-linear structure which commutes with the G_K-action. The L_0-space $D_{st}^L(V)$ is additionally equipped with an $L_0 \otimes_{\mathbb{Q}_p} E$-module structure, and V is L-semistable if and only if $D_{st}^L(V)$ is free of rank $\dim_E V$. For the rest of the section we assume that V is L-semistable. The Frobenius endomorphism of $\mathbb{B}_{st}$ induces an automorphism φ on $D_{st}^L(V)$ which is semilinear with respect to the automorphism $\tau \otimes 1_E$ of $L_0 \otimes_{\mathbb{Q}_p} E$, where τ is the absolute Frobenius of L_0. The monodromy operator N of $\mathbb{B}_{st}$ induces an $L_0 \otimes_{\mathbb{Q}_p} E$-linear nilpotent endomorphism N on $D_{st}^L(V)$ such that $N\varphi = p\varphi N$. We equip $L \otimes_{L_0} D_{st}(V)$ with the filtration induced by the injection $L \otimes_{L_0} D_{st}^L(V) \to D_{dR}(V)$. It has the properties that $\mathrm{Fil}^j \left(L \otimes_{L_0} D_{st}^L(V)\right) = 0$ for $j \gg 0$ and $\mathrm{Fil}^j \left(L \otimes_{L_0} D_{st}^L(V)\right) = L \otimes_{L_0} D_{st}^L(V)$ for $j \ll 0$. The module $D_{st}^L(V)$ is also equipped with an L_0-semilinear, E-linear action of $G = \mathrm{Gal}(L/K)$ which commutes with φ and N and preserves the filtration. The discussion above motivates the following.

Definition 1.1 *A rank n filtered $(\varphi, N, L/K, E)$-module is a free module D of rank n over $L_0 \otimes_{\mathbb{Q}_p} E$ equipped with*

- an L_0-semilinear, E-linear automorphism φ;
- an $L_0 \otimes_{\mathbb{Q}_p} E$-linear nilpotent endomorphism N such that $N\varphi = p\varphi N$;
- a decreasing filtration on $D_L = L \otimes_{L_0} D$ such that $\mathrm{Fil}^j D_L = 0$ for $j \gg 0$ and $\mathrm{Fil}^j D_L = D_L$ for $j \ll 0$, and
- an L_0-semilinear, E-linear action of $G = \mathrm{Gal}(L/K)$ which commutes with φ and N and preserves the filtration of D_L.

A morphism of filtered $(\varphi, N, L/K, E)$-modules is an $L_0 \otimes_{\mathbb{Q}_p} E$-linear map h which commutes with φ, N and the $\mathrm{Gal}(L/K)$-action, and is such that the $L \otimes_{\mathbb{Q}_p} E$-linear map $h_L = 1_{L \otimes_{\mathbb{Q}_p} E} \otimes h$ preserves the filtrations. A filtered $(\varphi, N, L/K, E)$-module is called weakly admissible if it is weakly admissible as a filtered (φ, N, E)-module in the sense of [BM02, Cor. 3.1.2.1]. The Galois action plays no role in weak admissibility. We have the following fundamental theorem essentially due to Colmez and Fontaine (cf. [BM02, Cor. 3.1.1.3]).

Theorem 1.2 *Let $k \geq 1$ be any integer. The category of L-semistable E-representations of G_K with Hodge-Tate weights in the range $\{0, 1, ..., k-1\}$ is equivalent to the category of weakly admissible filtered $(\varphi, N, L/K, E)$-modules D such that $\mathrm{Fil}^0(D_L) = D_L$ and $\mathrm{Fil}^k(D_L) = 0$.*

2 Rank two filtered $(\varphi, N, L/K, E)$-modules

Throughout the paper p will be a fixed prime number and L/K any finite Galois extension, with K any finite extension of $\mathbb{Q}_p$. The coefficient field E will be any finite, large enough extension of L. We denote by m the degree of L over $\mathbb{Q}_p$, by $f = [L_0 : \mathbb{Q}_p]$ the absolute inertia degree of L, and by $e = [L : L_0]$ the absolute ramification index of L. As in the introduction we denote by L_0 the maximal unramified extension of $\mathbb{Q}_p$ inside L. Let τ be the absolute Frobenius of L_0. We fix an embedding $\iota_{L_0} : L_0 \hookrightarrow E$ and we let $\tau_j = \iota_{L_0} \circ \tau^j$ for all $j = 0, 1, ..., f-1$. We fix once and for all the f-tuple of embeddings $\mathcal{S}_{L_0} := (\tau_0, \tau_1, ..., \tau_{f-1})$. The map

$$\xi_{L_0} : L_0 \otimes_{\mathbb{Q}_p} E \to \prod_{\mathcal{S}_{L_0}} E \ : \ \xi_{L_0}(x \otimes y) = (\tau_i(x)y)_{\tau_i}$$

is a ring isomorphism (cf. [SAV05, Lemma 2.2]). Let $E^{|\mathcal{S}_{L_0}|} := \prod_{\mathcal{S}_{L_0}} E$ and $(E^\times)^{|\mathcal{S}_{L_0}|} := \prod_{\mathcal{S}_{L_0}} E^\times$. The ring automorphism $\tau \otimes 1_E : L_0 \otimes_{\mathbb{Q}_p} E \to L_0 \otimes_{\mathbb{Q}_p} E$ transforms via ξ_{L_0} to the ring automorphism $\varphi : E^{|\mathcal{S}_{L_0}|} \to E^{|\mathcal{S}_{L_0}|}$ with $\varphi(x_0, x_1, ..., x_{f-1})$
$= (x_1, ..., x_{f-1}, x_0)$. A filtered $(\varphi, N, L/K, E)$-module may therefore be viewed as a module over $E^{|\mathcal{S}_{L_0}|}$. The automorphism $\varphi : D \to D$ is semilinear with respect to the automorphism φ of $E^{|\mathcal{S}_{L_0}|}$ defined above, and the monodromy N is $E^{|\mathcal{S}_{L_0}|}$-linear. The Galois action of $G = \mathrm{Gal}(L/K)$ on $E^{|\mathcal{S}_{L_0}|}$ will be described in Section 2.2.2. We let $e_{\tau_j} := (0, ..., 1_{\tau_j}, ..., 0) \in E^{|\mathcal{S}_{L_0}|}$ for any $j \in \{0, 1, ..., f-1\}$, and set up some more notation which will remain fixed throughout.

Notation 1 *For each $J \subset \{0, 1, ..., f-1\}$ we write $f_J = \sum_{i \in J} e_{\tau_i}$. If $\vec{x} \in E^{|\mathcal{S}_{L_0}|}$, we define $Nm_\varphi(\vec{x}) := \prod_{i=0}^{f-1} \varphi^i(\vec{x})$ and $Tr_\varphi(\vec{x}) := \sum_{i=0}^{f-1} \varphi^i(\vec{x})$. For any $\vec{x} \in E^{|\mathcal{S}_{L_0}|}$ we denote by x_i the i-th component of $\vec{x}$, and for any matrix $M \in M_2(E^{|\mathcal{S}_{L_0}|})$ we write $Nm_\varphi(M) = M\varphi(M) \cdots \varphi^{f-1}(M)$, with φ acting on each entry of M.*

2.1 Canonical forms for Frobenius and the monodromy operator

We start by putting the matrix of Frobenius of a rank two φ-module in a convenient form. The matrix of any (semi)linear operator T on D with respect to an ordered basis $\underline{e}$ will be denoted by $[T]_{\underline{e}}$ throughout. The following elementary lemma will be used frequently.

Lemma 2.1 *(1) The operator $Nm_\varphi : (E^\times)^{|\mathcal{S}_{L_0}|} \to (E^\times)^{|\mathcal{S}_{L_0}|}$ is multiplicative;*

(2) Let $\vec{\alpha}, \vec{\beta} \in (E^{\times})^{|\mathcal{S}_{L_0}|}$. The equation $\vec{\alpha}\cdot\vec{\gamma} = \vec{\beta}\cdot\varphi(\vec{\gamma})$ has nonzero solutions $\vec{\gamma} \in E^{|\mathcal{S}_{L_0}|}$ if and only if $Nm_{\varphi}(\vec{\alpha}) = Nm_{\varphi}(\vec{\beta})$. In this case, all the solutions are $\vec{\gamma} = \gamma\left(1, \frac{\alpha_0}{\beta_0}, \frac{\alpha_0\alpha_1}{\beta_0\beta_1}, ..., \frac{\alpha_0\alpha_1\cdots\alpha_{f-2}}{\beta_0\beta_1\cdots\beta_{f-2}}\right)$ for any $\gamma \in E$.

Proof. Straightforward. ■

Let D be a rank two φ-module over $E^{|\mathcal{S}_{L_0}|}$ and let $\underline{\eta}$ and $\underline{e}$ be ordered bases. Then $(\eta_1, \eta_2) = (e_1, e_2)M$ for some matrix $M \in GL_2\left(E^{|\mathcal{S}_{L_0}|}\right)$, and we write $M = [1]^{\underline{e}}_{\underline{\eta}}$. It follows from Section 2 that $[\varphi]_{\underline{e}} = M[\varphi]_{\underline{\eta}}\varphi(M)^{-1}$. The main observation of this section is the following proposition.

Proposition 2.2 *Let D be a rank two φ-module over $E^{|\mathcal{S}_{L_0}|}$. After enlarging E if necessary, there exists an ordered basis $\underline{\eta}$ of D with respect to which the matrix of Frobenius takes one of the following forms:*

(1) $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha\cdot\vec{1}, \delta\cdot\vec{1})$ for some $\alpha, \delta \in E^{\times}$ with $\alpha^f \neq \delta^f$, or

(2) $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha\cdot\vec{1}, \alpha\cdot\vec{1})$ for some $\alpha \in E^{\times}$, or

(3) $[\varphi]_{\underline{\eta}} = \begin{pmatrix} \alpha\cdot\vec{1} & \vec{0} \\ \vec{1} & \alpha\cdot\vec{1} \end{pmatrix}$ for some $\alpha \in E^{\times}$.

To prove Proposition 2.2, we use the following lemma.

Lemma 2.3 *Let D be as in Proposition 2.2. After enlarging E if necessary, the following hold:*

(1) If φ^f is not an $E^{\times}$-scalar times the identity map, then there exists an ordered basis $\underline{\eta}$ of D such that $[\varphi]_{\underline{\eta}} = \begin{pmatrix} \vec{\varepsilon} & \vec{0} \\ \vec{\eta} & \vec{\theta} \end{pmatrix}$, with the additional properties that:

(a) If $Nm_{\varphi}(\vec{\varepsilon}) \neq Nm_{\varphi}(\vec{\theta})$, then $\vec{\eta} = \vec{0}$ and

(b) If $Nm_{\varphi}(\vec{\varepsilon}) = Nm_{\varphi}(\vec{\theta})$, then $\vec{\varepsilon} = \vec{\theta}$ and $\vec{\eta}_{\varphi} = \vec{1}$, where $\vec{\eta}_{\varphi}$ is the $(2,1)$ entry of the matrix $Nm_{\varphi}\left([\varphi]_{\underline{\eta}}\right)$.

(2) If $\varphi^f = \alpha\cdot\vec{id}$ for some $\alpha \in E^{\times}$, then there exists an ordered basis $\underline{\eta}$ of D such that $[\varphi]_{\underline{\eta}} = \mathrm{diag}((\alpha, 1, ..., 1), (\alpha, 1, ..., 1))$.

Proof. (1) Since φ^f is an $E^{|\mathcal{S}_{L_0}|}$-linear isomorphism, extending E if necessary, there exists an ordered basis $\underline{e}$ of D such that $[\varphi^f]_{\underline{e}} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\delta} \end{pmatrix}$. With the convention of Notation 1 we have $\alpha_i\delta_i \neq 0$ for all $i \in I_0$ (because φ is an automorphism), and the basis can be chosen so that $\gamma_i = 0$ whenever $\alpha_i \neq \delta_i$ and $\gamma_i \in \{0, 1\}$ whenever $\alpha_i = \delta_i$. We repeatedly act by φ on the equation $(\varphi(e_1), \varphi(e_2)) = (e_1, e_2)[\varphi]_{\underline{e}}$ and get $\left(\varphi^f(e_1), \varphi^f(e_2)\right) = (e_1, e_2)Nm_{\varphi}([\varphi]_{\underline{e}})$. Let $P = [\varphi]_{\underline{e}} = (P_0, P_1, ..., P_{f-1})$ and $Q = Nm_{\varphi}(P) = (Q_0, Q_1, ..., Q_{f-1})$. Since $Q = P\varphi(Q)P^{-1}$, we have $Q_i = P_iQ_{i+1}P_i^{-1}$ and $\{\alpha_{i+1}, \delta_{i+1}\} = \{\alpha_i, \delta_i\}$ for all i. Since for all i, $\alpha_i\delta_i = \det Q_0 = d$, we have $\{\alpha_{i+1}, d\alpha_{i+1}^{-1}\} = \{\alpha_i, d\alpha_i^{-1}\}$. Let $\alpha = d\alpha_0^{-1}$. Then $\alpha_i \in \{\alpha, d\alpha^{-1}\}$ for all i, and $Nm_{\varphi}(P) = \begin{pmatrix} (\alpha_0, ..., \alpha_{f-1}) & (0\ ,\ .\ .\ .\ ,\ 0) \\ (\gamma_0, ..., \gamma_{f-1}) & (\delta_0, ..., \delta_{f-1}) \end{pmatrix}$ with $\delta_i = d\alpha_i^{-1}$. If $\alpha^2 \neq d$ then, $\vec{\gamma} = \vec{0}$ and if

$\alpha^2 = d$, then $\gamma_i \in \{0,1\}$ for all i. We conjugate by the matrix $R = (R_0, R_1, ..., R_{f-1})$, where $R_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ depending on whether $\alpha_i = d\alpha^{-1}$ or α respectively, and get $RQR^{-1} = \begin{pmatrix} d\alpha^{-1}\cdot\vec{1} & \vec{\gamma} \\ \vec{0} & \alpha\cdot\vec{1} \end{pmatrix}$. If $\alpha^2 \neq d$, then $RQR^{-1} = \mathrm{diag}((d\alpha^{-1}, ..., d\alpha^{-1}), (\alpha, \alpha, ..., \alpha))$. If $\alpha^2 = d$, then $Nm(P) = \begin{pmatrix} \alpha\cdot\vec{1} & \vec{1} \\ \vec{0} & \alpha\cdot\vec{1} \end{pmatrix}$. Indeed, since $P\varphi(Q)P^{-1} = Q$, if $\gamma_j = 0$ for some j then $\gamma_{j+1} = 0$ and $\varphi^f = \alpha\cdot\vec{id}$ a contradiction. Therefore $\vec{\gamma} = \vec{1}$. We have proved that there exists some ordered basis $\underline{\eta}$ of D over $E^{|\mathcal{S}_{L_0}|}$ such that $[\varphi^f]_{\underline{\eta}} = \begin{pmatrix} \alpha\cdot\vec{1} & \vec{0} \\ \gamma\cdot\vec{1} & \frac{d}{\alpha}\cdot\vec{1} \end{pmatrix}$ for some $\alpha \in E^\times$ and some $\gamma \in E$ with $\gamma = 0$ if $\alpha^2 \neq d$ and $\gamma = 1$ if $\alpha^2 = d$. We compute the matrix of φ with respect to that basis $\underline{\eta}$. The relations $Nm_\varphi\left([\varphi]_{\underline{\eta}}\right) = [\varphi^f]_{\underline{\eta}}$ and $[\varphi]_{\underline{\eta}}\varphi\left(Nm_\varphi\left([\varphi]_{\underline{\eta}}\right)\right) = Nm_\varphi\left([\varphi]_{\underline{\eta}}\right)[\varphi]_{\underline{\eta}}$ and a direct computation imply that: (1) If $\alpha^2 \neq d$, then the non diagonal entries of $[\varphi]_{\underline{\eta}}$ are $\vec{0}$, and (2) If $\alpha^2 = d$, then the $(1,2)$ entry of $[\varphi]_{\underline{\eta}}$ is $\vec{0}$ and the diagonal entries are equal. This concludes the proof of part (1). Part (2) follows immediately from the fact that the matrix of φ^f is basis-independent combined with the following claim. ■

Claim 1 *Let $P \in GL_2(E^{|\mathcal{S}_{L_0}|})$ be such that $Nm_\varphi(P) = \mathrm{diag}\ (\alpha\cdot\vec{1}, \alpha\cdot\vec{1})$ for some $\alpha \in E^\times$. Then there exists some matrix $Q^* \in GL_2(E^{|\mathcal{S}_{L_0}|})$ such that*

$$Q^*P\varphi(Q^*)^{-1} = \mathrm{diag}((\alpha, 1, .., 1), (\alpha, 1, .., 1)).$$

Proof. As above we write $P = (P_0, P_1, ..., P_{f-1})$. We easily see that there exist matrices $Q_i \in GL_2(E)$ such that the matrix $Q = (Q_0, Q_1, ..., Q_{f-1})$ has the property $QP\varphi(Q)^{-1} = (T_0, T_1, ..., T_{f-2}, T_{f-1})$ for some triangular matrices $T_i = \begin{pmatrix} \alpha_i & 0 \\ \gamma_i & \delta_i \end{pmatrix}$ for $i = 0, 1, ..., f-2$, and some matrix $T_{f-1} = \begin{pmatrix} \alpha_{f-1} & \beta_{f-1} \\ \gamma_{f-1} & \delta_{f-1} \end{pmatrix} \in GL_2(E)$. In the proof of this claim, the entries $\alpha_i, \beta_i, \gamma_i$ and δ_i are having independent meaning and should not be confused with those used before. The equation $Nm_\varphi(QP\varphi(Q)^{-1}) = \mathrm{diag}(\alpha\cdot\vec{1}, \alpha\cdot\vec{1})$ implies that $\prod\limits_{i=0}^{f-1}\alpha_i = \alpha$ and $(\prod\limits_{i=0}^{f-2}\alpha_i)\beta_{f-1} = 0$. Hence $\beta_{f-1} = 0$ and $QP\varphi(Q)^{-1} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\delta} \end{pmatrix}$ with $Nm_\varphi(\vec{\alpha}) = Nm_\varphi(\vec{\delta}) = \alpha\cdot\vec{1}$. Let $\vec{x} = (1, \alpha_0\alpha^{-1}, \alpha_0\alpha_1\alpha^{-1}, ..., \alpha_0\alpha_1\cdots\alpha_{f-2}\alpha^{-1})$, $\vec{y} = (1, \delta_0\alpha^{-1}, \delta_0\delta_1\alpha^{-1}, ..., \delta_0\delta_1\cdots\delta_{f-2}\alpha^{-1})$ and $R = \mathrm{diag}(\vec{x}, \vec{y})\cdot Q$. A computation shows that

$$RP\varphi(R)^{-1} = \begin{pmatrix} (\alpha, 1, .., 1) & \vec{0} \\ \vec{\zeta} & (\alpha, 1, .., 1) \end{pmatrix}$$

for some $\vec{\zeta} \in (E)^{|\mathcal{S}_{L_0}|}$. Since $Nm_\varphi(RP\varphi(R)^{-1}) = \mathrm{diag}(\alpha\cdot\vec{1}, \alpha\cdot\vec{1})$ we have $\zeta_0 + \alpha\sum\limits_{i=1}^{f-1}\zeta_i = 0$. Let $S = \begin{pmatrix} (1, 1, ..., 1) & (0, 0, ..., 0) \\ (z_0, z_1, ..., z_{f-1}) & (1, 1, ..., 1) \end{pmatrix}$, where $z_0 = 1$, $z_1 = 1 - \zeta_1 - \zeta_2 - \cdots - \zeta_{f-1}$, $z_2 = 1 - \zeta_2 - \cdots - \zeta_{f-1}, ..., z_{f-2} = 1 - \zeta_{f-2} - \zeta_{f-1}$ and $z_{f-1} = 1 - \zeta_{f-1}$, and let $Q^* = SR$. The fact that $\zeta_0 + \alpha\sum\limits_{i=1}^{f-1}\zeta_i = 0$ and a simple computation yield that $Q^*P\varphi(Q^*)^{-1} = \mathrm{diag}\ ((\alpha, 1, .., 1), (\alpha, 1, .., 1))$.

■ ***Proof of Proposition 2.2.*** Again, the notations in the proof of this lemma are having independent meaning and should not be confused with those of previous sections. Choose $\underline{\eta}$ as in Lemma 2.3. In case (1)(a) so that $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\vec{\varepsilon}, \vec{\theta})$ with $Nm_\varphi(\vec{\varepsilon}) \neq Nm_\varphi(\vec{\theta})$, let $\alpha_1, \delta_1 \in E^\times$ be such that $Nm_\varphi(\vec{\varepsilon}) = \alpha_1^f \cdot \vec{1}$ and $Nm_\varphi(\vec{\theta}) = \delta_1^f \cdot \vec{1}$. By Lemma 2.1 there exists a matrix $M \in GL_2(E^{|\mathcal{S}_{L_0}|})$ such that $M\left([\varphi]_{\underline{\eta}}\right)\varphi(M)^{-1} = \mathrm{diag}(\alpha_1 \cdot \vec{1}, \delta_1 \cdot \vec{1})$, and clearly $\alpha_1^f \neq \delta_1^f$. This gives the first possibility of the proposition. In case (1)(b) of Lemma 2.3, let α_1 an f-th root of α. By Lemma 2.1 there exists a matrix $M \in GL_2(E^{|\mathcal{S}_{L_0}|})$ such that $M\left([\varphi]_{\underline{\eta}}\right)\varphi(M)^{-1} = \begin{pmatrix} \alpha_1 \cdot \vec{1} & \vec{0} \\ \vec{\gamma} & \alpha_1 \cdot \vec{1} \end{pmatrix}$. Since $[\varphi^f]_{\underline{\eta}} = \begin{pmatrix} \alpha_1^f \cdot \vec{1} & \vec{0} \\ \alpha_1^{f-1} Tr_\varphi(\vec{\gamma}) & \alpha_1^f \cdot \vec{1} \end{pmatrix}$ and $[\varphi^f]_{\underline{e}} = \begin{pmatrix} \alpha \cdot \vec{1} & \vec{0} \\ \vec{1} & \alpha \cdot \vec{1} \end{pmatrix}$, we have $Tr_\varphi(\vec{\gamma}) \neq \vec{0}$. Let $M^* = \begin{pmatrix} f \cdot \vec{1} & \vec{0} \\ \vec{z} & Tr_\varphi(\vec{\gamma}) \end{pmatrix}$, where

$$\vec{z} = (0, 1, ..., f-1)\, Tr_\varphi(\vec{\gamma}) - f\,(\gamma_0, \gamma_0 + \gamma_1, ..., \gamma_0 + \gamma_1 + \cdots \gamma_{f-2}).$$

Then $\begin{pmatrix} \alpha_1 \cdot \vec{1} & \vec{0} \\ \vec{\gamma} & \alpha_1 \cdot \vec{1} \end{pmatrix} \varphi(M^*) = M^* \begin{pmatrix} \alpha_1 \cdot \vec{1} & \vec{0} \\ \vec{1} & \alpha_1 \cdot \vec{1} \end{pmatrix}$. This gives the third possibility of the proposition. Finally, in case (2)(b) of Lemma 2.3, let $\alpha_1 \in E^\times$ be an f-th root of α and proceed as in case (1). This gives the second possibility of the proposition and concludes the proof. ■

Definition 2.4 *A φ-module D is called* F*-semisimple,* F*-scalar or non-*F*-semisimple if and only if the $E^{|\mathcal{S}_{L_0}|}$-linear map φ^f has the corresponding property.*

One easily sees that D is F-semisimple if and only if there exists some ordered basis with respect to which the matrix of Frobenius is as in cases (1) or (2) of Proposition 2.2, with D being non F-scalar in case (1) and F-scalar in case (2). The φ-module D is not F-semisimple if and only if there exists an ordered basis with respect to which the matrix of Frobenius is as in case (3). A basis of D in which Frobenius is normalized as in Proposition 2.2 will be called standard. Unless otherwise stated, the matrix of any operator on D will be considered with respect to a fixed standard basis. In the next proposition we determine the matrix of the monodromy operator with respect to a standard basis $\underline{\eta}$.

Proposition 2.5 *Let D be a rank two (φ, N, E)-module.*

1. *If D is* F*-semisimple and $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha \cdot \vec{1}, \delta \cdot \vec{1})$, then the monodromy operator is as follows:*
   (a) *If $\alpha^f \neq p^{\pm f}\delta^f$, then $N = 0$;*
   (b) *If $\alpha^f = p^f \delta^f$, then $[N]_{\underline{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{n} & \vec{0} \end{pmatrix}$, where $\vec{n} = n(1, \zeta, \zeta^2, ..., \zeta^{f-1})$, with $\zeta = \frac{\alpha}{p\delta}$ and $n \in E$;*
   (c) *If $\delta^f = p^f \alpha^f$, then $[N]_{\underline{\eta}} = \begin{pmatrix} \vec{0} & \vec{n} \\ \vec{0} & \vec{0} \end{pmatrix}$, where $\vec{n} = n(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{f-1})$, with $\varepsilon = \frac{\delta}{p\alpha}$ and $n \in E$.*
2. *If D is non-*F*-semisimple, then $N = 0$.*

Proof. The condition $N\varphi = p\varphi N$ is equivalent to $[N]_{\underline{\eta}}[\varphi]_{\underline{\eta}} = p\ [\varphi]_{\underline{\eta}}\varphi([N]_{\underline{\eta}})$. The proposition follows by a short computation, using Lemma 2.1 and taking into account that N is nilpotent. ■

Corollary 2.6 *Let D be a rank two (φ, N, E)-module with nontrivial monodromy. There exists an ordered basis $\underline{\eta}$ with respect to which $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha \cdot \vec{1}, \delta \cdot \vec{1})$ for some $\alpha, \delta \in E^\times$ with $\alpha = p\delta$, and $[N]_{\underline{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{1} & \vec{0} \end{pmatrix}$.*

Proof. If $\alpha^f = p^f \delta^f$, change the basis to $\underline{\eta}'$ with $\eta_1' = \eta_1$ and $\eta_2' = \vec{n} \cdot \eta_2$. If $\delta^f = p^f \alpha^f$, first swap the basis elements, and then proceed as in the previous case. ■

When the monodromy operator is nontrivial our standard bases will always be as in the corollary above.

2.2 Galois descent data

In this section we determine the action of the Galois group $\mathrm{Gal}(L/K)$ on an arbitrary rank two filtered $(\varphi, N, L/K, E)$-module D.

2.2.1 The Galois action on $L \otimes_{\mathbb{Q}_p} E$

Since E is assumed to be large enough, each embedding τ_j of L_0 into E extends to an embedding of L into E in exactly $e = [L : L_0]$ different ways. For each $j \in \{0, 1, ..., f-1\}$, let $h_{ij} : L \to E$ with $i \in \{0, 1, ..., e-1\}$ be any numbering of the distinct extensions of $\tau_j : L_0 \to E$ to L. Each index $s \in \{0, 1, ..., m-1\}$ can be written uniquely in the form $s = fi + j$ with $i \in \{0, 1, ..., e-1\}$ and $j \in \{0, 1, ..., f-1\}$. For each $s = 0, 1, ..., m-1$, let $\sigma_s := h_{ij}$. These are all the distinct embeddings of L into E and we fix the m-tuple of embeddings $\mathcal{S}_L := (\sigma_0, \sigma_1, ..., \sigma_{m-1})$ once and for all. Recall the notation $E^{|\mathcal{S}_L|} := \prod\limits_{\mathcal{S}_L} E$. The map

$$\xi_L : L \otimes_{\mathbb{Q}_p} E \to E^{|\mathcal{S}_L|} : x \otimes y \mapsto (\sigma(x)y)_\sigma$$

is a ring isomorphism. A simple computation shows that $\xi_L(1 \otimes \alpha) = \xi_{L_0}(\alpha)^{\otimes e}$ for any $\alpha \in L_0 \otimes_{\mathbb{Q}_p} E$, where ξ_{L_0} is the isomorphism of Section 2. For each vector $\vec{a} \in E^{|\mathcal{S}_{L_0}|}$ we denote $\vec{a}^{\otimes e}$ the vector of $E^{|\mathcal{S}_L|}$ gotten by e copies of $\vec{a}$, removing the inner parentheses. For each $g \in G = \mathrm{Gal}(L/K)$ consider the permutation $\pi(g)$ on $\{0, 1, ..., m-1\}$ defined by $\sigma_i \cdot g = \sigma_{\pi(g)(i)}$ for any $g \in G$ and any embedding σ_i. The map $\rho : G \to S_m$ with $\rho(g) = \pi(g)^{-1}$ is a group monomorphism. We define an E-linear G-action on $E^{|\mathcal{S}_L|}$ by setting $g\xi_L(\alpha) = \xi_L(g\alpha)$ for all g and α. If $x \otimes y \in L \otimes_{\mathbb{Q}_p} E$ and $g \in G$, then $g\xi_L(x \otimes y) = (\sigma_{\pi(g)(i)}(x)y)_{\sigma_i}$, therefore $g(\sigma_0(x)y, \sigma_1(x)y, ..., \sigma_{m-1}(x)y) = (\sigma_{\pi(g)(0)}(x)y, ..., \sigma_{\pi(g)(m-1)}(x)y)$ for any $x \otimes y \in L \otimes_{\mathbb{Q}_p} E$ (with indices viewed modulo m). From this we easily deduce that

$$g(x_0, x_1, ..., x_{m-1}) = (x_{\pi(g)(0)}, ..., x_{\pi(g)(m-1)})$$

for any $(x_0, x_1, ..., x_{m-1}) \in E^{|\mathcal{S}_L|}$ and $g \in G$.

2.2.2 The Galois action on $L_0 \otimes_{\mathbb{Q}_p} E$

We use the isomorphism ξ_{L_0} of Section 2 to define an E-linear G-action on $E^{|\mathcal{S}_{L_0}|}$ by setting $g\xi_{L_0}(x) = \xi_{L_0}(gx)$ for all $g \in G$ and $x \in L_0 \otimes_{\mathbb{Q}_p} E$. For each $g \in G$ there exists a unique integer $n(g) \in$

$\{0,1,...,f-1\}$ such that $g\mid_{L_0}=\tau^{n(g)}$. One easily sees that $g\vec{\alpha}=(\alpha_{n(g)},\alpha_{n(g)+1},...,\alpha_{n(g)+f-1})$ for all g and $\vec{\alpha}=(\alpha_0,\alpha_1,...,\alpha_{f-1})$. We write ${}^g\vec{\alpha}$ instead of $g\vec{\alpha}$ and it is obvious that $Nm_\varphi({}^g\vec{\alpha})=Nm_\varphi(\vec{\alpha})$. Clearly $\xi_L(g(1\otimes\alpha))=\xi_{L_0}(g\alpha)^{\otimes e}$ for any $g\in G$ and $\alpha\in L_0\otimes_{\mathbb{Q}_p}E$, and this implies that $g(\vec{\alpha}^{\otimes e})=(g\vec{\alpha})^{\otimes e}$. In the next proposition we determine the matrix of the Galois action with respect to a standard basis. Recall that when the monodromy is nontrivial, standard bases are as in the comment succeeding Corollary 2.6.

Proposition 2.7 *Let D be a rank two $(\varphi,N,L/K,E)$-module and let $\underline{\eta}$ be a standard basis of D.*

1. *If D is* F*-semisimple and non-scalar,*
   - *(a) If the monodromy N is nontrivial, then there exists some $E^\times$-valued character χ of G such that $[g]_{\underline{\eta}}=\mathrm{diag}(\chi(g)\cdot\vec{1},\chi(g)\cdot\vec{1})$ for all $g\in G$;*
   - *(b) If the monodromy N is trivial, then there exist some $E^\times$-valued characters χ,ψ of G such that $[g]_{\underline{\eta}}=\mathrm{diag}(\chi(g)\cdot\vec{1},\psi(g)\cdot\vec{1})$ for all $g\in G$.*
2. *If D is* F*-scalar, then there exists some group homomorphism*

   $\lambda:G\to GL_2(E)$ *such that* $[g]_{\underline{\eta}}=\lambda(g)\cdot\mathrm{diag}(\vec{1},\vec{1})$ *for all* $g\in G$.
3. *If D is not* F*-semisimple, then there exist some $E^\times$-valued character χ of G such that $[g]_{\underline{\eta}}=\mathrm{diag}(\chi(g)\cdot\vec{1},\chi(g)\cdot\vec{1})$ for all $g\in G$.*

Proof. For G to act on D we must have $[g_1g_2]_{\underline{\eta}}=[g_1]_{\underline{\eta}}\left({}^{g_1}[g_2]_{\underline{\eta}}\right)$ for any $g_1,g_2\in G$. We determine the shape of the matrices $[g]_{\underline{\eta}}$ utilizing the fact that the Galois action commutes with Frobenius and the monodromy operators. That happens if and only if $[\varphi]_{\underline{\eta}}\varphi\left([g]_{\underline{\eta}}\right)=[g]_{\underline{\eta}}({}^g[\varphi]_{\underline{\eta}})$ and $[N]_{\underline{\eta}}[g]_{\underline{\eta}}=[g]_{\underline{\eta}}\left({}^g[N]_{\underline{\eta}}\right)$ for all $g\in G$. The proof of the proposition is a tedious calculation and we only give the details in Case (3). For any g, we write $[g]_{\underline{\eta}}=\begin{pmatrix}\vec{\alpha}(g) & \vec{\beta}(g)\\ \vec{\gamma}(g) & \vec{\delta}(g)\end{pmatrix}$. In this case the monodromy operator is trivial. Let $[\varphi]_{\underline{\eta}}=\begin{pmatrix}\alpha\cdot\vec{1} & \vec{0}\\ \vec{1} & \alpha\cdot\vec{1}\end{pmatrix}$ for some $\alpha\in E^\times$. The equation $[\varphi]_{\underline{\eta}}\varphi\left([g]_{\underline{\eta}}\right)=[g]_{\underline{\eta}}({}^g[\varphi]_{\underline{\eta}})$ implies that for all $g\in G$, $[g]_{\underline{\eta}}=\begin{pmatrix}\alpha(g)\cdot\vec{1} & \vec{0}\\ \gamma(g)\cdot\vec{1} & \alpha(g)\cdot\vec{1}\end{pmatrix}$ for some functions $\alpha,\gamma:G\to E$. The equation $[g_1g_2]_{\underline{\eta}}=[g_1]_{\underline{\eta}}\left({}^{g_1}[g_2]_{\underline{\eta}}\right)$ implies that $\alpha:G\to E^\times$ is a character, and that $\gamma(g_1g_2)=\alpha(g_1)\gamma(g_2)+\alpha(g_2)\gamma(g_1)$ for all g_1 and g_2. By induction, $\gamma(g^n)=n\alpha(g^{n-1})\gamma(g)$ for any $g\in G$ and any non negative integer n. Since $\gamma(1)=0$ and $\alpha(g)\neq 0$ for all g, we have $\gamma(g)=0$ because G is finite. ∎

3 Galois-stable filtrations

In this section we describe the shape of the filtrations of rank two filtered modules and construct those which are stable under the Galois action. The notion of a labeled Hodge-Tate weight will be important.

3.1 Labeled Hodge-Tate weights

If D is a rank n filtered $(\varphi, N, L/K, E)$-module, $D_L = L \otimes_{L_0} D$ may be viewed as a module over $E^{|\mathcal{S}_L|}$ via the ring isomorphism ξ_L of Section 2.2.1. For each embedding σ of L into E, let $e_\sigma := (0, ..., 0, 1_\sigma, 0, ..., 0) \in E^{|\mathcal{S}_L|}$ and $D_{L,\sigma} := e_\sigma D_L$. We have the decomposition

$$D_L = \bigoplus_{\sigma \in S_L} D_{L,\sigma}.$$

Since D_L is free of rank n over $L \otimes_{\mathbb{Q}_p} E$, the components $D_{L,\sigma}$ are equidimensional over E, each of dimension n. We remark that the $E^{|\mathcal{S}_L|}$-modules $e_\sigma D_L$ are not necessarily free. We filter each component $D_{L,\sigma} = e_\sigma D_L$ be setting $\mathrm{Fil}^j D_{L,\sigma} := e_\sigma \mathrm{Fil}^j D_L$. An integer j is called a labeled Hodge-Tate weight of D_L (or of D) with respect to the embedding σ if and only if $\mathrm{Fil}^{-j} D_{L,\sigma} \neq \mathrm{Fil}^{-j+1} D_{L,\sigma}$. It is counted with multiplicity $\dim_E \left(\mathrm{Fil}^{-j} D_{L,\sigma} / \mathrm{Fil}^{-j+1} D_{L,\sigma}\right)$. Since the components $D_{L,\sigma}$ are equidimensional over E, there are n labeled Hodge-Tate weights for each embedding σ, counting multiplicities. The labeled Hodge-Tate weights of D are by definition the m-tuple of multiset $(W_i)_{\sigma_i}$, where each such multiset W_i contains n integers, the opposites of the jumps of the filtration of D_{L,σ_i}. From now on we restrict attention to rank two filtered modules with labeled Hodge-Tate weights $(\{0, -k_i\})_{\sigma_i}$, with k_i non negative integers. When the labeled Hodge-Tate weights are arbitrary, we can always shift them into this range, after twisting by some appropriate rank one weakly admissible filtered φ-module. Indeed, since $\mathrm{Fil}^j (D_1 \otimes D_2) = \sum\limits_{j_1 + j_2 = j} \mathrm{Fil}^{j_1} D_1 \otimes \mathrm{Fil}^{j_2} D_2$ for any filtered modules D_1 and D_2 and any integer j, the claim follows easily using the shape of the rank-one weakly admissible filtered φ-modules given in the Appendix and the definition of a labeled Hodge-Tate weight.

Notation 2 *Let $k_0, k_1, ..., k_{m-1}$ be non negative integers which we call weights. Assume that after ordering them and omitting possibly repeated weights we get $w_0 < w_1 < ... < w_{t-1}$, where w_0 is the smallest weight, w_1 the second smallest weight, ..., w_{t-1} is the largest weight and $1 \leq t \leq m$. For convenience we define $w_{-1} = 0$. Let $I_0 = \{0, 1, ..., m-1\}$, $I_1 = \{i \in I_0 : k_i > w_0\}, ..., I_{t-1} = \{i \in I_0 : k_i > w_{t-2}\} = \{i \in I_0 : k_i = w_{t-1}\}$, $I_t = \varnothing$ and $I_0^+ = \{i \in I_0 : k_i > 0\}$. Notice that $\sum\limits_{i=0}^{t-1} w_i(|\, I_i \,| - |\, I_{i+1} \,|) = \sum\limits_{i=0}^{m-1} k_i$. If $\vec{x} \in E^{|\mathcal{S}_L|}$, we write $J_{\vec{x}} = \{i \in I_0 : x_i \neq 0\}$. For any $J \subset I_0$, we let $f_J := \sum\limits_{i \in J} e_{\sigma_i}$. If A is a matrix with entries in $E^{|\mathcal{S}_{L_0}|}$ we write $A^{\otimes e}$ for the matrix with entries in $\prod\limits_{\mathcal{S}_L} E$ obtained by replacing each entry $\vec{\alpha}$ of A by $\vec{a}^{\otimes e}$, where $\vec{a}^{\otimes e}$ is as in Section 2.2.1.*

3.2 The shape of the filtrations

Let D_L be a filtered φ-module with labeled Hodge-Tate weights $(\{-k_i, 0\})_{\sigma_i}$ and let $\underline{\eta} = (\eta_1, \eta_2)$ be any ordered basis of D over $E^{|\mathcal{S}_{L_0}|}$. By the definition of a labeled Hodge-Tate weight we have

$$\mathrm{Fil}^j(D_{L,\sigma_i}) = \begin{cases} e_{\sigma_i} D_L & \text{if } j \leq 0, \\ D_L^i & \text{if } 1 \leq j \leq k_i, \\ 0 & \text{if } j \geq 1 + k_i, \end{cases}$$

where $D_L^i = (E^{|\mathcal{S}_L|}) \left(\vec{x}^i (1 \otimes \eta_1) + \vec{y}^i (1 \otimes \eta_2)\right) e_{\sigma_i}$, for some vectors $\vec{x}^i = (x_0^i, x_1^i, ...,$

$x^i_{m-1})$ and $\vec{y}^i = (y^i_0, y^i_1, ..., y^i_{m-1}) \in E^{|\mathcal{S}_L|}$, with the additional condition that $(x^i_i, {y_i}^i) \neq (0,0)$ whenever $k_i > 0$. Since one may choose the x^i_i and y^i_i arbitrarily when $k_i = 0$, we may assume that $(x^i_i, y^i_i) \neq (0,0)$ for all $i \in I_0$. From now on we always make this assumption. Since $\mathrm{Fil}^j(D_L) = \bigoplus_{i=0}^{m-1} e_{\sigma_i}\mathrm{Fil}^j(D_L)$, we have $\mathrm{Fil}^j D_L = D_L$ for $j \leq 0$ and $\mathrm{Fil}^j D_L = 0$ for $j \geq 1 + w_{t-1}$. Let $1 + w_{r-1} \leq j \leq w_r$ for some $r \in \{0, 1, ..., t-1\}$ (recall that $w_{-1} = 0$), then $\mathrm{Fil}^j D_L = \bigoplus_{i \in I_r} D^i_L$. If $\vec{x} = (x^0_0, x^1_1, ..., x^{m-1}_{m-1})$ and $\vec{y} = (y^0_0, y^1_1, ..., y^{m-1}_{m-1})$, then $(x^i_i, y^i_i) \neq (0,0)$ for all $i \in I_0$ and

$$\mathrm{Fil}^j(D_L) = \begin{cases} D_L \quad \text{if} \quad j \leq 0, \\ (E^{|\mathcal{S}_L|}) f_{I_0}\left(\vec{x}(1\otimes\eta_1) + \vec{y}(1\otimes\eta_2)\right) \quad \text{if} \quad 1 \leq j \leq w_0, \\ (E^{|\mathcal{S}_L|}) f_{I_1}\left(\vec{x}(1\otimes\eta_1) + \vec{y}(1\otimes\eta_2)\right) \quad \text{if} \quad 1 + w_0 \leq j \leq w_1, \\ \cdots\cdots\cdots \\ (E^{|\mathcal{S}_L|}) f_{I_{t-1}}\left(\vec{x}(1\otimes\eta_1) + \vec{y}(1\otimes\eta_2)\right) \quad \text{if} \quad 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \quad \text{if} \quad j \geq 1 + w_{t-1}. \end{cases}$$

Remark 3.1 *The filtration of D_L can be put into the shape above (for appropriate vectors $\vec{x}$ and $\vec{y}$) with respect to any ordered basis of D_L. Two filtrations of D_L are called equivalent if one is obtained from the other by replacing $\vec{x}$ by $\vec{t}\cdot\vec{x}$ and $\vec{y}$ by $\vec{t}\cdot\vec{y}$, for some $\vec{t} \in (E^{\times})^{|\mathcal{S}_L|}$. Filtrations will be considered up to equivalence and one may assume that $\vec{y} = f_{J_{\vec{y}}}$. If $\underline{\eta} = (\eta_1, \eta_2)$ is a standard basis of D, the filtration of D_L will be considered with respect to the basis $1 \otimes \underline{\eta} = (1\otimes\eta_1, 1\otimes\eta_2)$. We denote $E^{|S_L|_J} := (E^{|\mathcal{S}_L|}) \cdot f_J$, for any $J \subset I_0$.*

3.3 Galois-stable filtrations in the non-F-scalar case

We now assume that D is not F-scalar and we construct the filtrations of D_L which are stable under the action of $G = \mathrm{Gal}(L/K)$. We define a right action of G on I_0 by letting $i \cdot g := \pi(g)(i)$, where π is as in Section 2.2.1. Each orbit has cardinality equal to $\#G$, hence there are $\nu := [K : \mathbb{Q}_p]$ orbits which we denote by $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_\nu$. Since the homomorphism ρ of Section 2.2.1 is injective, the G-action on I_0 is free. Let $[g]_{\underline{\eta}} = (\chi(g)\cdot\vec{1}, \psi(g)\cdot\vec{1})$ with the characters χ and ψ as in Proposition 2.7, and let the filtration of D_L be

$$\mathrm{Fil}^j(D_L) = \begin{cases} D_L \quad \text{if} \quad j \leq 0, \\ \left(E^{|S_L|_{I_r}}\right)\left(\vec{x}(1\otimes\eta_1) + \vec{y}(1\otimes\eta_2)\right) \text{ if} \\ 1 + w_{r-1} \leq j \leq w_r, \text{ for } r = 0, ..., t-1, \\ 0 \quad \text{if} \quad j \geq 1 + w_{t-1}, \end{cases} \tag{3.1}$$

for some vectors $\vec{x}, \vec{y} \in E^{|\mathcal{S}_L|}$ with $(x_i, y_i) \neq (0,0)$ for all $i \in I_0$. We must have that $g(\mathrm{Fil}^j D_L) \subset \mathrm{Fil}^j D_L$ for any $g \in G$ and $j \in \mathbb{Z}$. For any $r \in \{0, 1, ..., t-1\}$ there must exist some vector $\vec{t} = \vec{t}(r, g) \in E^{|\mathcal{S}_L|}$ such that the following equations hold:

$$\chi(g)({}^g f_{I_r \cap J_{\vec{x}}}) \cdot ({}^g\vec{x}) = \vec{t} \cdot f_{I_r \cap J_{\vec{x}}} \cdot \vec{x} \text{ and } \psi(g)({}^g f_{I_r \cap J_{\vec{y}}}) \cdot {}^g\,\vec{y} = \vec{t} \cdot f_{I_r \cap J_{\vec{y}}} \cdot \vec{y}. \tag{3.2}$$

Notation 3 *If $g \in G$ and $J \subset I_0$ we denote by ${}^gJ$ the set $\{j \cdot g,\ j \in J\}$.*

For any $J, J_1, J_2 \subset I_0$, any $g \in G$ and any $\vec{x} \in E^{|\mathcal{S}_L|}$ the following equations are trivial to check:

$$\begin{gathered} f_{J_1} \cdot f_{J_2} = f_{J_1 \cap J_2},\ {}^g(f_I) = f_{({}^gI)},\ ({}^g f_{J_1}) \cdot f_{J_2} = f_{({}^gJ_1)\cap J_2}, {}^g J_{\vec{x}} = J_{{}^g\vec{x}} \\ \text{and } {}^g(J_1 \cap J_2) = ({}^gJ_1) \cap ({}^gJ_2). \end{gathered} \tag{3.3}$$

Since $\chi(g) \neq 0$ for all g, the equation $\chi(g)({}^{g}f_{I_r \cap J_{\vec{x}}}) \cdot ({}^{g}\vec{x}) = \vec{t} \cdot f_{I_r \cap J_{\vec{x}}} \cdot \vec{x}$ implies ${}^{g}(I_r \cap J_{\vec{x}}) \cap J_{{}^{g}\vec{x}} \subset I_r \cap J_{\vec{x}}$. This is equivalent to ${}^{g}(I_r \cap J_{\vec{x}}) \subset I_r \cap J_{\vec{x}}$ and therefore to ${}^{g}(I_r \cap J_{\vec{x}}) = I_r \cap J_{\vec{x}}$ for all $g \in G$. Similarly, ${}^{g}(I_r \cap J_{\vec{y}}) = I_r \cap J_{\vec{y}}$ for all $g \in G$. The latter (for $r = 0$ combined with Formulae (3.3)) imply that the sets $J_{\vec{x}}$ and $J_{\vec{y}}$ are G-stable and therefore unions of G-orbits of I_0. Since $J_{\vec{x}} \cup J_{\vec{y}} = I_0$, each set I_r is G-stable and therefore a union of G-orbits as well. For a fixed g, equations (3.2) hold for any $r = 0, 1, ..., t-1$ if and only if they hold for $r = 0$, they are therefore equivalent to the existence of some vector $\vec{t} = \vec{t}(g) \in E^{|\mathcal{S}_L|}$ such that

$$\left(x_{\pi(g)(i_j)}, y_{\pi(g)(i_j)}\right) = \left(t(g)_{i_j} \cdot x_{i_j}, t(g)_{i_j} \cdot y_{i_j}\right) \cdot \operatorname{diag}\left(\chi(g)^{-1}, \psi(g)^{-1}\right) \text{ for all } g \in G.$$

Since $J_{\vec{x}} \cup J_{\vec{y}} = I_0$ all the coordinates of $\vec{t}(g)$ are non zero and by Remark 3.1 we may assume that $\vec{t}(g) = \vec{1}$ for all $g \in G$. Let i_j be any index in the orbit $\mathcal{O}_j$, with $1 \leq j \leq \nu$, and let $(x_{i_j}, y_{i_j}) \in E \times E$ with $(x_{i_j}, y_{i_j}) \neq (0,0)$. Since G acts freely on I_0, for each index $\ell \in I_0$ there exist unique $j \in \{1, 2, ..., \nu\}$ and $g \in G$ such that $\ell = i_j \cdot g$. Let $\vec{x}, \vec{y} \in E^{|\mathcal{S}_L|}$ be the vectors with coordinates $(x_\ell, y_\ell) := \left(x_{i_j}, y_{i_j}\right) \cdot \operatorname{diag}\left(\chi(g)^{-1}, \psi(g)^{-1}\right)$ for all $g \in G$. Clearly

$$\vec{x} = \sum_{j=1}^{\nu} \left\{ \sum_{g \in G} x_{i_j} \cdot \chi(g^{-1}) \cdot e_{\pi(g)(i_j)} \right\} \text{ and } \vec{y} = \sum_{j=1}^{\nu} \left\{ \sum_{g \in G} y_{i_j} \cdot \psi(g^{-1}) \cdot e_{\pi(g)(i_j)} \right\}.$$

By the discussion above we have the following proposition.

Proposition 3.2 *The filtration in (3.1) with vectors $\vec{x}$ and $\vec{y}$ as above is G-stable if and only if the sets I_r are unions of G-orbits of I_0 for all $1 \leq r \leq t-1$. Conversely, any G-stable filtration of D_L is equivalent to a filtration of this form.*

Example 3.3 *Let $K = \mathbb{Q}_p$ and let L be any finite Galois extension of $\mathbb{Q}_p$. The action of G on I_0 is free and transitive. Since the sets I_r are unions of G-orbits, $I_r = \varnothing$ for all $r \geq 1$ and all the labeled Hodge-Tate weights are equal to some non negative integer k. Since the sets $J_{\vec{x}}$ and $J_{\vec{y}}$ are unions of G-orbits, the only possibilities are $(J_{\vec{x}}, J_{\vec{y}}) = (\varnothing, I_0)$, $(I_0, \varnothing)$, (I_0, I_0). The only G-stable filtrations (up to equivalence) are*

$$\operatorname{Fil}^j(D_L) = \begin{cases} D_L & \text{if } j \leq 0, \\ (E^{|\mathcal{S}_L|})\,(\vec{x}(1 \otimes \eta_1) + \vec{y}(1 \otimes \eta_2)) & \text{if } 1 \leq j \leq k, \\ 0 & \text{if } j \geq 1 + k, \end{cases}$$

with $(\vec{x}, \vec{y}) = (\vec{0}, \vec{1})$ if $(J_{\vec{x}}, J_{\vec{y}}) = (\varnothing, I_0)$, $(\vec{x}, \vec{y}) = (\vec{1}, \vec{0})$ if $(J_{\vec{x}}, J_{\vec{y}}) = (I_0, \varnothing)$ and

$$(\vec{x}, \vec{y}) = \left(x_0 \left(1, \frac{\psi(\sigma)}{\chi(\sigma)}, \left(\frac{\psi(\sigma)}{\chi(\sigma)}\right)^2, ..., \left(\frac{\psi(\sigma)}{\chi(\sigma)}\right)^{m-1} \right),\ \vec{1} \right)$$

for any $x_0 \in E^\times$, if $(J_{\vec{x}}, J_{\vec{y}}) = (I_0, I_0)$.

3.4 Galois-stable filtrations in the F-scalar case

Let λ be the homomorphism of Proposition 2.7 and let $\lambda(g) = \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix}$. The Galois action preserves the filtration if and only if for any $g \in G$ and any $0 \leq r \leq t-1$, there exists some vector

$\vec{t} = \vec{t}(g,r) \in E^{|\mathcal{S}_L|}$ such that

$$ {}^g f_{I_r}\{\alpha(g)\cdot({}^g\vec{x}) + \beta(g)\cdot({}^g\vec{y})\} = \vec{t}\cdot\vec{x}\cdot f_{I_r}, $$
$$ {}^g f_{I_r}\{\gamma(g)\cdot({}^g\vec{x}) + \delta(g)\cdot({}^g\vec{y})\} = \vec{t}\cdot\vec{y}\cdot f_{I_r}. $$

Suppose that there exists some $i \in {}^gI_r$ with $i \notin I_r$. Then $\left(x_{\pi(g)(i)}, y_{\pi(g)(i)}\right)\cdot\lambda(g) = (0,0)$, and since $\det\lambda(g) \neq 0$ we have $\left(x_{\pi(g)(i)}, y_{\pi(g)(i)}\right) = (0,0)$ a contradiction. Therefore ${}^gI_r = I_r$ for all g. Then $g\left(\mathrm{Fil}^i D_L\right) \subset \mathrm{Fil}^j D_L$ if and only if there exists some vector $\vec{t} = \vec{t}(g,0) \in E^{|\mathcal{S}_L|}$ such that $({}^g\vec{x}, {}^g\vec{y}) = \left(\vec{t}\cdot\vec{x}, \vec{t}\cdot\vec{y}\right)\left(\lambda(g^{-1})\cdot\mathrm{diag}\left(\vec{1},\vec{1}\right)\right)$. This is equivalent to $\left(x_{\pi(g)(i_j)}, y_{\pi(g)(i_j)}\right) = \left(t(g)_{i_j}\cdot x_{i_j}, t(g)_{i_j}\cdot y_{i_j}\right)\cdot\lambda(g^{-1})$ for all $g \in G$. Arguing as in Section 3.3 one sees that $\vec{t}(g,0) \in (E^\times)^{|\mathcal{S}_L|}$ for all g. By Remark 3.1 we may assume that $\vec{t}(g) = \vec{1}$ for all $g \in G$. Let i_j be any index in the orbit $\mathcal{O}_j$, with $1 \leq j \leq \nu$, and let $(x_{i_j}, y_{i_j}) \in E \times E$ with $(x_{i_j}, y_{i_j}) \neq (0,0)$. Since G acts freely on I_0, for each index $\ell \in I_0$ there exist unique $j \in \{1,2,...,\nu\}$ and $g \in G$ such that $\ell = i_j \cdot g$. Let $\vec{x}, \vec{y} \in E^{|\mathcal{S}_L|}$ be the vectors with coordinates $(x_\ell, y_\ell) := \left(x_{i_j}, y_{i_j}\right)\cdot\lambda(g^{-1})$ for all $g \in G$. Clearly

$$ \vec{x} = \sum_{j=1}^{\nu}\left\{\sum_{g\in G} x_{\pi(g)(i_j)}\cdot e_{\pi(g)(i_j)}\right\} \text{ and } \vec{y} = \sum_{j=1}^{\nu}\left\{\sum_{g\in G} y_{\pi(g)(i_j)}\cdot e_{\pi(g)(i_j)}\right\}. $$

By the discussion above we have the following proposition.

Proposition 3.4 *The filtration in (3.1) with vectors $\vec{x}$ and $\vec{y}$ as above is G-stable if and only if the sets I_r are unions of G-orbits of I_0 for all $1 \leq r \leq t-1$. Conversely, any G-stable filtration of D_L is equivalent to a filtration of this form.*

4 Hodge and Newton invariants

In this section we compute Hodge and Newton invariants of rank two filtered φ-modules (D,φ). We thank the referee for pointing out a mistake in the computation of Newton invariants. The same mistake had been pointed out by David Savitt to whom we extend our thanks.

Let v_p be the valuation of $\bar{\mathbb{Q}}_p$ normalized so that $v_p(p) = 1$ and let $\mathrm{val}_L(x) = ev_p(x)$ for any $x \in L$. Following [BS06, §3], we define

$$ t_N(D) := \frac{1}{[L:\mathbb{Q}_p]}\mathrm{val}_L\left(\det_{L_0}\varphi^f\right) \tag{4.1} $$

and

$$ t_H(D_L) := \sum_{\sigma\in S_L}\sum_{j\in\mathbb{Z}}\left(\mathrm{Fil}^j D_{L,\sigma}/\mathrm{Fil}^{j+1}D_{L,\sigma}\right). \tag{4.2} $$

Recall that the map φ^f is $L_0\otimes_{\mathbb{Q}_p}E$-linear. The filtered φ-module (D,φ) is weakly admissible if $t_H(D_L) = t_N(D)$ and $t_H(D'_L) \leq t_N(D')$ for any φ-stable L_0-subspace $D' \subseteq D$, where $D'_L = L\otimes_{L_0} D'$, and D'_L is equipped with the induced filtration. By [BM02, Prop. 3.1.1.5] (with trivial modifications adopted to our definitions of the Hodge and Newton invariants), one may only check the inequalities above for φ-stable $L_0\otimes_{\mathbb{Q}_p}E$-submodules D' of D. We first determine the $L_0\otimes_{\mathbb{Q}_p}E$-submodules of D which are stable under Frobenius and the monodromy.

Proposition 4.1 *Let $\underline{\eta} = (\eta_1, \eta_2)$ be an ordered basis with respect to which the matrix of Frobenius has the form $[\varphi]_{\underline{\eta}} = \begin{pmatrix} \vec{\alpha} & \vec{0} \\ \vec{\gamma} & \vec{\delta} \end{pmatrix}$. All the φ-stable $L_0 \otimes_{\mathbb{Q}_p} E$-submodules of D are 0, D, $D_2 = (E^{|\mathcal{S}_{L_0}|})\eta_2$, or of the form $D_{\vec{\theta}} = (E^{|\mathcal{S}_{L_0}|})(\eta_1 + \vec{\theta}\eta_2)$ for some vector $\vec{\theta} \in E^{|\mathcal{S}_{L_0}|}$.*

Proof. Let M be a φ-stable submodule of D. Case (1). If $M \cap (E^{|\mathcal{S}_{L_0}|})\eta_2 \neq 0$. Let $\vec{x}\eta_2 \in M$ with $\vec{x} \neq \vec{0}$. Then $\sum\limits_{i \in J_{\vec{x}}} e_{\tau_i}\eta_2 \in M$, and after multiplying by e_{τ_i} for some $i \in J_{\vec{x}}$ we get $e_{\tau_i}\eta_2 \in M$ for some (in fact all) $i \in J_{\vec{x}}$. We repeatedly act by φ and see that $e_{\tau_i}\eta_2 \in M$ for all i, which implies that $\eta_2 \in M$. If $\vec{x}\eta_1 + \vec{y}\eta_2 \in M$ for some $\vec{x} \neq \vec{0}$, then $\vec{x}\eta_1 \in M$. Arguing as before, given that $\eta_2 \in M$, we see that $\eta_1 \in M$ therefore $M = D$. Hence in this case $M = (E^{|\mathcal{S}_{L_0}|})\eta_2$ or $M = D$. Case (2). If $M \cap (E^{|\mathcal{S}_{L_0}|})\eta_2 = 0$. Assume that $M \neq 0$ and let $\vec{x}\eta_1 + \vec{y}\eta_2 \in M$ with $\vec{x} \neq \vec{0}$. Then $(\sum\limits_{i \in J_{\vec{x}}} e_{\tau_i})\eta_1 + \vec{y}_1\eta_2 \in M$ for some $\vec{y}_1 \in E^{|\mathcal{S}_{L_0}|}$and $e_{\tau_i}\eta_1 + \vec{y}_2\eta_2 \in M$ for some index $i \in J_{\vec{x}}$ and some vector $\vec{y}_2$. We repeatedly act by φ and use the fact that M is φ-stable to get that $\eta_1 + \vec{\theta}\eta_2 \in M$ for some vector $\vec{\theta}$. We will show that $M = (E^{|\mathcal{S}_{L_0}|})(\eta_1 + \vec{\theta}\eta_2)$. Every nonzero element of M has the form $\vec{\alpha}\eta_1 + \vec{\beta}\eta_2$ for some vectors $\vec{\alpha} \neq \vec{0}$ and $\vec{\beta}$. Since $\vec{\alpha}\eta_1 + \vec{\alpha}\cdot\vec{\theta}\eta_2 \in M$, we see that $(\vec{\alpha}\cdot\vec{\theta} - \vec{\beta})\eta_2 \in M$ which implies that $\vec{\alpha}\cdot\vec{\theta} = \vec{\beta}$. Then $\vec{\alpha}\eta_1 + \vec{\beta}\eta_2 = \vec{\alpha}\eta_1 + \vec{\alpha}\cdot\vec{\theta}\eta_2 = \vec{\alpha}(\eta_1 + \vec{\theta}\eta_2)$. ■

We now determine the vectors $\vec{\theta}$ for which $D_{\vec{\theta}} = (E^{|\mathcal{S}_{L_0}|})(\eta_1 + \vec{\theta}\eta_2)$ is φ-stable. We have the following cases.

Case (1). If D is F-semisimple and non-scalar. In this case $D_{\vec{\theta}}$ is φ-stable if and only if there exists $\vec{t} \in E^{|\mathcal{S}_{L_0}|}$ such that $\varphi(\eta_1 + \vec{\theta}\eta_2) = \vec{t}(\eta_1 + \vec{\theta}\ \eta_2)$. We repeatedly act by φ and get $\varphi^f(\eta_1) + \vec{\theta}\varphi^f(\eta_2) = Nm_{\varphi}(\vec{t})(\eta_1 + \vec{\theta}\eta_2)$. This implies $Nm_{\varphi}(\alpha\cdot\vec{1}) = Nm_{\varphi}(\vec{t})$ and $\vec{0} = (\alpha^f - \delta^f)\cdot\vec{\theta}$. Since $\alpha^f \neq \delta^f$, the only nontrivial φ-stable submodules of D are $D_1 = (E^{|\mathcal{S}_{L_0}|})\eta_1$ and $D_2 = (E^{|\mathcal{S}_{L_0}|})\eta_2$.

Case (2). If D is F-scalar we easily see that $D_{\vec{\theta}}$ is φ-stable if and only if $\vec{\theta} = \theta\cdot\vec{1}$ for some $\theta \in E^{\times}$.

Case (3). If D is not F-semisimple $D_{\vec{\theta}}$ is never φ-stable.

Note that the submodules D_1, D_2 and D_θ are pairwise complementary in D, and so are D_{θ_1} and D_{θ_2} whenever $\theta_1 \neq \theta_2$. Combining the results of the previous paragraph with those of Proposition 2.5, we get the following proposition.

Proposition 4.2 *Let $\underline{\eta}$ be a standard basis of a (φ, N)-module D. The submodules of D fixed by Frobenius and the monodromy are*

1. 0, D, $D_1 = (E^{|\mathcal{S}_{L_0}|})\eta_1$ *and* $D_2 = (E^{|\mathcal{S}_{L_0}|})\eta_2$ *if D is F-semisimple, non-*F*-scalar;*
2. 0, D, D_1, D_2 *and* $D_\theta = (E^{|\mathcal{S}_{L_0}|})(\eta_1 + \theta\cdot\vec{1}\cdot\eta_2)$, *for any $\theta \in E^{\times}$ if D is* F*-scalar;*
3. 0, D *and* D_2 *if D is* F*-semisimple.*

We proceed to compute Hodge invariants. We retain the notation of Proposition 4.2 and we write $D_{i,L} := L \otimes_{L_0} D_i$ for $i = 1, 2$ and $D_{\theta,L} := L \otimes_{L_0} D_\theta$ for any $\theta \in E^{\times}$.

Proposition 4.3 *The Hodge invariants of the filtered modules D_L, $D_{i,L}$ and $D_{\theta,L}$ are*

$$t_H(D_L) = \sum_{i \in I_0} k_i, \quad t_H(D_{1,L}) = \sum_{\{i \in I_0 \ :\ y_i = 0\}} k_i, \ t_H(D_{2,L}) = \sum_{\{i \in I_0 \ :\ x_i = 0\}} k_i$$

and

$$t_H(D_\theta) = \sum_{\{i \in J_{\vec{x}} \ \cap \ J_{\vec{y}} \ : \ x_i\theta = y_i\}} k_i.$$

Proof. The formula for $t_H(D_L)$ follows immediately form Formula (4.2) since

$$\dim_E(E^{|\mathcal{S}_L|})f_J\left(\vec{x}(1 \otimes \eta_1) + f_{J_{\vec{y}}}(1 \otimes \eta_2)\right) = \mid J \mid$$

for any $J \subset I_0$ (recall that $(x_i, y_i) \neq (0,0)$ for all i). By definition,

$$Fil^j(D_{2,L}) = D_{2,L} \cap \mathrm{Fil}^j(D_L)$$

for all j. Let $1 + w_{r-1} \leq j \leq w_r$ for some $1 \leq r \leq t-1$. We have $\vec{t}(1 \otimes \eta_2) = \vec{\xi} \cdot f_{I_r} \cdot \vec{x}((1 \otimes \eta_1) + \vec{y}(1 \otimes \eta_2))$ if and only if $\vec{\xi} \cdot \vec{x} \cdot f_{I_r} = \vec{0}$ and $\vec{\xi} \cdot \vec{y} \cdot f_{I_r} = \vec{t}$. For all $i \in I_r$ with $x_i \neq 0$ we have $\xi_i = 0$. If $x_i = 0$, then $y_i \neq 0$ and as $\vec{\xi}$ varies in $E^{|\mathcal{S}_L|}$ the vector $\vec{\xi} \cdot \vec{y} \cdot f_{I_r}$ can be any element of $f_{I_r \cap \ J'_{\vec{x}}}(E^{|\mathcal{S}_L|})$, where $J'_{\vec{x}}$ is the complement of $J_{\vec{x}}$ in I_0. Let $I_{r,\vec{x}} = I_r \cap J'_{\vec{x}}$. For all $1 + w_{r-1} \leq j \leq w_r$, one has $\mathrm{Fil}^j(D_{2,L}) = (E^{|\mathcal{S}_L|})f_{I_{r,\vec{x}}}(1 \otimes \eta_2)$ and therefore

$$\mathrm{Fil}^j(D_{2,L}) = \begin{cases} D_{2,L} & \text{if } \ j \leq 0, \\ \left(E^{|S_L|_{I_{i,\vec{x}}}}\right)(1 \otimes \eta_2), & \text{if} \\ 1 + w_{i-1} \leq j \leq w_i, & \text{for } i = 0, 1, ..., t-1, \\ 0 & \text{if } \ j \geq 1 + w_{t-1}. \end{cases}$$

Clearly $t_H(D_{2,L}) = \sum\limits_{i=0}^{t-1} w_i(\mid I_{i,\vec{x}} \mid - \ I_{i+1,\vec{x}} \mid)$ (with $I_{t,\vec{x}} = \varnothing$). Since $\mid I_{i,\vec{x}} \mid - \mid I_{i+1,\vec{x}} \mid = \#\{j \in I_0 : k_j = w_i \text{ and } x_j = 0\}$, we have

$$t_H(D_{2,L}) = \sum_{\{i \in I_0 : \ x_i = 0\}} k_i.$$

The computation for $t_H(D_{1,L})$ is identical. Last, for any $\theta \in E^\times$,

$$\mathrm{Fil}^j(D_\theta) = D_\theta \cap \mathrm{Fil}^j(D).$$

Let $1 + w_{r-1} \leq j \leq w_r$ for some $1 \leq r \leq t-1$ and let $\vec{t}(\eta_1 + \theta \cdot \vec{1}\eta_2) = \vec{\xi} \cdot f_{I_r}(\vec{x}\eta_1 + \vec{y}\eta_2) \in \mathrm{Fil}^j(D_\theta)$. One easily sees that t_i can be any elements of E as ξ_i varies in E if and only if $y_i = x_i\theta$, and $t_i = 0$ in any other case. Therefore $\mathrm{Fil}^j D_\theta = \left(E^{|S_L|_{I_r(\theta)}}\right)(\eta_1 + \theta \cdot \vec{1}\eta_2)$, where $I_r(\theta) := I_r \cap J_{\vec{x}} \cap J_{\vec{y}} \cap \{i \in I_0 : x_i\theta = y_i \ \}$ for all $1 + w_{r-1} \leq j \leq w_r$. This implies $t_H(D_\theta) = \sum\limits_{i=0}^{t-1} w_i\#\{i \in I_0 : w_i = k_i, \ x_iy_i \neq 0 \text{ and } \theta = x_i^{-1} \cdot y_i\} = \sum\limits_{\{i \in J_{\vec{x}} \cap J_{\vec{y}} : \ x_i\theta = y_i\}} k_i$. ■

For the Newton invariants of D, D_i, and D_θ we have the following proposition.

Proposition 4.4 *If the diagonal entries of the matrix of φ with respect to a standard basis are $\alpha \cdot \vec{1}$ and $\delta \cdot \vec{1}$, then $t_N(D) = efv_p(\alpha\delta)$, $\ t_N(D_2) = efv_p(\delta)$, $\ t_N(D_1) = efv_p(\alpha)$ and $t_N(D_\theta) = efv_p(\alpha)$.*

Proof. Follows easily from Formula (4.1) in the beginning of the section. ■

5 The weakly admissible rank two filtered modules.

We summarize the results of the previous sections and list the rank two weakly admissible filtered $(\varphi, N, L/K, E)$-modules. Before doing so, we briefly digress to recall some well known facts about Galois types ([CDT99, App.B]).

5.1 Galois types

Let $\rho : G_K \to GL(V)$ be an L-semistable n-dimensional E-representation of G_K, as in the introduction. Let W_L be the Weil group of L and W_K the Weil group of K. Recall that $W_K/W_L = \mathrm{Gal}(L/K)$. The Frobenius endomorphism φ of $D^L_{st}(V)$ defines an E-linear isomorphism

$$\varphi : e_{\tau_{i+1}} D^L_{st}(V) \to e_{\tau_i} D^L_{st}(V),$$

for each embedding τ_i of L_0 in E. If e_K is the absolute ramification index K, we define an L_0-linear action of $g \in W_K$ on $D^L_{st}(V)$ given by $(g \bmod W_L) \circ \varphi^{-\alpha(g) e_K}$, were the image of g in $\mathrm{Gal}(\bar{k}_K/k_K)$ is the $\alpha(g)$-th power of the q_K-th power map, with k_K being the residue field of K and q_K its cardinality. Since V is L-semistable, each component $e_{\tau_i} D^L_{st}(V)$ is an E-vector spaces of dimension n with an induced action of (W_K, N). Its isomorphism class is independent of the choice of the embedding τ_i (cf. [BM02, Lemme 2.2.1.2]), and this unique isomorphism class is the Weil-Deligne representation $WD(\rho)$ attached to ρ.

Definition 5.1 *A Galois type of degree* 2 *is an equivalence class of representations $\tau : I_K \to GL_2(\bar{\mathbb{Q}}_p)$ with open kernel which extend to W_K. We say that a two-dimensional potentially semistable representation has Galois type τ if $WD(\rho)\mid_{I_K} \simeq \tau$.*

We have the following lemma.

Lemma 5.2 *Assume that $p > 2$ and let τ be a Galois type of degree* 2*. Then τ has one of the following forms:*

(1) $\tau \simeq \chi_1 \mid_{I_K} \oplus \chi_2 \mid_{I_K}$*, where χ_1 and χ_2 are characters of W_K finite on I_K;*

(2) $\tau \simeq \mathit{Ind}^{W_K}_{W_{K'}}(\chi) \mid_{I_K} \simeq \chi \mid_{I_K} \oplus \chi^h \mid_{I_K}$*, where K' is the quadratic unramified extension of K, χ is a character of $W_{K'}$ finite on $I_{K'}$ which does not extend to W_K, and h a generator of Gal(K'/K);*

(3) $\tau \simeq \mathit{Ind}^{W_K}_{W_{K'}}(\chi) \mid_{I_K}$*, where K' is a ramified quadratic extension of K and χ a character of $W_{K'}$, finite on $I_{K'}$, such that $\chi \mid_{I_{K'}}$ which does not extend to I_K.*

For Galois types we have the following three possibilities:

- $N \neq 0$ and τ is a scalar (special or Steinberg case);
- $N = 0$ and τ as in (1) of Lemma 5.2 (principal series case);
- $N = 0$ and τ as in (2) or (3) of Lemma 5.2 (supercuspidal case).

Notice that in the unramified supercuspidal case (Case (2) of Lemma 5.2), τ is reducible and the characters $\chi \mid_{I_K}$ and $\chi^h \mid_{I_K}$ are necessarily distinct, while in the ramified supercuspidal case (Case (3) of Lemma 5.2), τ is irreducible.
We now provide the list of rank two weakly admissible filtered $(\varphi, N, L/K, E)$-modules and comment on the Galois type of the corresponding potentially semistable representation, understanding that

the above mentioned terminology applies only in case that p is odd, an assumption not necessary in this paper.

Recall from Section 3.3 that there is a right action of $G = \mathrm{Gal}(L/K)$ on I_0 defined by $i \cdot g := \pi(g)(i)$, where π is as in Section 2.2.1. This action has orbits $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_\nu$, where $\nu = [K : \mathbb{Q}_p]$. Let i_j be any fixed index in the orbit $\mathcal{O}_j$ for any $1 \leq j \leq \nu$, and choose any fixed pair $(x_{i_j}, y_{i_j}) \in E \times E$ with $(x_{i_j}, y_{i_j}) \neq (0, 0)$. Assume that the labeled Hodge-Tate weights are $(\{-k_i, 0\})_{\sigma_i}$, with k_i non negative integers.

5.2 The F-semisimple, non-scalar case

There exists an ordered basis
$\underline{\eta} = (\eta_1, \eta_2)$ of D over $E^{|\mathcal{S}_{L_0}|}$ such that:

- The Frobenius endomorphism φ of D is given by $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha \cdot \vec{1}, \delta \cdot \vec{1})$ with $\alpha, \delta \in E^\times$ and $\alpha^f \neq \delta^f$;
- The Galois action is given by $[g]_{\underline{\eta}} = \mathrm{diag}(\chi_1(g) \cdot \vec{1}, \chi_2(g) \cdot \vec{1})$ for some characters $\chi_i : G \to E^\times$;
- The Galois-stable filtrations are equivalent to

$$\mathrm{Fil}^j(D_L) = \begin{cases} D_L \ \text{ if } \ j \leq 0, \\ \left(E^{|S_L|}\right)(\vec{x}(1 \otimes \eta_1) + \vec{y}(1 \otimes \eta_2)) & \text{if } 1 \leq j \leq w_0, \\ \left(E^{|S_L|_{I_1}}\right)((\vec{x}1 \otimes \eta_1) + \vec{y}(1 \otimes \eta_2)\ ) & \text{if } \ 1 + w_0 \leq j \leq w_1, \\ \cdots\cdots\cdots \\ \left(E^{|S_L|_{I_{t-1}}}\right)(\vec{x}(1 \otimes \eta_1) + \vec{y}(1 \otimes \eta_2)) & \text{if } \ 1 + w_{t-2} \leq j \leq w_{t-1}, \\ 0 \ \text{ if } \ j \geq 1 + w_{t-1}, \end{cases}$$

with $\vec{x} = \sum\limits_{j=1}^{\nu} \left\{ \sum\limits_{g \in G} x_{i_j} \cdot \chi_1(g^{-1}) \cdot e_{\pi(g)(i_j)} \right\}, \quad \vec{y} = \sum\limits_{j=1}^{\nu} \left\{ \sum\limits_{g \in G} y_{i_j} \cdot \chi_2(g^{-1}) \cdot e_{\pi(g)(i_j)} \right\},$

where the sets I_r are unions of G-orbits of I_0 for all r.

5.2.1 The potentially crystalline case

- The Frobenius-stable submodules are $0,\ D,\ D_1 = (E^{|\mathcal{S}_{L_0}|})\eta_1$ and $D_2 = (E^{|\mathcal{S}_{L_0}|})\eta_2$;
- The filtered $(\varphi, L/K, E)$-module D is weakly admissible if and only if

$$\begin{gathered} \text{(i) } efv_p(\alpha\delta) = \sum_{i \in I_0} k_i \\ \text{(ii) } efv_p(\alpha) \geq \sum_{\{i \in I_0:\ y_i = 0\}} k_i \text{ and (iii) } efv_p(\delta) \geq \sum_{\{i \in I_0:\ x_i = 0\}} k_i, \end{gathered} \tag{5.1}$$

where e is the absolute ramification index and f the absolute inertia degree of L. Assuming that D is weakly admissible,

1. It is irreducible if and only if both inequalities (ii) and (iii) in (5.1) are strict;
2. It is reducible, non-split if and only if exactly one of the inequalities in (5.1) is strict. If inequality (ii) is strict, the only nontrivial weakly admissible submodule is D_2, while if inequality (iii) is strict the only weakly admissible submodule is D_1;
3. It is split-reducible if and only if $I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$. The only nontrivial weakly admissible submodules are D_1 and D_2.

The corresponding potentially crystalline representation is a principal series.

5.2.2 The potentially semistable, noncrystalline case

In this case, there exists a basis $\underline{\eta}$ so that $\alpha = p\delta$ and $[N]_{\underline{\eta}} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{1} & \vec{0} \end{pmatrix}$. Moreover,

- The characters χ_1 and χ_2 are equal;
- The submodules fixed by Frobenius and the monodromy are 0, D and D_2;

The filtered $(\varphi, N, L/K)$-module D is weakly admissible if and only if

$$2efv_p(\delta) + ef = \sum_{i \in I_0} k_i \text{ and } efv_p(\delta) \geq \sum_{\{i \in I_0 :\ x_i = 0\}} k_i. \tag{5.2}$$

Assuming that D is weakly admissible, it is reducible, non-split if and only if the inequality in (5.2) is equality. In this case, the only nontrivial weakly admissible submodule stable under Frobenius and the monodromy is D_2. In any other case D is irreducible.
The corresponding potentially semistable representation is a special series.

5.3 The F-scalar case

There exists an ordered basis $\underline{\eta}$ of D over $E^{|S_{L_0}|}$ such that $[\varphi]_{\underline{\eta}} = \mathrm{diag}(\alpha \cdot \vec{1}, \alpha \cdot \vec{1})$ with $\alpha \in E^{\times}$.

- The monodromy operator N is trivial;
- There exists a group homomorphism $\lambda : G \to GL_2(E)$ such that $[g]_{\underline{\eta}} = \lambda(g) \cdot \mathrm{diag}(\vec{1}, \vec{1})$ for all $g \in G$;
- The Galois-stable filtrations are as in the non-F-scalar case with

$$\vec{x} = \sum_{j=1}^{\nu} \left\{ \sum_{g \in G} x_{\pi(g)(i_j)} \cdot e_{\pi(g)(i_j)} \right\}, \quad \vec{y} = \sum_{j=1}^{\nu} \left\{ \sum_{g \in G} y_{\pi(g)(i_j)} \cdot e_{\pi(g)(i_j)} \right\},$$

  where $\left(x_{\pi(g)(i_j)}, y_{\pi(g)(i_j)}\right) = \left(x_{i_j}, y_{i_j}\right) \cdot \lambda(g^{-1})$ for all $g \in G$;

- The Frobenius-stable submodules are 0, D, D_1, D_2, with D_1 and D_2 as in the previous cases, and $D_\theta = (E^{|\mathcal{S}_{L_0}|})(\eta_1 + \theta \cdot \vec{1}\eta_2)$ for any $\theta \in E^\times$.

For each $c \in E^\times$, let $k(c) := \sum_{\{i \in J_{\vec{x}} \cap J_{\vec{y}}:\ x_i^{-1} y_i = c\}} k_i$, where x_i and y_i are the coordinates of the vectors $\vec{x}$ and $\vec{y}$. Let k be the maximum of the integers $k(c)$. The filtered φ-module D is weakly admissible if and only if

$$\begin{gathered}
\text{(i) } 2efv_p(\alpha) = \sum_{i \in I_0} k_i, \ \text{(ii) } efv_p(\alpha) \geq \sum_{\{i \in I_0:\ y_i = 0\}} k_i, \\
\text{(iii) } efv_p(\alpha) \geq \sum_{\{i \in I_0:\ x_i = 0\}} k_i, \ \text{and (iv) } efv_p(\alpha) \geq k.
\end{gathered} \tag{5.3}$$

Assuming that D is weakly admissible,

1. It is irreducible if and only if all inequalities (ii), (iii) and (iv) in (5.3) are strict.
2. It is reducible, non-split if and only if either exactly one of the inequalities (ii) and (iii) is equality and inequality (iv) is strict, or both inequalities (ii) and (iii) above are strict, inequality (iv) is equality and the maximum is attained for precisely one constant c. The only φ-stable weakly admissible submodules are D_1, D_2 and D_c respectively.
3. It is split-reducible if and only if either $x_i^{-1} y_i$ is a constant c for all $i \in I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}}$ (including the case $I_0^+ \cap J_{\vec{x}} \cap J_{\vec{y}} = \varnothing$ in which we define $c = 0$) and one of the inequalities (ii) and (iii) above is equality, or there exist two distinct constants c_1, c_2 such that $k(c_1) = k(c_2)$. The only weakly admissible submodules are D_1 and D_c, or D_2 and D_c, or D_{c_1} and D_{c_2} respectively, and all these pairs of submodules are complementary in D.

The corresponding potentially crystalline representation is supercuspidal or principal series, depending on λ.

5.4 The non-F-semisimple case

There exists an ordered basis $\underline{\eta}$ of D over $E^{|\mathcal{S}_{L_0}|}$ such that $[\varphi]_{\underline{\eta}} = \begin{pmatrix} \alpha \cdot \vec{1} & \vec{0} \\ \vec{1} & \alpha \cdot \vec{1} \end{pmatrix}$, with $\alpha \in E^\times$. In this case the monodromy operator N is trivial.

- The Galois action is given by $[g]_{\underline{\eta}} = \mathrm{diag}(\chi(g) \cdot \vec{1}, \chi(g) \cdot \vec{1})$ for some character $\chi : G \to E^\times$, and the G-stable filtrations are as in the F-semisimple, non-scalar case;
- The Frobenius-fixed submodules are 0, D, D_2;

The filtered φ-module D is weakly admissible if and only if

$$2efv_p(\alpha) + ef = \sum_{i \in I_0} k_i \text{ and } efv_p(\alpha) \geq \sum_{\{i \in I_0:\ x_i = 0\}} k_i. \tag{5.4}$$

Assuming that D is weakly admissible, it is reducible, non-split if and only if the inequality in (5.4) is equality. In this case, the only nontrivial weakly admissible submodule is D_2. In any other case D is irreducible.
The corresponding potentially crystalline representation is a principal series.

6 Isomorphism classes

Let (D_i, φ_i, N_i), $i = 1, 2$ be isomorphic filtered $(\varphi, N, L/K, E)$-modules with labeled Hodge-Tate weights $(\{-k_\sigma, 0\})_\sigma$, where k_σ are non negative integers. Let $\underline{\eta}^i = (\eta_1^i, \eta_2^i)$, $i = 1, 2$ be standard bases and let $h : D_1 \to D_2$ be an isomorphism. We denote by $[h]_{\underline{\eta}^1}^{\underline{\eta}^2}$ the matrix of h with respect to the bases $\underline{\eta}^i$ and by $[h]_{1\otimes\underline{\eta}^1}^{1\otimes\underline{\eta}^2}$ the matrix of $h_L = 1_{L\otimes_{\mathbb{Q}_p}E} \otimes h$ with respect to the bases $1 \otimes \underline{\eta}^i$. If all the weights k_σ equal zero, compatibility of h with the filtrations holds trivially and the corresponding sections should be ignored.

6.1 The F-semisimple, non-scalar case

Let $[\varphi_i]_{\underline{\eta}^i} = \mathrm{diag}(\alpha_i \cdot \vec{1}, \delta_i \cdot \vec{1})$, with $\alpha_i^f \neq \delta_i^f$ and $\alpha_i = p\delta_i \neq 0$ if the monodromy operators are nontrivial. In the next proposition we determine when the isomorphism h commutes with the Frobenius operators. We write $Q = [h]_{\underline{\eta}^1}^{\underline{\eta}^2} = \begin{pmatrix} \vec{a} & \vec{b} \\ \vec{c} & \vec{d} \end{pmatrix}$, and by Section 2.2.1 it is clear that $\left([h_L]_{1\otimes\underline{\eta}^1}^{1\otimes\underline{\eta}^2}\right) = Q^{\otimes e} = \begin{pmatrix} \vec{a}^{\otimes e} & \vec{b}^{\otimes e} \\ \vec{c}^{\otimes e} & \vec{d}^{\otimes e} \end{pmatrix} =: \begin{pmatrix} \vec{a}_1 & \vec{b}_1 \\ \vec{c}_1 & \vec{d}_1 \end{pmatrix}$.

Proposition 6.1 *The isomorphism h commutes with Frobenius endomorphisms if and only if either*

1. $\alpha_1^f = \alpha_2^f$ *and* $\delta_1^f = \delta_2^f$, *in which case* $[h]_{\underline{\eta}^1}^{\underline{\eta}^2} = diag(a \cdot \vec{a}_0, d \cdot \vec{d}_0)$, *where* $\vec{a}_0 = (1, \mu_1, \mu_1^2, ..., \mu_1^{f-1})$, $\vec{d}_0 = (1, \mu_2, \mu_2^2, ..., \mu_2^{f-1})$, *with* $\mu_1 = \frac{\alpha_1}{\alpha_2}$, $\mu_2 = \frac{\delta_1}{\delta_2}$ *and* $a, d \in E^\times$, *or*
2. $\alpha_1^f = \delta_2^f$ *and* $\delta_1^f = \alpha_2^f$, *in which case* $[h]_{\underline{\eta}^1}^{\underline{\eta}^2} = \begin{pmatrix} \vec{0} & b \cdot \vec{b}_0 \\ c \cdot \vec{c}_0 & \vec{0} \end{pmatrix}$, *where* $\vec{b}_0 = (1, \xi_1, \xi_1^2, ..., \xi_1^{f-1})$, $\vec{c}_0 = (1, \xi_2, \xi_2^2, ..., \xi_2^{f-1})$, *with* $\xi_1 = \frac{\delta_1}{\alpha_2}$, $\xi_2 = \frac{\alpha_1}{\delta_2}$ *and* $b, c \in E^\times$.

Proof. We need $([\varphi_2]_{\underline{\eta}^2}) \cdot \varphi(Q) = Q \cdot ([\varphi_1]_{\underline{\eta}^1})$, or equivalently $\alpha_1\vec{a} = \alpha_2\varphi(\vec{a}), \delta_1\vec{b} = \alpha_2\varphi(\vec{b}), \alpha_1\vec{c} = \delta_2\varphi(\vec{c})$ and $\delta_1\vec{d} = \delta_2\varphi(\vec{d})$.If $\alpha_1^f \notin \{\alpha_2^f, \delta_2^f\}$, then Lemma 2.1 implies $\vec{a} = \vec{c} = \vec{0}$ a contradiction. Hence $\alpha_1^f \in \{\alpha_2^f, \delta_2^f\}$, and similarly $\delta_1^f \in \{\alpha_2^f, \delta_2^f\}$. Since $\alpha_i^f \neq \delta_i^f$ for $i = 1, 2$ we have the following cases: Case (1). If $\alpha_1^f = \alpha_2^f$ and $\delta_1^f = \delta_2^f$. By Lemma 2.1, $Q = \mathrm{diag}(\vec{a}, \vec{d})$, where $\vec{a} = a(1, \mu_1, \mu_1^2, ..., \mu_1^{f-1})$, $\vec{d} = d(1, \mu_2, \mu_2^2, ..., \mu_2^{f-1})$ with $\mu_1 = \frac{\alpha_1}{\alpha_2}$, $\mu_2 = \frac{\delta_1}{\delta_2}$ and $a, d \in E^\times$. Case (2). If $\alpha_1^f = \delta_2^f$ and $\delta_1^f = \alpha_2^f$. Arguing as in Case (1), $Q = \begin{pmatrix} \vec{0} & \vec{b} \\ \vec{c} & \vec{0} \end{pmatrix}$, with $\vec{b} = b(1, \xi_1, \xi_1^2, ..., \xi_1^{f-1})$, $\vec{c} = c(1, \xi_2, \xi_2^2, ..., \xi_2^{f-1})$, where $\xi_1 = \frac{\delta_1}{\alpha_2}$, $\xi_2 = \frac{\alpha_1}{\delta_2}$ and $b, c \in E^\times$. ■

We now determine when h commutes with the monodromy operators.

Proposition 6.2 *The isomorphism h commutes with the monodromy operators if and only if either both the monodromies are trivial or the matrix $[h]_{\underline{\eta}^1}^{\underline{\eta}^2}$ is as in Case (1) of Proposition 6.1, $a = d$ and $\alpha_1\delta_2 = \alpha_2\delta_1$.*

Proof. Clearly the monodromy operator of one of the filtered modules is trivial if and only if the monodromy operator of the other is. The monodromy operators commute with h if and only if

$\left([h]_{\underline{\eta}^1}^{\underline{\eta}^2}\right)[N_1]_{\underline{\eta}^1} = [N_2]_{\underline{\eta}^2}\left([h]_{\underline{\eta}^1}^{\underline{\eta}^2}\right)$. The proposition follows by a straightforward computation using Corollary 2.6 and Proposition 6.1. ■

Proposition 6.3 *Let* $[g]_{\underline{\eta}^1} = diag(\chi_1(g)\cdot\vec{1}, \chi_2(g)\cdot\vec{1})$ *and* $[g]_{\underline{\eta}^2} = diag(\psi_1(g)\cdot\vec{1}, \psi_2(g)\cdot\vec{1})$.

(1) *If the matrix of h is as in Case (1) of Proposition 6.1, then h commutes with the Galois actions if and only if $\chi_1(g) = \mu_1^{n(g)}\psi_1(g)$ and $\chi_2(g) = \mu_2^{n(g)}\psi_2(g)$ for all $g \in G$.*

(2) *If the matrix of h is as in Case (2) of Proposition 6.1, then h commutes with the Galois actions if and only if $\chi_1(g) = \xi_2^{n(g)}\psi_2(g)$ and $\chi_2(g) = \xi_1^{n(g)}\psi_1(g)$ for all $g \in G$.*

Proof. A straightforward computation, using that the Galois actions commutes with h if and only if $\left([h]_{\underline{\eta}^1}^{\underline{\eta}^2}\right)[g]_{\underline{\eta}^1} = [g]_{\underline{\eta}^2}\left({}^g[h]_{\underline{\eta}^1}^{\underline{\eta}^2}\right)$. ■

6.1.1 Compatibility with the filtrations

Throughout this section we assume that at least one weight k_σ is positive. Suppose that for $i = 1, 2$ we have

$$\mathrm{Fil}^j(D_{i,L}) = \begin{cases} D_{i,L} \ \text{ if } \ j \le 0, \\ \left(E^{|S_L|_{I_r}}\right)(\vec{x}_i(1\otimes\eta_1) + \vec{y}_i(1\otimes\eta_2)) \ \text{ if} \\ 1 + w_{r-1} \le j \le w_r, \ \text{for } r = 0, ..., t-1, \\ 0 \ \text{ if } \ j \ge 1 + w_{t-1}, \end{cases}$$

We need

$$h_L(\mathrm{Fil}^j D_{1,L}) = \mathrm{Fil}^j D_{2,L} \tag{6.1}$$

for all j and we have the following cases: (1) If $Q = \mathrm{diag}(\vec{a}, \vec{d})$ is as in Case (1) of Proposition 6.1, let $Q^{\otimes e} = \mathrm{diag}(\vec{a}_1, \vec{d}_1)$, where $\vec{a}_1 = \vec{a}^{\otimes e}$ and $\vec{d}_1 = \vec{d}^{\otimes d}$. Since h_L is $\left(E^{|S_L|}\right)$-linear. Condition (6.1) is equivalent to

$$\left(E^{|S_L|}\right)(f_{J_{\vec{x}_1}}\cdot\vec{x}_1\cdot\vec{a}_1(1\otimes\eta_1^1) + f_{J_{\vec{y}_1}}\cdot\vec{d}_1(1\otimes\eta_2^1)) = \left(E^{|S_L|}\right)(f_{J_{\vec{x}_2}}\cdot\vec{x}_2\cdot((1\otimes\eta_1^2) + f_{J_{\vec{y}_2}}((1\otimes\eta_2^2)),$$

and the latter equivalent to the system of equations

$$\text{(i)} \quad \left\{ \begin{array}{l} f_{J_{\vec{x}_1}}\cdot\vec{a}_1\cdot\vec{x}_1 = \vec{t}\cdot f_{J_{\vec{x}_2}}, \\ f_{J_{\vec{y}_1}}\cdot\vec{d}_1\cdot\vec{x}_2 = \vec{t}\cdot f_{J_{\vec{y}_2}}, \end{array}\right\} \quad \text{and} \quad \text{(ii)} \quad \left\{ \begin{array}{l} f_{J_{\vec{x}_2}} = f_{J_{\vec{x}_1}}\cdot\vec{t}_1\cdot\vec{a}_1, \\ f_{J_{\vec{y}_2}} = f_{J_{\vec{y}_1}}\cdot\vec{t}_1\cdot\vec{d}_1, \end{array}\right\} \tag{6.2}$$

for some vectors $\vec{t}, \vec{t}_1 \in E^{|S_L|}$. We easily see that (6.2) implies

$$f_{J_{\vec{x}_1}\cap J_{\vec{y}_2}}\cdot\vec{a}_1\cdot\vec{x}_1 = f_{J_{\vec{x}_2}\cap J_{\vec{y}_1}}\cdot\vec{d}_1\cdot\vec{x}_2.$$

Since $\vec{a}_1 \in (E^\times)^{|S_L|}$, the first equation of (6.2)(i) implies that $J_{\vec{x}_1} \subset J_{\vec{x}_2}$ and the first equation of (6.2)(ii) that $J_{\vec{x}_2} \subset J_{\vec{x}_1}$, therefore $J_{\vec{x}_1} = J_{\vec{x}_2}$.
Similarly, since $\vec{d}_1 \in (E^\times)^{|S_L|}$, we have $J_{\vec{y}_1} = J_{\vec{y}_2}$. Conversely, if the equations

$$J_{\vec{x}_1} = J_{\vec{x}_2}; \ J_{\vec{y}_1} = J_{\vec{y}_2} \text{ and } f_{J_{\vec{x}_1}\cap J_{\vec{y}_2}}\cdot\vec{a}_1\cdot\vec{x}_1 = f_{J_{\vec{x}_2}\cap J_{\vec{y}_1}}\cdot\vec{d}_1\cdot\vec{x}_2$$

hold, then it is easy to see that the system of equations (6.2) has solutions in $\vec{t}$ and $\vec{t}_1$. Hence, h preserves the filtrations if and only if

$$J_{\vec{x}_1} = J_{\vec{x}_2};\ J_{\vec{y}_1} = J_{\vec{y}_2} \text{ and } f_{J_{\vec{x}_1}\cap J_{\vec{y}_1}} \cdot \vec{a}_1 \cdot \vec{x}_1 = f_{J_{\vec{x}_2}\cap J_{\vec{y}_2}} \cdot \vec{d}_1 \cdot \vec{x}_2 \tag{6.3}$$

We have the following subcases:
(a) When the monodromies are trivial: In this case, the third equation in (6.3) can be replaced by

$$f_{J_{\vec{x}}\cap J_{\vec{y}}} \cdot (\vec{a}_0)^{\otimes e} \cdot \vec{x}_1 = f_{J_{\vec{x}}\cap J_{\vec{y}}} \cdot (\vec{d}_0)^{\otimes e} \cdot \vec{x}_2 \text{ in the projective space } \mathbb{P}^{m-1}(E), \tag{6.4}$$

where $\vec{a}_0 = (1, \mu_1, \mu_1^2, ..., \mu_1^{f-1})$ and $\vec{d}_0 = (1, \mu_2, \mu_2^2, ..., \mu_2^{f-1})$.
Conversely, if $\alpha_1^f = \alpha_2^f$, $\delta_1^f = \delta_2^f$ and equation (6.4) holds, then (after scaling one of the vectors $\vec{a}_0$ or $\vec{d}_0$ if necessary) $Q = \left([h]_{\bar{\eta}^1}^{\eta^2}\right) = \mathrm{diag}(\vec{a}_0, \vec{d}_0)$ defines an isomorphism of filtered $(\varphi, N, L/K, E)$-modules $h : (D_1, \varphi_1) \to (D_2, \varphi_2)$.
(b) When the monodromies are nontrivial: By Proposition 6.2 we have $\vec{a} = \vec{d}$ and (6.3) is equivalent to

$$J_{\vec{x}_1} = J_{\vec{x}_2};\ J_{\vec{y}_1} = J_{\vec{y}_2} \text{ and } f_{J_{\vec{x}}\cap J_{\vec{y}}} \cdot \vec{x}_1 = f_{J_{\vec{x}}\cap J_{\vec{y}}} \cdot \vec{x}_2. \tag{6.5}$$

Conversely, if $\alpha_1^f = \alpha_2^f$, $\delta_1^f = \delta_2^f$, and $\alpha_1\delta_2 = \alpha_2\delta_1$, if the monodromy operators are non-trivial, and if equations (6.5) hold, then the $E^{|S_{L_0}|}$-linear map $h : (D_1, \varphi_1) \to (D_2, \varphi_2)$ defined by $Q = [h]_{\bar{\eta}^1}^{\eta^2} = \mathrm{diag}(\vec{a}_0, \vec{a}_0)$ is an isomorphism of filtered $(\varphi, N, L/K, E)$-modules.
(2) If $Q = \begin{pmatrix} \vec{0} & \vec{b} \\ \vec{c} & \vec{0} \end{pmatrix}$, then both the monodromy operators are zero. Arguing before we see that h_L preserves the filtrations if and only if

$$\begin{gathered} J_{\vec{x}_1} = J_{\vec{y}_2};\ J_{\vec{y}_1} = J_{\vec{x}_2} \text{ and} \\ f_{J_{\vec{x}_1}\cap J_{\vec{y}_1}} \cdot (\vec{b}_0)^{\otimes e} = f_{J_{\vec{y}_2}\cap J_{\vec{x}_2}} \cdot (\vec{c}_0)^{\otimes e} \cdot \vec{x}_1 \cdot \vec{x}_2 \text{ in } \mathbb{P}^{m-1}(E). \end{gathered} \tag{6.6}$$

Conversely, if $\alpha_1^f = \delta_2^f$, $\delta_1^f = \alpha_2^f$ and equations (6.6) hold, then the $E^{|S_{L_0}|}$-linear map $h : (D_1, \varphi_1) \to (D_2, \varphi_2)$ defined by $Q = \left([h]_{\bar{\eta}^1}^{\eta^2}\right) = \begin{pmatrix} \vec{0} & \vec{b}_0 \\ \vec{c}_0 & \vec{0} \end{pmatrix}$ (after scaling one of the vectors $\vec{b}_0$ or $\vec{c}_0$ if necessary) is an isomorphism of filtered $(\varphi, N, L/K, E)$-modules.

6.2 The F-scalar case

Suppose that

$$[\varphi_i]_{\underline{\eta}^i} = \mathrm{diag}(\alpha_i \cdot \vec{1}, \alpha_i \cdot \vec{1}) \text{ and } [g]_{\underline{\eta}^i} = \lambda_i(g) \cdot \mathrm{diag}(\vec{1}, \vec{1})$$

for some group homomorphisms $\lambda_i : G \to GL_2(E)$, $i = 1, 2$. Arguing as in the non-F-scalar case, one easily sees that an isomorphism h commuting with Frobenius exists if and only if $\alpha_1^f = \alpha_2^f$. Then, $Q = [h]_{\underline{\eta}^1}^{\eta^2} = R{\cdot}\mathrm{diag}(\vec{1}, \vec{1})$ for some $R \in GL_2(E)$, and h commutes with the Galois action if and only if $\lambda_2(g) = R\lambda_1(g)R^{-1}$ for all $g \in G$. Let $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since h_L is an $E^{|S_L|}$-linear isomorphism, it preserves the filtrations if and only if $h_L\left(\mathrm{Fil}^1 D_{1,L}\right) = \mathrm{Fil}^1 D_{2,L}$, or equivalently

$$E^{|S_L|}\left(a \cdot \vec{x}_1 + b \cdot f_{J_{\vec{y}_1}}\right) = E^{|S_L|}\vec{x}_2 \text{ and } E^{|S_L|}\left(c \cdot \vec{x}_1 + d \cdot f_{J_{\vec{y}_1}}\right) = E^{|S_L|} f_{J_{\vec{y}_2}}$$

which we write in assorted form as

$$\left(E^{|S_L|}\right)\left(\vec{x}_1,\ f_{J_{\vec{y}_1}}\right) \cdot \left(R \cdot \mathrm{diag}(\vec{1}, \vec{1})\right) = \left(\ E^{|S_L|}\right)\left(\vec{x}_2,\ f_{J_{\vec{y}_2}}\right). \tag{6.7}$$

Conversely, if $\alpha_1^f = \alpha_2^f$, if there exists some $R \in GL_2(E)$ such that $\lambda_2(g) = R\lambda_1(g)R^{-1}$ for all $g \in G$ and (6.7) holds, then the $E^{|S_{L_0}|}$-linear map

$$h : D_1 \to D_2 \text{ defined by } [h]_{\underline{\eta}^1}^{\underline{\eta}^2} = R \cdot \mathrm{diag}(\vec{1}, \vec{1})$$

is an isomorphism of filtered $(\varphi, N, L/K, E)$-modules.

6.3 The non-F-semisimple case

Let

$$[\varphi_i]_{\underline{\eta}^i} = \begin{pmatrix} \alpha_i \cdot \vec{1} & \vec{0} \\ \vec{1} & \alpha_i \cdot \vec{1} \end{pmatrix} \text{ and } [g]_{\underline{\eta}^i} = \mathrm{diag}(\chi_i(g) \cdot \vec{1}, \chi_i(g) \cdot \vec{1})$$

for some characters $\chi_i : G \to E^\times$. Let $Q = [h]_{\underline{\eta}^1}^{\underline{\eta}^2} = \begin{pmatrix} \vec{a} & \vec{b} \\ \vec{c} & \vec{d} \end{pmatrix}$.

The isomorphism h commutes with the Frobenius endomorphisms if and only if

$$([\varphi_2]_{\underline{\eta}^2}) \cdot \varphi(Q) = Q \cdot ([\varphi_1]_{\underline{\eta}^1}). \tag{6.8}$$

This implies that $Nm_\varphi([\varphi_2]_{\underline{\eta}^2}) \cdot Q = Q \cdot\ Nm_\varphi([\varphi_1]_{\underline{\eta}^1})$, and this combined with Lemma 2.1 that $\alpha_1^f = \alpha_2^f,\ \vec{b} = \vec{0}$ and $\vec{a} = \vec{d} = a \cdot \left(1, \frac{\alpha_2}{\alpha_1}, \left(\frac{\alpha_2}{\alpha_1}\right)^2, ..., \left(\frac{\alpha_2}{\alpha_1}\right)^{f-1}\right)$ for some $a \in E^\times$. Then by equation (6.8), the coordinates of $\vec{c}$ satisfy

$$c_i = \mu_1^i \left\{(c_0 - a\mu_1^{-1} + a) - \sum_{j=1}^{i-1}\left(\mu_1^{-2j-1} - \mu_1^{-2j}\right)\right\} \text{ for } i = 1, 2, ..., f-1,$$

where $c_0 \in E$ is arbitrary. Arguing as in Section 6.1.1 we see that h is preserves the filtrations if and only if

$$J_{\vec{x}_1} = J_{\vec{x}_2} \text{ and } f_{J_{\vec{x}}} \cdot \vec{x} \cdot \vec{x}_1 \cdot \vec{c}^{\otimes e} = \left(f_{J_{\vec{x}} \cap J_{\vec{y}_1}} \cdot \vec{x} - f_{J_{\vec{x}} \cap J_{\vec{y}_1}} \cdot \vec{x}_1\right) \cdot \vec{a}^{\otimes e}. \tag{6.9}$$

It is straightforward to see that h commutes with the Galois actions if and only if $\chi_1(g) = \mu_1^{n(g)} \cdot \chi_2(g)$ and ${}^g\vec{c} = \mu_1^{n(g)} \cdot \vec{c}$ for all g. The latter equation holds if and only if either $\alpha_1 = \alpha_2$, or $\sum\limits_{j=0}^{n(g)-1}\left(\frac{\alpha_2}{\alpha_1}\right)^{2j} = 0$ for all $g \in G$. Conversely, assume that $\alpha_1^f = \alpha_2^f$ and $\sum\limits_{j=0}^{n(g)-1}\left(\frac{\alpha_2}{\alpha_1}\right)^{2j} = 0$ for all $g \in G$, in case that $\alpha_1 \neq \alpha_2$. In addition, assume that $\chi_1(g) = \mu_1^{n(g)} \cdot \chi_2(g)$ for all g. If the first two equations in (6.9) hold and there exist $a \in E^\times$ and $c_0 \in E$ such that the third equation in (6.9) holds, then the $E^{|S_{L_0}|}$-linear map $h : D_1 \to D_2$ defined by $[h]_{\underline{\eta}^1}^{\underline{\eta}^2} = \begin{pmatrix} \vec{a} & \vec{0} \\ \vec{c} & \vec{a} \end{pmatrix}$ is an isomorphism of filtered $(\varphi, L/K, E)$-modules. We now list the isomorphism classes of rank two filtered $(\varphi, N, L/K, E)$-modules.

6.4 The list of isomorphism classes

Let $(D_i, \varphi_i, N_i, L/K, E)$ be filtered modules with labeled Hodge-Tate weights $(\{-k_\sigma, 0\})_\sigma$, with k_σ non negative integers. Let $\underline{\eta}^i$, $i = 1, 2$, be standard bases, and suppose that the filtrations are given by

$$\mathrm{Fil}^j(D_{i,L}) = \begin{cases} D_{i,L} \ \text{ if } \ j \leq 0, \\ \left(E^{|S_L|_{I_r}}\right)\left(\vec{x}_i(1 \otimes \eta_1^i) + \vec{y}_i(1 \otimes \eta_2^i)\right) \ \text{if} \\ 1 + w_{r-1} \leq j \leq w_r, \ \text{for } r = 0, ..., t-1, \\ 0 \ \text{ if } \ j \geq 1 + w_{t-1}, \end{cases}$$

for some vectors $\vec{x}_i, \vec{y}_i \in E^{|S_L|}$ whose coordinates do not vanish simultaneously. Throughout this section, any equation involving the sets $J_{\vec{x}}$ and $J_{\vec{y}}$ should be ignored if all the weights k_σ equal zero. Recall the definition of $n(g)$ from Section 2.2.2.

6.4.1 The F-semisimple case

Let $[\varphi_i]_{\underline{\eta}^i} = \mathrm{diag}(\alpha_i \cdot \vec{1},\ \delta_i \cdot \vec{1})$ with $\alpha_i, \delta_i \in E^\times$ such that $\alpha_i^f \neq \delta_i^f$ and $[g]_{\underline{\eta}^1} = \mathrm{diag}(\chi_1(g) \cdot \vec{1}, \chi_2(g) \cdot \vec{1})$, $[g]_{\underline{\eta}^2} = \mathrm{diag}(\psi_1(g) \cdot \vec{1}, \psi_2(g) \cdot \vec{1})$ for some $E^\times$-valued characters χ_i and ψ_i of $G = \mathrm{Gal}(L/K)$. When the monodromy operators are nontrivial, the bases are chosen so that $\alpha_i = p\delta_i$ and $[N_i]_{\underline{\eta}^i} = \begin{pmatrix} \vec{0} & \vec{0} \\ \vec{1} & \vec{0} \end{pmatrix}$.

6.4.2 The potentially crystalline case

If both the monodromy operators are trivial, then $(D_1, \varphi_1, L/K, E) \simeq (D_2, \varphi_2, L/K, E)$ if and only if either

$$\left\{ \begin{array}{c} \alpha_1^f = \alpha_2^f \\ \delta_1^f = \delta_2^f \end{array} \right\}, \quad \left\{ \begin{array}{c} J_{\vec{x}_1} = J_{\vec{x}_2} \\ J_{\vec{y}_1} = J_{\vec{y}_2} \end{array} \right\}, \quad \left\{ \begin{array}{c} \chi_1(g) = \mu_1^{n(g)} \psi_1(g) \\ \chi_2(g) = \mu_2^{n(g)} \psi_2(g) \end{array} \right\}$$

for all $g \in G$ and

$$\vec{a} \cdot f_{J_{\vec{x}_1} \cap J_{\vec{y}_1}} \cdot \vec{x}_1 = \vec{d} \cdot f_{J_{\vec{x}_2} \cap J_{\vec{y}_2}} \cdot \vec{x}_2 \ \text{in } \mathbb{P}^{m-1}(E),$$

with $\vec{a} = \left(1, \mu_1, \mu_1^2, ..., \mu_1^{f-1}\right)^{\otimes e}$ and $\vec{d} = \left(1, \mu_2, \mu_2^2, ..., \mu_2^{f-1}\right)^{\otimes e}$, where $\mu_1 = \frac{\alpha_1}{\alpha_2}$ and $\mu_2 = \frac{\delta_1}{\delta_2}$, or

$$\left\{ \begin{array}{c} \alpha_1^f = \delta_2^f \\ \delta_1^f = \alpha_2^f \end{array} \right\}, \quad \left\{ \begin{array}{c} J_{\vec{x}_1} = J_{\vec{y}_2} \\ J_{\vec{y}_1} = J_{\vec{x}_2} \end{array} \right\}, \quad \left\{ \begin{array}{c} \chi_1(g) = \xi_2^{n(g)} \psi_2(g) \\ \chi_2(g) = \xi_1^{n(g)} \psi_1(g) \end{array} \right\}$$

for all $g \in G$ and

$$\vec{b} \cdot f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} = \vec{c} \cdot f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} \cdot \vec{x}_1 \cdot \vec{x}_2 \ \text{in } \mathbb{P}^{m-1}(E),$$

with $\vec{b} = \left(1, \xi_1, \xi_1^2, ..., \xi_1^{f-1}\right)^{\otimes e}$ and $\vec{c} = \left(1, \xi_2, \xi_2^2, ..., \xi_2^{f-1}\right)^{\otimes e}$, where $\xi_1 = \frac{\delta_1}{\alpha_2}$ and $\xi_2 = \frac{\alpha_1}{\delta_2}$.

6.4.3 The potentially semistable, noncrystalline case

If both the monodromies are nontrivial, then $(D_1, \varphi_1, N_1, L/K, E) \simeq (D_2, \varphi_2, N_2, L/K, E)$ if and only if

$$\left\{ \begin{array}{c} \alpha_1^f = \alpha_2^f \\ \alpha_1\delta_2 = \alpha_2\delta_1 \end{array} \right\}, \quad \left\{ \begin{array}{c} J_{\vec{x}_1} = J_{\vec{x}_2} \\ J_{\vec{y}_1} = J_{\vec{y}_2} \end{array} \right\}, \quad \left\{ \begin{array}{c} \chi_1(g) = \mu_1^{n(g)}\psi_1(g) \text{ for all } g \in G \text{ and} \\ f_{J_{\vec{x}_1} \cap\, J_{\vec{y}_1}} \cdot \vec{x}_1 = f_{J_{\vec{x}_1} \cap\, J_{\vec{y}_1}} \cdot \vec{x}_2 \text{ in } \mathbb{A}^m(E) \end{array} \right\},$$

where $\mu_1 = \frac{\alpha_1}{\alpha_2}$.

6.4.4 The F-scalar case

Let $[\varphi_i]_{\underline{\eta}^i} = \mathrm{diag}(\alpha_i \cdot \vec{1}, \alpha_i \cdot \vec{1})$ and $[g]_{\underline{\eta}^i} = \lambda_i(g) \cdot \mathrm{diag}(\vec{1}, \vec{1})$ for some group homomorphisms $\lambda_i : G \to GL_2(E)$, $i = 1, 2$. Then

$$(D_1, \varphi_1, L/K, E) \simeq (D_2, \varphi_2, L/K, E)$$

if and only if $\alpha_1^f = \alpha_2^f$ and there exists some matrix $R \in GL_2(E)$ such that $\lambda_2(g) = R\lambda_1(g)R^{-1}$ for all g and (with the notation of Section 6.2)

$$\left(E^{|S_L|}\right)\left(\vec{x}_1,\ f_{J_{\vec{y}_1}}\right) \cdot \left(R \cdot \mathrm{diag}(\vec{1}, \vec{1})\right) = \left(\ E^{|S_L|}\right)\left(\vec{x}_2,\ f_{J_{\vec{y}_2}}\right).$$

6.4.5 The non-F-semisimple case

Let

$$[\varphi_i]_{\underline{\eta}^i} = \begin{pmatrix} \alpha_i \cdot \vec{1} & \vec{0} \\ \vec{1} & \alpha_i \cdot \vec{1} \end{pmatrix} \text{ with } \alpha_i \in E^\times \text{and } [g]_{\underline{\eta}^i} = \mathrm{diag}(\chi_i(g) \cdot \vec{1}, \chi_i(g) \cdot \vec{1})$$

for some characters $\chi_i : G \to E^\times$. Then $(D_1, \varphi_1, L/K, E) \simeq (D_2, \varphi_2, L/K, E)$ if and only if

(1) $\alpha_1^f = \alpha_2^f$ and in case that $\alpha_1 \neq \alpha_2$, $\sum_{j=0}^{n(g)-1} \mu_1^{-2j} = 0$ for all $g \in G$, where $\mu_1 = \frac{\alpha_1}{\alpha_2}$;

(2) $\chi_1(g) = \mu_1^{n(g)} \cdot \chi_2(g)$ for all $g \in G$;

(3) $J_{\vec{x}_1} = J_{\vec{x}_2}$ and there exist $a \in E^\times$ and $c_0 \in E$ such that

$$f_{J_{\vec{x}}} \cdot \vec{x} \cdot \vec{x}_1 \cdot \vec{c}^{\otimes e} = \left(f_{J_{\vec{x}} \cap J_{\vec{y}_1}} \cdot \vec{x} - f_{J_{\vec{x}} \cap J_{\vec{y}_1}} \cdot \vec{x}_1\right) \cdot \vec{a}^{\otimes e} \text{ in } \mathbb{A}^m(E),$$

where $\vec{a} = a \cdot \left(1, \mu_1^{-1}, \mu_1^{-2}, ..., \mu_1^{-(f-1)}\right)$ and $\vec{c} = (c_0, c_1, ..., c_{f-1})$ with

$$c_i = \mu_1^i \left\{ (c_0 - a\mu_1^{-1} + a) - \sum_{j=1}^{i-1} \left(\mu_1^{-2j-1} - \mu_1^{-2j}\right) \right\} \text{ for } i = 1, 2, ..., f-1.$$

7 Some consequences for crystalline representations

Let K be any finite extension of $\mathbb{Q}_p$ of absolute ramification index e and absolute inertia degree f. We apply the results of the previous sections to study 2-dimensional crystalline E-representations of G_K. Let V be such a representation and let (D, φ) be the corresponding weakly admissible filtered φ-module. Recall that the map φ^f is $K_0 \otimes E$-linear. We call characteristic polynomial of V the characteristic polynomial of φ^f, and throughout this section we assume that V is F-semisimple, meaning that φ^f has the same property. Let $\underline{\eta}$ be a standard basis so that $[\varphi]_{\underline{\eta}} = \mathrm{diag}\Big(\alpha \cdot \vec{1}, \delta \cdot \vec{1}\Big)$ with $\alpha, \delta \in E^\times$ and $\alpha^f \neq \delta^f$, and let

$$\mathrm{Fil}^j(D_K) = \left\{ \begin{array}{c} D_K \ \text{ if } \ \ j \leq 0, \\ \left(E^{|S_K|_{I_r}}\right)(\vec{x} \cdot \eta_1 + \vec{y} \cdot \eta_2) \ \text{ if } \\ 1 + w_{r-1} \leq j \leq w_r, \ \text{for } r = 0, ..., t-1, \\ 0 \ \text{ if } \ \ j \geq 1 + w_{t-1}. \end{array} \right. \tag{7.1}$$

for some vectors $\vec{x}, \vec{y} \in E^m$, where m is the degree of K over $\mathbb{Q}_p$, whose coordinates do not vanish simultaneously. In practice it is often desirable to allow for a more flexible shape of Frobenius, at the cost of adding extra rigidity to the filtrations. By Remark 3.1 we may assume that $\vec{y} = f_{J_{\vec{y}}}$, and by considering the ordered basis $\underline{\zeta} = (\zeta_1, \zeta_2)$ with $\zeta_1 = (\sum\limits_{i \in J'_{\vec{x}}} e_{\tau_i} + \sum\limits_{i \in J_{\vec{x}}} x_i^{-1} e_{\tau_i})\eta_1$ and $\zeta_2 = \eta_2$, we may further assume that $\vec{x} = f_{J_{\vec{x}}}$ and $\vec{y} = f_{J_{\vec{y}}}$. In such a basis the matrix of Frobenius remains diagonal of the form $[\varphi]_{\underline{\zeta}} = \mathrm{diag}\Big(\vec{\alpha}, \vec{\delta}\Big)$ for some vectors $\vec{\alpha}, \vec{\delta} \in (E^\times)^{|S_{K_0}|}$ with $Nm_\varphi(\vec{\alpha}) \neq Nm_\varphi(\vec{\delta})$. The results of Section 6.4.2 take the form of the following proposition.

Proposition 7.1 *Let (D_i, φ_i) be filtered φ-modules with $[\varphi_i]_{\underline{\eta}^i} = diag(\vec{\alpha}_i, \vec{\delta}_i)$, $i = 1, 2$ and filtrations as in Section 6.4, with $\vec{x}_i = f_{J_{\vec{x}_i}}$ and $\vec{y}_i = f_{J_{\vec{y}_i}}$, $i = 1, 2$. The F-semisimple filtered φ-modules (D_i, φ_i) are isomorphic if and only if either*

$$\left\{ \begin{array}{c} Nm_\varphi(\vec{\alpha}_1) = Nm_\varphi(\vec{\alpha}_2), \\ Nm_\varphi(\vec{\delta}_1) = Nm_\varphi(\vec{\delta}_2) \end{array} \right\}, \ \left\{ \begin{array}{c} J_{\vec{x}_1} = J_{\vec{x}_2}, \\ J_{\vec{y}_1} = J_{\vec{y}_2} \end{array} \right\}$$

and $f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} \cdot \vec{a} = f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} \cdot \vec{d}$ viewed in the projective space $\mathbb{P}^{m-1}(E)$, where

$$\vec{a} = \left(1, \frac{\alpha_0^1}{\alpha_0^2}, \frac{\alpha_0^1\alpha_1^1}{\alpha_0^2\alpha_1^2}, \ldots, \frac{\alpha_0^1\alpha_1^1\cdots\alpha_{f-2}^1}{\alpha_0^2\alpha_1^2\cdots\alpha_{f-2}^2}\right)^{\otimes e} \ \ \textit{and} \ \vec{d} = \left(1, \frac{\delta_0^1}{\delta_0^2}, \frac{\delta_0^1\delta_1^1}{\delta_0^2\delta_1^2}, \ldots, \frac{\delta_0^1\delta_1^1\cdots\delta_{f-2}^1}{\delta_0^2\delta_1^2\cdots\delta_{f-2}^2}\right)^{\otimes e},$$

or

$$\left\{ \begin{array}{c} Nm_\varphi(\vec{\alpha}_1) = Nm_\varphi(\vec{\delta}_2), \\ Nm_\varphi(\vec{\delta}_1) = Nm_\varphi(\vec{\alpha}_2) \end{array} \right\}, \ \left\{ \begin{array}{c} J_{\vec{x}_1} = J_{\vec{y}_2}, \\ J_{\vec{y}_1} = J_{\vec{x}_2} \end{array} \right\}$$

and $f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} \cdot \vec{b} = f_{J_{\vec{x}_1} \cap \ J_{\vec{y}_1}} \cdot \vec{c}$ viewed in the projective space $\mathbb{P}^{m-1}(E)$, where

$$\vec{b} = \left(1, \frac{\delta_0^1}{\alpha_0^2}, \frac{\delta_0^1\delta_1^1}{\alpha_0^2\alpha_1^2}, \ldots, \frac{\delta_0^1\delta_1^1\cdots\delta_{f-2}^1}{\alpha_0^2\alpha_1^2\cdots\alpha_{f-2}^2}\right)^{\otimes e} \ \ \textit{and} \ \vec{c} = \left(1, \frac{\alpha_0^1}{\delta_0^2}, \frac{\alpha_0^1\alpha_1^1}{\delta_0^2\delta_1^2}, \ldots, \frac{\alpha_0^1\alpha_1^1\cdots\alpha_{f-2}^1}{\delta_0^2\delta_1^2\cdots\delta_{f-2}^2}\right)^{\otimes e}.$$

If all the k_i are 0, any equation involving the sets $J_{\vec{x}_i}, \ J_{\vec{y}_i}$ should be ignored.

The two cases of Proposition 7.1 occur due to the isomorphism of any rank two filtered module which swaps its basis elements. For our current normalization the results of Section 5.2.1 should be slightly modified: One should only replace $efv_p(\alpha\delta)$ by $ev_p(Nm_\varphi(\vec{\alpha})Nm_\varphi(\vec{\delta}))$, $efv_p(\alpha)$ by $ev_p(Nm_\varphi(\vec{\alpha}))$ and $ev_p(\delta)$ by $ev_p(Nm_\varphi(\vec{\delta}))$, where for a vector $\vec{a}$ we denote by $v_p(Nm_\varphi(\vec{a}))$ the valuation of the product of its coordinates. For the rest of the section we assume that our bases are standard with Frobenius as in Proposition 7.1 and filtrations as in (7.1) with $\vec{x} = f_{J_{\vec{x}}}$ and $\vec{y} = f_{J_{\vec{y}}}$. To avoid trivialities we assume that at least one of the non negative weights k_i is strictly positive. The following corollary follows easily.

Corollary 7.2 *Let (D, φ) be an F-semisimple, weakly admissible filtered φ-module of rank two over $K_0 \otimes E$ with labeled Hodge-Tate weights $(\{-k_i, 0\})_{\sigma_i}$.*

(1) *If $Tr(\varphi^f) \in \mathcal{O}_E^\times$ then the corresponding crystalline representation is reducible;*

(2) *There exist infinite families of weakly admissible non isomorphic F-semisimple rank two filtered φ-modules sharing the same characteristic polynomial and filtration with (D, φ) if and only if $\mid J_{\vec{x}} \cap J_{\vec{y}} \mid > 1$.*

Let $k := \sum_{i=0}^{m-1} k_i$, and let $\pi \in E^\times$ be an e-th root of p. Let $\alpha \in m_E$ with $\alpha^2 \neq 4\pi^k$ so that the roots $\varepsilon_0, \varepsilon_1$ of $X^2 - \alpha X + \pi^k$ be distinct. Consider the rank two filtered φ-modules $D\left(\vec{\lambda}, \vec{\mu}\right)$, with $\vec{\lambda}, \vec{\mu} \in (E^\times)^{f-1}$, with Frobenius endomorphisms given by

$$[\varphi]_{\underline{\eta}} = \mathrm{diag}\left(\left(\lambda_0, \lambda_1, ..., \lambda_{f-2}, \frac{\varepsilon_0}{\lambda_0\lambda_1 \cdots \lambda_{f-2}}\right), \left(\mu_0, \mu_1, ..., \mu_{f-2}, \frac{\varepsilon_1}{\mu_0\mu_1 \cdots \mu_{f-2}}\right)\right),$$

and filtrations as in (7.1) with $\vec{x} = \vec{y} = \vec{1}$. We have the following corollary.

Corollary 7.3 (1) *For any $\vec{\lambda}, \vec{\mu} \in (E^\times)^{f-1}$, the filtered modules $D\left(\vec{\lambda}, \vec{\mu}\right)$ are irreducible and weakly admissible;*

(2) $D\left(\vec{\lambda}, \vec{\mu}\right) \simeq D\left(\vec{\lambda}_1, \vec{\mu}_1\right)$ *if and only if $\vec{\lambda} \cdot \vec{\mu}_1 = \vec{\lambda}_1 \cdot \vec{\mu}$;*

(3) *The filtered modules $D\left(\vec{1}, \vec{\mu}\right)$ with $\vec{\mu} \in (E^\times)^{f-1}$ are representatives of the distinct isomorphism classes of all rank two weakly admissible filtered modules with fixed characteristic polynomial $X^2 - \alpha X + \pi^k$ and filtration as in (7.1), with $\vec{x} = \vec{y} = \vec{1}$.*

Corollary 7.4 *If $K \neq \mathbb{Q}_p$ there exist (infinitely many) disjoint infinite families of irreducible 2-dimensional crystalline E-representations of G_K, sharing the same characteristic polynomial and filtration.*

Appendix

The potentially crystalline $E^\times$-valued characters of G_K. Let $k_0, k_1, ..., k_{m-1}$ be arbitrary integers. Assume that there exists $\varpi \in E^\times$ such that $\varpi^{em} = p^{\sum_{i=0}^{m-1} k_i}$. The weakly admissible rank one filtered $(\varphi, L/K, E)$-modules with labeled Hodge-Tate weights $(-k_i)_{\sigma_i}$ are of the form $D = (\prod_{S_{L_0}} E)\eta$ with $\varphi(\eta) = u(\varpi, \varpi, ..., \varpi)\eta$ for some $u \in E^\times$ with $v_p(u) = 0$ and, $g(\eta) = (\chi(g) \cdot \vec{1})\eta$ for some character

$\chi : \mathrm{Gal}(L/K) \to E^\times$. Their filtrations are given by

$$\mathrm{Fil}^j(D_L) = \begin{cases} \left(E^{|S_L|}\right)(1 \otimes \eta) & \text{if } j \le w_0, \\ \left(E^{|S_L|_{I_1}}\right)(1 \otimes \eta) & \text{if } \ 1 + w_0 \le j \le w_1, \\ \qquad \cdots\cdots\cdots & \\ \left(E^{|S_L|_{I_{t-1}}}\right)(1 \otimes \eta) & \text{if } 1 + w_{t-2} \le j \le w_{t-1}, \\ \quad 0 & \text{if } j \ge 1 + w_{t-1}, \end{cases}$$

where the sets I_r are unions of $\mathrm{Gal}(L/K)$-orbits for all r. Denote such a filtered module by (D_u, χ). Then $(D_u, \chi) \simeq (D_v, \psi)$ if and only if (i) $u^f = v^f$ and (ii) $\chi(g) = \varepsilon^{n(g)}\psi(g)$ for all $g \in G$, where $\varepsilon = uv^{-1}$.

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